FIFTH EDITION

Physics

RESNICK · HALLIDAY · KRANE

Volume 1

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Instructor's Manual by J. RICHARD CHRISTMAN, U.S. Coast Guard Academy. This manual includes suggested syllabi, lecture notes, a list of the problems that appear in the Student Solutions Manual, a complete list of answers to the problems, a comparison of the problems with the Fourth Edition, and a list of computer projects.

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Student Solutions Manual by PAUL STANLEY, California Lutheran University. This manual provides students with complete worked-out solutions to 25 percent of the problems found at the end of each chapter of the text.

VOLUME ONE

PHYSICS

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Fifth Edition

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PREFACE TO VOLUME 1

This is the fifth edition of the textbook first published in 1960 as *Physics for Students of Science and Engineering* by David Halliday and Robert Resnick. For four decades this book has provided the standard for the calculus-based introductory survey course and has been known for the clarity and completeness of its presentation. In the present edition we have striven to increase accessibility without sacrificing the level or the rigor of its content. The text has been substantially rewritten to make the material flow more smoothly and to ease the student's entry into new subjects. We have attempted to provide more practical examples and to proceed from the particular to the general when new topics are introduced.

This edition features significant changes in the pedagogy as well as in the order of the chapters. Those who are familiar with the fourth edition of this text will find the same topics but in a revised order. In making these revisions, we have sought the advice of users of past editions and have taken into consideration the results of physics education research. Among the changes we have made in this edition are the following:

1. We have continued the effort (begun in the previous edition) to achieve a more coherent approach to energy, especially one that bridges the gap between mechanics and thermodynamics. The need for a new approach to energy has been indicated from a variety of sources. Persistent student difficulties with energy concepts have been revealed through physics education research (for example, see the work of Lillian McDermott and co-workers*). The need to promote a greater understanding of Newton's laws has led Priscilla Laws** to propose a re-ordering of topics in introductory mechanics in which conservation of mechanical energy is introduced only after a full study of vector mechanics, including systems of particles and momentum conservation. A survey pointing out some difficulties with the conventional presentations of energy conservation has been given by Arnold Arons.*** Based in part on these ideas, in this edition we have chosen to develop the energy concept following the presentation of vector mechanics (in both translational and rotational forms). This approach allows for a more unified and coherent treatment of energy and the law of conservation of energy, and it also permits a "spiral" approach in which we can apply energy techniques to problems already solved using laws of vector mechanics. Energy concepts are introduced in this edition in Chapters 11-13, which then provide the critical background necessary for the extensive use of energy and its conservation in the remainder of this volume.

2. The chapter on vectors in the fourth edition has been eliminated. Instead, vector techniques are introduced as needed, beginning with vector addition and components of vectors in Chapter 2 (kinematics) and continuing with the cross product in Chapters 8 and 9 (rotational kinematics and dynamics) and the dot product in Chapter 11 (work and energy). In this way students find presentations of vector techniques as they are needed and immediately applied. In each case we have provided end-of-chapter exercises to help students become familiar with the concepts and techniques. A new appendix gives a summary of important vector concepts and formulas.

3. Again based in part on the findings of Priscilla Laws and other physics education researchers, we have changed the ordering of introductory topics to: one-dimensional kinematics, one-dimensional dynamics, and then two-dimensional kinematics and dynamics. We need not reproduce here the many arguments that support this change, but we feel that at minimum it helps to deal with the persistent student confusion in associating acceleration with velocity rather than with

*"Student Understanding of the Work-Energy and Impulse-Momentum Theorems," by Ronald A. Lawson and Lillian C. McDermott, *American Journal of Physics,* September 1987, p. 811.

**"A New Order for Mechanics," by Priscilla W. Laws, in *Conference on the Introductory Physics Course,* John Wiley & Sons, 1997, p. 125.

***"Development of Energy Concepts in Introductory Physics Courses," by Arnold Arons, *American Journal of Physics,* December 1999, p. 1063; see also *Teaching Introductory Physics,* by Arnold Arons, John Wiley & Sons, 1997, chapter 5.

force; for example, our new ordering allows us to introduce centripetal force upon the first presentation of uniform circular motion (rather than one or two chapters later, as in the previous ordering), and it allows the association between gravitational force and gravitational acceleration to be made at an earlier stage to dispel some of the errors that students commonly make in identifying the magnitude and direction of the acceleration in projectile motion.

4. The chapter on oscillations, which preceded gravitation and fluid mechanics in the previous edition, now follows those topics and serves as a natural introduction to wave motion.

5. The material in the fourth edition on equilibrium (Chapter 14) has been largely incorporated into the chapter on rotational dynamics (Chapter 9) in the present edition.

6. Thermodynamics, which occupied five chapters in the previous edition, has been recast into four chapters in this edition. A new chapter (22) on the molecular properties of gases incorporates topics from kinetic theory and statistical mechanics (Chapters 23 and 24 of the fourth edition) as they relate to the properties of the ideal gas. Topics relating to work and energy in the ideal gas then fall naturally into Chapter 23 of this edition (the first law of thermodynamics). Chapter 24 (entropy and the second law) differs considerably from the corresponding chapter in the fourth edition in that here we give entropy its appropriate and more prominent role as fundamental to an understanding of the second law.

7. In the fourth edition, topics from modern physics were "sprinkled" throughout the text, generally in sections labeled as "optional." In this edition we continue to use examples from modern physics where appropriate throughout the text, but the separate sections on modern physics have been consolidated into Chapter 20 (special relativity) in this volume and Chapters 45-52 in volume 2 (which treat topics from quantum physics and its applications to atoms, solids, and nuclei). We strongly believe that relativity and quantum physics are essential parts of an introductory survey course at this level, but that justice to these subjects is done better by a coherent, unified presentation rather than a collection of isolated expositions. As was the case in the fourth edition, we continue to place the chapter on special relativity among the classical mechanics chapters in volume 1, which reflects our strong belief that special relativity belongs squarely among the kinematics and mechanics chapters dealing with classical physics. (However, instructors who with to delay the presentation of this material can easily postpone coverage of Chapter 20 until later in the course.)

The end-of-chapter material in this edition differs significantly from that of the previous edition. The previous problem sets (which were all keyed to chapter sections) have been carefully edited and placed into two groups: exercises and problems. Exercises, which are keyed to text sections, generally represent direct applications of the material in the associated section. Their purpose is usually to help students become familiar with the concepts, important formulas, units and dimensions, and so forth. Problems, which are not keyed to text sections, often require use of concepts from different sections or even from previous chapters. Some problems call for the student to estimate or independently locate the data needed to solve the problem. In editing and grouping the exercises and problems, we have also eliminated some problems from the previous edition. Within the next year we shall offer a problem supplement that will incorporate most of the missing problems as well as a selection of new exercises and problems. As before, answers to odd-numbered exercises and problems are given in the text and those to the even-numbered exercises and problems can be found in the instructor's manual that accompanies the text.

Multiple-choice questions and computer problems have also been added to the end-of-chapter material. The multiple-choice questions are generally conceptual in nature and often call for unusual insights into the material. Answers to the multiple-choice questions can be found in the instructor's manual. The computer problems may require familiarity with spread-sheet techniques or with symbolic manipulation routines such as Maple or Mathematica.

We have striven to develop a textbook that offers as complete and rigorous a survey of introductory physics as is possible at this level. It is, however, important to assert that *few (if any) instructors will want to follow the entire text from start to finish,* especially in a one-year course. There are many alternate pathways through this text. The instructor who wishes to treat fewer topics in greater depth (often called the "less is more" approach) will be able to select from among these pathways. Some sections or subsections are explicitly labeled as "optional," indicating that they can be skipped without loss of continuity. Depending on the course design, other sections or even entire chapters can be skipped or treated lightly. The Instructor's Manual, available as a companion volume, offers suggestions for abbreviating the coverage. Even so, the complete presentation remains in the text where the curious student can seek out the omitted topics and be rewarded with a broader view of the subject. We hope that the text can thus be regarded as a sort of "road map" through physics; many roads, scenic or direct, can be taken, and all roads need not be utilized on the first journey. The eager traveler may be encouraged to return to the map to explore areas missed on previous journeys.

The text is available in two volumes. The present volume covers kinematics, mechanics, and thermodynamics; volume 2 covers electromagnetism, optics, and quantum physics and its applications. Supplements available include:

Instructor's Solutions Manual	Student Solutions Manual
Instructor's Manual	Student Study Guide
Instructor's Resource CD	Physics Simulations
Test Bank	eGrade Homework Management System

In preparing this edition, we have benefitted from the advice of a dedicated team of reviewers who have, individually or collectively, carefully offered comments and criticisms on nearly every page of the text:

Richard Bukrey, Loyola University
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Harvey Picker, Trinity College
Anthony Pitucco, Pima Community College
Robert Scherrer, Ohio State University
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John Toutonghi, Seattle University

We are deeply indebted to these individuals for their efforts and for the insights they have provided to the authors. We would also like to acknowledge the advice of the Physics Education Group at the University of Washington, especially Paula Heron and Lillian McDermott.

We would like to extend special appreciation to two individuals whose tireless efforts and exceptional contributions have been essential to the success of this project and who have set high standards for the quality of the finished product. J. Richard Christman has been a long-time contributor whose careful review of the text and contributions to the supplements have now extended over three editions. His insistence on careful explanations and correct pedagogy throughout the text has in a multitude of instances kept us on the proper track. Paul Stanley is a new addition to the team whose primary responsibility has been the end-of-chapter questions and problems. He has brought to the project a wealth of creative ideas and clever insights that will challenge students (as well as instructors) to extend their understanding of the material.

The staff at John Wiley & Sons has provided constant support for this project, for which we are exceptionally grateful. We would especially like to thank Stuart Johnson for his management of this project and his dedication to its completion. Essential contributions to the quality of this text have been made by production editor Elizabeth Swain, photo editor Hilary Newman, illustration editor Anna Melhorn, and designer Karin Kincheloe. Without the skill and efforts of these individuals this project would not have been possible.

Despite the best efforts of authors, reviewers, and editors, it is inevitable that errors may appear in the text, and we welcome communication from users with corrections or comments on the content or pedagogy. We read all of these communications and respond to as many as possible, but we regret not being able to respond to all of them. Nevertheless, we encourage readers' comments, which can be sent to www.wiley.com/college/hrk.

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CHAPTER 1

MEASUREMENT

Despite the mathematical beauty of some of its most complex and abstract theories, physics is above all an experimental science. It is therefore critical that those who make precise measurements be able to agree on standards in which to express the results of those measurements, so that they can be communicated from one laboratory to another and verified.

In this chapter we begin our study of physics by introducing some of the basic units of physical quantities and the standards that have been accepted for their measurement. We consider the proper way to express the results of calculations and measurements, including the appropriate dimensions and number of significant figures. We discuss and illustrate the importance of paying attention to the dimensions of the quantities that appear in our equations. Later in the text, other basic units and many derived units are introduced as they are needed.

1-1 PHYSICAL QUANTITIES, STANDARDS, AND UNITS

The laws of physics are expressed in terms of many different quantities: mass, length, time, force, speed, density, resistance, temperature, luminous intensity, magnetic field strength, and many more. Each of these terms has a precise meaning, and they form part of the common language that physicists and other scientists use to communicate with one another—when a physicist uses a term such as "kinetic energy," all other physicists will immediately understand what is meant. Each of these terms also represents a quantity that can be measured in the laboratory, and just as there must be agreement on the meaning of these terms, there must also be agreement about the units used to express their values. Without such agreement, it would not be possible for scientists to communicate their results to one another or to compare the results of experiments from different laboratories.

Such comparisons require the development and acceptance of a set of *standards* for units of measurement. For example, if a measurement of length is quoted as 4.3 meters, it means that the measured length is 4.3 times as long as the value accepted for a standard length defined to be "one meter." If two laboratories base their measurements on the same accepted standard for the meter, then presumably their results can be easily compared. For this to be possible, the accepted standards must be *accessible* to those who need to calibrate their secondary standards, and they must be *invariable* to change with the passage of time or with changes in their environment (temperature, humidity, etc.).

Maintaining and developing standards for measurement is an active branch of science. In the United States, the National Institute of Standards and Technology* (NIST) has the primary responsibility for this development. However, it is also necessary to have wide international agreement about standards, which has been accomplished through a series of international meetings of the General Conference on Weights and Measures (known by their French acroynm

* See http://physics.nist.gov/cuu for information about NIST's role in maintaining standards.

CGPM) beginning in 1889; the twenty-first meeting was held in 1999.*

Fortunately, it is not necessary to establish a measurement standard for every physical quantity—some quantities can be regarded as fundamental, and the standards for other quantities can be derived from the fundamental ones. For example, length and time were once regarded as fundamental quantities with their individual established standards (respectively the meter and the second); the measurement standard for speed (= length/time) could then be derived in terms of those standards. However, in more recent years the speed of light has been measured to a precision exceeding that of the former standard meter; as a result, today we still use a fundamental standard for the second, but we define the standard for length (the meter) in terms of the speed of light and the second (see Section 1-4). This case illustrates how measurements of increasing precision can change the established standards and how rapidly such standards evolve. Since the publication of the first edition of this textbook, the precision of the standard unit for time (the second) has improved by more than a factor of 1000.

The basic problem therefore is to choose a system involving the smallest number of physical quantities as fundamental and to agree on accessible and invariable standards for their measurement. In the next sections of this chapter we discuss the internationally accepted system and some of its fundamental quantities.

1-2 THE INTERNATIONAL SYSTEM OF UNITS**

At its various meetings, the General Conference on Weights and Measures selected as *base units* the seven quantities displayed in Table 1-1. This is the basis of the International System of Units, abbreviated SI from the French *Le Système International d'Unités.* SI is the modern form of what is known generally as the metric system.

Throughout the book we give many examples of SI derived units, such as speed, force, and electric resistance, that follow from Table 1-1. For example, the SI unit of force, called the *newton* (abbreviation N), is defined in terms of the SI base units as

$$1\ \text{N} = 1\ \text{kg}\cdot\text{m/s}^2,$$

as we shall make clear in Chapter 3.

If we express physical properties such as the output of a power plant or the time interval between two nuclear events in SI units, we often find very large or very small numbers. For convenience, the General Conference on Weights and Measures recommended the prefixes shown in Table 1-2.

TABLE 1-1 SI Base Units

	SI Unit	
Quantity	*Name*	*Symbol*
Time	second	s
Length	meter	m
Mass	kilogram	kg
Amount of substance	mole	mol
Thermodynamic temperature	kelvin	K
Electric current	ampere	A
Luminous intensity	candela	cd

Thus we can write the output of a typical electrical power plant, 1.3×10^9 watts, as 1.3 gigawatts or 1.3 GW. Similarly, we can write a time interval of the size often encountered in nuclear physics, 2.35×10^{-9} seconds, as 2.35 nanoseconds or 2.35 ns. As Table 1-1 shows, the kilogram is the only SI base unit that *already* incorporates one of the prefixes displayed in Table 1-2. Thus 10^3 kg is *not* expressed as 1 kilokilogram; instead, $10^3\ \text{kg} = 10^6\ \text{g} = 1\ \text{Mg}$ (megagram).

To fortify Table 1-1 we need seven sets of operational procedures that tell us how to produce the seven SI base units in the laboratory. We explore those for time, length, and mass in the next three sections.

Two other major systems of units compete with the International System (SI). One is the Gaussian system, in terms of which much of the literature of physics is expressed. We do not use the Gaussian system in this book. Appendix G gives conversion factors to SI units.

The other is the British system, still in daily use in the United States. The basic units, in mechanics, are length (the foot), force (the pound), and time (the second). Again Appendix G gives conversion factors to SI units. We use SI units in this book, but we sometimes give the British equivalents, to help those who are unaccustomed to SI units to acquire more familiarity with them. The United States continues to be the only developed country that, so far, has not adopted SI as its official unit system. However, SI is standard in all U.S. government laboratories and in many industries, especially those involved in foreign trade. The loss of the *Mars Climate Orbiter* spacecraft in September 1999 has

TABLE 1-2 SI Prefixes[a]

Factor	*Prefix*	*Symbol*	*Factor*	*Prefix*	*Symbol*
10^{24}	yotta-	Y	10^{-1}	deci-	d
10^{21}	zetta-	Z	10^{-2}	**centi-**	c
10^{18}	exa-	E	10^{-3}	**milli-**	m
10^{15}	peta-	P	10^{-6}	**micro-**	μ
10^{12}	tera-	T	10^{-9}	**nano-**	n
10^{9}	**giga-**	G	10^{-12}	**pico-**	p
10^{6}	**mega-**	M	10^{-15}	femto-	f
10^{3}	**kilo-**	k	10^{-18}	atto-	a
10^{2}	hecto-	h	10^{-21}	zepto-	z
10^{1}	deka-	da	10^{-24}	yocto-	y

[a] In all cases, the first syllable is accented, as in na′-no-me′-ter. Prefixes commonly used in this book are shown in boldface type.

* See http://www.bipm.fr for the recommendations of this conference.

** See "SI: The International System of Units," by Robert A. Nelson (American Association of Physics Teachers, 1981). The "official" U.S. guide to the SI system can be found in Special Publication 811 of the National Institute of Standards and Technology (1995 edition).

been traced to the fact that the manufacturer reported some of the *Orbiter*'s characteristics in British units, which the NASA navigation team mistakenly took to be SI units. Careful attention to units can be very important!

SAMPLE PROBLEM 1-1. Any physical quantity can be multiplied by 1 without changing its value. For example, 1 min = 60 s, so 1 = 60 s/1 min; similarly, 1 ft = 12 in., so 1 = 1 ft/12 in. Using appropriate conversion factors, find (*a*) the speed in meters per second equivalent to 55 miles per hour, and (*b*) the volume in cubic centimeters of a tank that holds 16 gallons of gasoline.

Solution (*a*) For our conversion factors, we need (see Appendix G) 1 mi = 1609 m (so that 1 = 1609 m/1 mi) and 1 h = 3600 s (so 1 = 1 h/3600 s). Thus

$$\text{speed} = 55\,\frac{\cancel{\text{mi}}}{\cancel{\text{h}}} \times \frac{1609\ \text{m}}{1\ \cancel{\text{mi}}} \times \frac{1\ \cancel{\text{h}}}{3600\ \text{s}} = 25\ \text{m/s}.$$

(*b*) One fluid gallon is 231 cubic inches, and 1 in. = 2.54 cm. Thus

$$\text{volume} = 16\ \cancel{\text{gal}} \times \frac{231\ \cancel{\text{in.}}^3}{1\ \cancel{\text{gal}}} \times \left(\frac{2.54\ \text{cm}}{1\ \cancel{\text{in.}}}\right)^3 = 6.1 \times 10^4\ \text{cm}^3.$$

Note in these two calculations how the unit conversion factors are inserted so that the unwanted units appear in one numerator and one denominator, and thus cancel.

1-3 THE STANDARD OF TIME

The measurement of time has two aspects. For civil and for some scientific purposes we want to know the time of day so that we can order events in sequence. In most scientific work we want to know how long an event lasts (the time interval). Thus any time standard must be able to answer the questions "At what time does it occur?" and "How long does it last?" Table 1-3 shows the range of time intervals that can be measured. They vary by a factor of about 10^{63}.

We can use any phenomenon that repeats itself as a measure of time. The measurement consists of counting the repetitions, including the fractions thereof. We could use an oscillating pendulum, a mass–spring system, or a quartz crystal, for example. Of the many repetitive phenomena in nature the rotation of the Earth on its axis, which determines the length of the day, was used as a time standard for centuries. One (mean solar) second was defined to be 1/86,400 of a (mean solar) day.

Quartz crystal clocks based on the electrically sustained periodic vibrations of a quartz crystal serve well as secondary time standards. A quartz clock can be calibrated against the rotating Earth by astronomical observations and used to measure time in the laboratory. The best of these have kept time with a precision of about 1 second in 200,000 years, but even this precision is not sufficient for the demands of modern science, technology, and commerce.

In 1967, the 13th General Conference on Weights and Measures adopted a definition of the second based on a characteristic frequency of the radiation emitted by a cesium atom. In particular, they stated that

The second is the duration of 9,192,631,770 vibrations of a (specified) radiation emitted by a (specified) isotope of the cesium atom.

Figure 1-1 shows the current national frequency standard, a so-called *cesium fountain clock* developed at the National Institute of Standards and Technology (NIST). Its precision is about 1 second in 20 million years.

TABLE 1-3 Some Measured Time Intervals[a]

Time Interval	*Seconds*
Lifetime of proton	$>10^{40}$
Half-life of double beta decay of ^{82}Se	3×10^{27}
Age of universe	5×10^{17}
Age of pyramid of Cheops	1×10^{11}
Human life expectancy (U.S.)	2×10^{9}
Time of Earth's orbit around the Sun (1 year)	3×10^{7}
Time of Earth's rotation about its axis (1 day)	9×10^{4}
Period of typical low-orbit Earth satellite	5×10^{3}
Time between normal heartbeats	8×10^{-1}
Period of concert-A tuning fork	2×10^{-3}
Period of oscillation of 3-cm microwaves	1×10^{-10}
Typical period of rotation of a molecule	1×10^{-12}
Shortest light pulse produced (1990)	6×10^{-15}
Lifetime of least stable particles	$<10^{-23}$

[a] Approximate values.

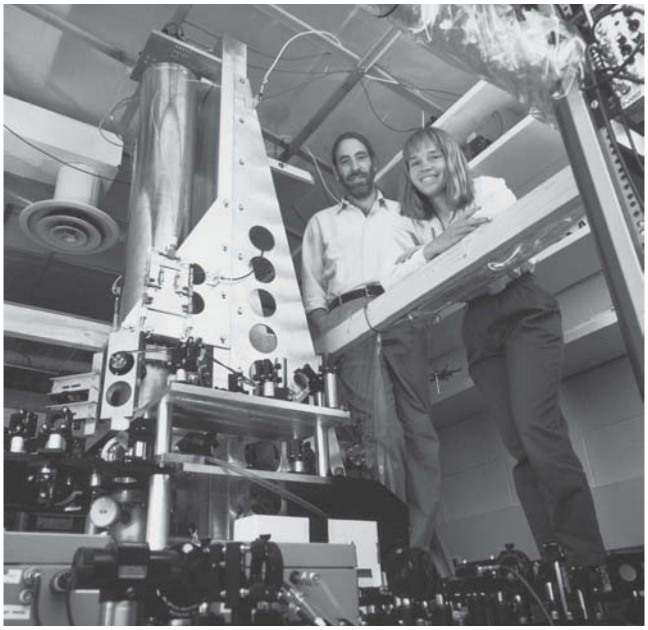


FIGURE 1-1. The National Frequency Standard NIST-F1, a so-called cesium fountain clock, developed at the National Institute of Standards and Technology. It is shown with its developers, Steve Jefferts and Dawn Meekhof. In this device extremely slow-moving cesium atoms are projected upward, covering a distance of about a meter before falling back under gravity to their launch position in about 1 second. Hence the label "fountain." The small speeds of these projected atoms make possible precise observation of the frequency of the atomic radiation that they emit. For more information, see http://www.nist.gov/public_affairs/releases/n99-22.htm.

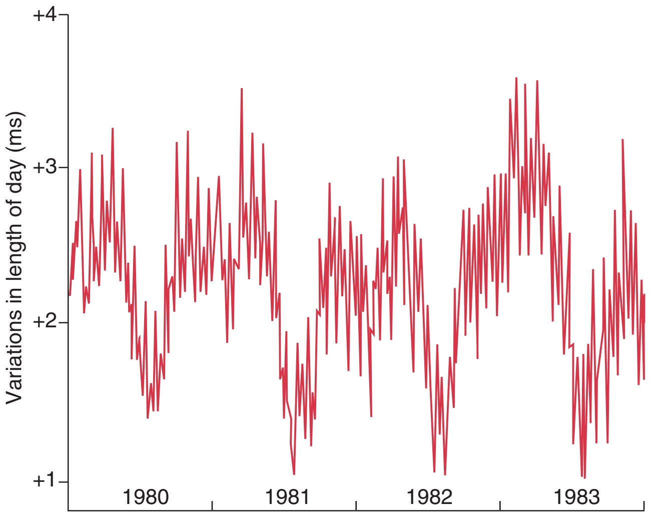


FIGURE 1-2. The variation in the length of the day over a 4-year period. Note that the vertical scale is only 3 ms = 0.003 s. See "The Earth's Rotation Rate," by John Wahr, *American Scientist,* January–February 1985, p. 41.

Cesium clocks housed in satellites form the basis of the Global Positioning System. Portable cesium clocks the size of a suitcase are commercially available. It is also possible to purchase desk-top clocks or wrist watches that, automatically and periodically updated by radio time signals from NIST, display "atomic time." Figure 1-2 shows, by comparison with a cesium clock, variations in the rate of rotation of the Earth over a 4-year period. These data suggest what a relatively poor time standard the Earth's rotation rate provides for precise work. Figure 1-3 shows the impressive record of improvements in time-keeping that have occurred over the past 300 years or so, starting with the invention of the pendulum clock by Christian Huygens in 1665.

The maintenance of timekeeping standards in the United States is the responsibility of the U.S. Naval Observatory (USNO) in Washington, DC. The USNO Master Clock represents the combined output of an assembly of cesium clocks and hydrogen masers housed in 20 separate, environmentally controlled vaults.*

1-4 THE STANDARD OF LENGTH**

The first international standard of length was a bar of a platinum–iridium alloy called the standard meter, which was kept at the International Bureau of Weights and Measures near Paris. The distance between two fine lines engraved near the ends of the bar, when the bar was held at a temperature of 0°C and supported mechanically in a prescribed way, was defined to be one meter. Historically, the meter was intended to be one ten-millionth of the distance from the north pole to the equator along the meridian line through Paris. However, accurate measurements showed that the standard meter bar differs slightly (about 0.023%) from this value.

Because the standard meter is not very accessible, accurate master copies of it were made and sent to standardized laboratories throughout the world. These secondary standards were used to calibrate other, still more accessible, measuring rods. Thus, until recently, every measuring rod or device derived its authority from the standard meter through a complicated chain of comparisons using micro-

* Information about time services provided by the USNO is available on the Internet at http://tycho.usno.navy.mil/ and by telephone at (202) 762-1401.

** See "The New Definition of the Meter," by P. Giacomo, *American Journal of Physics,* July 1984, p. 607.

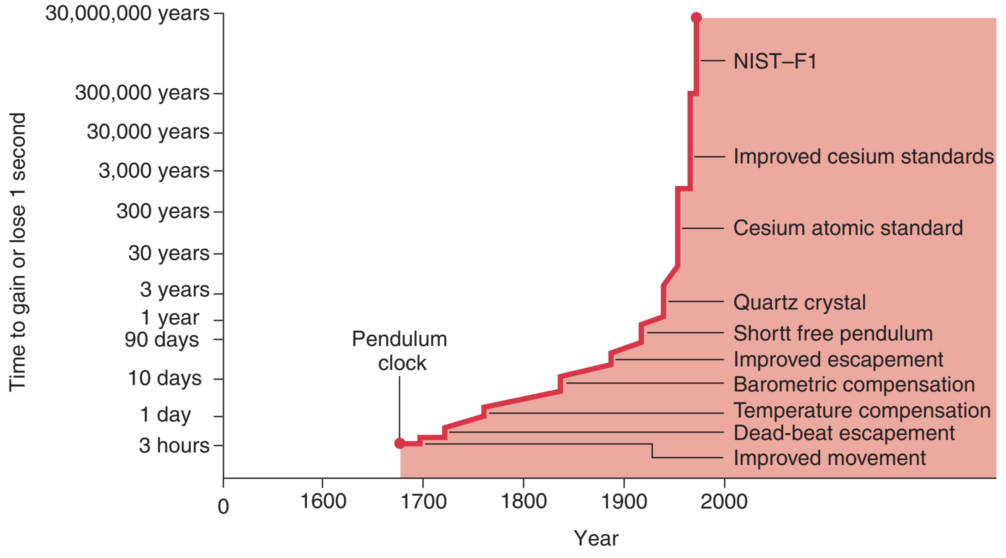


FIGURE 1-3. The improvement in timekeeping over the centuries. Early pendulum clocks gained or lost a second every few hours; present cesium clocks would do so only after several million years.

scopes and dividing engines. Since 1959 this statement had also been true for the yard, whose legal definition in the United States was adopted in that year to be

$$1 \text{ yard} = 0.9144 \text{ meter} \qquad \text{(exactly)},$$

which is equivalent to

$$1 \text{ inch} = 2.54 \text{ centimeters} \qquad \text{(exactly)}.$$

The accuracy with which the necessary intercomparisons of length can be made by the technique of comparing fine scratches using a microscope is no longer satisfactory for modern science and technology. A more precise and reproducible standard of length was obtained when the American physicist Albert A. Michelson in 1893 compared the length of the standard meter with the wavelength of the red light emitted by atoms of cadmium. Michelson carefully measured the length of the meter bar and found that the standard meter was equal to 1,553,163.5 of those wavelengths. Identical cadmium lamps could easily be obtained in any laboratory, and thus Michelson found a way for scientists around the world to have a precise standard of length without relying on the standard meter bar.

Despite this technological advance, the metal bar remained the official standard until 1960, when the 11th General Conference on Weights and Measures adopted an atomic standard for the meter. This standard was based on the wavelength in vacuum of a certain orange-red light emitted by atoms of the isotope of krypton with mass number 86, identified by the symbol ^{86}Kr.* Specifically, one meter was defined to be 1,650,763.73 wavelengths of this light. Using this standard, it became possible to compare lengths to a precision below 1 part in 10^9.

By 1983, the demands for higher precision had reached such a point that even the ^{86}Kr standard could not meet them and in that year a bold step was taken. The meter was redefined as the distance traveled by a light wave in a specified time interval. In the words of the 17th General Conference on Weights and Measures:

> *The meter is the length of the path traveled by light in vacuum during a time interval of 1/299,792,458 of a second.*

This is equivalent to saying that the speed of light c is now *defined* as

$$c = 299{,}792{,}458 \text{ m/s} \qquad \text{(exactly)}.$$

This new definition of the meter was necessary because measurements of the speed of light had become so precise that the reproducibility of the ^{86}Kr meter itself became the limiting factor. In view of this, it then made sense to adopt the speed of light as a defined quantity and to use it along with the precisely defined standard of time (the second) to redefine the meter.

Table 1-4 shows the range of measured lengths that can be compared with the standard.

TABLE 1-4 Some Measured Lengths[a]

Length	*Meters*
Distance to the farthest observed quasar	2×10^{26}
Distance to the Andromeda galaxy	2×10^{22}
Radius of our galaxy	6×10^{19}
Distance to the nearest star (Proxima Centauri)	4×10^{16}
Mean orbit radius for the most distant planet (Pluto)	6×10^{12}
Radius of the Sun	7×10^{8}
Radius of the Earth	6×10^{6}
Height of Mt. Everest	9×10^{3}
Height of a typical person	2×10^{0}
Thickness of a page in this book	1×10^{-4}
Size of a typical virus	1×10^{-6}
Radius of a hydrogen atom	5×10^{-11}
Effective radius of a proton	1×10^{-15}

[a] Approximate values.

Sample Problem 1-2. A light-year is a measure of length (not a measure of time) equal to the distance that light travels in 1 year. Compute the conversion factor between light-years and meters, and find the distance to the star Proxima Centauri (4.0×10^{16} m) in light-years.

Solution The conversion factor from years to seconds is

$$1 \text{ y} = 1 \text{ y} \times \frac{365.25 \text{ d}}{1 \text{ y}} \times \frac{24 \text{ h}}{1 \text{ d}} \times \frac{60 \text{ min}}{1 \text{ h}} \times \frac{60 \text{ s}}{1 \text{ min}}$$
$$= 3.16 \times 10^7 \text{ s}.$$

The speed of light is, to three significant figures, 3.00×10^8 m/s. Thus in 1 year, light travels a distance of

$$(3.00 \times 10^8 \text{ m/s})(3.16 \times 10^7 \text{ s}) = 9.48 \times 10^{15} \text{ m},$$

so that

$$1 \text{ light-year} = 9.48 \times 10^{15} \text{ m}.$$

The distance to Proxima Centauri is

$$(4.0 \times 10^{16} \text{ m}) \times \frac{1 \text{ light-year}}{9.48 \times 10^{15} \text{ m}} = 4.2 \text{ light-years}.$$

Light from Proxima Centauri thus takes about 4.2 years to travel to Earth.

1-5 THE STANDARD OF MASS

The SI standard of mass is a platinum–iridium cylinder kept at the International Bureau of Weights and Measures and assigned, by international agreement, a mass of 1 kilogram. Secondary standards are sent to standardizing laboratories in other countries and the masses of other

* The mass number is the total number of protons plus neutrons in the nucleus. Naturally occurring krypton has several different isotopes, corresponding to atoms with different mass numbers. It is important to specify a particular isotope for the standard, because the wavelength of the chosen radiation will vary from one isotope to another by about 1 part in 10^5, which is unacceptably large in comparison with the precision of the standard.

bodies can be found by an equal-arm balance technique to a precision of 1 part in 10^8.

The U.S. copy of the international standard of mass, known as Prototype Kilogram No. 20, is housed in a vault at the National Institute of Standards and Technology (see Fig. 1-4). It is removed no more than once a year for checking the values of tertiary standards. Since 1889 Prototype No. 20 has been taken to France twice for recomparison with the master kilogram. When it is removed from the vault two people are always present, one to carry the kilogram in a pair of forceps, the second to catch the kilogram if the first person should fall.

Table 1-5 shows some measured masses. Note that they vary by a factor of about 10^{83}. Most masses have been measured in terms of the standard kilogram by indirect methods. For example, we can measure the mass of the Earth (see Section 14-3) by measuring in the laboratory the gravitational force of attraction between two lead spheres and comparing it with the attraction of the Earth for a known mass. The masses of the spheres must be known by direct comparison with the standard kilogram.

On the atomic scale we have a second standard of mass, which is not an SI unit. It is the mass of the ^{12}C atom, which, by international agreement, has been assigned an atomic mass of 12 unified atomic mass units (abbreviation u), exactly and by definition. We can find the masses of other atoms to considerable accuracy by using a mass spectrometer (see Section 32-2). Table 1-6 shows some selected atomic masses, including the estimated uncertainties of measurement. We need a second standard of mass because present laboratory techniques permit us to compare atomic masses

FIGURE 1-4. The National Standard Prototype Kilogram No. 20, resting in its double bell jar at the U.S. National Institute of Standards and Technology.

TABLE 1-5 Some Measured Masses[a]

Object	*Kilograms*
Known universe (estimate)	10^{53}
Our galaxy	2×10^{43}
Sun	2×10^{30}
Earth	6×10^{24}
Moon	7×10^{22}
Ocean liner	7×10^{7}
Elephant	4×10^{3}
Person	6×10^{1}
Grape	3×10^{-3}
Speck of dust	7×10^{-10}
Virus	1×10^{-15}
Penicillin molecule	5×10^{-17}
Uranium atom	4×10^{-26}
Proton	2×10^{-27}
Electron	9×10^{-31}

[a] Approximate values.

with each other to greater precision than we can presently compare them with the standard kilogram. However, development of an atomic mass standard to replace the standard kilogram is well under way. The relationship between the present atomic standard and the primary standard is approximately

$$1 \text{ u} = 1.661 \times 10^{-27} \text{ kg}.$$

A related SI unit is the *mole,* which measures the quantity of a substance. One mole of ^{12}C atoms has a mass of exactly 12 grams and contains a number of atoms numerically equal to the Avogadro constant N_A:

$$N_A = 6.02214199 \times 10^{23} \text{ per mole}.$$

This is an experimentally determined number, with an uncertainty of about one part in a million. One mole of any other substance contains the same number of elementary entities (atoms, molecules, or whatever). Thus 1 mole of helium gas contains N_A atoms of He, 1 mole of oxygen contains N_A molecules of O_2, and 1 mole of water contains N_A molecules of H_2O.

To relate an atomic unit of mass to a bulk unit, it is necessary to use the Avogadro constant. Replacing the standard kilogram with an atomic standard will require an improvement of at least two orders of magnitude in the precision of the measured value of N_A to obtain masses with precisions of 1 part in 10^8.

TABLE 1-6 Some Measured Atomic Masses

Isotope	*Mass (u)*	*Uncertainty (u)*
^{1}H	1.00782503214	0.00000000035
^{12}C	12.00000000	(exact)
^{64}Cu	63.9297679	0.0000015
^{109}Ag	108.9047551	0.0000032
^{137}Cs	136.9070836	0.0000032
^{208}Pb	207.9766358	0.0000031
^{238}Pu	238.0495534	0.0000022

1-6 PRECISION AND SIGNIFICANT FIGURES

As we improve the quality of our measuring instruments and the sophistication of our techniques, we can carry out experiments at ever increasing levels of precision; that is, we can extend the measured results to more and more *significant figures* and correspondingly reduce the *experimental uncertainty* of the result. Both the number of significant figures and the uncertainty tell something about our estimate of the precision of the result. That is, the result $x = 3$ m implies that we know less about x than the value $x = 3.14159$ m. When we declare $x = 3$ m, we mean that we are reasonably certain that x lies between 2 m and 4 m, whereas expressing x as 3.14159 m means that x probably lies between 3.14158 m and 3.14160 m. If you express x as 3 m when in fact you really believe that x is 3.14159 m, you are withholding information that might be important. On the other hand, if you express $x = 3.14159$ m when you really have no basis for knowing anything other than $x = 3$ m, you are being somewhat dishonest by claiming to have more information than you really do. Attention to significant figures is important when presenting the results of measurements and calculations, and it is equally as wrong to include too many as too few.

There are a few simple rules to follow in deciding how many significant figures to keep:

Rule 1. Counting from the left and ignoring leading zeros, keep all digits up to the first doubtful one. That is, $x = 3$ m has only one significant figure, and expressing this value as $x = 0.003$ km does not change the number of significant figures. If we instead wrote $x = 3.0$ m (or, equivalently, $x = 0.0030$ km), we would imply that we know the value of x to two significant figures. In particular, don't write down all 9 or 10 digits of your calculator display if they are not justified by the precision of the input data! Most calculations in this text are done with two or three significant figures.

Be careful about ambiguous notations: $x = 300$ m does not indicate whether there are one, two, or three significant figures; we don't know whether the zeros are carrying information or merely serving as place holders. Instead, we should write $x = 3 \times 10^2$ or 3.0×10^2 or 3.00×10^2 to specify the precision more clearly.

Rule 2. When multiplying or dividing, the number of significant figures in the product or quotient should be no greater than the number of significant figures in the least precise of the factors. Thus

$$2.3 \times 3.14159 = 7.2.$$

A bit of good judgment is occasionally necessary when applying this rule:

$$9.8 \times 1.03 = 10.1$$

because, even though 9.8 technically has only two significant figures, it is very close to being a number with three significant figures. The product should therefore be expressed with three significant figures.

Rule 3. In adding or subtracting, the least significant digit of the sum or difference occupies the same relative position as the least significant digit of the quantities being added or subtracted. In this case the *number* of significant figures is not important; it is the *position* that matters. For example, suppose we wish to find the total mass of three objects as follows:

$$\begin{array}{r l}
103.\mathbf{9} & \text{kg} \\
2.1\mathbf{0} & \text{kg} \\
0.31\mathbf{9} & \text{kg} \\
\hline
106.\mathbf{319} & \text{or} \quad 106.\mathbf{3}\ \text{kg}
\end{array}$$

The least significant or first doubtful digit is shown in **boldface.** By rule 1, we should include only one doubtful digit; thus the result should be expressed as 106.3 kg, for if the "3" is doubtful, then the following "19" gives no information and is useless.

Sample Problem 1-3. You wish to weigh your pet cat, but all you have available is an ordinary home platform scale. It is a digital scale, which displays your weight in a whole number of pounds. You therefore use the following scheme: you determine your own weight to be 119 lbs, and then holding the cat you find your combined weight to be 128 lbs. What is the fractional or percentage uncertainty in your weight and in the weight of your cat?

Solution The least significant digit is the units digit, and so your weight is uncertain by about 1 pound. That is, your scale would read 119 lb for any weight between 118.5 and 119.5 lb. The fractional uncertainty is therefore

$$\frac{1\ \text{lb}}{119\ \text{lb}} = 0.008 \qquad \text{or} \qquad 0.8\%.$$

The weight of the cat is 128 lb − 119 lb = 9 lb. However, the uncertainty in the cat's weight is still about 1 lb, and so the fractional uncertainty is

$$\frac{1\ \text{lb}}{9\ \text{lb}} = 0.11 = 11\%.$$

Although the *absolute* uncertainty in your weight and the cat's weight is the same (1 lb), the *relative* uncertainty in your weight is an order of magnitude smaller than the relative uncertainty in the cat's weight. If you tried to weigh a 1-lb kitten by this method, the relative uncertainty in its weight would be 100%. This illustrates a commonly occurring danger in the subtraction of two numbers that are nearly equal: the relative or percentage uncertainty in the difference can be very large.

1-7 DIMENSIONAL ANALYSIS

Associated with every measured or calculated quantity is a *dimension.* For example, both the absorption of sound by an enclosure and the probability for nuclear reactions to occur have the dimensions of an area. The units in which the quantities are expressed do not affect the dimension of the quantities: an area is still an area whether it is expressed in m^2 or ft^2 or acres or sabins (sound absorption) or barns (nuclear reactions).

Just as we defined our measurement standards earlier in this chapter as fundamental quantities, we can choose a set of fundamental dimensions based on independent measurement standards. For mechanical quantities, mass, length, and time are elementary and independent, so they can serve as fundamental dimensions. They are represented respectively by M, L, and T.

Any equation must be *dimensionally consistent;* that is, the dimensions on both sides must be the same. Attention to dimensions can often keep you from making errors in writing equations. For example, the distance x covered in a time t by an object starting from rest and moving subject to a constant acceleration a will be shown in the next chapter to be $x = \frac{1}{2}at^2$. Acceleration is measured in units such as m/s^2. We use square brackets [ ] to denote "the dimension of," so that $[x] = L$ or $[t] = T$. It follows that $[a] = L/T^2$ or LT^{-2}. Keeping the units, and therefore the dimension, of acceleration in mind, you will therefore never be tempted to write $x = \frac{1}{2}at$ or $x = \frac{1}{2}at^3$.

The analysis of dimensions can often help in working out equations. The following two sample problems illustrate this procedure.

SAMPLE PROBLEM 1-4. To keep an object moving in a circle at constant speed requires a force called the "centripetal force." (Circular motion is discussed in Chapter 4.) Do a dimensional analysis of the centripetal force.

Solution We begin by asking "On which mechanical variables could the centripetal force F depend?" The moving object has only three properties that are likely to be important: its mass m, its speed v, and the radius r of its circular path. The centripetal force F must therefore be given, apart from any dimensionless constants, by an equation of the form

$$F \propto m^a v^b r^c,$$

where the symbol $\propto$ means "is proportional to" and where a, b, and c are numerical exponents to be determined from analyzing the dimensions. As we wrote in Section 1-2 (and as we shall discuss in Chapter 3), force has units of $kg \cdot m/s^2$, and so its dimensions are $[F] = MLT^{-2}$. We can therefore write the centripetal force equation in terms of dimensions as

$$\begin{aligned}[F] &= [m^a]\,[v^b]\,[r^c] \\ MLT^{-2} &= M^a\,(L/T)^b\,L^c \\ &= M^a\,L^{b+c}\,T^{-b}.\end{aligned}$$

Dimensional consistency means that the fundamental dimensions must be the same on each side. Thus, equating the exponents,

exponents of M: $a = 1$;

exponents of T: $b = 2$;

exponents of L: $b + c = 1$, so $c = -1$.

The resulting expression is

$$F \propto \frac{mv^2}{r}.$$

The actual expression for centripetal force, derived from Newton's laws and the geometry of circular motion, is $F = mv^2/r$. The dimensional analysis gives us the exact dependence on the mechanical variables! This is really a happy accident, because dimensional analysis can't tell us anything about constants that do not have dimensions. In this case the constant happens to be 1.

SAMPLE PROBLEM 1-5. An important milestone in the evolution of the universe just after the Big Bang is the Planck time t_P, the value of which depends on three fundamental constants: (1) the speed of light (the fundamental constant of relativity), $c = 3.00 \times 10^8$ m/s; (2) Newton's gravitational constant (the fundamental constant of gravity), $G = 6.67 \times 10^{-11}\ m^3/s^2 \cdot kg$; and (3) Planck's constant (the fundamental constant of quantum mechanics), $h = 6.63 \times 10^{-34}\ kg \cdot m^2/s$. Based on a dimensional analysis, find the value of the Planck time.

Solution Using the units given for the three constants, we can obtain their dimensions:

$$\begin{aligned}[c] &= [m/s] = LT^{-1} \\ [G] &= [m^3/s^2 \cdot kg] = L^3T^{-2}M^{-1} \\ [h] &= [kg \cdot m^2/s] = ML^2T^{-1}\end{aligned}$$

Let the Planck time depend on these constants as

$$t_P \propto c^i\,G^j\,h^k,$$

where i, j, and k are exponents to be determined. The dimensions of this expression are

$$\begin{aligned}[t_P] &= [c^i]\,[G^j]\,[h^k] \\ T &= (LT^{-1})^i\,(L^3T^{-2}M^{-1})^j\,(ML^2T^{-1})^k \\ &= L^{i+3j+2k}\,T^{-i-2j-k}\,M^{-j+k}.\end{aligned}$$

Equating powers on both sides gives

exponents of L: $0 = i + 3j + 2k$

exponents of T: $1 = -i - 2j - k$

exponents of M: $0 = -j + k$

and solving these three equations for the three unknowns, we find

$$i = -\tfrac{5}{2}, \qquad j = \tfrac{1}{2}, \qquad k = \tfrac{1}{2}.$$

Thus

$$\begin{aligned} t_P &\propto c^{-5/2}G^{1/2}h^{1/2} \\ &= \sqrt{\frac{Gh}{c^5}} = \sqrt{\frac{(6.67 \times 10^{-11}\ m^3/s^2 \cdot kg)(6.63 \times 10^{-34}\ kg \cdot m^2/s)}{(3.00 \times 10^8\ m/s)^5}} \\ &= 1.35 \times 10^{-45}\ s.\end{aligned}$$

As commonly defined, the Planck time differs from this value by a factor of $(2\pi)^{-1/2}$. Such dimensionless factors cannot be found by this technique.

In similar fashion, we can determine the Planck length and the Planck mass, which also have very fundamental interpretations (see Exercises 32 and 33).

MULTIPLE CHOICE

1-1 Physical Quantities, Standards, and Units

1-2 The International System of Units

1-3 The Standard of Time

1-4 The Standard of Length

1-5 The Standard of Mass

1-6 Precision and Significant Figures

1. A student is calculating the surface area of a single sheet of paper. He measures the length to be $l = 27.9$ cm; he measures the width to be $w = 21.6$ cm. The student should record the area of the paper as
(A) 602.64 cm^2. (B) 602.6 cm^2.
(C) 602 cm^2. (D) 603 cm^2.

2. A student is calculating the thickness of a single sheet of paper. She measures the thickness of a stack of 80 sheets with vernier calipers, and finds the thickness to be $l = 1.27$ cm. To calculate the thickness of a single sheet she divides. Which of the following answers has the correct number of significant digits?
(A) 0.15875 mm. (B) 0.159 mm.
(C) 0.16 mm. (D) 0.2 mm.

1-7 Dimensional Analysis

3. The period of oscillation of a nonlinear oscillator depends on the mass m, with dimensions of M; a restoring force constant k with dimensions of $ML^{-2}T^{-2}$, and the amplitude A, with dimensions of L. Dimensional analysis shows that the period of oscillation should be proportional to
(A) $A\sqrt{m/k}$. (B) $A^2 m/k$.
(C) $A^{-1}\sqrt{m/k}$. (D) $A^2 k^3/m$.

QUESTIONS

1. How would you criticize this statement: "Once you have picked a standard, by the very meaning of 'standard' it is invariable"?

2. List characteristics other than accessibility and invariability that you would consider desirable for a physical standard.

3. Can you imagine a system of base units (Table 1-1) in which time was not included?

4. Of the seven base units listed in Table 1-1, only one—the kilogram—has a prefix (see Table 1-2). Would it be wise to redefine the mass of that platinum–iridium cylinder at the International Bureau of Weights and Measures as 1 g rather than 1 kg?

5. What does the prefix "micro-" signify in the words "microwave oven"? It has been proposed that food that has been irradiated by gamma rays to lengthen its shelf life be marked "picowaved." What do you suppose that means?

6. Many capable investigators, on the evidence, believe in the reality of extrasensory perception. Assuming that ESP is indeed a fact of nature, what physical quantity or quantities would you seek to define to describe this phenomenon quantitatively?

7. Name several repetitive phenomena occurring in nature that could serve as reasonable time standards.

8. You could define "1 second" to be one pulse beat of the current president of the American Association of Physics Teachers. Galileo used his pulse as a timing device in some of his work. Why is a definition based on the atomic clock better?

9. What criteria should be satisfied by a good clock?

10. Five clocks are being tested in a laboratory. Exactly at noon, as determined by the WWV time signal, on the successive days of a week the clocks read as follows:

Clock	Sun.	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
A	12:36:40	12:36:56	12:37:12	12:37:27	12:37:44	12:37:59	12:38:14
B	11:59:59	12:00:02	11:59:57	12:00:07	12:00:02	11:59:56	12:00:03
C	15:50:45	15:51:43	15:52:41	15:53:39	15:54:37	15:55:35	15:56:33
D	12:03:59	12:02:52	12:01:45	12:00:38	11:59:31	11:58:24	11:57:17
E	12:03:59	12:02:49	12:01:54	12:01:52	12:01:32	12:01:22	12:01:12

How would you arrange these five clocks in the order of their relative value as good timekeepers? Justify your choice.

11. From what you know about pendulums, cite the drawbacks to using the period of a pendulum as a time standard.

12. How did Galileo know that the pendulum swings at the same frequency regardless of the amplitude? Note: since pendulums were crucial to the building of the first clocks, Galileo couldn't have used a clock to find the answer!

13. What is the uncertainty in a good sand-based egg timer? What about an hourglass? What about the candles used for marking time at night?

14. On June 30, 1981, the minute extending from 10:59 to 11:00 A.M. was arbitrarily lengthened to contain 61 s. The last day of 1989 also was lengthened by 1 s. Such a *leap second* is occasionally introduced to compensate for the fact that, as measured by our atomic time standard, Earth's rotation rate is slowly decreasing. Why is it desirable to readjust our clocks in this way?

15. A radio station advertises that it is "at 89.5 on your FM dial." What does this number mean?

16. Why are there no SI base units for area or volume?

17. The meter was originally intended to be one ten-millionth of the meridian line from the north pole to the equator that passes through Paris. This definition disagrees with the meter bar (adopted at that early date as a standard) by 0.023%. Does this mean that the standard meter bar is inaccurate to this extent?

18. The original meter's definition did *not* involve directly measuring the entire distance from the north pole to the equator. How was it done? Discuss any experimental uncertainty.

19. Can length be measured along a curved line? If so, how?

20. When the meter bar was taken to be the standard of length, its temperature was specified. Can length be called a fundamental property if another physical quantity, such as temperature, must be specified in choosing a standard?

21. In redefining the meter in terms of the speed of light, why did the delegates to the 1983 General Conference on Weights and Measures not simplify matters by defining the speed of light to be 3×10^8 m/s exactly? For that matter, why did they not define it to be 1 m/s exactly? Were both of these possibilities open to them? If so, why did they reject them?

22. The kilogram was originally defined so that the density of water was 1000 in metric units. Can a "metric" version of $\pi = 3.1415926535 \ldots$ be redefined so that it is exactly equal to 22/7? What about saving considerable computation trouble and defining it as $\pi = 3$?

23. Suggest a way to measure (*a*) the radius of the Earth, (*b*) the distance between the Sun and the Earth, and (*c*) the radius of the Sun.

24. Suggest a way to measure (*a*) the thickness of a sheet of paper, (*b*) the thickness of a soap bubble film, and (*c*) the diameter of an atom.

25. If someone told you that every dimension of every object had shrunk to half its former value overnight, how could you refute this statement?

26. Is the current standard kilogram of mass accessible, invariable, reproducible, and indestructible? Does it have simplicity for comparison purposes? Would an atomic standard be better in any respect? Why don't we adopt an atomic standard, as we do for length and time?

27. Why do we find it useful to have two standards of mass, the kilogram and the ^{12}C atom?

28. How does one obtain the relation between the mass of the standard kilogram and the mass of the ^{12}C atom?

29. Suggest practical ways by which one could determine the masses of the various objects listed in Table 1-5.

30. Suggest objects whose masses would fall in the wide range in Table 1-5 between that of an ocean liner and the Moon, and estimate their masses.

31. Critics of the metric system often cloud the issue by saying things such as: "Instead of buying 1 lb of butter you will have to ask for 0.454 kg of butter." The implication is that life would be more complicated. How might you refute this?

EXERCISES

1-1 Physical Quantities, Standards, and Units

1-2 The International System of Units

1. Use the prefixes in Table 1-2 to express (*a*) 10^6 phones, (*b*) 10^{-6} phones, (*c*) 10^1 cards, (*d*) 10^9 lows, (*e*) 10^{12} bulls, (*f*) 10^{-1} mates, (*g*) 10^{-2} pedes, (*h*) 10^{-9} Nannettes, (*i*) 10^{-12} boos, (*j*) 10^{-18} boys, (*k*) 2×10^2 withits, (*l*) 2×10^3 mockingbirds. Now that you have the idea, invent a few more similar expressions. (See p. 61 of *A Random Walk in Science,* compiled by R. L. Weber; Crane, Russak & Co., New York, 1974.)

2. Some of the prefixes of the SI units have crept into everyday language. (*a*) What is the weekly equivalent of an annual salary of 36K (= 36 k\$)? (*b*) A lottery awards 10 megabucks as the top prize, payable over 20 years. How much is received in each monthly check? (*c*) The hard disk of a computer has a capacity of 30 GB (= 30 gigabytes). At 8 bytes/word, how many words can it store?

1-3 The Standard of Time

3. Enrico Fermi once pointed out that a standard lecture period (50 min) is close to 1 microcentury. How long is a microcentury in minutes, and what is the percentage difference from Fermi's approximation?

4. New York and Los Angeles are about 3000 mi apart; the time difference between these two cities is 3 h. Calculate the circumference of the Earth.

5. A convenient substitution for the number of seconds in a year is π times 10^7. To within what percentage error is this correct?

6. (*a*) A unit of time sometimes used in microscopic physics is the *shake.* One shake equals 10^{-8} s. Are there more shakes in a second than there are seconds in a year? (*b*) Humans have existed for about 10^6 years, whereas the universe is about 10^{10} years old. If the age of the universe is taken to be 1 day, for how many seconds have humans existed?

7. In two different track meets, the winners of the mile race ran their races in 3 min 58.05 s and 3 min 58.20 s. In order to conclude that the runner with the shorter time was indeed faster, what is the maximum tolerable error, in feet, in laying out the distances?

8. A certain pendulum clock (with a 12-h dial) happens to gain 1 min/day. After setting the clock to the correct time, how long must one wait until it again indicates the correct time?

9. The age of the universe is about 5×10^{17} s; the shortest light pulse produced in a laboratory (1990) lasted for only 6×10^{-15} s (see Table 1-3). Identify a physically meaningful time interval approximately halfway between these two on a logarithmic scale.

10. Assuming that the length of the day uniformly increases by 0.001 s in a century, calculate the cumulative effect on the measure of time over 20 centuries. Such a slowing down of the Earth's rotation is indicated by observations of the occurrences of solar eclipses during this period.

11. The time it takes the Moon to return to a given position as seen against the background of fixed stars, 27.3 days, is called a sidereal month. The time interval between identical phases of the Moon is called a lunar month. The lunar month is longer than a sidereal month. Why and by how much?

1-4 The Standard of Length

12. Your French pen pal Pierre writes to say that he is 1.9 m tall. What is his height in British units?

13. (*a*) In track meets both 100 yards and 100 meters are used as distances for dashes. Which is longer? By how many meters is it longer? By how many feet? (*b*) Track and field records are kept for the mile and the so-called metric mile (1500 meters). Compare these distances.

14. The stability of the cesium clock used as an atomic time standard is such that two cesium clocks would gain or lose 1 s with respect to each other in about 300,000 y. If this same precision were applied to the distance between New York and San Francisco (2572 mi), by how much would successive measurements of this distance tend to differ?

15. Antarctica is roughly semicircular in shape with a radius of 2000 km. The average thickness of the ice cover is 3000 m. How many cubic centimeters of ice does Antarctica contain? (Ignore the curvature of the Earth.)

16. A unit of area, often used in expressing areas of land, is the *hectare,* defined as 10^4 m^2. An open-pit coal mine consumes 77 hectares of land, down to a depth of 26 m, each year. What volume of earth, in cubic kilometers, is removed in this time?

17. Earth is approximately a sphere of radius 6.37×10^6 m. (*a*) What is its circumference in kilometers? (*b*) What is its surface area in square kilometers? (*c*) What is its volume in cubic kilometers?

18. The approximate maximum speeds of various animals follow, but in different units of speed. Convert these data to m/s, and thereby arrange the animals in order of increasing maximum speed: squirrel, 19 km/h; rabbit, 30 knots; snail, 0.030 mi/h; spider, 1.8 ft/s; cheetah, 1.9 km/min; human, 1000 cm/s; fox, 1100 m/min; lion, 1900 km/day.

19. A certain spaceship has a speed of 19,200 mi/h. What is its speed in light-years per century?

20. A new car is equipped with a "real-time" dashboard display of fuel consumption. A switch permits the driver to toggle back and forth between British units and SI units. However, the British display shows mi/gal while the SI version is the inverse, L/km. What SI reading corresponds to 30.0 mi/gal?

21. Astronomical distances are so large compared to terrestrial ones that much larger units of length are used for easy comprehension of the relative distances of astronomical objects. An *astronomical unit* (AU) is equal to the average distance from Earth to the Sun, 1.50×10^8 km. A *parsec* (pc) is the distance at which 1 AU would subtend an angle of 1 second of arc. A *light-year* (ly) is the distance that light, traveling through a vacuum with a speed of 3.00×10^5 km/s, would cover in 1 year. (*a*) Express the distance from Earth to the Sun in parsecs and in light-years. (*b*) Express a light-year and a parsec in kilometers. Although the light-year is much used in popular writing, the parsec is the unit preferred by astronomers.

22. The effective radius of a proton is about 1×10^{-15} m; the radius of the observable universe (given by the distance to the farthest observable quasar) is 2×10^{26} m (see Table 1-4). Identify a physically meaningful distance that is approximately halfway between these two extremes on a logarithmic scale.

1-5 The Standard of Mass

23. Using conversions and data in the chapter, determine the number of hydrogen atoms required to obtain 1.00 kg of hydrogen.

24. One molecule of water (H_2O) contains two atoms of hydrogen and one atom of oxygen. A hydrogen atom has a mass of 1.0 u and an atom of oxygen has a mass of 16 u. (*a*) What is the mass in kilograms of one molecule of water? (*b*) How many molecules of water are in the oceans of the world? The oceans have a total mass of 1.4×10^{21} kg.

25. In continental Europe, one "pound" is half a kilogram. Which is the better buy: one Paris pound of coffee for \$9.00 or one New York pound of coffee for \$7.20?

26. A room has dimensions of 21 ft $\times$ 13 ft $\times$ 12 ft. What is the mass of the air it contains? The density of air at room temperature and normal atmospheric pressure is 1.21 kg/m^3.

27. A typical sugar cube has an edge length of 1 cm. If you had a cubical box that contained 1 mole of sugar cubes, what would its edge length be?

28. A person on a diet loses 0.23 kg (corresponding to about 0.5 lb) per week. Express the mass loss rate in milligrams per second.

1-6 Precision and Significant Figures

29. For the period 1960–1983, the meter was defined to be 1,650,763.73 wavelengths of a certain orange-red light emitted by krypton atoms. Compute the distance in nanometers corresponding to one wavelength. Express your result using the proper number of significant figures.

30. (*a*) Evaluate 37.76 + 0.132 to the correct number of significant figures. (*b*) Evaluate 16.264 − 16.26325 to the correct number of significant figures.

1-7 Dimensional Analysis

31. Porous rock through which groundwater can move is called an aquifer. The volume V of water that, in time t, moves through a cross section of area A of the aquifer is given by

$$V/t = KAH/L,$$

where H is the vertical drop of the aquifer over the horizontal distance L; see Fig. 1-5. This relation is called Darcy's law. The quantity K is the hydraulic conductivity of the aquifer. What are the SI units of K?

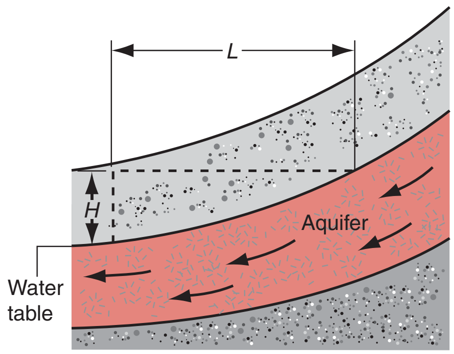


FIGURE 1-5. Exercise 31.

32. In Sample Problem 1-5, the constants h, G, and c were combined to obtain a quantity with the dimensions of time. Repeat the derivation to obtain a quantity with the dimensions of length, and evaluate the result numerically. Ignore any dimensionless constants. This is the Planck length, the size of the observable universe at the Planck time.

33. Repeat the procedure of Exercise 32 to obtain a quantity with the dimensions of mass. This gives the Planck mass, the mass of the observable universe at the Planck time.

PROBLEMS

1. Shortly after the French Revolution, as part of their introduction of the metric system, the revolutionary National Convention made an attempt to introduce decimal time. In this plan, which was not successful, the day—starting at midnight—was divided into 10 decimal hours consisting of 100 decimal minutes each. The hands of a surviving decimal pocket watch are stopped at 8 decimal hours, 22.8 decimal minutes. What time is it? See Fig. 1-6.

FIGURE 1-6. Problem 1.

2. The average distance of the Sun from the Earth is 390 times the average distance of the Moon from the Earth. Now consider a total eclipse of the Sun (Moon between Earth and Sun; see Fig. 1-7) and calculate (*a*) the ratio of the Sun's diameter to the Moon's diameter, and (*b*) the ratio of the Sun's volume to the Moon's volume. (*c*) The angle intercepted at the eye by the Moon is 0.52° and the distance between the Earth and the Moon is 3.82×10^5 km. Calculate the diameter of the Moon.

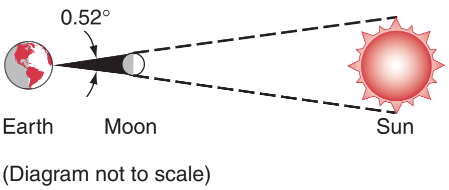


FIGURE 1-7. Problem 2.

3. The navigator of an oil tanker uses the satellites of the Global Positioning System (GPS/NAVSTAR) to find latitude and longitude; see Fig. 1-8. These are 43°36′25.3″ N and 77°31′48.2″ W. If the accuracy of these determinations is ±0.5″, what is the uncertainty in the tanker's position measured along (*a*) a north-south line (meridian of longitude) and (*b*) an east-west line (parallel of latitude)? (*c*) Where is the tanker?

4. In October, 1707 four British warships ran aground because of error in position, sparking the effort to produce a marine clock that would be accurate to locate a position within 30 miles after sailing from England to the West Indies and back again. (*a*) What daily accuracy for such a clock would be required? (*b*) What daily accuracy in a clock is required to fix position to within 0.5 miles after 1 year at sea? (See *Longitude* by Dava Sobel, Penguin, Baltimore, 1995.)

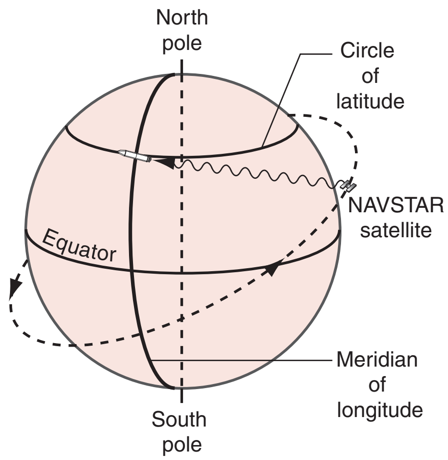


FIGURE 1-8. Problem 3.

5. During the night each breath in contains about 0.3 L of oxygen (O_2, 1.43 g/L at room temperature and pressure). Each breath out contains 0.3 L of carbon dioxide (CO_2, 1.96 g/L at room temperature and pressure). In the course of an 8-hour sleep how much weight in pounds is lost from breathing?

6. Suppose that it takes 12 h to drain a container of 5700 m^3 of water. What is the mass flow rate (in kg/s) of water from the container? The density of water is 1000 kg/m^3.

7. The grains of fine California beach sand have an average radius of 50 μm. What mass of sand grains would have a total surface area equal to the surface area of a cube exactly 1 m on an edge? Sand is made of silicon dioxide, 1 m^3 of which has a mass of 2600 kg.

8. The standard kilogram (see Fig. 1-4) is in the shape of a circular cylinder with its height equal to its diameter. Show that, for a circular cylinder of fixed volume, this equality gives the smallest surface area, thus minimizing the effects of surface contamination and wear.

9. The distance between neighboring atoms, or molecules, in a solid substance can be estimated by calculating twice the radius of a sphere with volume equal to the volume per atom of the material. Calculate the distance between neighboring atoms in (*a*) iron and (*b*) sodium. The densities of iron and sodium are 7870 kg/m^3 and 1013 kg/m^3, respectively; the mass of an iron atom is 9.27×10^{-26} kg, and the mass of a sodium atom is 3.82×10^{-26} kg.

10. (*a*) A rectangular metal plate has a length of 8.43 cm and a width of 5.12 cm. Calculate the area of the plate to the correct number of significant figures. (*b*) A circular metal plate has a radius of 3.7 cm. Calculate the area of the plate to the correct number of significant figures.

CHAPTER 2

MOTION IN ONE DIMENSION

Mechanics, the oldest of the physical sciences, is the study of the motion of objects. The calculation of the path of a baseball or of a space probe sent to Mars is among its problems, as is the analysis of the tracks of elementary particles formed following collisions in our largest accelerators. When we describe *motion, we are dealing with the part of mechanics called* kinematics *(from the Greek word for motion, as also in cinema). When we analyze the* causes *of motion we are dealing with* dynamics *(from the Greek word for force, as in dynamite). In this chapter, we deal mostly with kinematics in one dimension. Chapter 3 introduces one-dimensional dynamics, and Chapter 4 extends these concepts to two and three dimensions.*

2-1 KINEMATICS WITH VECTORS

A scouting party has become trapped in a forest away from their field camp (Fig. 2-1). Based on their explorations, the scouts know that they are 2.0 km from the field camp in a direction 30° west of north. They also know that the field camp is located 3.0 km from the base camp in a direction 45° north of east. They wish to radio their position to the base camp so that food and supplies can be dropped by air as close to their position as possible. How can they pinpoint their location relative to the base camp?

Although there are several ways to solve this problem, the most compact way is in terms of *vectors*. Vectors are quantities that have both magnitude and direction and that follow a certain set of mathematical rules for processes such as addition and multiplication. In Fig. 2-1, the position vector $\vec{\mathbf{r}}_1$ (of length 3.0 km in a direction 45° N of E) locates the field camp relative to the base camp. The position vector $\vec{\mathbf{r}}_2$ (of length 2.0 km in a direction 30° W of N) locates the scouting party relative to the field camp.

We wish to find the vector $\vec{\mathbf{r}}$ that locates the scouting party relative to the base camp. Mathematically, we write this relationship as $\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 + \vec{\mathbf{r}}_2$, but the + sign in this equation means something different from its meaning in ordinary arithmetic or algebra. Clearly it does not mean we should add 3.0 km + 2.0 km to find the distance from the base camp to the scouting party; moreover, this equation must also convey some directional information to help locate the scouting party. Note that the equation $\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 + \vec{\mathbf{r}}_2$ does *not* tell us that the distance from the base camp to the scouting party along $\vec{\mathbf{r}}$ is the same as the sum of the distances along $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$. Instead, it tells us

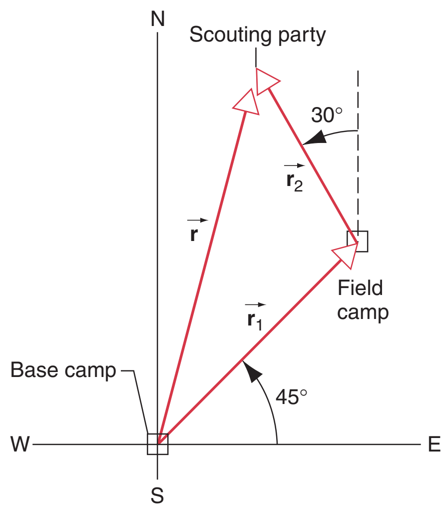


FIGURE 2-1. The relative locations of the base camp, field camp, and scouting party can be specified using vectors.

that we can reach our goal of traveling from the base camp to the scouting party along two equivalent paths, where "equivalent" means that we end up in the same final location.

Position is only one of many quantities in physics that can be represented by vectors. Others include velocity, acceleration, force, momentum, and electromagnetic fields. In contrast to vectors, quantities that can be completely described by specifying only a number (and its units) are called *scalars.* Examples of scalars are mass, time, temperature, and energy.

Kinematics

In this chapter we begin our study of the motion of objects by introducing the terms that are used to describe the motion and showing how they are related to one another. This branch of physics is called *kinematics.* By specifying the *position, velocity,* and *acceleration* of an object, we can describe how the object moves, including the direction of its motion, how that direction changes with time, whether the object speeds up or slows down, and so forth.

For simplicity we will in this chapter consider only the motion of *particles.* By "particle" we often mean a single mass point, such as an electron, but we also can use "particle" to describe an object whose parts all move in exactly the same way. Even a complex object can be treated as a particle if there are no internal motions such as rotations or vibrations of its parts. For example, a rolling wheel cannot be treated as a particle, because a point on its rim moves differently from a point on its axle. (However, a *sliding* wheel *can* be treated as a particle.) Thus an object may be considered a particle for some calculations but not for others. For now we will neglect all internal motions and consider an electron and a freight train on the same basis—as examples of particle motion. Within this approximation, particles can execute a variety of motions: speed up, slow down, even stop and reverse direction or move in curved paths such as circles or parabolas. As long as we can regard these objects as particles, we can use the same set of kinematic equations to describe them.

Many of the laws of physics can be expressed most compactly as relations between quantities expressed as vectors. When a law is written in vector form, it is often easier to understand and to manipulate. Position, velocity, and acceleration, which are the quantities of kinematics, are all vector quantities, and the rules that define them and relate them to one another are vector laws. In this chapter we develop these laws and apply them to motion in a straight line. A more complete demonstration of the usefulness of these vector laws will come in Chapter 4, when we consider two- and three-dimensional motion in curved paths.

In the next section we summarize some of the basic properties of vectors that we will need in kinematics. Further details about the properties of vectors may be found in Appendix H.

2-2 PROPERTIES OF VECTORS

To represent a vector on a diagram we draw an arrow. The length of the arrow is drawn to be proportional to the magnitude of the vector using any convenient scale. Other vectors that are part of the same problem are drawn using the same scale, so that the relative sizes of the arrows are the same as the relative magnitudes of the vectors. (For example, the vector $\vec{\mathbf{r}}_1$ in Fig. 2-1, which represents 3.0 km, is drawn to be 1.5 times as long as the vector $\vec{\mathbf{r}}_2$, which represents 2.0 km.) The direction of the arrow corresponds to the direction of the vector, with the arrowhead giving the sense of the direction. Vectors are represented in this book using boldface type with an arrow, such as $\vec{\mathbf{a}}$ or $\vec{\mathbf{b}}$. In your handwritten work, you can write vectors by placing an arrow above the symbol, such as $\vec{a}$ or $\vec{b}$.

The magnitude or length of a vector is indicated by $|\vec{\mathbf{a}}|$, which gives us no directional information at all about the vector $\vec{\mathbf{a}}$. We will usually write the magnitude of a vector by a single italic symbol such as a, which has the same meaning as $|\vec{\mathbf{a}}|$.

Components of Vectors

We can specify a vector by giving its length and direction, as for example the position vectors of Fig. 2-1. It is often more useful, however, to describe a vector in terms of its *components.* Figure 2-2*a* shows the vector $\vec{\mathbf{a}}$. Its magnitude or length is a and its direction is specified by the angle ϕ, which is measured with respect to the positive x axis. The x and y components of $\vec{\mathbf{a}}$ are defined by

$$a_x = a\cos\phi \qquad \text{and} \qquad a_y = a\sin\phi. \tag{2-1}$$

Although the magnitude a is always positive, the components a_x and a_y may be positive or negative, depending on

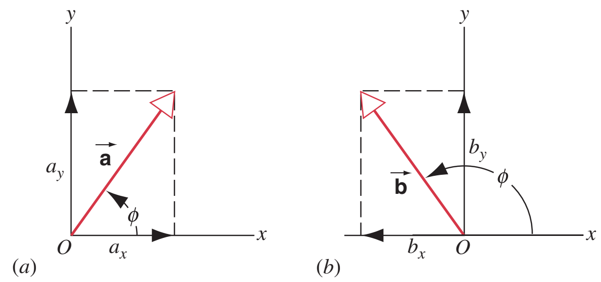


FIGURE 2-2. (*a*) The vector $\vec{\mathbf{a}}$ has component a_x in the x direction and component a_y in the y direction. (*b*) The vector $\vec{\mathbf{b}}$ has a negative x component.

the angle ϕ. For example, as shown in Fig. 2-2*b*, the vector $\vec{b}$ is located by an angle ϕ that is greater than 90° but less than 180° (ϕ always being measured from the *positive x* axis). For this case, b_x is negative and b_y is positive.

If we know a and ϕ, we can find the components according to Eqs. 2-1. We can also reverse this process—given a_x and a_y, we can find the magnitude of the vector and the angle ϕ by

$$a = \sqrt{a_x^2 + a_y^2} \quad \text{and} \quad \tan\phi = a_y/a_x. \tag{2-2}$$

The quadrant in which $\vec{a}$ is located, and thus the value of ϕ, can be determined from the signs of a_x *and* a_y. For example, both −45° (or 315°) and 135° have tangents equal to −1, and thus $a_y/a_x = -1$ for both. For $\phi = -45°$, a_x is positive and a_y is negative, whereas for $\phi = 135°$, a_x is negative and a_y is positive. Knowing the signs of a_x and a_y would allow us to distinguish between these two possibilities. (See Sample Problem 2-3 for another discussion of this problem.)

A more formal way to write a vector in terms of its components is based on a set of *unit vectors.* Unit vectors are vectors of length 1 in the direction of each of the coordinate axes. In the Cartesian coordinate system, the x and y unit vectors are indicated by $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, as shown in Fig. 2-3*a*. Using the unit vectors, we can write the vector $\vec{a}$ as

$$\vec{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}. \tag{2-3}$$

Unit vectors are dimensionless; the dimensions of $\vec{a}$ in Eq. 2-3 are carried by the components a_x and a_y.

The vector relation of Eq. 2-3 is exactly equivalent to the two scalar relations of Eq. 2-1. Sometimes we refer to $a_x\hat{\mathbf{i}}$ and $a_y\hat{\mathbf{j}}$ as the *vector components* of $\vec{a}$. Figure 2-3*b* shows the vector $\vec{a}$ and its vector components. Because the physical effect of a vector is identical to the combined physical effects of its vector components, we may occasionally want to analyze problems by replacing a vector by its vector components. Usually when we speak of components, however, we will mean the scalar components of Eq. 2-1.

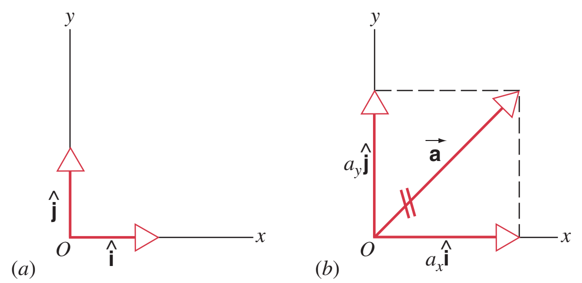


Figure 2-3. (*a*) The unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. (*b*) The *vector* components of $\vec{a}$. When we want to replace $\vec{a}$ by its vector components, it is helpful to draw a double line through the original vector, as shown; this helps to remind us not to consider the original vector any more.

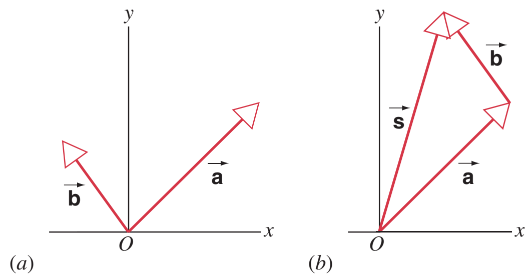


Figure 2-4. (*a*) Vectors $\vec{a}$ and $\vec{b}$. (*b*) To find the sum $\vec{s}$ of vectors $\vec{a}$ and $\vec{b}$, we slide $\vec{b}$ without changing its magnitude or direction until its tail is on the head of $\vec{a}$. Then the vector $\vec{s} = \vec{a} + \vec{b}$ is drawn from the tail of $\vec{a}$ to the head of $\vec{b}$.

When we write an equation involving vectors, such as $\vec{a} = \vec{b}$, we mean that the two vectors are precisely the same: they have the same magnitude and direction. This can occur only if $a_x = b_x$ and $a_y = b_y$. That is,

Two vectors are equal to each other only if their corresponding components are equal.

Adding Vectors

As in the case of Fig. 2-1, we often want to add two or more vectors to find their sum. Consider the two vectors $\vec{a}$ and $\vec{b}$ in Fig. 2-4*a*. We wish to find the vector $\vec{s}$ such that $\vec{s} = \vec{a} + \vec{b}$.

Figure 2-4*b* shows a graphical construction that allows us to find $\vec{a} + \vec{b}$. We first draw the vector $\vec{a}$. Instead of drawing $\vec{b}$ with its tail at the origin, as in Fig. 2-4*a*, we move $\vec{b}$ until its tail coincides with the head of $\vec{a}$. As long as we don't change its magnitude or direction, we can move a vector in this way. The vector $\vec{s}$, representing the sum $\vec{a} + \vec{b}$, is now drawn from the tail of $\vec{a}$ to the head of $\vec{b}$. If we are adding more than two vectors, we can continue placing them tail to head in this way, and the sum vector is drawn from the tail of the first to the head of the last. Often we can use geometric or trigonometric relationships to find the magnitude and direction of the sum vector.

Another way to add vectors is to add their components. That is, $\vec{s} = \vec{a} + \vec{b}$ means

$$\begin{aligned} s_x\hat{\mathbf{i}} + s_y\hat{\mathbf{j}} &= (a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}) + (b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}}) \\ &= (a_x + b_x)\hat{\mathbf{i}} + (a_y + b_y)\hat{\mathbf{j}}. \end{aligned}$$

Setting the x components on the left side of this equation equal to those on the right, and doing the same for the y components, we obtain

$$s_x = a_x + b_x \quad \text{and} \quad s_y = a_y + b_y. \tag{2-4}$$

To add vectors in this way, we resolve each vector into its components and then add the components (taking into account their algebraic signs) to find the components of the sum vector. Figure 2-5 illustrates this addition. Once we

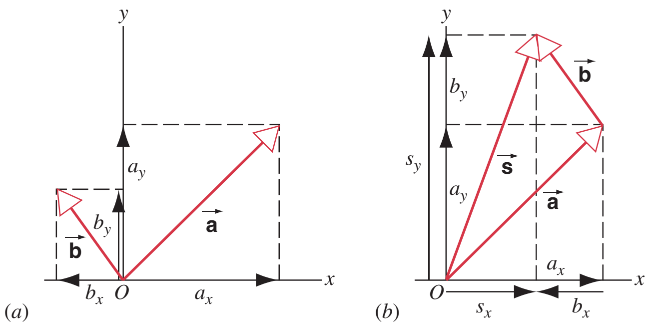


FIGURE 2-5. (*a*) The components of the vectors $\vec{a}$ and $\vec{b}$. (*b*) The sum vector $\vec{s} = \vec{a} + \vec{b}$ can be found by adding the components of $\vec{a}$ to the components of $\vec{b}$. Note that b_x is negative, so that $s_x = a_x + b_x$ involves a subtraction.

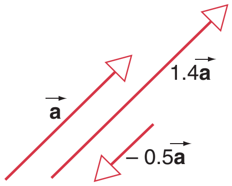


FIGURE 2-6. Multiplication of a vector $\vec{a}$ by a scalar c gives a vector $c\vec{a}$ whose magnitude is c times the magnitude of $\vec{a}$. The vector $c\vec{a}$ has the same direction as $\vec{a}$ if c is positive and the opposite direction if c is negative. Examples are illustrated for $c = +1.4$ and $c = -0.5$.

know the components of the sum vector, we can easily find its magnitude and direction using Eq. 2-2.

Multiplying a Vector by a Scalar

The product of a scalar c and a vector $\vec{a}$, written $c\vec{a}$, is defined to be a new vector whose magnitude is the magnitude of c times the magnitude of $\vec{a}$. Equivalently, the components of this new vector are ca_x and ca_y. The scalar may be a pure number, or it may be a physical quantity with dimensions and units, so that the new vector $c\vec{a}$ represents a physical quantity that is different from $\vec{a}$. To divide a vector by a scalar, as in $\vec{a}/c$, we simply multiply the vector $\vec{a}$ by $1/c$.

Multiplication by a scalar does not change the direction of a vector, except that it may reverse the direction if the scalar is negative. Figure 2-6 shows the effect of multiplying a vector $\vec{a}$ by a positive scalar and a negative scalar.

If we multiply a vector $\vec{b}$ by the scalar -1, we obtain the vector $-\vec{b}$, which is a vector with the same magnitude as $\vec{b}$ but the opposite direction. The components of $-\vec{b}$ are $-b_x$ and $-b_y$. We can use this property to find the difference between two vectors, $\vec{d} = \vec{a} - \vec{b}$, by first writing $\vec{d} = \vec{a} + (-\vec{b})$ and in effect adding the vectors $\vec{a}$ and $-\vec{b}$. Figure 2-7 illustrates the graphical method

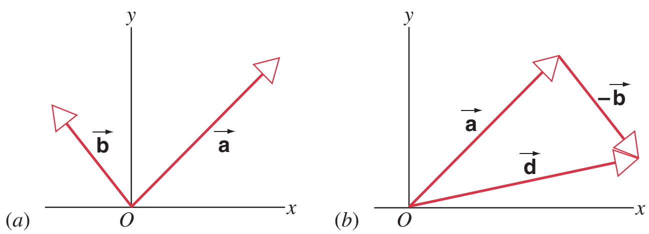


FIGURE 2-7. (*a*) Vectors $\vec{a}$ and $\vec{b}$. (*b*) The difference $\vec{d} = \vec{a} - \vec{b}$ is obtained by adding $-\vec{b}$ to $\vec{a}$.

for adding $\vec{a}$ and $-\vec{b}$ to find $\vec{d} = \vec{a} - \vec{b}$. In analogy with Eq. 2-4, the components of $\vec{d}$ are $d_x = a_x - b_x$ and $d_y = a_y - b_y$.

SAMPLE PROBLEM 2-1. An airplane travels 209 km on a straight course making an angle of 22.5° east of due north. How far north and how far east does the plane travel from its starting point?

Solution We choose the positive x direction to be east and the positive y direction to be north. Next, we draw a displacement vector (Fig. 2-8) from the origin (starting point), making an angle of 22.5° with the y axis (north) inclined along the positive x direction (east). The length of the vector represents a magnitude of 209 km. If we call this vector $\vec{d}$, then d_x gives the distance traveled east of the starting point and d_y gives the distance traveled north of the starting point. We have

$$\phi = 90.0^\circ - 22.5^\circ = 67.5^\circ,$$

so that (see Eqs. 2-1)

$$d_x = d\cos\phi = (209\ \text{km})(\cos 67.5^\circ) = 80.0\ \text{km},$$

and

$$d_y = d\sin\phi = (209\ \text{km})(\sin 67.5^\circ) = 193\ \text{km}.$$

In solving this problem, we have assumed that the surface of the Earth can be represented as the xy plane. We know, however,

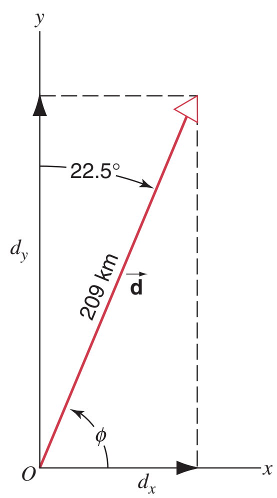


FIGURE 2-8. Sample Problem 2-1.

that the surface of the Earth is not flat but curved, with a radius of about 6400 km. Over small distances, the Earth's surface is approximately flat and we can safely use the *xy* coordinates. Can you estimate the distance *d* that the airplane must fly before the use of flat Cartesian coordinates causes a 5% error in the calculated distance that the airplane travels north and east of its starting point?

SAMPLE PROBLEM 2-2. An automobile travels due east on a level road for 32 km. It then turns due north at an intersection and travels 47 km before stopping. Find the vector that indicates the resulting location of the car.

Solution We choose a coordinate system fixed with respect to the Earth, with the positive *x* direction pointing east and the positive *y* direction pointing north. The two successive journeys, represented by vectors $\vec{a}$ and $\vec{b}$, are then drawn as shown in Fig. 2-9. The resultant $\vec{s}$ is obtained from $\vec{s} = \vec{a} + \vec{b}$. Since $\vec{b}$ has no *x* component and $\vec{a}$ has no *y* component, using Eq. 2-4 we obtain

$$s_x = a_x + b_x = 32 \text{ km} + 0 = 32 \text{ km},$$

$$s_y = a_y + b_y = 0 + 47 \text{ km} = 47 \text{ km}.$$

The magnitude and direction of $\vec{s}$ are then (see Eqs. 2-2)

$$s = \sqrt{s_x^2 + s_y^2} = \sqrt{(32 \text{ km})^2 + (47 \text{ km})^2} = 57 \text{ km},$$

$$\tan \phi = \frac{s_y}{s_x} = \frac{47 \text{ km}}{32 \text{ km}} = 1.47, \qquad \phi = \tan^{-1}(1.47) = 56°.$$

The resultant vector $\vec{s}$ has a magnitude of 57 km and makes an angle of 56° north of east.

SAMPLE PROBLEM 2-3. Three vectors in the *xy* plane are expressed with respect to the coordinate system as

$$\vec{a} = 4.3\hat{i} - 1.7\hat{j},$$

$$\vec{b} = -2.9\hat{i} + 2.2\hat{j},$$

and

$$\vec{c} = -3.6\hat{j},$$

in which the components are given in arbitrary units. Find the vector $\vec{s}$, which is the sum of these vectors.

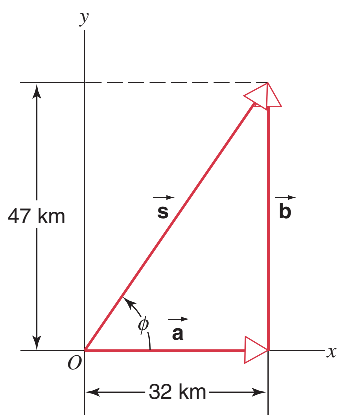


FIGURE 2-9. Sample Problem 2-2.

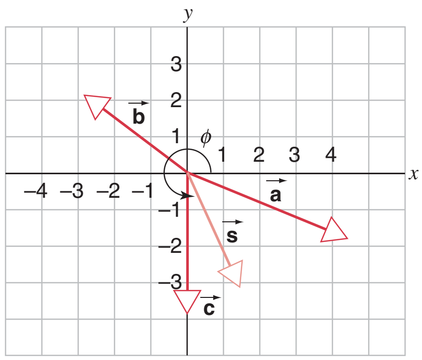


FIGURE 2-10. Sample Problem 2-3.

Solution Generalizing Eqs. 2-4 to the case of three vectors, we have

$$s_x = a_x + b_x + c_x = 4.3 - 2.9 + 0 = 1.4,$$

and

$$s_y = a_y + b_y + c_y = -1.7 + 2.2 - 3.6 = -3.1.$$

Thus

$$\vec{s} = s_x\hat{i} + s_y\hat{j} = 1.4\hat{i} - 3.1\hat{j}.$$

Figure 2-10 shows the four vectors. From Eqs. 2-2 we can calculate that the magnitude of $\vec{s}$ is 3.4 and that the angle ϕ that $\vec{s}$ makes with the positive *x* axis, measured counterclockwise from that axis, is

$$\phi = \tan^{-1}(-3.1/1.4) = 294°.$$

Most pocket calculators return angles between +90° and −90° for the arctan. In this case, −66° (which our calculator gives) is equivalent to 294°. However, we would obtain the same angle if we asked for $\tan^{-1}(3.1/-1.4)$, which should give an angle in the second (upper left) quadrant. In applying Eq. 2-2 to find ϕ, the *individual signs* of the components must be considered—it is not sufficient to deal only with the sign of their ratio. Drawing a sketch similar to Fig. 2-10 will keep you from going too far wrong, and if necessary you can convert your calculator's value into a result in the correct quadrant by using the identity $\tan(-\phi) = \tan(180° - \phi)$.

2-3 POSITION, VELOCITY, AND ACCELERATION VECTORS

In kinematics, we describe the motion of a particle using vectors to specify its position, velocity, and acceleration. So far all of our examples have been in two dimensions (the *xy* plane). Now we consider motion in three dimensions (using an *xyz* coordinate system).

Figure 2-11 shows a particle moving along an arbitrary path in three dimensions. At any particular time *t*, the particle can be located by its *x*, *y*, and *z* coordinates, which are the three components of the *position vector* $\vec{r}$:

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \qquad (2\text{-}5)$$

where $\hat{i}$, $\hat{j}$, and $\hat{k}$ are the Cartesian unit vectors, as shown in Fig. 2-11.

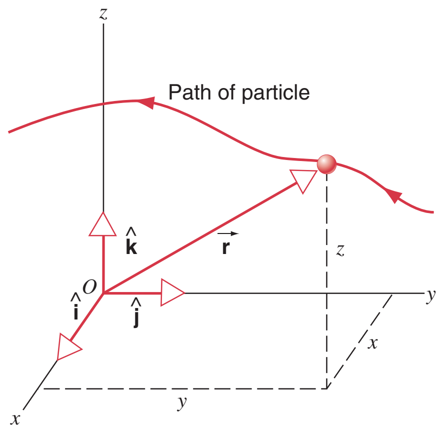


FIGURE 2-11. The position of a particle moving on its path is located by the position vector $\vec{\mathbf{r}}$, which has components x, y, and z. Also shown are the three Cartesian unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$.

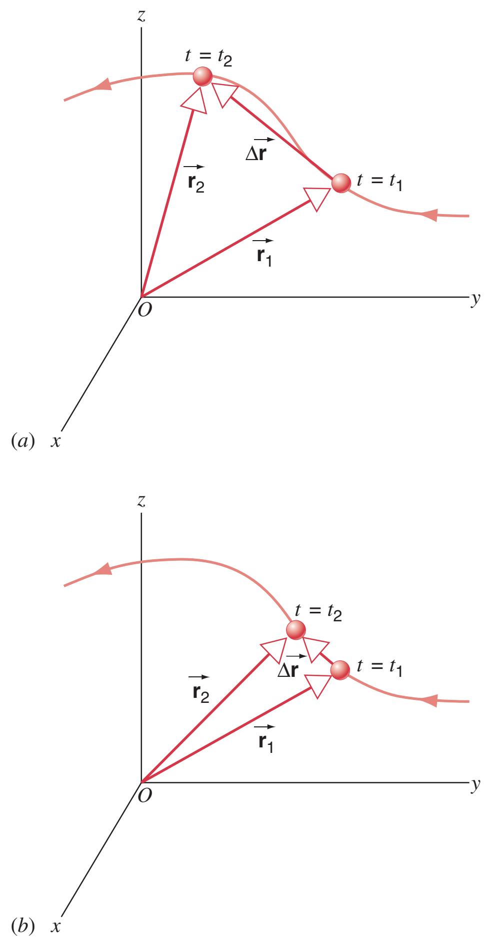


FIGURE 2-12. (a) In the interval Δt from t_1 to t_2, the particle moves from position $\vec{\mathbf{r}}_1$ to position $\vec{\mathbf{r}}_2$. Its displacement in that interval is $\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1$. ($b$) As the interval grows smaller, the displacement vector approaches the actual path of the particle.

Suppose the particle is located at position $\vec{\mathbf{r}}_1$ at time t_1, and it moves along its path to position $\vec{\mathbf{r}}_2$ at time t_2, as shown in Fig. 2-12a. We define the *displacement vector* $\Delta\vec{\mathbf{r}}$ as the change in position that occurs in this interval:

$$\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1. \tag{2-6}$$

The three vectors $\vec{\mathbf{r}}_1$, $\Delta\vec{\mathbf{r}}$, and $\vec{\mathbf{r}}_2$ have the same relationship as the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{s}}$ in Fig. 2-4b. That is, using the graphical head-to-tail addition method, $\Delta\vec{\mathbf{r}}$ added to $\vec{\mathbf{r}}_1$ gives the sum $\vec{\mathbf{r}}_2$. Thus $\vec{\mathbf{r}}_2 = \vec{\mathbf{r}}_1 + \Delta\vec{\mathbf{r}}$, which gives Eq. 2-6.

Note from Fig. 2-12a that the displacement is not the same as the distance traveled by the particle. The displacement is determined only by the starting and ending points of the interval and not by the path traveled between them.

Velocity

The *average velocity* in any interval is defined to be the displacement (change in position) divided by the time interval during which the displacement occurs, or

$$\vec{\mathbf{v}}_{\text{av}} = \frac{\Delta\vec{\mathbf{r}}}{\Delta t}, \tag{2-7}$$

where $\Delta t = t_2 - t_1$. In this equation, the vector $\Delta\vec{\mathbf{r}}$ is multiplied by the positive scalar $1/\Delta t$, *so the vector* $\vec{\mathbf{v}}_{\text{av}}$ *points in the same direction as* $\Delta\vec{\mathbf{r}}$.

Like the displacement, the average velocity in any interval depends only on the locations of the particle at the beginning and end of the interval; it does not depend on whether it speeds up or slows down or even reverses direction in that interval. Note especially that if a particle returns to its starting point, then according to the definition of Eq. 2-7 the average velocity is zero. According to this definition, the average velocity of a race car in the Indianapolis 500 is zero!

Average velocity may be helpful in considering the overall behavior of a particle during some interval, but in describing the details of its motion it would be more useful to have a mathematical function that gives the velocity at every point in the motion. This is the *instantaneous velocity* $\vec{\mathbf{v}}$. When we use the term "velocity," we mean the instantaneous velocity.

To find the instantaneous velocity, we reduce the size of the interval Δt; as we do so, the vector $\Delta\vec{\mathbf{r}}$ approaches the actual path (as in Fig. 2-12b), and it becomes tangent to the path in the limit $\Delta t \to 0$. In this case, the average velocity approaches the instantaneous velocity $\vec{\mathbf{v}}$:

$$\vec{\mathbf{v}} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\mathbf{r}}}{\Delta t}. \tag{2-8}$$

The direction of $\vec{\mathbf{v}}$ is tangent to the path of the particle, indicating the direction in which the particle is moving at that instant of time.

Equation 2-8 resembles the definition of a derivative, and we can write it as

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt}. \tag{2-9}$$

The derivative of a vector is found by taking the derivative of each of its components:

$$\frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}. \tag{2-10}$$

The unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are constant in both magnitude and direction, and so they can be treated as constants in taking the derivative; in other coordinate systems (such as cylindrical or spherical polar systems) the unit vectors may change direction with time and so those vectors do not pass unchanged through the derivative.

The vector $\vec{\mathbf{v}}$ can also be written in terms of its components as

$$\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}. \tag{2-11}$$

Since two vectors can be equal only if their corresponding components are equal, equating Eqs. 2-10 and 2-11 gives three equations:

$$v_x = \frac{dx}{dt}, \qquad v_y = \frac{dy}{dt}, \qquad v_z = \frac{dz}{dt}. \tag{2-12}$$

The three-dimensional vector relation of Eq. 2-9 is equivalent to the three one-dimensional relations of Eq. 2-12.

The term *speed* (taken to mean instantaneous speed) usually refers to the magnitude of the instantaneous velocity, with no indication of direction; that is, the speed v is $|\vec{\mathbf{v}}|$. The speedometer of a car indicates its speed, not its velocity, because it does not specify a direction. Speed is a scalar, since it lacks directional information. We can also define the *average speed:*

$$\text{average speed} = \frac{\text{total distance traveled}}{\text{elapsed time}}. \tag{2-13}$$

It is important to note that the average speed is generally *not* related to the magnitude of the average velocity. For example, the Indianapolis 500 race car with an average velocity of zero (because it starts and ends the race at the same location) certainly does not have an average speed of zero! Equation 2-13 is a scalar equation—the total distance traveled does not provide any information about the direction of the journey.

Both velocity and speed have dimensions of length divided by time, so their SI unit is meters per second (m/s). Other convenient units include miles per hour (mi/h or mph), kilometers per hour (km/h), and so forth.

Acceleration

The velocity of a particle may change in magnitude or direction as it moves. The change in velocity with time is called *acceleration.* In analogy with Eq. 2-7, we can define the *average acceleration* in this interval as the change in velocity per unit time, or

$$\vec{\mathbf{a}}_{\text{av}} = \frac{\Delta\vec{\mathbf{v}}}{\Delta t}. \tag{2-14}$$

The change in velocity $\Delta\vec{\mathbf{v}}$ means $\vec{\mathbf{v}}_{\text{final}} - \vec{\mathbf{v}}_{\text{initial}}$. As in the case of average velocity, the average acceleration tells us nothing about the variation of $\vec{\mathbf{v}}$ during the interval Δt. The direction of $\vec{\mathbf{a}}_{\text{av}}$ is the same as the direction of $\Delta\vec{\mathbf{v}}$.

The *instantaneous acceleration* $\vec{\mathbf{a}}$ is obtained from the limit of Eq. 2-14 for vanishingly small time intervals:

$$\vec{\mathbf{a}} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\mathbf{v}}}{\Delta t}. \tag{2-15}$$

Once again this can be expressed as a derivative:

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} \tag{2-16}$$

and by analogy with Eqs. 2-10 and 2-11 we can write the components of the instantaneous acceleration vector as

$$a_x = \frac{dv_x}{dt}, \qquad a_y = \frac{dv_y}{dt}, \qquad a_z = \frac{dv_z}{dt}. \tag{2-17}$$

Note that vector equations such as Eq. 2-16 can simplify notation, and they also remind us how to separate the components (for example, a_x has no effect on v_y or v_z).

In general the direction of the acceleration has no relation to the direction of $\vec{\mathbf{v}}$. It is possible for $\vec{\mathbf{v}}$ and $\vec{\mathbf{a}}$ to be parallel, antiparallel, or perpendicular to each other, or at any other relative angle.

Because $\vec{\mathbf{v}}$ is a vector quantity, a change in its direction gives an acceleration, even if its magnitude is unchanged. Thus motion at constant speed can be accelerated motion. For example, the components of $\vec{\mathbf{v}}$ can change in such a way that the magnitude of $\vec{\mathbf{v}}$ $(= \sqrt{v_x^2 + v_y^2 + v_z^2})$ remains constant. This situation is found most commonly in uniform circular motion, which we discuss in Section 4-5.

Because acceleration is defined as velocity divided by time, its dimensions are length/time^2. In the SI system, the units of acceleration are m/s^2. Sometimes you may see this written as m/s/s, which is read as "meters per second per second." This emphasizes that acceleration is change in velocity per unit time.

Sample Problem 2-4. A particle moves in the xy plane so that its x and y coordinates vary with time according to $x(t) = At^3 + Bt$ and $y(t) = Ct^2 + D$, where $A = 1.00$ m/s^3, $B = -32.0$ m/s, $C = 5.0$ m/s^2, and $D = 12.0$ m. Find the position, velocity, and acceleration of the particle when $t = 3$ s.

Solution The position is given by Eq. 2-5, with the expressions given for $x(t)$ and $y(t)$:

$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} = (At^3 + Bt)\hat{\mathbf{i}} + (Ct^2 + D)\hat{\mathbf{j}}.$$

Evaluating this expression at $t = 3$ s, we obtain

$$\vec{\mathbf{r}} = (-69\ \text{m})\hat{\mathbf{i}} + (57\ \text{m})\hat{\mathbf{j}}.$$

The velocity components are found from Eq. 2-12:

$$v_x = \frac{dx}{dt} = \frac{d}{dt}(At^3 + Bt) = 3At^2 + B$$

$$v_y = \frac{dy}{dt} = \frac{d}{dt}(Ct^2 + D) = 2Ct.$$

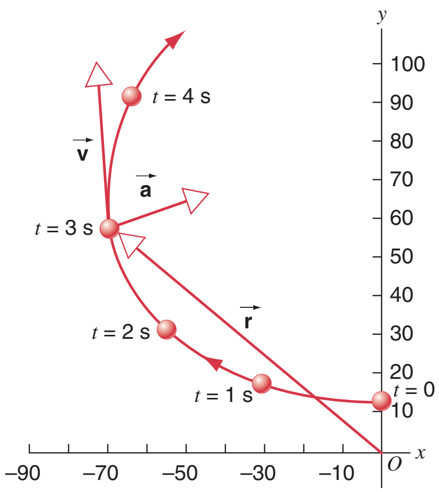


FIGURE 2-13. Sample Problem 2-4. The path of the moving particle is shown, and its positions at $t = 0, 1, 2, 3$, and 4 s are indicated. At $t = 3$ s, the vectors representing its position, velocity, and acceleration are shown. Note that there is no particular relationship between the directions of $\vec{\mathbf{r}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{a}}$ or the lengths of the vectors representing them.

Evaluating the components at $t = 3$ s and using Eq. 2-11, we obtain

$$\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} = (-5\text{ m/s})\hat{\mathbf{i}} + (30\text{ m/s})\hat{\mathbf{j}}.$$

The components of the acceleration are

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}(3At^2 + B) = 6At$$

$$a_y = \frac{dv_y}{dt} = \frac{d}{dt}(2Ct) = 2C.$$

At $t = 3$ s, the acceleration is

$$\vec{\mathbf{a}} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} = (18\text{ m/s}^2)\hat{\mathbf{i}} + (10\text{ m/s}^2)\hat{\mathbf{j}}.$$

Figure 2-13 shows the path of the particle from $t = 0$ to $t = 4$ s. The position, velocity, and acceleration vectors at $t = 3$ s are drawn. (Because $\vec{\mathbf{r}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{a}}$ are expressed in different units, the lengths of the arrows drawn in Fig. 2-13 have no relationship to one another.) The vector $\vec{\mathbf{r}}$ locates the particle relative to the origin. The vector $\vec{\mathbf{v}}$ is tangent to the path of the particle, as shown at $t = 3$ s. The acceleration $\vec{\mathbf{a}}$ represents the change in velocity, and the direction of $\vec{\mathbf{a}}$ at $t = 3$ s is reasonable based on how the direction of $\vec{\mathbf{v}}$ changes in the interval around $t = 3$ s.

2-4 ONE-DIMENSIONAL KINEMATICS

Now that we have established the definitions of the important quantities for describing motion, let's look at some examples of how they can be applied. For simplicity, we consider motion only in one dimension, so we will use only one component of Eqs. 2-5, 2-12, and 2-17.

In one-dimensional kinematics, a particle can move only along a straight line. It may change its speed or even reverse direction, but its motion is always along the line. Within this limitation, we can consider many different physical situations, such as a falling stone, an accelerating train, a braking car, a sliding hockey puck, a crate being pulled up a ramp, or a fast-moving electron in an x-ray tube.

We can describe the motion of a particle in two ways: with mathematical equations and with graphs. Either way gives information about the problem, and often we will want to use both methods. The mathematical approach is usually better for solving problems, because it permits greater precision than the graphical method. On the other hand, the graphical method often provides more physical insight than a set of equations.

Here are some possible kinds of motion with the equations or graphs that describe them.

1. *No motion at all.* Here the particle occupies the same position at all times. Suppose the particle is on the x axis at the coordinate A, so that (at all times)

$$x(t) = A. \tag{2-18}$$

A graph of this "motion" is shown in Fig. 2-14*a*. The velocity, which remains constant at zero, is shown in Fig. 2-14*b*. The physical situation represented here might be, for example, a bead that can slide freely along a straight wire. In this case the bead is at rest at $x = A$.

Often in problems in kinematics we want to know how the position and velocity depend on the time as a particle moves. For this reason we write the position coordinate here as a function of the time as $x(t)$. For the same reason we plot the graph in Fig. 2-14 with x as the dependent variable (on the vertical axis) and t as the independent variable (on the horizontal axis). Placing x on the vertical axis does *not* mean that the particle moves vertically; in this situation, the wire on which the bead slides can have any direction.

2. *Motion at constant velocity.* For motion in one dimension (which we choose to be the x direction), the veloc-

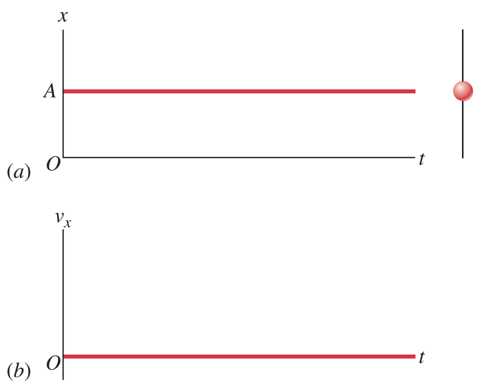


FIGURE 2-14. (*a*) The position and (*b*) the velocity of a bead on a wire at rest $x = A$.

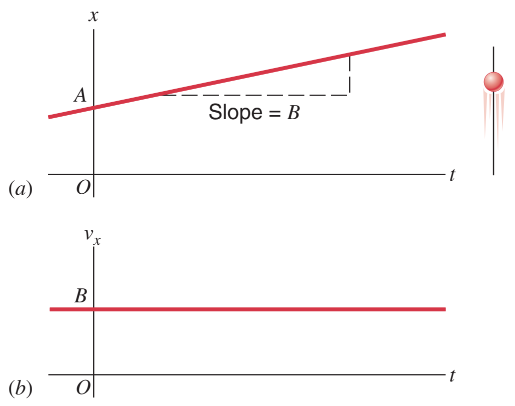


FIGURE 2-15. (*a*) The position and (*b*) the velocity of a bead sliding in one dimension along a wire with constant velocity. The velocity is equal to the slope B of the graph of $x(t)$. The graph of $v_x(t)$ is the horizontal line $v_x = B$.

ity v_x can be positive, if the particle is moving in the direction of increasing x, or negative, if it is moving in the opposite direction. When the velocity is constant, the graph of position against time is a straight line. As Eq. 2-12 ($v_x = dx/dt$) shows, the rate of change of the position is the velocity. In the graph of x against t, the rate of change is the slope of the graph; the greater the velocity, the greater the slope. Figure 2-15*a* shows this graph, whose mathematical form can be expressed as

$$x(t) = A + Bt, \tag{2-19}$$

which is in the customary form as the equation of a straight line (more commonly expressed as $y = mx + b$) with slope B. Note that $v_x = dx/dt = B$; Fig. 2-15*b* shows the constant velocity.

As shown by Fig. 2-15 and also by Eq. 2-19, the particle is at the position $x = A$ when $t = 0$. It is moving in the direction of increasing x, so the slope B (and correspondingly the velocity v_x) is positive.

3. *Accelerated motion.* With acceleration defined as the rate of change of the velocity, accelerated motion then corresponds to motion in which the velocity changes. Since the velocity is the slope of the graph of $x(t)$, the slope must change in accelerated motion. These graphs are therefore curves rather than straight lines. Two examples of accelerated motion are:

$$x(t) = A + Bt + Ct^2, \tag{2-20}$$

$$x(t) = D\cos\omega t. \tag{2-21}$$

In the first case (Fig. 2-16*a*), assuming C is positive, the slope is increasing as the particle moves, corresponding to an increase in the positive velocity of the particle. According to Eqs. 2-12 and 2-17, $a_x = dv_x/dt = d^2x/dt^2$. For Eq. 2-20, $d^2x/dt^2 = 2C$, so the acceleration is constant. In the second case (Fig. 2-16*b*), the particle oscillates between $x = +D$ and $x = -D$; its velocity changes from positive to negative as the derivative of Eq. 2-21 changes sign.

Often the complete descriptions of motion are more complicated than these simple examples. Here are some additional cases to consider:

4. *Accelerating and braking car.* A car is at rest for a time and then begins accelerating until it reaches a certain velocity. It then moves for a time at that velocity, after which the brakes are applied to bring the car to rest again. Figure 2-17*a* shows the acceleration in the various time intervals; for simplicity we assume the acceleration is constant during the intervals when the car is speeding up and slowing down. The acceleration is zero when the car is at rest or when it is moving with constant velocity (as suggested by Eq. 2-17, when v_x is a constant then $a_x = dv_x/dt = 0$).

Figure 2-17*b* shows the velocity in each interval. Where $a_x = 0$, v_x is constant. Where a_x is a positive or negative constant, v_x is represented by a straight line with a corresponding positive or negative slope. At any point, the value of a_x can be found from the slope of the graph of v_x against t.

In a similar way we can obtain the position versus time graph $x(t)$, which is shown in Fig. 2-17*c*. Note that the slope of $x(t)$ gives $v_x(t)$, as required by Eq. 2-12 ($v_x = dx/dt$). For example, in the interval when the car is speeding up, the slope of $x(t)$ is gradually increasing, corresponding exactly to the increase in velocity. In the interval when the car is moving at constant velocity, $x(t)$ is represented by a straight line with a constant slope.

These graphs are somewhat idealized; a real car cannot go instantly from a state of rest to a state of constant acceleration. In practice, the sudden jumps in $a_x(t)$ would instead be connected by a continuous curve, and the sharp bends in the graph of $v_x(t)$ would become rounded.

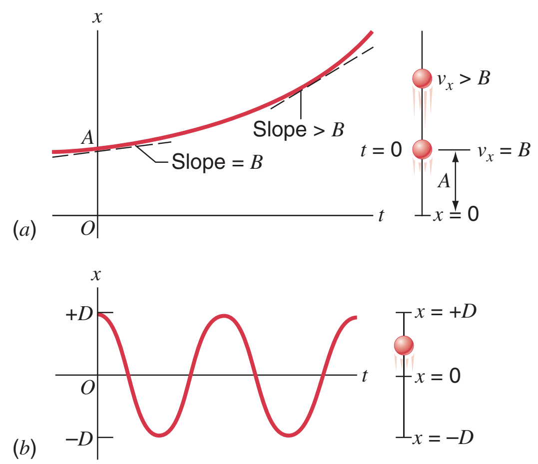


FIGURE 2-16. (*a*) A bead sliding along a wire in one dimension moves in the positive x direction with ever increasing velocity. The velocity is equal to the slope of the curve describing the particle's motion; you can see how the slope of the curve continually increases. (*b*) A bead sliding along a wire in one dimension oscillates between $x = +D$ and $x = -D$.

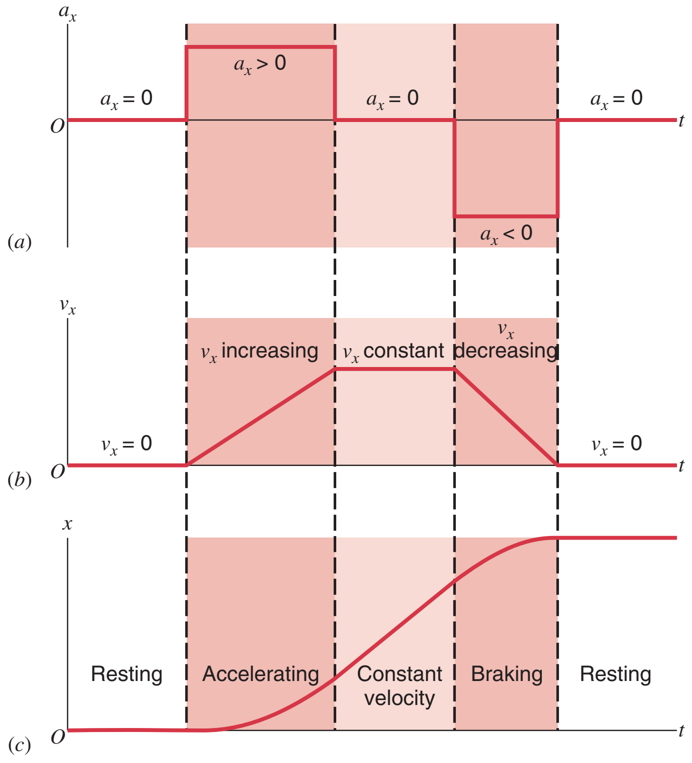


FIGURE 2-17. (*a*) The acceleration, (*b*) the velocity, and (*c*) the position of a car that starts at rest, accelerates for an interval, then moves with constant velocity, and then brakes with negative acceleration to rest again. In reality, we cannot instantly change the acceleration of a car from one value to another; both $a_x(t)$ and $v_x(t)$ for a real car would be smooth and continuous. Smooth curves would connect the flat $a_x(t)$ segments, and the sharp bends in $v_x(t)$ would become rounded.

5. *A falling object.* As we will discuss later in this chapter, when an object falls near the Earth's surface, it experiences a constant downward acceleration due to gravity. In this problem we take the y axis to be vertical and choose the upward direction as positive, so that the acceleration has a negative y component a_y. Figure 2-18*a* shows the acceleration $a_y(t)$ equal to a constant negative value.

If an object is released from rest, its downward (negative) velocity increases in magnitude due to this acceleration. Since a_y is a constant, $v_y(t)$ must be a straight line whose (negative) slope is equal to a_y, as shown in Fig. 2-18*b*. The vertical coordinate y begins at some positive value corresponding to the height H at which the object was originally released, and $y(t)$ decreases gradually to zero as the object falls to the ground, as shown in Fig. 2-18*c*. Initially the slope of $y(t)$ is zero, because $v_y(t)$ is initially zero. As v_y becomes more negative, the corresponding slope of $y(t)$ becomes more negative, as you can see by drawing lines tangent to $y(t)$ at various locations.

6. *A rebounding ball bearing.* Consider now a small steel ball bearing dropped from rest onto a hard surface from which it rebounds. We again choose the vertical direction as the y axis for this problem, and we take the upward direction to be positive.

After the ball bearing strikes the surface, it rebounds upward. We assume that the velocity simply changes direction upon contact with the surface—its magnitude remains unchanged. (In reality, there is a small loss of speed because the ball bearing and the surface are not perfectly elastic.)

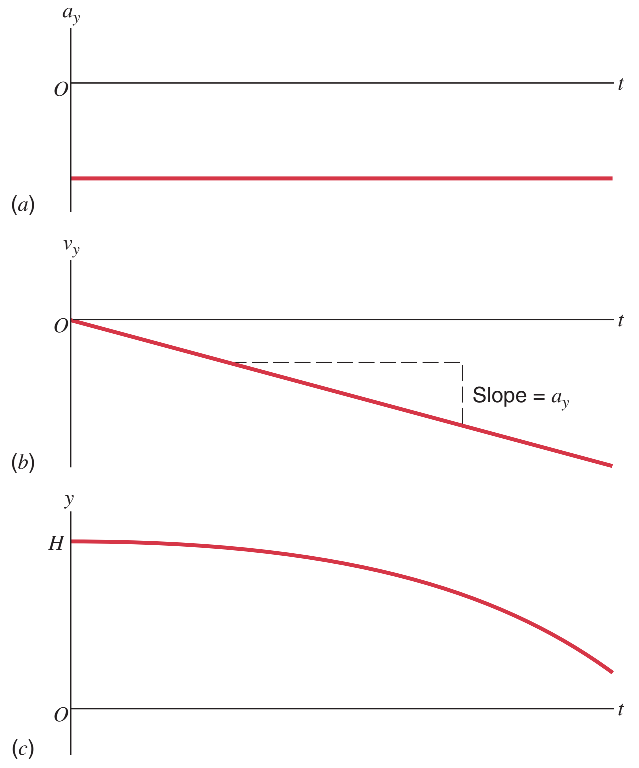


FIGURE 2-18. (*a*) The acceleration, (*b*) velocity, and (*c*) position of an object dropped from rest and accelerated downward by the Earth's gravity. The acceleration is a negative constant, which is equal to the slope of $v_y(t)$.

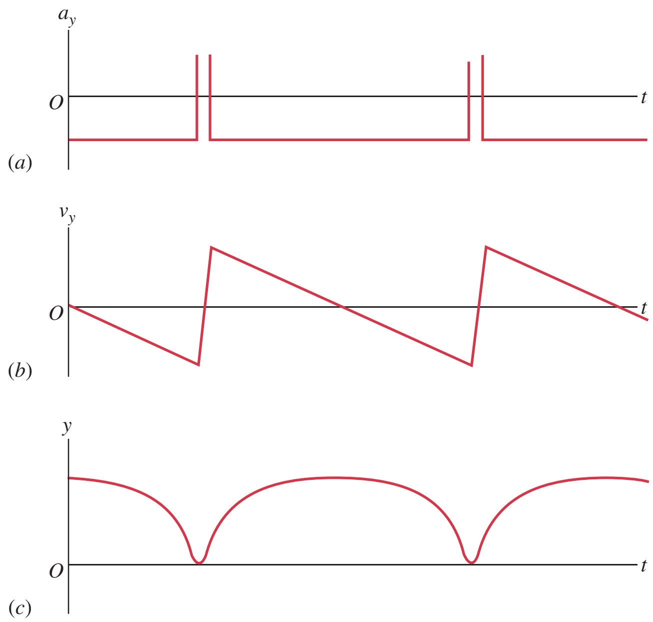


FIGURE 2-19. (*a*) The acceleration, (*b*) velocity, and (*c*) position of a ball bearing that falls and rebounds from a hard surface. The sudden jumps in $v_y(t)$, corresponding to the short intervals when the ball bearing is in contact with the surface, have been drawn as if the acceleration is equal to a large positive constant in those intervals. Note that $y(t)$ reaches its maximum value and that the tangent to $y(t)$ is horizontal when $v_y(t)$ goes through zero.

During the very short interval of time that the ball bearing is in contact with the surface, a large upward (positive) acceleration must act to reverse the direction of its velocity. Note that an acceleration is present even though the magnitude of the velocity is unchanged; an acceleration must occur whenever the magnitude *or the direction* of a velocity is changed.

As the ball bearing rises after the bounce, there is again a (constant) downward acceleration due to gravity, which eventually brings the ball bearing to rest for an instant, after which it begins falling toward the surface again.

Figure 2-19 shows the acceleration, velocity, and position for the ball bearing. Once, again $a_y(t)$ corresponds to the slope of $v_y(t)$, and $v_y(t)$ corresponds to the slope of $y(t)$.

This is also a somewhat idealized representation of this motion. The exact behavior during the instant of collision may be very complicated and will certainly not be characterized by a constant acceleration, as we have assumed here. Nevertheless, the overall behavior should look very similar to Fig. 2-19.

Equations of One-Dimensional Kinematics

We can directly apply the vector equations of Section 2-3 to motion in one dimension, which we take to be the x direction. Let the particle start at location x_1 at time t_1 and move to location x_2 at time t_2. In the time interval $\Delta t = t_2 - t_1$, the displacement of the particle is $\Delta x = x_2 - x_1$. According to Eq. 2-7, its average velocity is

$$v_{\text{av},x} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}. \quad (2\text{-}22)$$

The average velocity tells us about the average behavior during the time interval Δt. It depends only on the initial and final locations of the particle and not on the path of the particle between x_1 and x_2. If we assume that Δt is positive (that is, our clocks are running forward), then according to Eq. 2-22 and the rule for multiplying a vector by a scalar, the sign of $v_{\text{av},x}$ is determined by the sign of Δx. If $v_{\text{av},x} > 0$, then on the average the particle moves from a smaller x coordinate to a larger x coordinate (that is, it moves in the positive x direction). For example, the particle might move from $x_1 = -8$ m to $x_2 = -4$ m, or from $x_1 = -3$ m to $x_2 = +1$ m, or from $x_1 = +2$ m to $x_2 = +6$ m. In each case, $\Delta x = +4$ m, and so $v_{\text{av},x} > 0$. If $v_{\text{av},x} < 0$, the particle on the average moves in the negative x direction, for example from $x_1 = +5$ m to $x_2 = +2$ m or from $x_1 = -3$ m to $x_2 = -6$ m (both of which have $\Delta x = -3$ m).

The instantaneous velocity follows from Eq. 2-12:

$$v_x = \frac{dx}{dt}. \quad (2\text{-}23)$$

Equation 2-23 allows us to analyze the examples of one-dimensional motion discussed previously in this section. For example, for the motion at constant velocity shown in Fig. 2-15*a*, taking $x(t) = A + Bt$ gives $v_x = dx/dt = B$, as shown in Fig. 2-15*b*.

In the case of accelerated motion in which the particle is moving with velocity v_{1x} at time t_1 and accelerates to velocity v_{2x} at time t_2, Eq. 2-14 gives

$$a_{\text{av},x} = \frac{\Delta v_x}{\Delta t} = \frac{v_{2x} - v_{1x}}{t_2 - t_1}. \quad (2\text{-}24)$$

The sign of the average acceleration is determined by the sign of Δv_x; for example, a change in velocity from $v_{1x} = -9$ m/s to $v_{2x} = -4$ m/s or from $v_{1x} = +4$ m/s to $v_{2x} = +9$ m/s both correspond to $\Delta v_x = +5$ m/s and to positive $a_{\text{av},x}$, whereas a change from $v_{1x} = +9$ m/s to $v_{2x} = +4$ m/s or from $v_{1x} = -4$ m/s to $v_{2x} = -9$ m/s both give $\Delta v_x = -5$ m/s and negative $a_{\text{av},x}$.

As in the case of average velocity, the average acceleration depends only on the difference between the starting and ending velocities in the interval and not on the detailed motion during the interval. All motions that result in the same Δv_x in the interval Δt will give the same average acceleration.

The instantaneous acceleration is found in Eq. 2-17:

$$a_x = \frac{dv_x}{dt}. \quad (2\text{-}25)$$

For example, in the case shown in Fig. 2-16*a* with $x(t) = A + Bt + Ct^2$, Eq. 2-23 gives $v_x = B + 2Ct$ and Eq. 2-25 gives $a_x = 2C$.

Using Eqs. 2-23 and 2-25, you should review the motions graphed in Figs. 2-17, 2-18, and 2-19 to be sure you understand how position, velocity, and acceleration are related. Note especially that the acceleration is the slope of the graph of $v(t)$. Compare Figs. 2-17*a* and *b* to see that where the slope of $v_x(t)$ is zero (the horizontal line segments) then $a_x = 0$; where $v_x(t)$ is increasing (line segment with positive slope), a_x is a positive constant.

The acceleration may be positive or negative independent of whether the velocity is positive or negative. For example, let's consider an elevator moving vertically; we'll call that the *y* direction and take up as positive. If the elevator is moving upward but slowing down, v_y is positive but a_y is negative; in this case the velocity vector points in the $+y$ direction and the acceleration vector in the $-y$ direction. If the elevator is moving downward and braking, then v_y is negative but a_y is positive. These two cases, in which the velocity and acceleration vectors have opposite directions so that the speed (the magnitude of the velocity) is decreasing, are often referred to as *deceleration.*

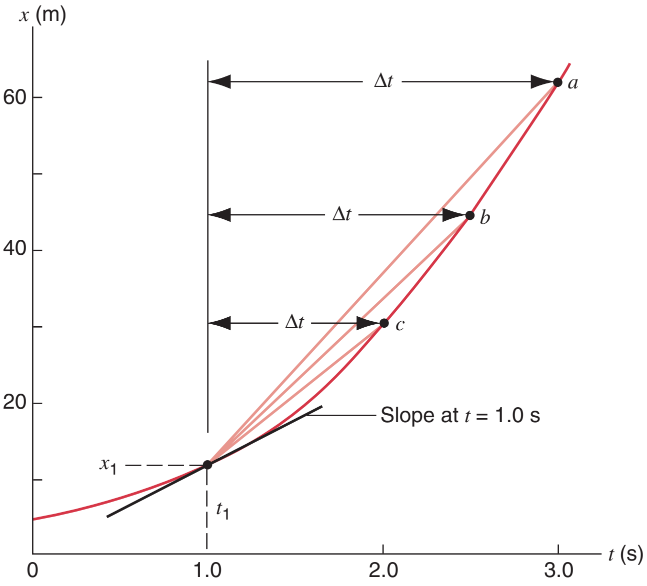


FIGURE 2-20. The interval Δt grows smaller, in this case as we keep t_1 fixed and move the other endpoint t_2 closer to t_1. In the limit, the interval goes to zero and the chord becomes a tangent. The three values of x_2 and t_2 shown here, labeled *a*, *b*, and *c*, correspond to the first three lines of Table 2-1.

The Limiting Process

It is interesting to observe how the average velocity approaches the instantaneous velocity as Δt approaches zero. As an example, we consider the case $x(t) = 5.0 + 1.0t + 6.0t^2$, where *x* is in meters and *t* is in seconds. We arbitrarily select the point $x_1 = 12$ m, $t_1 = 1.0$ s and calculate the value of $v_{av,x}$ using Eq. 2-22 by choosing a series of points x_2, t_2 that approach closer and closer to x_1, t_1 to simulate the limiting process. The calculated values are listed in Table 2-1. Note that they seem to be approaching the value $v_{av,x} = 13$ m/s. To compare this limiting value of $v_{av,x}$ with the instantaneous value, we use Eq. 2-23 to find $v_x(t) = 1.0 + 12.0t$, which does indeed evaluate to $v_x = 13$ m/s when $t = 1.0$ s.

Figure 2-20 gives a graphical illustration of this limiting process. As point 2 moves closer to point 1, the line connecting the two points, whose slope represents the average velocity in that interval, approaches the tangent at point 1. This indicates that the average velocity approaches the instantaneous velocity in this limit, as our mathematical calculation showed.

TABLE 2-1 The Limiting Process

Initial Point		*Final Point*		*Average Velocity*
x_1 (m)	t_1 (s)	x_2 (m)	t_2 (s)	$\Delta x/\Delta t$ (m/s)
12	1.0	62	3.0	25
12	1.0	45	2.5	22
12	1.0	31	2.0	19
12	1.0	20	1.5	16
12	1.0	13.4	1.1	13.6
12	1.0	12.7	1.05	13.3
12	1.0	12.1	1.01	13.06

SAMPLE PROBLEM 2-5. You drive your BMW down a straight road for 5.2 mi at 43 mi/h, at which point you run out of gas. You walk 1.2 mi farther, to the nearest gas station, in 27 min. What is your average velocity from the time that you start your car to the time that you arrive at the gas station?

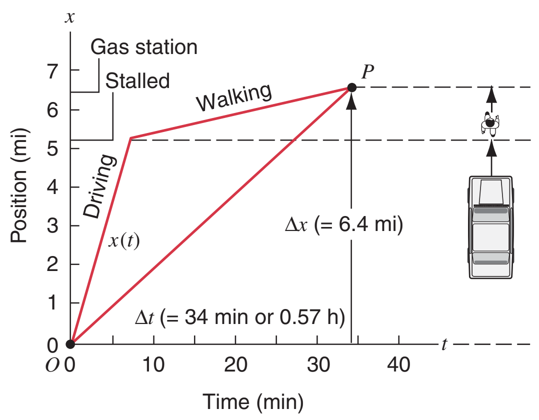


FIGURE 2-21. Sample Problem 2-5. The lines marked "Driving" and "Walking" show motions at different constant velocities for the two portions of the trip. The average velocity is the slope of the line *OP*.

Solution You can find your average velocity from Eq. 2-22 if you know both Δx, the net displacement, and Δt, the corresponding elapsed time. These quantities are

$$\Delta x = 5.2 \text{ mi} + 1.2 \text{ mi} = 6.4 \text{ mi}$$

and

$$\Delta t = \frac{5.2 \text{ mi}}{43 \text{ mi/h}} + 27 \text{ min}$$
$$= 7.3 \text{ min} + 27 \text{ min} = 34 \text{ min} = 0.57 \text{ h}.$$

From Eq. 2-22 we then have

$$v_{\text{av},x} = \frac{\Delta x}{\Delta t} = \frac{6.4 \text{ mi}}{0.57 \text{ h}} = 11.2 \text{ mi/h}.$$

The $x(t)$ plot of Fig. 2-21 helps us to visualize the problem. Points O and P define the interval for which we want to find the average velocity, this quantity being the slope of the straight line connecting these points.

SAMPLE PROBLEM 2-6. Figure 2-22*a* shows six successive "snapshots" of a particle moving along the x axis. At $t = 0$ it is at position $x = +1.00$ m to the right of the origin; at $t = 2.5$ s it has come to rest at $x = +5.00$ m; at $t = 4.0$ s it has returned to $x = 1.4$ m. Figure 2-22*b* is a plot of position x versus time t for this motion, and Figs. 2-22*c* and 2-22*d* show the corresponding velocity and acceleration of the particle. (*a*) Find the average velocity for the intervals AD and DF. (*b*) Estimate the slope of $x(t)$ at points B and F and compare with the corresponding points on the $v_x(t)$ curve. (*c*) Find the average acceleration in the intervals AD and AF. (*d*) Estimate the slope of $v_x(t)$ at point D and compare with the corresponding value of $a_x(t)$.

Solution (*a*) From Eq. 2-22,

$$AD: \quad v_{\text{av},x} = \frac{\Delta x_{AD}}{\Delta t_{AD}} = \frac{x_D - x_A}{t_D - t_A}$$
$$= \frac{5.0 \text{ m} - 1.0 \text{ m}}{2.5 \text{ s} - 0.0 \text{ s}} = +1.6 \text{ m/s}$$

$$DF: \quad v_{\text{av},x} = \frac{\Delta x_{DF}}{\Delta t_{DF}} = \frac{x_F - x_D}{t_F - t_D}$$
$$= \frac{1.4 \text{ m} - 5.0 \text{ m}}{4.0 \text{ s} - 2.5 \text{ s}} = -2.4 \text{ m/s}$$

In the interval AD, the positive sign for $v_{\text{av},x}$ tells us that, on the average, the particle moves in the direction of increasing x (that is, to the right in Fig. 2-22*a*) during that interval. The negative sign for $v_{\text{av},x}$ in the interval DF tells us that the particle is, on the average, moving in the direction of decreasing x (to the left in Fig. 2-22*a*) during that interval.

(*b*) From the tangents to $x(t)$ drawn at points B and F in Fig. 2-22*b* we estimate the following:

$$\text{point } B: \quad \text{slope} = \frac{4.5 \text{ m} - 2.8 \text{ m}}{1.5 \text{ s} - 0.5 \text{ s}} = +1.7 \text{ m/s}$$

$$\text{point } F: \quad \text{slope} = \frac{1.4 \text{ m} - 4.5 \text{ m}}{4.0 \text{ s} - 3.5 \text{ s}} = -6.2 \text{ m/s}$$

From $v_x(t)$ in Fig. 2-22*c* we estimate $v_x = +1.7$ m/s at point B and $v_x = -6.2$ m/s at part F, in agreement with the slopes of $x(t)$. As expected, $v_x(t) = dx/dt$.

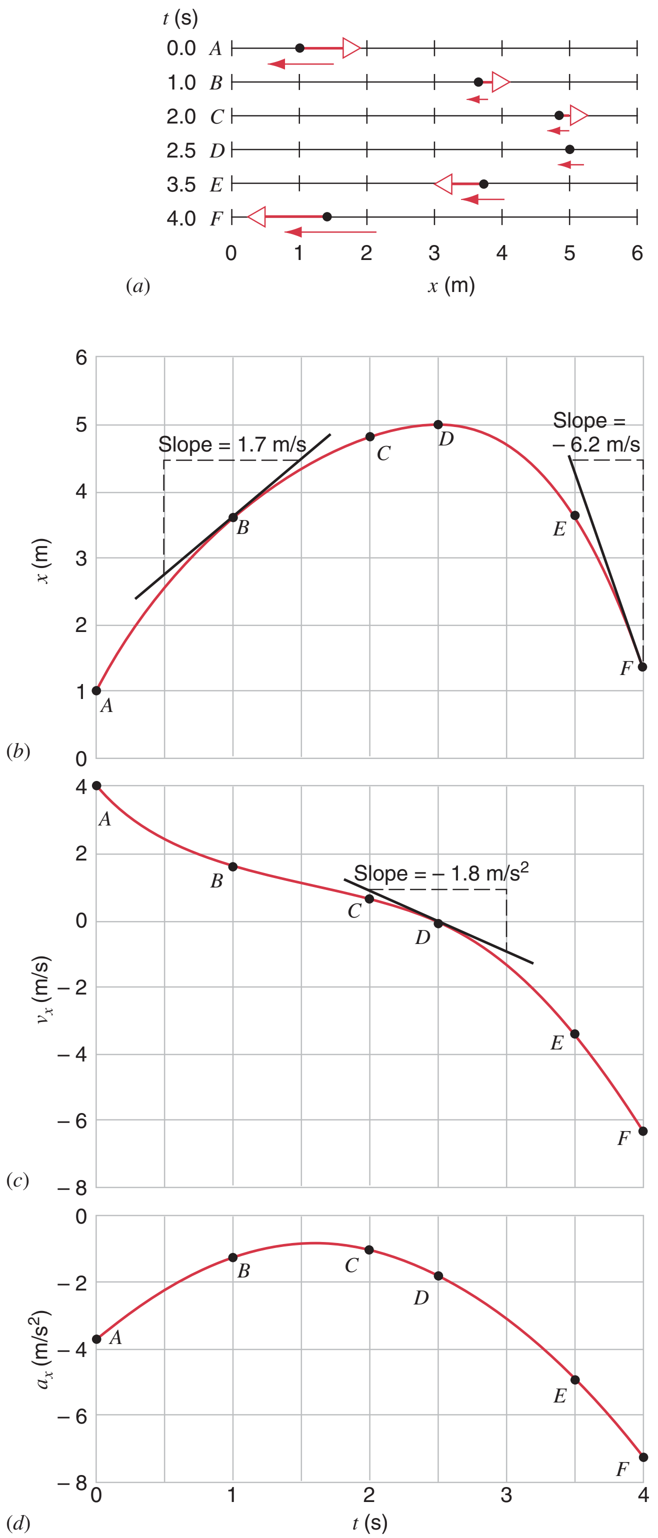


FIGURE 2-22. Sample Problem 2-6. (*a*) Six consecutive "snapshots" of a particle moving along the x axis. The arrow *through* the particle shows its instantaneous velocity, and the arrow *below* the particle shows its instantaneous acceleration. (*b*) A plot of $x(t)$ for the motion of the particle. The six points A–F correspond to the six snapshots. (*c*) A plot of $v_x(t)$. (*d*) A plot of $a_x(t)$.

(c) From Eq. 2-24,

$$AD: \quad a_{\mathrm{av},x} = \frac{\Delta v_{AD}}{\Delta t_{AD}} = \frac{v_D - v_A}{t_D - t_A}$$

$$= \frac{0.0\ \mathrm{m/s} - 4.0\ \mathrm{m/s}}{2.5\ \mathrm{s} - 0.0\ \mathrm{s}} = -1.6\ \mathrm{m/s^2}$$

$$AF: \quad a_{\mathrm{av},x} = \frac{\Delta v_{AF}}{\Delta t_{AF}} = \frac{v_F - v_A}{t_F - t_A}$$

$$= \frac{-6.2\ \mathrm{m/s} - 4.0\ \mathrm{m/s}}{4.0\ \mathrm{s} - 0.0\ \mathrm{s}} = -2.6\ \mathrm{m/s^2}$$

(d) From the line drawn tangent to $v_x(t)$ at D, we estimate the following:

$$\text{slope} = \frac{-0.9\ \mathrm{m/s} - 0.9\ \mathrm{m/s}}{3.0\ \mathrm{s} - 2.0\ \mathrm{s}} = -1.8\ \mathrm{m/s^2}.$$

At point D on the $a_x(t)$ graph we see $a_x = -1.8\ \mathrm{m/s^2}$. Thus $a_x = dv_x/dt$. Examining the $v_x(t)$ graph of Fig. 2-22c, we see that its slope is negative at all times covered by the graph, and thus $a_x(t)$ should be negative. Figure 2-22d bears this out.

2-5 MOTION WITH CONSTANT ACCELERATION

It is fairly common to encounter motion with constant (or nearly constant) acceleration: the examples already cited of objects falling near the Earth's surface or braking cars are typical. In this section we derive a set of equations for analyzing this special case of one-dimensional kinematics with constant acceleration. Keep in mind, however, that these results can be applied *only* when the acceleration is constant and therefore do *not* apply to such situations as a swinging pendulum bob, a rocket blasting off toward Earth orbit, or a raindrop falling against air resistance. (Later in this book we will discuss methods for analyzing these situations.)

Let's assume our motion is along the x direction. We let a_x represent the x component of the acceleration vector; that is, a_x can be positive or negative. The particle's initial ($t = 0$) velocity is v_{0x} and its initial position is x_0, both of which are also the x components of vectors and can be independently positive or negative. At a later time t, the particle has velocity v_x and is located at position x. Our goal is to find the position and velocity at time t.

For constant acceleration, the instantaneous and average accelerations are everywhere equal, and so we can use Eq. 2-14 to write

$$a_x = a_{\mathrm{av},x} = \frac{\Delta v_x}{\Delta t} = \frac{v_x - v_{0x}}{t - 0}$$

or, solving for v_x,

$$v_x = v_{0x} + a_x t. \tag{2-26}$$

This important result allows us to find the velocity at all times, but *only for constant acceleration.* Equation 2-26 gives the velocity as a function of the time, which we could write as $v_x(t)$ but which we usually write simply as v_x. Note that Eq. 2-26 is in the form of $y = mx + b$, which describes the graph of a straight line. Thus the plot of v_x against t gives a straight line of slope a_x and intercept v_{0x} (the value of v_x at $t = 0$). This line is plotted in Fig. 2-23b. It is apparent that Eq. 2-26 satisfies Eq. 2-25 ($a_x = dv_x/dt$).

Let us now find how the position varies with time. In this special case in which $v_x(t)$ is a straight line, the average velocity in any interval (as we have defined it in Eq. 2-22) is also equal to the average of the initial and final velocities in that interval. For the time interval from 0 to t,

$$v_{\mathrm{av},x} = \tfrac{1}{2}(v_x + v_{0x}). \tag{2-27}$$

You can see that this must be true from the straight-line graph in Fig. 2-23b. Combining Eqs. 2-22, 2-26, and 2-27, we can eliminate v_x and solve for x to obtain

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2, \tag{2-28}$$

which gives us the position x at all times. We can also write Eq. 2-28 to find the displacement $x - x_0$ (the net change in position in the interval). Figure 2-23c shows a graph of x against t, which has the form of a parabola. Equations 2-27 and 2-28 are valid *only for constant acceleration.*

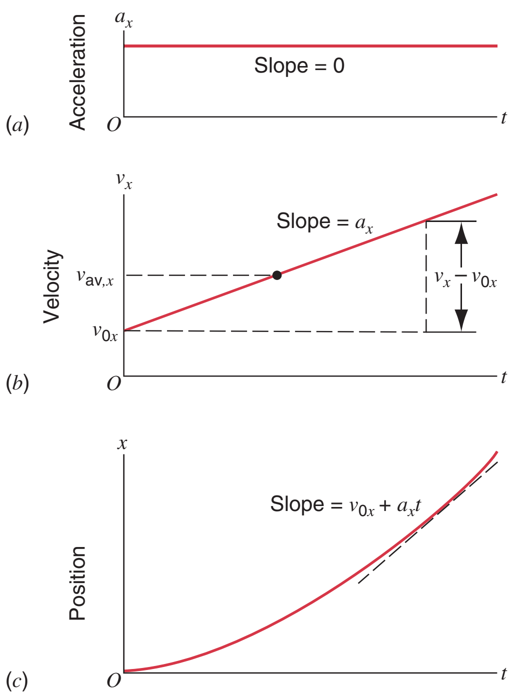


FIGURE 2-23. (a) The constant acceleration of a particle, equal to the (constant) slope of $v_x(t)$. (b) Its velocity $v_x(t)$, given at each point by the slope of the $x(t)$ curve. The average velocity $v_{\mathrm{av},x}$, which in the case of constant acceleration is equal to the average of v_x and v_{0x}, is indicated. (c) The position $x(t)$ of a particle moving with constant acceleration. The curve is drawn for initial position $x_0 = 0$.

The instantaneous velocity and position should be related by $v_x = dx/dt$. Equation 2-28 satisfies that relationship, as we can show:

$$\frac{dx}{dt} = \frac{d}{dt}(x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2) = v_{0x} + a_x t = v_x.$$

Students who are already familiar with integral calculus will recognize that, just as Eq. 2-26 can be obtained from Eq. 2-28 by differentiating, Eq. 2-28 can be obtained from Eq. 2-26 by integrating. We will demonstrate this process at the end of this section.

Equations 2-26 and 2-28 will serve as our basic equations for analyzing motion with constant acceleration. If we specify the *initial conditions* (the initial values of the position x_0 and the velocity v_0) and the acceleration (which later we will learn comes from the interaction of the particle with its environment), we can find the position and velocity for all values of the time t.

Keep in mind that in these equations v_x and x represent the x components of the velocity and position vectors. As is always the case in problems involving vectors, we can locate our coordinate axes anywhere we like and in any orientation. In these problems we must select the origin of the coordinate system (often chosen so that $x_0 = 0$, which simplifies the problem) and the direction of the positive x axis, so that all displacements, velocities, and accelerations in that direction are positive and those in the opposite direction are negative. Once you have chosen the origin and direction of your coordinate system, they must remain in effect throughout the solution of that problem.

Integrating the Equations of Motion (Optional)*

Equations 2-26 and 2-28, our basic expressions for kinematics with constant acceleration, can also be obtained using the methods of integral calculus. We start with the definition of acceleration, $a_x = dv_x/dt$, which we can write as

$$dv_x = a_x dt.$$

We take the integral of both sides of this equation:

$$\int dv_x = \int a_x dt = a_x \int dt,$$

where the last step, taking the acceleration out of the integral, can be made because the acceleration is constant. Carrying out the integrals, we obtain

$$v_x = a_x t + C,$$

where C is a constant of integration. We can determine the constant C from one of the initial conditions: at $t = 0$, the velocity is v_{0x}. Substituting these values into the above equation, we find $C = v_{0x}$, and so we obtain $v_x = v_{0x} + a_x t$, in agreement with Eq. 2-26.

To find $x(t)$ by integration, we begin with the definition of velocity, $v_x = dx/dt$, which we write as

$$dx = v_x dt.$$

We now substitute Eq. 2-26 for v_x and integrate on both sides:

$$\int dx = \int (v_{0x} + a_x t)\, dt = v_{0x}\int dt + a_x \int t\, dt.$$

Carrying out the integrals, we obtain

$$x = v_{0x}t + \tfrac{1}{2}a_x t^2 + C',$$

where C' is another constant of integration. To find this constant, we use the second initial condition: at $t = 0$, $x = x_0$. Substituting these values, we find $C' = x_0$ which gives $x(t)$ in agreement with Eq. 2-28. ■

*Students who are not yet familiar with integral calculus may want to delay reading this section.

SAMPLE PROBLEM 2-7. An alpha particle (the nucleus of a helium atom) travels along the inside of an evacuated straight tube 2.0 m long that forms part of a particle accelerator. The alpha particle enters the tube (at $t = 0$) moving at a velocity 9.5×10^5 m/s and emerges from the other end at time $t = 8.0 \times 10^{-7}$ s. (*a*) If the particle's acceleration is constant, what is the acceleration? (*b*) What is its velocity when it leaves the tube?

Solution (*a*) We choose the x axis to be along the tube with its positive direction to be that of the motion of the particle, and we take the origin to be at the entrance of the tube so that $x_0 = 0$. We can find the acceleration by solving Eq. 2-28 for a_x:

$$a_x = \frac{x - v_{0x}t}{\tfrac{1}{2}t^2}$$

$$= \frac{2.0\text{ m} - (9.5 \times 10^5\text{ m/s})(8.0 \times 10^{-7}\text{ s})}{0.5(8.0 \times 10^{-7}\text{ s})^2}$$

$$= +3.9 \times 10^{12}\text{ m/s}^2.$$

The positive sign tells us that the particle is speeding up as it passes through the tube.

(*b*) To find the velocity as the particle leaves the tube, we use Eq. 2-26:

$$v_x = v_{0x} + a_x t$$

$$= (9.5 \times 10^5\text{ m/s}) + (3.9 \times 10^{12}\text{ m/s}^2)(8.0 \times 10^{-7}\text{ s})$$

$$= +4.1 \times 10^6\text{ m/s}.$$

Consistent with the positive acceleration, the particle's velocity does increase.

SAMPLE PROBLEM 2-8. You brake your Porsche with constant acceleration from a velocity of 23.6 m/s (about 53 mph, well below the speed limit, of course) to 12.5 m/s over a distance of 105 m. (*a*) How much time elapses during this interval? (*b*) What is the acceleration? (*c*) If you were to continue braking with the same constant acceleration, how much longer would it take for you to stop and how much additional distance would you cover?

Solution (*a*) We select the positive direction for our coordinate system to be the direction of the velocity and choose the origin so that $x_0 = 0$ when you begin braking. Then the initial velocity is $v_{0x} = +23.6$ m/s at $t = 0$, and the final velocity and position are $v_x = +12.5$ m/s and $x = 105$ m at time t. Because the acceleration is constant, the average velocity in the interval can be found from the average of the initial and final velocities according to Eq. 2-27:

$$v_{\mathrm{av},x} = \tfrac{1}{2}(v_x + v_{0x}) = \tfrac{1}{2}(12.5\ \mathrm{m/s} + 23.6\ \mathrm{m/s}) = 18.05\ \mathrm{m/s}.$$

The average velocity can also be expressed as $v_{\mathrm{av},x} = \Delta x/\Delta t$. With $\Delta x = 105$ and $\Delta t = t - 0$, we can solve for t:

$$t = \frac{\Delta x}{v_{\mathrm{av},x}} = \frac{105\ \mathrm{m}}{18.05\ \mathrm{m/s}} = 5.81\ \mathrm{s}.$$

(*b*) We can now find the acceleration from Eq. 2-26:

$$a_x = \frac{v_x - v_{0x}}{t} = \frac{12.5\ \mathrm{m/s} - 23.6\ \mathrm{m/s}}{5.81\ \mathrm{s}} = -1.91\ \mathrm{m/s^2}.$$

The acceleration is negative, which means that, as expected, the positive velocity is growing smaller as you brake.
(*c*) Now with a known acceleration, we can find the total time for the car to go from velocity $v_{0x} = 23.6$ m/s to $v_x = 0$. Solving Eq. 2-26 for t, we find

$$t = \frac{v_x - v_{0x}}{a_x} = \frac{0 - 23.6\ \mathrm{m/s}}{-1.91\ \mathrm{m/s^2}} = 12.4\ \mathrm{s}.$$

The total distance covered is found from Eq. 2-28 for this interval of time with $x_0 = 0$:

$$x = v_{0x}t + \tfrac{1}{2}a_x t^2$$
$$= (23.6\ \mathrm{m/s})(12.4\ \mathrm{s}) + \tfrac{1}{2}(-1.91\ \mathrm{m/s^2})(12.4\ \mathrm{s})^2 = 146\ \mathrm{m}.$$

From the time you originally began braking to the time your car came to rest, you covered a total distance of 146 m in a total time of 12.4 s. The change from 23.6 m/s to 12.5 m/s covered a distance of 105 m in 5.8 s, so the change from 12.5 m/s to 0 covered a distance of 146 m − 105 m = 41 m and lasted for a time of 12.4 s − 5.8 s = 6.6 s.

2-6 FREELY FALLING BODIES

The most common example of motion with (nearly) constant acceleration is that of a falling body near the Earth's surface. If we neglect air resistance, we find a remarkable fact: at any given point near the Earth's surface, *all bodies, regardless of their size, shape, or composition, fall with the same acceleration.* This acceleration, denoted by the symbol g, is called the *free-fall acceleration* (or sometimes the *acceleration due to gravity*). Although the acceleration depends on the distance from the center of the Earth (as we shall show in Chapter 14), if the distance of fall is small compared with the Earth's radius (6400 km) we can regard the acceleration as constant throughout the fall.

Near the Earth's surface the magnitude of g is approximately 9.8 m/s^2, a value that we use throughout the text unless we specify otherwise. The direction of the free-fall acceleration at any point establishes what we mean by the word "down" at that point.

Although we speak of *falling* bodies, bodies in upward motion experience the same free-fall acceleration (magnitude *and* direction). That is, no matter whether a particle is moving up or down, the direction of its acceleration under the influence of the Earth's gravity is always down.

The exact value of the free-fall acceleration varies with latitude and with altitude. There are also significant variations caused by differences in the local density of the Earth's crust. We discuss these variations in Chapter 14.

The equations of constant acceleration (Eqs. 2-26 and 2-28) can be applied to free fall. For this purpose, we first make two small changes: (1) We label the direction of free fall as the y axis and take its positive direction to be upward. Later, in Chapter 4, we shall consider motion in two dimensions, and we shall want to use the x label for horizontal motion. (2) We replace the constant acceleration a with $-g$, since our choice of the positive y direction to be upward means that the downward acceleration is negative. By choosing $a_y = -g$, we will always have g as a positive number.

With these small changes, the equations describing freely falling bodies are

$$v_y = v_{0y} - gt, \tag{2-29}$$

$$y = y_0 + v_{0y}t - \tfrac{1}{2}gt^2. \tag{2-30}$$

Just as we used Eqs. 2-26 and 2-28 to solve problems involving constant acceleration, we can use Eqs. 2-29 and 2-30 as the basic formulas for solving problems involving free fall.

SAMPLE PROBLEM 2-9. A body is dropped from rest and falls freely. Determine the position and velocity of the body after 1.0, 2.0, 3.0, and 4.0 s have elapsed.

Solution We choose the starting point as the origin so that $y_0 = 0$. We know the initial speed (zero) and the acceleration, and we are given the time. To find the position, we use Eq. 2-30 with $y_0 = 0$ and $v_{0y} = 0$:

$$y = -\tfrac{1}{2}gt^2.$$

Putting $t = 1.0$ s, we obtain

$$y = -\tfrac{1}{2}(9.8\ \mathrm{m/s^2})(1.0\ \mathrm{s})^2 = -4.9\ \mathrm{m}.$$

To find the velocity, we use Eq. 2-29, again with $v_{0y} = 0$:

$$v_y = -gt = -(9.8\ \mathrm{m/s^2})(1.0\ \mathrm{s}) = -9.8\ \mathrm{m/s}.$$

After falling for 1.0 s, the body is 4.9 m *below* (y is negative) its starting point and is moving *downward* (v_y is negative) with a speed of 9.8 m/s. Continuing in this way, we can find the positions and velocities at $t = 2.0$, 3.0, and 4.0 s, which are shown in Fig. 2-24.

Note that the change in velocity in each second is −9.8 m/s, and that the average velocity during each one-second interval (equal to the displacement in that interval divided by the time interval) is equal to half the sum of the initial and final velocities in the interval, as required by Eq. 2-27.

SAMPLE PROBLEM 2-10. A ball is thrown vertically upward from the ground with a speed of 25.2 m/s. (*a*) How long does it take to reach its highest point? (*b*) How high does it rise? (*c*) At what times will it be 27.0 m above the ground?

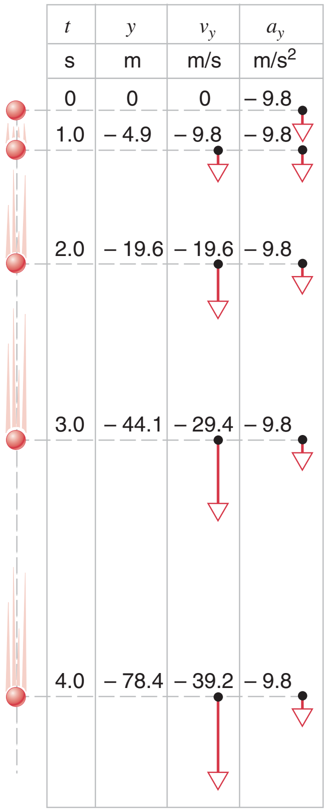


FIGURE 2-24. Sample Problem 2-9. The height, velocity, and acceleration of a body in free fall are shown.

Solution (*a*) At its highest point its velocity passes through the value zero. Given v_{0y} and $v_y (= 0)$, we wish to find t and we therefore choose Eq. 2-29, which we solve for t:

$$t = \frac{v_{0y} - v_y}{g} = \frac{25.2 \text{ m/s} - 0}{9.8 \text{ m/s}^2} = 2.57 \text{ s}.$$

(*b*) Now that we have found the time for the ball to reach its maximum height, Eq. 2-30, with y_0 assigned as 0, allows us to solve for y when we know the other quantities:

$$y = v_{0y}t - \tfrac{1}{2}gt^2$$
$$= (25.2 \text{ m/s})(2.57 \text{ s}) - \tfrac{1}{2}(9.8 \text{ m/s}^2)(2.57 \text{ s})^2 = 32.4 \text{ m}.$$

(*c*) Equation 2-30 is useful for this case, because t is the only unknown. Since we wish to solve for t, let us rewrite Eq. 2-30, with $y_0 = 0$, in the usual form of a quadratic equation:

$$\tfrac{1}{2}gt^2 - v_{0y}t + y = 0,$$

or, inserting the numerical values,

$$(4.9 \text{ m/s}^2)t^2 - (25.2 \text{ m/s})t + 27.0 \text{ m} = 0.$$

Solving this quadratic equation, we find $t = 1.52$ s and $t = 3.62$ s. At $t = 1.52$ s, the velocity of the ball is

$$v_y = v_{0y} - gt = 25.2 \text{ m/s} - (9.8 \text{ m/s}^2)(1.52 \text{ s}) = 10.3 \text{ m/s}.$$

At $t = 3.62$ s, the velocity is

$$v_y = v_{0y} - gt = 25.2 \text{ m/s} - (9.8 \text{ m/s}^2)(3.62 \text{ s}) = -10.3 \text{ m/s}.$$

The two velocities have identical magnitudes but opposite directions. You should be able to convince yourself that, in the absence of air resistance, the ball will take as long to rise to its maximum height as to fall the same distance, and that at each point it will have the same speed going up that it has coming down. Note that the answer to part (*a*) for the time to reach the highest point, 2.57 s, is exactly midway between the two times found in part (*c*). Can you explain this? Can you predict qualitatively the effect of air resistance on the times of rise and fall?

Figure 2-25 illustrates the motion of the ball. Note especially the symmetry of the upward and downward motions.

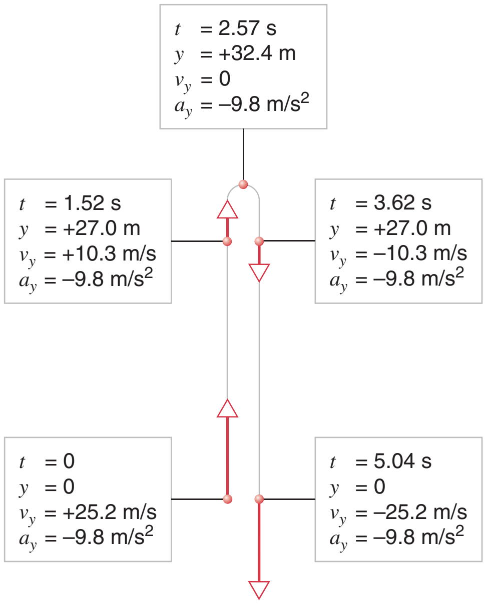


FIGURE 2-25. Sample Problem 2-10. The height, velocity, and acceleration at various points are shown.

SAMPLE PROBLEM 2-11. A rocket is launched from rest from an underwater base a distance of 125 m below the surface of a body of water. It moves vertically upward with an unknown but assumed constant acceleration (the combined effect of its engines, Earth's gravity, and the buoyancy and drag of the water), and it reaches the surface in a time of 2.15 s. When it breaks the surface its engines automatically shut off (to make it more difficult to detect) and it continues to rise. What maximum height does it reach? (Ignore any effects at the surface.)

Solution As with any projectile in free fall, we could analyze the motion of the rocket during the portion of its motion in the air if we knew the initial velocity of that part of the motion. The plan of attack in this problem is therefore to analyze the underwater portion of the motion to find the velocity when the rocket reaches the surface, and then to treat that velocity as the initial velocity of the free-fall portion. These parts must be done separately, because the acceleration changes at the surface of the water.

For the underwater motion, which is accelerated but not free fall, we can find the acceleration from Eq. 2-28 (replacing x by y) with $y - y_0 = 125$ m and $v_{0y} = 0$:

$$a_y = \frac{2(y - y_0)}{t^2} = \frac{2(125 \text{ m})}{(2.15 \text{ s})^2} = 54.1 \text{ m/s}^2.$$

Now Eq. 2-26 (again with x replaced by y) gives the final velocity for this portion of the motion:

$$v_y = v_{0y} + a_yt = 0 + (54.1 \text{ m/s}^2)(2.15 \text{ s}) = 116 \text{ m/s}.$$

The velocity at the surface is 116 m/s upward. We now analyze the free-fall portion of the motion, taking this velocity to be the *initial* velocity v_{0y}. At the highest point, the rocket comes instantaneously to rest ($v_y = 0$); we use Eq. 2-29 to find the time at which this occurs:

$$t = -\frac{v_y - v_{0y}}{g} = -\frac{(0 - 116\text{ m/s})}{9.8\text{ m/s}^2} = 11.8\text{ s}.$$

The height at this time is, from Eq. 2-30 with $y_0 = 0$:

$$y = v_{0y}t - \tfrac{1}{2}gt^2 = (116\text{ m/s})(11.8\text{ s}) - \tfrac{1}{2}(9.8\text{ m/s}^2)(11.8\text{ s})^2$$

$$= 687\text{ m}.$$

To test your understanding of this problem, you should draw graphs of $y(t)$, $v_y(t)$, and $a_y(t)$ in a fashion similar to Fig. 2-17. Be sure to keep in mind which variables change continuously and smoothly and which do not in this idealized problem. How would the motion of a real rocket vary from this picture?

Measuring the Free-Fall Acceleration (Optional)

The nature of the motion of a falling object has long been of interest to scientists and philosophers. Aristotle (384–322 B.C.) thought that heavier objects would fall more rapidly because of their weight. That was the prevailing view for two millennia, until Galileo Galilei (1564–1642) made the correct assertion, that in the absence of air resistance all objects fall with the same speed. We can test this assertion by dropping a feather and a ball of lead in a vacuum, and we find that they do indeed fall at the same rate. In 1971, astronaut David Scott dropped a feather and a hammer on the (airless) Moon, and he observed that they reached the surface at about the same time.

In Galileo's time, there was no way to achieve a vacuum, and he lacked the equipment to make precise measurements of time intervals. (The famous story about Galileo dropping different objects from the Tower of Pisa and observing them to reach the ground at the same time is probably only legend. For a tower of that height, air resistance would have a considerable effect on objects unless they had exactly the same size and shape.) Galileo reduced the acceleration (making it easier to measure time intervals and also reducing the effect of air resistance) by rolling a ball down an incline instead of dropping it. He showed that in equal time intervals the ball covered distances that were proportional to the odd integers 1,3,5,7, ... The total distances covered for consecutive intervals were thus proportional to 1, 1 + 3 (= 4), 1 + 3 + 5 (= 9), 1 + 3 + 5 + 7 (= 16), and so on. He thus concluded that the distances increased as the *square* of the time, which we now know holds *only* in the case of constant acceleration. He also found that the same results were obtained regardless of the mass of the ball, and thus (in present-day terminology) he deduced that the free-fall acceleration is independent of the mass of the object.

Today the measurement of g is a standard exercise in the introductory laboratory. By timing the fall of an object over a distance of a meter or two (which takes about 0.5 s), it is possible to determine g to a precision of a few percent. Using even a crude pendulum (which "slows down" the motion in analogy with Galileo's incline) and measuring the time for one complete back-and-forth swing, you can (as we discuss in Chapter 17) determine g to a precision of about 0.1%. This level of precision is sufficient to observe the variation in g between sea level and a high mountain (3 km or 10,000 ft), or between the equator and the poles of the Earth.

With carefully designed apparatus, the pendulum method can be extended to a precision of 1 part in 10^6, sufficient to detect variations in g from one floor of a building to the next. To achieve even greater precision, investigators have refined the free-fall method. By dropping an object in vacuum and reflecting a laser beam from it as it falls, very precise determinations of the distance of fall can be made. Coupled with atomic clocks to measure the time of fall, the value of g can be determined to a precision of about 1 part in 10^9, which enables the variation in g over a vertical distance of 1 cm to be observed. Equivalently, such a gravity meter can detect the gravitational effect of the measuring scientist standing 1 m from the apparatus!

Falling-body gravimeters for making these precise measurements are now commercially available. In the latest model, the falling object is placed in an evacuated box and the object is projected upward, so that measurements can be taken as the object rises and as it falls, as suggested in Problem 33. Figure 2-26 shows a portable version of this type of apparatus.

FIGURE 2-26. A portable rise-and-fall gravimeter. Its uses include geophysical research, oil and mineral exploration, and inertial navigation. Photo provided courtesy of Dr. T.M. Niebauer, Micro-g Solutions. (See http://www.microgsolutions.com.)

Such accurate measurements of the free-fall acceleration permit detailed studies of the Earth's gravity, which has important practical consequences. Variations in g from place to place can reveal the presence of oil or minerals under the Earth's surface, and variations in g with time can reveal movement of the Earth's plates or seismic activity. Knowledge of the small variations in g due to irregularities in the Earth's gravity enables accurate calculations of the paths of ballistic missiles or Earth satellites. In addition to these practical applications, precise measurements of g can provide detailed tests of our understanding of the theory of gravitation, one of the basic forces of the universe. ■

MULTIPLE CHOICE

2-1 Kinematics with Vectors

2-2 Properties of Vectors

2-3 Position, Velocity, and Acceleration Vectors

1. An object is moving with velocity given by $\vec{\mathbf{v}}(t) = v_x(t)\hat{\mathbf{i}} + v_y(t)\hat{\mathbf{j}} + v_z(t)\hat{\mathbf{k}}$ where $v_z(t) = 0$. From this, one can conclude

(*a*) that the acceleration $\vec{\mathbf{a}}(t)$

(A) will have no components that are identically zero.
(B) may have some components that are identically zero.
(C) will have only a z component that is identically zero.
(D) will have an identically zero z component, and maybe an identically zero component in the x or y direction.

(*b*) and that the position $\vec{\mathbf{r}}(t)$

(A) will have no components that are identically zero.
(B) may have some components that are identically zero.
(C) will have only a z component that is identically zero.
(D) will have an identically zero z component, and maybe an identically zero component in the x or y direction.

2. An object is moving in the xy plane with the position as a function of time given by $\vec{\mathbf{r}} = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$. Point O is at $\vec{\mathbf{r}} = 0$. The object is definitely moving toward O when

(A) $v_x > 0, v_y > 0$. (B) $v_x < 0, v_y < 0$.
(C) $xv_x + yv_y < 0$. (D) $xv_x + yv_y > 0$.

2-4 One-Dimensional Kinematics

3. An object is launched straight up into the air from the ground with an initial vertical velocity of 30 m/s. The object rises to a highest point approximately 45 m above the ground in 3 seconds; it then falls back to the ground in 3 more seconds, impacting with a speed of 30 m/s.

(*a*) The average *speed* of the object during the 6-second interval is closest to

(A) 0 m/s. (B) 5 m/s. (C) 15 m/s. (D) 30 m/s.

(*b*) The magnitude of the average *velocity* during the 6-second interval is closest to

(A) 0 m/s. (B) 5 m/s. (C) 15 m/s. (D) 30 m/s.

4. A car travels 15 miles east at a constant speed of 20 mi/h, then continues east for 20 miles at a constant 30 mi/h. What can be concluded about the magnitude of the average velocity?

(A) $v_{av} < 25$ mi/h. (B) $v_{av} = 25$ mi/h.
(C) $v_{av} > 25$ mi/h.
(D) More information is needed to answer the question.

5. An object is moving along the x axis with position as a function of time given by $x = x(t)$. Point O is at $x = 0$. The object is definitely moving toward O when

(A) $dx/dt < 0$. (B) $dx/dt > 0$.
(C) $d(x^2)/dt < 0$. (D) $d(x^2)/dt > 0$.

6. An object starts from rest at $x = 0$ when $t = 0$. The object moves in the x direction with positive velocity after $t = 0$. The instantaneous velocity and average velocity are related by

(A) $dx/dt < x/t$. (B) $dx/dt = x/t$. (C) $dx/dt > x/t$.
(D) dx/dt can be larger than, smaller than, or equal to x/t.

7. Figure 2-27 shows several graphs with unlabeled axes. (*a*) Which graph would best represent velocity as a function of time for an object moving with constant speed? (*b*) Which graph best represents velocity as a function of time for acceleration given by $a = +3t$? (*c*) Which graph best represents distance as a function of time for a constant negative acceleration? (*d*) Which graph best represents velocity as a function of time if graph E shows distance as a function of time?

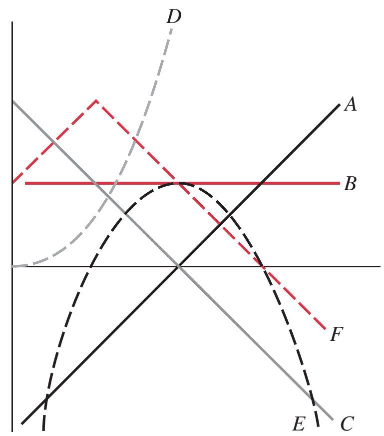


FIGURE 2-27. Multiple-choice question 7.

2-5 Motion with Constant Acceleration

8. An object is moving in the x direction with velocity $v_x(t)$, and dv_x/dt is a nonzero constant. With $v_x = 0$ when $t = 0$, then for $t > 0$ the quantity $v_x\, dv_x/dt$ is

(A) negative. (B) zero. (C) positive.
(D) not determined from the information given.

2-6 Freely Falling Bodies

9. A fellow student makes the following comment during a study session: "A freely falling object falls a greater distance during each second than the total distance fallen in all the previous seconds." This statement

(A) is always true.
(B) is true only for sufficiently short times.
(C) is true for sufficiently long times.
(D) is never true.

10. An object is tossed vertically into the air with an initial velocity of 8 m/s. Using the sign convention *up is positive,* how does the vertical component of the acceleration a_y of the object (after leaving the hand) vary during the flight of the object?

(A) On the way up $a_y > 0$, on the way down $a_y > 0$.
(B) On the way up $a_y < 0$, on the way down $a_y > 0$.
(C) On the way up $a_y > 0$, on the way down $a_y < 0$.
(D) On the way up $a_y < 0$, on the way down $a_y < 0$.

11. A boy jumps off a high diving board above a swimming pool. Halfway between the board and the water he tosses a ball upward. Ignoring air friction, the instant after the ball leaves his hands the vertical component of the acceleration of the ball
(A) is positive, but then decreases through zero to $-9.8\ \text{m/s}^2$.
(B) is zero, but then decreases to $-9.8\ \text{m/s}^2$.
(C) is between zero and $-9.8\ \text{m/s}^2$, but then decreases to $-9.8\ \text{m/s}^2$.
(D) is $-9.8\ \text{m/s}^2$.

12. A small toy looks like a pipe and shoots a marble out each end. The toy is dropped out of a tree and fires halfway to the ground. One marble shoots directly up, the other directly down. Consider the vertical component of the acceleration a_y of the marbles immediately after the marbles leave the toy, ignoring any air friction.
(A) The upward-moving marble has $a_y < -9.8\ \text{m/s}^2$.
(B) The upward-moving marble has $a_y = -9.8\ \text{m/s}^2$.
(C) The upward-moving marble has $a_y > -9.8\ \text{m/s}^2$.
(D) The downward-moving marble has $a_y > -9.8\ \text{m/s}^2$.

QUESTIONS

1. Can two vectors having different magnitudes be combined to give a zero resultant? Can three vectors?

2. Can a vector have zero magnitude if one of its components is not zero?

3. Can the sum of the magnitudes of two vectors ever be equal to the magnitude of the sum of these two vectors?

4. Can the magnitude of the difference between two vectors ever be greater than the magnitude of either vector? Can it be greater than the magnitude of their sum? Give examples.

5. Suppose that $\vec{\mathbf{d}} = \vec{\mathbf{d}}_1 + \vec{\mathbf{d}}_2$. Does this mean that we must have either $d \geq d_1$ or $d \geq d_2$? If not, explain why.

6. Can the speed of a particle ever be negative? If so, give an example; if not, explain why.

7. Does average velocity have a direction associated with it?

8. Each second a rabbit moves one-half the remaining distance from its nose to a head of lettuce. Does the rabbit ever get to the lettuce? What is the limiting value of the rabbit's average velocity? Draw graphs showing the rabbit's velocity and position as time increases.

9. Instead of the definition given in Eq. 2-13, we might have defined average speed as the magnitude of the average velocity. Are the definitions different? Give examples to support your answer.

10. A racing car, in a qualifying two-lap heat, covers the first lap with an average speed of 90 mi/h. The driver wants to speed up during the second lap so that the average speed of the two laps together will be 180 mi/h. Show that it cannot be done.

11. Bob beats Judy by 10 m in a 100-m dash. Bob, claiming to give Judy an equal chance, agrees to race her again but to start from 10 m behind the starting line. Does this really give Judy an equal chance?

12. When the velocity is constant, can the average velocity over any time interval differ from the instantaneous velocity at any instant? If so, give an example; if not, explain why.

13. Can the average velocity of a particle moving along the x axis ever be $\frac{1}{2}(v_{0x} + v_x)$ if the acceleration is not constant? Prove your answer with the use of graphs.

14. (*a*) Can an object have zero velocity and still be accelerating? (*b*) Can an object have a constant velocity and still have a varying speed? In each case, give an example if your answer is yes; explain why if your answer is no.

15. Can the velocity of an object reverse direction when its acceleration is constant? If so, give an example; if not, explain why.

16. Figure 2-28 shows Colonel John P. Stapp in his braking rocket sled; see Exercise 45. (*a*) His body is an accelerometer, not a speedometer. Explain. (*b*) Can you tell the direction of the acceleration from the figure?

FIGURE 2-28. Question 16 and Exercise 45.

17. Can an object be increasing in speed as the magnitude of its acceleration decreases? If so, give an example; if not, explain why.

18. Of the following situations, which one is impossible? (*a*) A body having velocity east and acceleration east; (*b*) a body having velocity east and acceleration west; (*c*) a body having zero velocity but acceleration not zero; (*d*) a body having constant acceleration and variable velocity; (*e*) a body having constant velocity and variable acceleration.

19. If a particle is released from rest ($v_{0x} = 0$) at $x_0 = 0$ at the time $t = 0$, Eq. 2-28 for constant acceleration says that it is at position x at two different times—namely, $+\sqrt{2x/a_x}$ and $-\sqrt{2x/a_x}$. What is the meaning of the negative root of this quadratic equation?

20. What happens to our kinematic equations (Eq. 2-26 or 2-28) under the operation of time reversal—that is, replacing t by $-t$? Explain.

21. We expect a truly general relation, such as Eqs. 2-26 and 2-28, to be valid regardless of the choice of coordinate system. By demanding that general equations be dimensionally consistent we ensure that the equations are valid regardless of the choice of units. Is there any need then for units or coordinate systems?

22. What are some examples of falling objects for which it would be unreasonable to neglect air resistance?

23. Figure 2-29 shows a shot tower in Baltimore, Maryland. It was built in 1829 and used to manufacture lead shot pellets by pouring molten lead through a sieve at the top of the tower. The lead pellets solidify as they fall into a tank of water at the bottom of the tower, 230 ft below. What are the advantages of manufacturing shot in this way?

FIGURE 2-29. Question 23.

24. A person standing on the edge of a cliff at some height above the ground throws one ball straight up with initial speed v_0 and then throws another ball straight down with the same initial speed. Which ball, if either, has the larger speed when it hits the ground? Neglect air resistance.

25. What is the downward acceleration of a projectile that is released from a missile accelerating upward at 9.8 m/s^2?

26. On another planet, the value of g is one-half the value on Earth. How is the time needed for an object to fall to the ground from rest on that planet related to the time required to fall the same distance on Earth?

27. (*a*) A stone is thrown upward with a certain speed on a planet where the free-fall acceleration is double that on Earth. How high does it rise compared to the height it rises on Earth? (*b*) If the initial speed were doubled, what change would that make?

28. Consider a ball thrown vertically up. Taking air resistance into account, would you expect the time during which the ball rises to be longer or shorter than the time during which it falls? Why?

29. Make a qualitative graph of speed v versus time t for a falling object (*a*) for which air resistance can be ignored and (*b*) for which air resistance cannot be ignored.

30. A second ball is dropped down an elevator shaft 1 s after the first ball is dropped. (*a*) What happens to the distance between the balls as time goes on? (*b*) How does the ratio v_1/v_2 of the speed of the first ball to the speed of the second ball change as time goes on? Neglect air resistance, and give qualitative answers.

31. Repeat Question 30 taking air resistance into account. Again, give qualitative answers.

32. If m is a light stone and M is a heavy one, according to Aristotle M should fall faster than m. Galileo attempted to show that Aristotle's belief was logically inconsistent by the following argument. Tie m and M together to form a double stone. Then, in falling, m should retard M, because it tends to fall more slowly, and the combination would fall faster than m but more slowly than M; but according to Aristotle the double body $(M + m)$ is heavier than M and, hence, should fall faster than M. If you accept Galileo's reasoning as correct, can you conclude that M and m must fall at the same rate? What need is there for experiment in that case? If you believe Galileo's reasoning is incorrect, explain why.

EXERCISES

2-1 Kinematics with Vectors

2-2 Properties of Vectors

1. Consider two displacements, one of magnitude 3 m and another of magnitude 4 m. Show how the displacement vectors may be combined to get a resultant displacement of magnitude (*a*) 7 m, (*b*) 1 m, and (*c*) 5 m.

2. A person walks in the following pattern: 3.1 km north, then 2.4 km west, and finally 5.2 km south. (*a*) Construct the vector diagram that represents this motion. (*b*) How far and in what direction would a bird fly in a straight line to arrive at the same final point?

3. Vector $\vec{a}$ has a magnitude of 5.2 units and is directed east. Vector $\vec{b}$ has a magnitude of 4.3 units and is directed 35° west of north. By constructing vector diagrams, find the magnitudes and directions of (a) $\vec{a} + \vec{b}$, and (b) $\vec{a} - \vec{b}$.

4. (*a*) What are the components of a vector $\vec{a}$ in the xy plane if its direction is 252° counterclockwise from the positive x axis and its magnitude is 7.34 units? (*b*) The x component of a certain vector is -25 units and the y component is $+43$ units. What are the magnitude of the vector and the angle between its direction and the positive x axis?

5. A person desires to reach a point that is 3.42 km from her present location and in a direction that is 35.0° north of east.

However, she must travel along streets that go either north-south or east-west. What is the minimum distance she could travel to reach her destination?

6. A ship sets out to sail to a point 124 km due north. An unexpected storm blows the ship to a point 72.6 km to the north and 31.4 km to the east of its starting point. How far, and in what direction, must it now sail to reach its original destination?

7. (*a*) What is the sum in unit-vector notation of the two vectors $\vec{a} = 5\hat{i} + 3\hat{j}$ and $\vec{b} = -3\hat{i} + 2\hat{j}$? (*b*) What are the magnitude and the direction of $\vec{a} + \vec{b}$?

8. Two vectors are given by $\vec{a} = 4\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} + 4\hat{k}$. Find (*a*) $\vec{a} + \vec{b}$, (*b*) $\vec{a} - \vec{b}$, and (*c*) a vector $\vec{c}$ such that $\vec{a} - \vec{b} + \vec{c} = 0$.

9. Given two vectors, $\vec{a} = 4.0\hat{i} - 3.0\hat{j}$ and $\vec{b} = 6.0\hat{i} + 8.0\hat{j}$, find the magnitudes and directions (with the $+x$ axis) of (*a*) $\vec{a}$, (*b*) $\vec{b}$, (*c*) $\vec{a} + \vec{b}$, (*d*) $\vec{b} - \vec{a}$, and (*e*) $\vec{a} - \vec{b}$.

10. Two vectors $\vec{a}$ and $\vec{b}$ have equal magnitudes of 12.7 units. They are oriented as shown in Fig. 2-30 and their vector sum is $\vec{r}$. Find (*a*) the x and y components of $\vec{r}$, (*b*) the magnitude of $\vec{r}$, and (*c*) the angle $\vec{r}$ makes with the $+x$ axis.

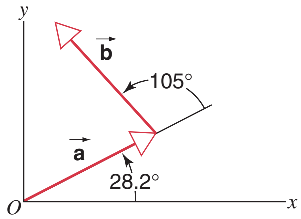


FIGURE 2-30. Exercise 10.

2-3 Position, Velocity, and Acceleration Vectors

11. A woman walks 250 m in the direction 35° east of north, then 170 m directly east. (*a*) Using graphical methods, find her final displacement from the starting point. (*b*) Compare the magnitude of her displacement with the distance she walked.

12. A car is driven east for a distance of 54 km, then north for 32 km, and then in a direction 28° east of north for 27 km. Draw the vector diagram and determine the total displacement of the car from its starting point.

13. The minute hand of a wall clock measures 11.3 cm from axis to tip. What is the displacement vector of its tip (*a*) from a quarter after the hour to half past, (*b*) in the next half hour, and (*c*) in the next hour?

14. A particle undergoes three successive displacements in a plane, as follows: 4.13 m southwest, 5.26 m east, and 5.94 m in a direction 64.0° north of east. Choose the x axis pointing east and the y axis pointing north and find (*a*) the components of each displacement, (*b*) the components of the resultant displacement, (*c*) the magnitude and direction of the resultant displacement, and (*d*) the displacement that would be required to bring the particle back to the starting point.

15. A radar station detects a missile approaching from the east. At first contact, the range to the missile is 12,000 ft at 40.0° above the horizon. The missile is tracked for another 123° in the east-west plane, the range at final contact being 25,800 ft; see Fig. 2-31. Find the displacement of the missile during the period of radar contact.

16. A plane flies 410 mi east from city A to city B in 45 min and then 820 mi south from city B to city C in 1 h 30 min. (*a*) What are the magnitude and direction of the displacement vector that represents the total trip? What are (*b*) the average velocity vector and (*c*) the average speed for the trip?

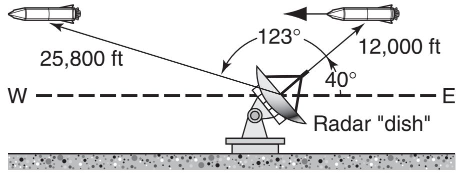


FIGURE 2-31. Exercise 15.

17. The position of a particle moving in an xy plane is given by $\vec{r} = [(2\text{ m/s}^3)t^3 - (5\text{ m/s})t]\hat{i} + [(6\text{ m}) - (7\text{ m/s}^4)t^4]\hat{j}$. Calculate (*a*) $\vec{r}$, (*b*) $\vec{v}$, and (*c*) $\vec{a}$ when $t = 2$ s.

18. In 3 h 24 min, a balloon drifts 8.7 km north, 9.7 km east, and 2.9 km in elevation from its release point on the ground. Find (*a*) the magnitude of its average velocity and (*b*) the angle its average velocity makes with the horizontal.

19. The velocity of a particle moving in the xy plane is given by $\vec{v} = [(6.0\text{ m/s}^2)t - (4.0\text{ m/s}^3)t^2]\hat{i} + (8.0\text{ m/s})\hat{j}$. Assume $t > 0$. (*a*) What is the acceleration when $t = 3$ s? (*b*) When (if ever) is the acceleration zero? (*c*) When (if ever) is the velocity zero? (*d*) When (if ever) does the speed equal 10 m/s?

20. A particle is moving in the xy plane with velocity $\vec{v}(t) = v_x(t)\hat{i} + v_y(t)\hat{j}$ and acceleration $\vec{a}(t) = a_x(t)\hat{i} + a_y(t)\hat{j}$. By taking the appropriate derivative, show that the magnitude of $\vec{v}$ can be constant only if $a_x v_x + a_y v_y = 0$.

2-4 One-Dimensional Kinematics

21. A plane flies round-trip between Los Angeles and Namulevu, Vanuavinaka. The plane takes off at 12:50 P.M. Los Angeles time and lands at 6:50 P.M. Namulevu time. On the return trip it takes off at 1:50 A.M. Namulevu time and lands at 6:50 P.M. Los Angeles time. Assume that the flight time is the same in both directions and that the plane flies in a straight line at an average speed of 520 mi/hr. (*a*) What length of time is the flight (one way, as measured by the passengers)? (*b*) What is the time difference between Namulevu and Los Angeles? (*c*) Approximately where on the globe is Namulevu located?

22. On April 15 an airplane takes off at 4:40 P.M. from Belém, Brazil bound for Villamil, Ecuador (in the Galapagos). The plane lands at 8:40 P.M. Villamil local time. The sun sets at 6:15 P.M. in Belém (local time), and 7:06 P.M. in Villamil (local time). At what time during the flight do the airplane passengers see the sun set?

23. How far does your car, moving at 70 mi/h (= 112 km/h) travel forward during the 1 s of time that you take to look at an accident on the side of the road?

24. New York Yankees pitcher Roger Clemens threw a fastball at a horizontal speed of 160 km/h, as verified by a radar gun. How long did it take for the ball to reach home plate, which is 18.4 m away?

25. Figure 2-32 shows the relation between the age of the oldest sediment, in millions of years, and the distance, in kilometers, at which the sediment was found from a particular ocean ridge. Seafloor material is extruded from this ridge and moves away from it at approximately uniform speed. Find the speed, in centimeters per year, at which this material recedes from the ridge.

26. Maurice Greene once ran the 100-m dash in 9.81 s (the wind was at his back), and Khalid Khannouchi ran the marathon (26 mi, 385 yd) in 2:05:42. (*a*) What are their average speeds?

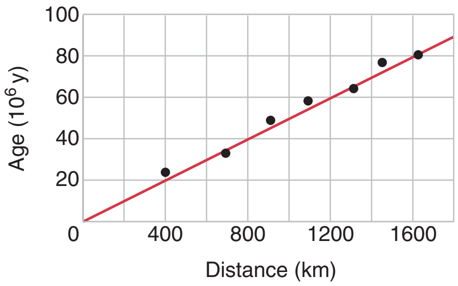


FIGURE 2-32. Exercise 25.

(*b*) If Maurice Greene could maintain his sprint speed during a marathon, how long would it take him to finish?

27. The legal speed limit on a highway is changed from 55 mi/h (= 88.5 km/h) to 65 mi/h (= 104.6 km/h). How much time is thereby saved on a trip from the Buffalo entrance to the New York City exit of the New York State Thruway for someone traveling at the higher speed over this 435-mi (= 700-km) stretch of highway?

28. A high-performance jet plane, practicing radar avoidance maneuvers, is in horizontal flight 35 m above the level ground. Suddenly, the plane encounters terrain that slopes gently upward at 4.3°, an amount difficult to detect; see Fig. 2-33. How much time does the pilot have to make a correction if the plane is to avoid flying into the ground? The airspeed is 1300 km/h.

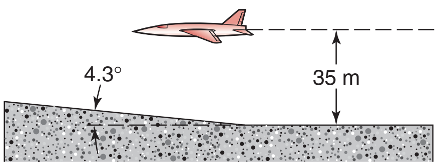


FIGURE 2-33. Exercise 28.

29. A car travels up a hill at the constant speed of 40 km/h and returns down the hill at the speed of 60 km/h. Calculate the average speed for the round trip.

30. Compute your average speed in the following two cases. (*a*) You walk 240 ft at a speed of 4.0 ft/s and then run 240 ft at a speed of 10 ft/s along a straight track. (*b*) You walk for 1.0 min at a speed of 4.0 ft/s and then run for 1.0 min at 10 ft/s along a straight track.

31. How far does the runner whose velocity-time graph is shown in Fig. 2-34 travel in 16 s?

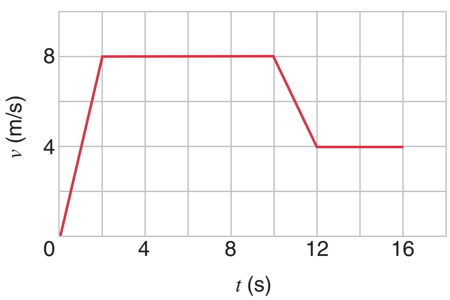


FIGURE 2-34. Exercises 31 and 32.

32. What is the acceleration of the runner in Exercise 31 at $t = 11$ s?

33. A particle had a velocity of 18 m/s in the $+x$ direction and 2.4 s later its velocity was 30 m/s in the opposite direction. What was the average acceleration of the particle during this 2.4-s interval?

34. An object moves in a straight line as described by the velocity-time graph in Fig. 2-35. Sketch a graph that represents the acceleration of the object as a function of time.

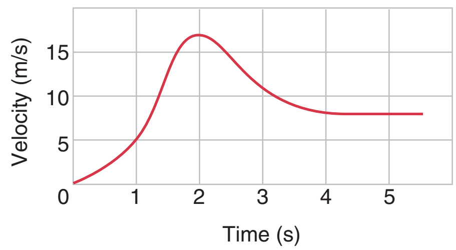


FIGURE 2-35. Exercise 34.

35. The graph of x versus t in Fig. 2-36*a* is for a particle in straight line motion. (*a*) State for each interval whether the velocity v_x is +, −, or 0, and whether the acceleration a_x is +, −, or 0. The intervals are *OA*, *AB*, *BC*, and *CD*. (*b*) From the curve, is there any interval over which the acceleration is obviously not constant? (Ignore the behavior at the endpoints of the intervals.)

36. Answer the previous questions for the motion described by the graph of Fig. 2-36*b*.

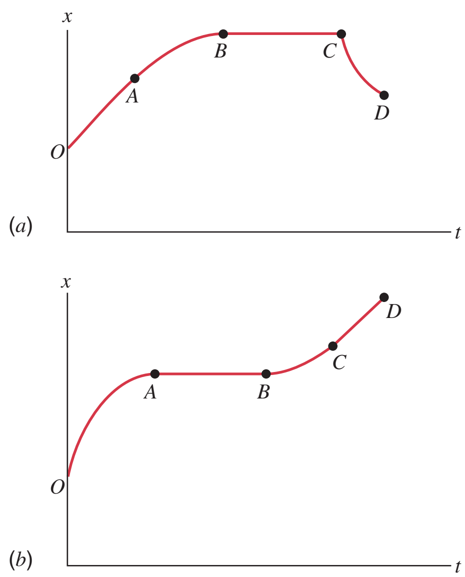


FIGURE 2-36. (*a*) Exercise 35 and (*b*) Exercise 36.

37. A particle moves along the x axis with a displacement versus time as shown in Fig. 2-37. Roughly sketch curves of velocity versus time and acceleration versus time for this motion.

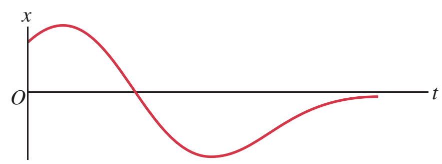


FIGURE 2-37. Exercise 37.

38. A particle moving along the positive x axis has the following positions at various times:

x(m)	0.080	0.050	0.040	0.050	0.080	0.13	0.20
t(s)	0	1	2	3	4	5	6

(*a*) Plot displacement (not position) versus time. (*b*) Find the average velocity of the particle in the intervals 0 to 1 s, 0 to 2 s, 0 to 3 s, 0 to 4 s. (*c*) Find the slope of the curve drawn in part (*a*) at the points $t = 0$, 1, 2, 3, 4, and 5 s. (*d*) Plot the slope (units?) versus time. (*e*) From the curve of part (*d*) determine the acceleration of the particle at times $t = 2$, 3, and 4 s.

39. The position of a particle along the x axis depends on the time according to the equation $x = At^2 - Bt^3$, where x is in meters and t is in seconds. (*a*) What SI units must A and B have? For the following, let their numerical values in SI units be 3.0 and 1.0, respectively. (*b*) At what time does the particle reach its maximum positive x position? (*c*) What total path-length does the particle cover in the first 4 seconds? (*d*) What is its displacement during the first 4 seconds? (*e*) What is the particle's velocity at the end of each of the first 4 seconds? (*f*) What is the particle's acceleration at the end of each of the first 4 seconds? (*g*) What is the average velocity for the time interval $t = 2$ to $t = 4$ s?

2-5 Motion with Constant Acceleration

40. A jumbo jet needs to reach a speed of 360 km/h (= 224 mi/h) on the runway for takeoff. Assuming a constant acceleration and a runway 1.8 km (= 1.1 mi) long, what minimum acceleration from rest is required?

41. A rocket ship in free space moves with constant acceleration equal to 9.8 m/s^2. (*a*) If it starts from rest, how long will it take to acquire a speed one-tenth that of light? (*b*) How far will it travel in so doing? (The speed of light is 3.0×10^8 m/s.)

42. The head of a rattlesnake can accelerate 50 m/s^2 in striking a victim. If a car could do as well, how long would it take for it to reach a speed of 100 km/h from rest?

43. A muon (an elementary particle) is shot with initial speed 5.20×10^6 m/s into a region where an electric field produces an acceleration of 1.30×10^{14} m/s^2 directed opposite to the initial velocity. How far does the muon travel before coming to rest?

44. An electron with initial velocity $v_0 = 1.5 \times 10^5$ m/s enters a region 1.2 cm long where it is electrically accelerated (see Fig. 2-38). It emerges with a velocity $v = 5.8 \times 10^6$ m/s. What was its acceleration, assumed constant? (Such a process occurs in the electron gun in a cathode-ray tube, used in television receivers and computer monitors.)

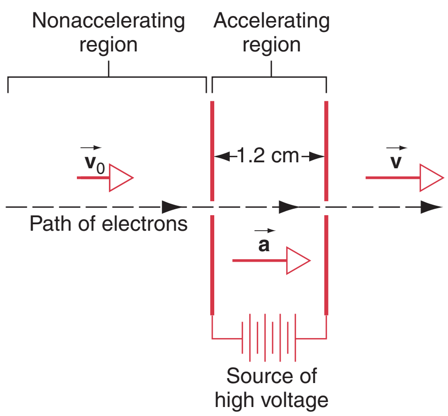


FIGURE 2-38. Exercise 44.

45. A world's land speed record was set by Colonel John P. Stapp when, on March 19, 1954, he rode a rocket-propelled sled that moved down a track at 1020 km/h. He and the sled were brought to a stop in 1.4 s; see Fig. 2-28. What acceleration did he experience? Express your answer in terms of g (= 9.8 m/s^2), the acceleration due to gravity. (Note that his body acts as an accelerometer, not a speedometer.)

46. The brakes on your automobile are capable of creating a deceleration of 17 ft/s^2. If you are going 85 mi/h and suddenly see a state trooper, what is the minimum time in which you can get your car under the 55 mi/h speed limit?

47. On a dry road a car with good tires may be able to brake with a deceleration of 11.0 mi/h/s (= 4.92 m/s^2). (*a*) How long does it take such a car, initially traveling at 55 mi/h (= 24.6 m/s), to come to rest? (*b*) How far does it travel in this time?

48. An arrow is shot straight up into the air and on its return strikes the ground at 260 ft/s, imbedding itself 9.0 in. into the ground. Find (*a*) the acceleration (assumed constant) required to stop the arrow, and (*b*) the time required for the ground to bring it to rest.

49. An elevator cab in the New York Marriott Marquis (see Fig. 2-39) has a total run of 624 ft. Its maximum speed is 1000 ft/min and its (constant) acceleration is 4.00 ft/s^2. (*a*) How far does it move while accelerating to full speed from rest? (*b*) How long does it take to make the run, starting and ending at rest?

FIGURE 2-39. Exercise 49.

50. An automobile traveling 35 mi/h (= 56 km/h) is 110 ft (= 34 m) from a barrier when the driver slams on the brakes. Four seconds later the car hits the barrier. (*a*) What was the automobile's constant deceleration before impact? (*b*) How fast was the car traveling at impact?

2-6 Freely Falling Bodies

51. Raindrops fall to the ground from a cloud 1700 m above Earth's surface. If they were not slowed by air resistance, how fast would the drops be moving when they struck the ground? Would it be safe to walk outside during a rainstorm?

52. The single cable supporting an unoccupied construction elevator breaks when the elevator is at rest at the top of a 120-m-high building. (*a*) With what speed does the elevator strike the ground? (*b*) For how long was it falling? (*c*) What was its speed when it passed the halfway point on the way down? (*d*) For how long was it falling when it passed the halfway point?

53. At a construction site a pipe wrench strikes the ground with a speed of 24.0 m/s. (*a*) From what height was it inadvertently dropped? (*b*) For how long was it falling?

54. (*a*) With what speed must a ball be thrown vertically up in order to rise to a maximum height of 53.7 m? (*b*) For how long will it be in the air?

55. A rock is dropped from a 100-m-high cliff. How long does it take to fall (*a*) the first 50.0 m and (*b*) the second 50.0 m?

56. Space explorers land on a planet in our solar system. They note that a small rock tossed at 14.6 m/s vertically upward takes 7.72 s to return to the ground. On which planet have they landed? (Hint: See Appendix C.)

57. A ball thrown straight up takes 2.25 s to reach a height of 36.8 m. (*a*) What was its initial speed? (*b*) What is its speed at this height? (*c*) How much higher will the ball go?

58. A ball is dropped from a height of 2.2 m and rebounds to a height of 1.9 m above the floor. Assume the ball was in contact with the floor for 96 ms and determine the average acceleration (magnitude and direction) of the ball during contact with the floor.

59. Two objects begin a free fall from rest from the same height 1.00 s apart. How long after the first object begins to fall will the two objects be 10.0 m apart?

60. A balloon is ascending at 12.4 m/s at a height of 81.3 m above the ground when a package is dropped. (*a*) With what speed does the package hit the ground? (*b*) How long did it take to reach the ground?

61. A dog sees a flowerpot sail up and then back down past a window 1.1 m high. If the total time the pot is in sight is 0.54 s, find the height above the top of the window to which the pot rises.

PROBLEMS

1. Rock *faults* are ruptures along which opposite faces of rock have moved past each other, parallel to the fracture surface. Earthquakes often accompany this movement. In Fig. 2-40, points A and B coincided before faulting. The component of the net displacement AB parallel to the horizontal surface fault line is called the *strike-slip* (AC). The component of the net displacement along the steepest line of the fault plane is the *dip-slip* (AD). (*a*) What is the net shift if the strike-slip is 22 m and the dip-slip is 17 m? (*b*) If the fault plane is inclined 52° to the horizontal, what is the net *vertical* displacement of B as a result of the faulting in (*a*)?

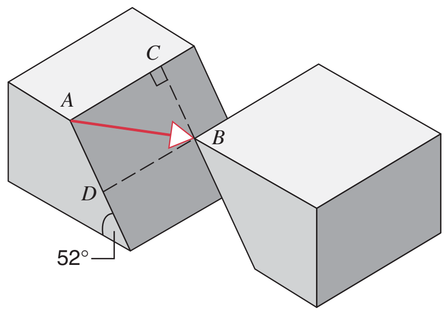


FIGURE 2-40. Problem 1.

2. A wheel with a radius of 45 cm rolls without slipping along a horizontal floor, as shown in Fig. 2-41. P is a dot painted on the rim of the wheel. At time t_1, P is at the point of contact between the wheel and the floor. At a later time t_2, the wheel has rolled through one-half of a revolution. What is the displacement of P during this interval?

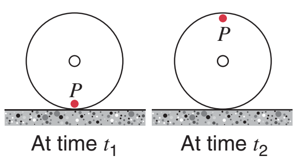


FIGURE 2-41. Problem 2.

3. A room has the dimensions 10 ft × 12 ft × 14 ft. A fly starting at one corner ends up at a diametrically opposite corner. (*a*) Find the displacement vector in a frame with coordinate axes parallel to the edges of the room. (*b*) What is the magnitude of the displacement? (*c*) Could the length of the path traveled by the fly be less than this distance? Greater than this distance? Equal to this distance? (*d*) If the fly walks rather than flies, what is the length of the shortest path it can take?

4. Two vectors of magnitudes a and b make an angle θ with each other when placed tail to tail. Prove, by taking components along two perpendicular axes, that the magnitude of their sum is

$$r = \sqrt{a^2 + b^2 + 2ab\cos\theta}.$$

5. You drive on Interstate 10 from San Antonio to Houston, one-half the time at 35.0 mi/h (= 56.3 km/h) and the other half at 55.0 mi/h (= 88.5 km/h). On the way back you travel one-half the distance at 35.0 mi/h and the other half at 55.0 mi/h. What is your average speed (*a*) from San Antonio to Houston, (*b*) from Houston back to San Antonio, and (*c*) for the entire trip?

6. The position of an object moving in a straight line is given by $x = At + Bt^2 + Ct^3$, where $A = 3.0$ m/s, $B = -4.0$ m/s^2, and $C = 1.0$ m/s^3. (*a*) What is the position of the object at $t = 0$, 1, 2, 3, and 4 s? (*b*) What is the object's displacement between $t = 0$ and $t = 2$ s? Between $t = 0$ and $t = 4$ s? (*c*) What is the average velocity for the time interval from $t = 2$ to $t = 4$ s? From $t = 0$ to $t = 3$ s?

7. Two trains, each having a speed of 34 km/h, are headed toward each other on the same straight track. A bird that can fly 58 km/h flies off the front of one train when they are 102 km apart and heads directly for the other train. On reaching the other train it flies directly back to the first train, and so forth. (*a*) How many trips can the bird make from one train to the other before the trains crash? (*b*) What is the total distance the bird travels?

8. An object, constrained to move along the x axis, travels a distance d_1 with constant velocity v_1 for a time t_1. It then instantaneously changes its velocity to a constant v_2 for a time t_2, traveling a distance d_2. (*a*) Show that

$$\frac{v_1 d_1 + v_2 d_2}{d_1 + d_2} \geq \frac{v_1 t_1 + v_2 t_2}{t_1 + t_2}.$$

(*b*) Under what conditions is this an equality?

9. The position of a particle moving along the x axis is given by $x = A + Bt^3$, where $A = 9.75$ cm and $B = 1.50$ cm/s^3. Consider the time interval $t = 2$ to $t = 3$ s and calculate (*a*) the average velocity; (*b*) the instantaneous velocity at $t = 2$ s; (*c*) the instantaneous velocity at $t = 3$ s; (*d*) the instantaneous velocity at $t = 2.5$ s; and (*e*) the instantaneous velocity when the particle is midway between its positions at $t = 2$ and $t = 3$ s.

10. For each of the following situations, sketch a graph that is a possible description of position as a function of time for a particle that moves along the x axis. At $t = 1$ s, the particle has (*a*) zero velocity and positive acceleration; (*b*) zero velocity and negative acceleration; (*c*) negative velocity and positive acceleration; (*d*) negative velocity and negative acceleration. (*e*) For which of these situations is the speed of the particle increasing at $t = 1$ s?

11. If the position of an object is given by $x = (2.0 \text{ m/s}^3)t^3$, find (*a*) the average velocity and the average acceleration between $t = 1$ and $t = 2$ s and (*b*) the instantaneous velocities and the instantaneous accelerations at $t = 1$ and $t = 2$ s. (*c*) Compare the average and instantaneous quantities and in each case explain why the larger one is larger.

12. An electron, starting from rest, has an acceleration that increases linearly with time; that is, $a = kt$, in which $k = 1.50$ m/s^3. (*a*) Plot a versus t during the first 10-s interval. (*b*) From the curve of part (*a*) plot the corresponding v versus t curve and estimate the electron's velocity 5 s after the motion starts. (*c*) From the v versus t curve of part (*b*) plot the corresponding x versus t curve and estimate how far the electron moves during the first 5 s of its motion.

13. Suppose that you were called upon to give some advice to a lawyer concerning the physics involved in one of her cases. The question is whether a driver was exceeding a 30-mi/h speed limit before he made an emergency stop, brakes locked and wheels sliding. The length of skid marks on the road was 19.2 ft. The police officer made the assumption that the maximum deceleration of the car would not exceed the acceleration of a freely falling body (= 32 ft/s^2) and did not give the driver a ticket. Was the driver speeding? Explain.

14. A train started from rest and moved with constant acceleration. At one time it was traveling at 33.0 m/s, and 160 m farther on it was traveling at 54.0 m/s. Calculate (*a*) the acceleration, (*b*) the time required to travel the 160 m, (*c*) the time required to attain the speed of 33.0 m/s, and (*d*) the distance moved from rest to the time the train had a speed of 33.0 m/s.

15. When a driver brings a car to a stop by braking as hard as possible, the stopping distance can be regarded as the sum of a "reaction distance," which is initial speed times reaction time, and "braking distance," which is the distance covered during braking. The following table gives typical values:

Initial Speed (m/s)	Reaction Distance (m)	Braking Distance (m)	Stopping Distance (m)
10	7.5	5.0	12.5
20	15	20	35
30	22.5	45	67.5

(*a*) What reaction time is the driver assumed to have? (*b*) What is the car's stopping distance if the initial speed is 25 m/s?

16. At the instant the traffic light turns green, an automobile starts with a constant acceleration of 2.2 m/s^2. At the same instant a truck, traveling with a constant speed of 9.5 m/s, overtakes and passes the automobile. (*a*) How far beyond the starting point will the automobile overtake the truck? (*b*) How fast will the car be traveling at that instant? (It is instructive to plot a qualitative graph of x versus t for each vehicle.)

17. A sprinter, in the 100-m dash, accelerates from rest to a top speed with a (constant) acceleration of 2.80 m/s^2 and maintains the top speed to the end of the dash. (*a*) What time elapsed and (*b*) what distance did the sprinter cover during the acceleration phase if the total time taken in the dash was 12.2 s?

18. A ball is tossed vertically into the air with an initial speed somewhere between $(25 - \epsilon)$ m/s and $(25 + \epsilon)$ m/s, where ϵ is a small number compared to 25. The total time of flight for the ball to return to the ground will be somewhere between $t - \tau$ and $t + \tau$. Find t and τ.

19. Figure 2-42 shows a simple device for measuring your reaction time. It consists of a strip of cardboard marked with a scale and two large dots. A friend holds the strip vertically with his thumb and forefinger at the upper dot and you position your thumb and forefinger at the lower dot, being careful not to touch the strip. Your friend releases the strip, and you try to pinch it as soon as possible after you see it begin to fall. The mark at the place where you pinch the strip gives your reaction time. How far from the lower dot should you place the 50-, 100-, 150-, 200-, and 250-ms marks?

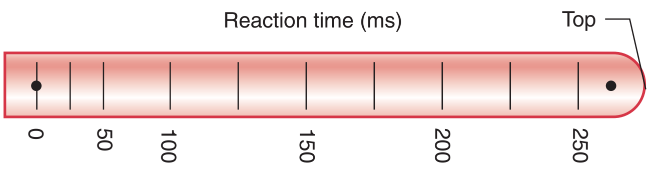


FIGURE 2-42. Problem 19.

20. While thinking of Isaac Newton, a person standing on a bridge overlooking a highway inadvertently drops an apple over the railing just as the front end of a truck passes directly below the railing. If the vehicle is moving at 55 km/h (= 34 mi/h) and is 12 m (= 39 ft) long, how far above the truck must the railing be if the apple just misses hitting the rear end of the truck?

21. A rocket is fired vertically and ascends with a constant vertical acceleration of 20 m/s^2 for 1.0 min. Its fuel is then all used and it continues as a free-fall particle. (*a*) What is the maximum altitude reached? (*b*) What is the total time elapsed from takeoff until the rocket strikes the Earth? (Ignore the variation of g with altitude).

22. A basketball player, about to "dunk" the ball, jumps 76 cm vertically. How much time does the player spend (*a*) in the top 15 cm of this jump and (*b*) in the bottom 15 cm? Does this help explain why such players seem to hang in the air at the tops of their jumps? See Fig. 2-43.

FIGURE 2-43. Problem 22.

23. A stone is thrown vertically upward. On its way up it passes point A with speed v, and point B, 3.00 m higher than A, with speed $v/2$. Calculate (*a*) the speed v and (*b*) the maximum height reached by the stone above point B.

24. The Zero Gravity Research Facility at the NASA Lewis Research Center includes a 145-m drop tower. This is an evacuated vertical tower through which, among other possibilities, a 1-m-diameter sphere containing an experimental package can be dropped. (*a*) For how long is the experimental package in free fall? (*b*) What is its speed at the bottom of the tower? (*c*) At the bottom of the tower, the sphere experiences an average acceleration of $25g$ as its speed is reduced to zero. Through what distance does it travel in coming to rest?

25. A woman fell 144 ft from the top of a building, landing on the top of a metal ventilator box, which she crushed to a depth of 18 in. She survived without serious injury. What acceleration (assumed uniform) did she experience during the collision? Express your answer in terms of g.

26. A certain computer hard disk drive is rated to withstand an acceleration of $100g$ without damage. Assuming the drive decelerates through a distance of 2 mm when it hits the ground, from how high can you drop the drive without ruining it?

27. As Fig. 2-44 shows, Clara jumps from a bridge, followed closely by Jim. How long did Jim wait after Clara jumped? Assume that Jim is 170 cm tall and that the jumping-off level is at the top of the figure. Make scale measurements directly on the figure.

FIGURE 2-44. Problem 27.

28. A parachutist after bailing out falls 52.0 m without friction. When the parachute opens, she decelerates at 2.10 m/s^2 and reaches the ground with a speed of 2.90 m/s. (*a*) How long is the parachutist in the air? (*b*) At what height did the fall begin?

29. A steel ball bearing is dropped from the roof of a building (the initial velocity of the ball is zero). An observer standing in front of a window 120 cm high notes that the ball takes 0.125 s to fall from the top to the bottom of the window. The ball bearing continues to fall, makes a completely elastic collision with a horizontal sidewalk, and reappears at the bottom of the window 2.0 s after passing it on the way down. How tall is the building? (The ball will have the same speed at a point going up as it had going down after a completely elastic collision.)

30. A juggler juggles 5 balls with two hands. Each ball rises 2 meters above her hands. Approximately how many times per minute does each hand toss a ball?

31. What is a reasonable estimate for the maximum number of objects a juggler can juggle with two hands if the height to which the objects are tossed above the hands is h?

32. Assume that Galileo had attempted to drop two objects off of the Tower of Pisa. (*a*) If he had released the objects from his hands, but dropped one slightly sooner than the second with a time difference of $\Delta t = 0.1$ s, then what would be the vertical separation of the two objects just before hitting the ground? (*b*) What release accuracy Δt would he need so that the two objects would have a vertical separation of less than 1 cm just prior to hitting the ground? (Ignore any effects of air friction.)

33. At the National Physical Laboratory in England (the British equivalent of our National Institute of Standards and Technology), a measurement of the acceleration g was made by throwing a glass ball straight up in an evacuated tube and letting it return, as in Fig. 2-45. Let Δt_L be the time interval between the two passages across the lower level, Δt_U the time interval between the two passages across the upper level, and H the distance between the two levels. Show that

$$g = \frac{8H}{(\Delta t_L)^2 - (\Delta t_U)^2}.$$

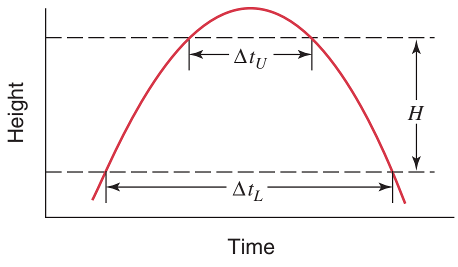


FIGURE 2-45. Problem 33.

COMPUTER PROBLEM

1. The velocity of an object is given by

$$v_x(t) = e^{-t^2/100}(t + 10 \sin \pi t).$$

Both v_x and x are zero when $t = 0$. (*a*) Numerically find the first time when v_x equals zero after the start and find the distance traveled from the origin. (*b*) Find the final position of the object as $t \to \infty$.

CHAPTER 3

FORCE AND NEWTON'S LAWS

In Chapter 2 we studied the motion of a particle. We did not ask what "caused" the motion; we simply described it in terms of the particle's position, velocity, and acceleration. In this chapter, we discuss the causes of motion, a field of study called dynamics.

The approach to dynamics we consider, which is generally known as classical mechanics, *was developed and successfully tested in the 17th and 18th centuries. More recent theories (special and general relativity and quantum mechanics) have indicated certain realms far from our ordinary experiences where classical mechanics fails to give predictions that agree with experiment, but these new theories reduce to classical mechanics in the limits of ordinary objects.*

Without reference to special or general relativity or to quantum mechanics, we can build great skyscrapers and study the properties of their construction materials; build airplanes that can carry hundreds of people and fly halfway around the world; and send space probes on complex missions to the comets, the planets, and beyond. This is the stuff of classical mechanics.

3-1 CLASSICAL MECHANICS

Ancient philosophers were perplexed by the motion of objects. They wrestled with questions such as: Do all motions require a cause? If so, what is the nature of this cause? Confusion about these issues persisted until the 17th century, when Galileo (1564–1642) and Isaac Newton (1642–1727) developed the approach to understanding these motions that we call "classical mechanics." Newton presented his three laws of motion in 1687 in his *Philosophiae Naturalis Principia Mathematica,* usually called the *Principia.* Until the 20th-century discoveries of quantum physics (which governs the behavior of microscopic particles such as electrons and atoms) and special relativity (which governs the behavior of objects moving at high speed) revealed their limitations, Newton's laws of classical mechanics formed the basis of our understanding of motion and its causes.

In classicial mechanics, we focus our attention on the motion of a particular object, which interacts with the surrounding objects (its *environment*) so that its velocity changes—an acceleration is produced. Table 3-1 shows some common accelerated motions and the object in the environment that is mostly responsible for the acceleration. The central problem of classical mechanics is: (1) An object with known physical properties (mass, volume, electric charge, etc.) is placed at a known initial location moving with known initial velocity. (2) We know (or can measure) all of the interactions of this object with its environment. (3) Can we predict the subsequent motion of the object? That is, can we find its position and velocity at all future times?

For this analysis we will begin by treating physical objects as *particles,* by which we mean bodies whose internal structures or motions can be ignored and whose parts all move in exactly the same way. Often we must analyze the motion of extended objects whose different parts may have different interactions with the environment. For example, a worker might be pushing on one side of a heavy crate while the bottom experiences friction as it slides along the floor. If

TABLE 3-1 Some Accelerated Motions and Their Major Causes

Object	*Change in Motion*	*Object in Environment*	*Type of Force*
Apple	Falls from tree	Earth	Gravitational
Car	Comes to a stop	Road	Frictional
Compass needle	Rotates toward north	Earth	Magnetic
Beam of ink drops in printer	Deflects	Capacitor	Electric
Helium balloon	Rises from land	Air	Buoyant

all parts of the crate move in the same way, we can treat the crate as a particle. As a result, it doesn't matter where the environment acts on the object; our main concern is with the *net effect* of all of the interactions with the environment. (Later in the text we will encounter situations in which it *does* matter where we apply the forces to an extended object, but for now we'll treat all objects as particles.)

We describe the interaction of a body with its environment in terms of a *force* $\vec{\mathbf{F}}$. A force is a push or a pull in a particular direction. Forces are described using vectors—for every force we must specify the direction in which it acts, and forces must be combined using the rules for vector addition. In this chapter we consider mostly situations involving one-dimensional motion, in which case we must specify the force component (positive or negative) relative to that one direction.

Each force that is exerted on an object is caused by a particular body in its environment. As you begin your study of classical mechanics, you may find it useful whenever you are analyzing the forces in a problem to describe each force by the body on which it acts and the body in the environment that is responsible for the force. For example, "pushing force *on* crate *by* worker" or "frictional force *on* crate *by* floor" or "gravitational force *on* crate *by* Earth." You will find this technique especially helpful when we discuss Newton's third law later in this chapter.

To carry out our scheme for classical mechanics, we begin by defining the magnitude of a force in terms of the acceleration of a particular standard body upon which that

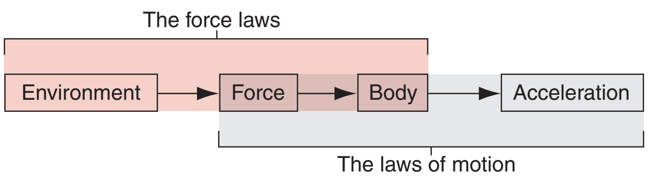


FIGURE 3-1. Our program for mechanics. The three boxes on the left suggest that force is an interaction between a body and its environment. The three boxes on the right suggest that a force acting on a body will accelerate it.

force acts. We then assign a *mass m* to a body by comparing the acceleration of that body with the acceleration of the standard body when the same force is applied to both. Finally, we develop *force laws* based on the properties of the body and its environment. Force thus appears in both the laws of motion (which tell us what acceleration an object will experience under the action of a given force) and in the force laws (which tell us how to calculate the force on a body in a certain environment). The laws of motion and the force laws together make up the laws of classical mechanics, as Fig. 3-1 suggests.

This program of mechanics cannot be tested piecemeal. We must view it as a whole and judge its success based on the answers to two questions: (1) Does the program yield results that agree with experiment? (2) Are the force laws simple and reasonable in form? It is the crowning glory of classical mechanics that we can answer an enthusiastic "yes" to both of these questions.

3-2 NEWTON'S FIRST LAW

Before Galileo's time most philosophers thought that some influence or "force" was needed to keep a body moving. They thought that a body was in its "natural state" when it was at rest. For a body to move in a straight line at constant speed, for example, they believed that some external agent had to continually propel it; otherwise it would "naturally" stop moving.

If we wanted to test these ideas experimentally, we would first have to find a way to free a body from all influences of its environment or from all forces. This is hard to do, but in certain cases we can make the forces very small. If we study the motion as we make the forces smaller and smaller, we can get an idea of what the motion would be like if the external forces were truly zero.

Let us place our test body—say, a block—on a rigid horizontal plane. If we let the block slide along this plane, we note that it gradually slows down and stops. This observation was used, in fact, to support the idea that motion stopped when the external force, in this case the hand initially pushing the block, was removed. We can argue against this idea, however, by reasoning as follows. Let us repeat our experiment, now using a smoother block and a smoother plane and providing a lubricant. We note that the velocity decreases more slowly than before. Let us use still smoother blocks and surfaces and better lubricants. We find that the block decreases in velocity at a slower and slower rate and travels farther each time before coming to rest. You may have experimented with an air track, on which objects can be made to float on a film of air; such a device comes close to the limit of no friction, as even a slight tap on one of the gliders can send it moving along the track at a slow and almost constant speed. We can now extrapolate and say that if all friction could be eliminated, the body would continue indefinitely in a straight line with constant speed. An external force is needed to set the body in motion, but *no external force is needed to keep a body moving with constant velocity.*

It is difficult to find a situation in which no external force acts on a body. The force of gravity acts on an object on or near the Earth, and resistive forces such as friction or air resistance oppose motion on the ground or in the air. Fortunately, we need not go to the vacuum of distant space to study motion free of external force, because, as far as the overall translational motion of a body is concerned, *there is no distinction between a body on which no external force acts and a body on which the sum or resultant of all the external forces is zero.* We usually refer to the resultant of all the forces acting on a body as the "net" force. For example, the push of our hand on the sliding block can exert a force that counteracts the force of friction on the block, and an upward force of the horizontal plane counteracts the force of gravity. The net force on the block can then be zero, and the block can move with constant velocity.

Note that, even though four forces act on the block, the *net force* can still be zero. The net force is determined by the *vector* sum of all the forces that act on the object. Forces of equal magnitude and opposite direction have a vector sum of zero. Thus we can achieve a condition of no net force on an object by arranging to apply forces that counteract other forces that act on the body, such as a push by a hand or an engine to overcome friction.

This principle was adopted by Newton as the first of his three laws of motion:

> *Consider a body on which no net force acts. If the body is at rest, it will remain at rest. If the body is moving with constant velocity, it will continue to do so.*

The First Law and Reference Frames

Suppose you are a passenger riding in a car and you are tightly held in your seat by the seat belt. When the brakes are applied, a book that was on the seat next to you begins to slide forward. There is no apparent force on the book that is pushing it forward, but relative to you it appears to start moving, in violation of Newton's first law. Your friend Bill, who is standing along the side of the road, sees you, the car, and the book all moving together, say at 22 m/s (about 50 mi/h). If you and the car suddenly slow to 20 m/s (about 45 mi/h), in the absence of friction with the seat the book continues to move at 22 mph according to Bill. Bill notices nothing unusual and detects no violation of Newton's first law.

For another example, you are tightly held by your seat belt in an airplane that encounters turbulence and suddenly drops in altitude by one meter. The glass on your tray table appears to you to leap one meter into the air, with no apparent force causing its motion. From your perspective, it appears that Newton's first law has been violated. Your friend Sally is flying at constant velocity in a plane just next to yours; her plane is not affected by the turbulence. Sally sees your glass moving in a straight line, while you and the plane suddenly drop by one meter. Sally detects no violation of Newton's first law in the motion of your glass.

Each observer—such as you in the car or airplane, Bill standing on the ground, and Sally in her airplane—defines a *reference frame.* A reference frame requires a coordinate system and a set of clocks, which enable an observer to measure positions, velocities, and accelerations in his or her particular reference frame. Observers in different reference frames may measure different velocities or accelerations.

Newton's first law, which may seem like an obvious result, is very important because it helps us to identify a set of special reference frames in which we can apply the laws of classical mechanics. In the example involving the car, you and Bill will reach different conclusions about the acceleration of the book—you conclude that it accelerates forward, while Bill concludes that its acceleration is zero. In general, the acceleration of a body depends on the reference frame relative to which it is measured. However, the laws of classical mechanics are valid only in a certain set of reference frames—namely, those in which *all* observers would measure the *same* acceleration for a moving body. Newton's first law allows us to choose this special family of reference frames if we express it as follows:

> *If the net force acting on a body is zero, then it is possible to find a set of reference frames in which that body has no acceleration.*

The tendency of a body to remain at rest or in uniform linear motion is called *inertia,* and Newton's first law is often called the *law of inertia.* The reference frames to which it applies are called *inertial frames.*

To test whether a particular frame of reference is an inertial frame, we place a test body at rest in the frame and ascertain that no net force acts on it. If the body does not remain at rest, the frame is not an inertial frame. Similarly, we can put the body (again subject to no net force) in motion at constant velocity; if its velocity changes, either in magnitude or direction, the frame is not an inertial frame. A frame in which these tests are everywhere passed is an inertial frame.

As a passenger in the decelerating car, your frame of reference is not an inertial frame, and you cannot directly apply the laws of mechanics as we formulate them. If Bill's frame of reference passes the tests as an inertial frame, he can successfully apply the laws of mechanics in his frame. He can measure the change in velocity of the car and thus deduce its acceleration (due to the force of friction with the road), but he concludes that the net force on the book is zero and thus that it should move with constant velocity. Similarly, if Sally's frame of reference passes the tests, she can successfully apply the laws of mechanics by associating your plane's sudden change in vertical velocity with a net vertical force (in this case, the difference between gravity and the upward "lift" force), and she can account for the motion of the glass by applying those same laws.

Inertial Reference Frames and Relative Motion

Suppose you are in a car moving down the road at 22 m/s (about 50 mi/h). Your friends in the same car are also moving at 22 m/s. You could toss a ball sideways into your friend's lap, and the toss is unaffected by the velocity of the car. As long as the car continues to move at constant velocity, the ball lands in your friend's lap.

If you are passed by your friend Sally who is in another car moving with a velocity of 27 m/s (about 60 mi/h), you observe the distance between your car and Sally's to increase at the rate of 27 m/s − 22 m/s = 5 m/s. That is, relative to your car, Sally's car is moving at 5 m/s. Take away the external clues—the scenery speeding by, the still air rushing past the moving car, the bumpiness of the road, and the noise of the engine—and consider only the two cars. You would have no way to decide which car was "really" moving. For example, Sally's car could be at rest and your car could be moving backward at 5 m/s; the observed result would be the same. One minute after she passes you, you observe the distance between you and Sally to be the relative velocity times the time interval: 5 m/s × 60 s = 300 m.

Now consider how this looks to your friend Bill by the side of the road (Fig. 3-2). Suppose Sally passes you just as you both pass Bill's position. According to Bill, 1 minute later your car has moved a distance of 22 m/s × 60 s = 1320 m, while Sally's car has moved a distance of 27 m/s × 60 s = 1620 m. Bill concludes that the distance between the cars is 1620 m − 1320 m = 300 m. Thus you and Bill agree on your conclusions about the distance between the cars.

Soon after passing you, Sally sees a police car and applies her brakes. Bill observes her to slow from 27 m/s to 20 m/s in a time of 3.5 s. According to Bill, her acceleration is (taking the direction of her motion to be the positive x direction and solving Eq. 2-26, $v_x = v_{0x} + a_x t$, for a_x)

$$a_x = \frac{v_x - v_{0x}}{t} = \frac{(20\text{ m/s}) - (27\text{ m/s})}{3.5\text{ s}} = -2.0\text{ m/s}^2.$$

According to your frame of reference, Sally's velocity is now 20 m/s − 22 m/s = −2 m/s; that is, you are now moving faster than Sally by 2 m/s. According to you, her velocity changes from +5 m/s to −2 m/s and thus her acceleration is

$$a_x = \frac{v_x - v_{0x}}{t} = \frac{(-2\text{ m/s}) - (5\text{ m/s})}{3.5\text{ s}} = -2.0\text{ m/s}^2.$$

You and Bill agree on the acceleration!

You and Bill are both inertial observers. You agree on the acceleration of Sally's car, and you will therefore agree on the force that was necessary to cause the acceleration. In fact, *all inertial observers agree on measurements of acceleration* (although they will not in general agree on measurements of position or velocity).

Consider the contrary case in which you also brake slightly when passing the police car, reducing your speed from 22 m/s to 21 m/s in the same 3.5-s interval. At the beginning of the interval, you determine Sally's velocity to be +5 m/s, as before. At the end of the braking interval, you would determine her velocity to be 20 m/s − 21 m/s = −1 m/s (in the backward direction). You would then conclude Sally's acceleration to be

$$a_x = \frac{v_x - v_{0x}}{t} = \frac{(-1\text{ m/s}) - (5\text{ m/s})}{3.5\text{ s}} = -1.7\text{ m/s}^2,$$

which differs from Bill's result of -2.0 m/s^2. You are no longer an inertial observer (because during the 3.5-s interval in which you were braking you could no longer pass the test of Newton's first law).

In this book we almost always apply the laws of classical mechanics from the point of view of an observer in an inertial frame. Occasionally, we discuss problems involving observers in noninertial reference frames, such as an accelerating car, a rotating merry-go-round, or an orbiting satellite. Even though the Earth is rotating, a reference frame attached to the Earth can be considered to be approximately an inertial reference frame for most practical purposes. For large-scale applications, such as analyzing the flight of ballistic missiles or studying wind and ocean currents, the noninertial character of the rotating Earth becomes important. (See Section 5-6.)

Note that there is no distinction in the first law between a body at rest and one moving with a constant velocity. Both motions are "natural" if the net force acting on the body is zero. This becomes clear when a body at rest in one inertial frame is viewed from a second inertial frame—that is, a frame moving with constant velocity with respect to the first. An observer in the first frame finds the body to be at rest; an observer in the second frame finds the same body to be moving with constant velocity. Both observers find the body to have no acceleration—that is, no change in velocity—and both may conclude from the first law that no net force acts on the body.

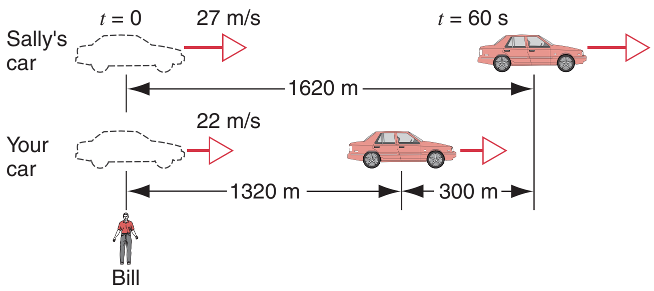


FIGURE 3-2. You and Bill agree that Sally's car is 300 m ahead of yours after 60 s.

3-3 FORCE

According to Newtons' first law, the *absence* of force leads to the *absence* of acceleration. What about the *presence* of force? Based on common experience, it is reasonable to as-

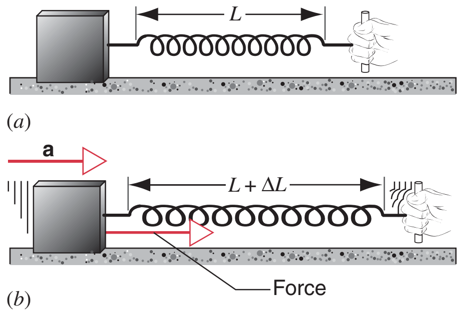


Figure 3-3. (*a*) A standard body at rest on a horizontal frictionless surface. (*b*) The body is accelerated by pulling to the right to stretch the spring by ΔL.

sume that a body will accelerate when a force is applied to it. We now develop our concept of force by defining it operationally in terms of the acceleration it produces when applied to a chosen standard body. Any object can serve as our "standard body," as long as it is clearly identifiable and reproducible. For example, we might choose a block of copper or glass of any specified dimensions.

Before we proceed with this measurement, we must first invoke Newton's first law to check that we are working in an inertial reference frame. If the body is at rest, does it remain at rest? If we start it moving with a constant velocity, does it remain in that state of motion? In attempting to answer the second question, we probably will discover that the body, once in motion, gradually slows down due to friction. By careful design of our apparatus, we create as nearly a friction-free environment as we can, perhaps by floating the body on a film of air or lubricating the surface on which it moves. We intend to apply a force to the body and measure its acceleration, and we want to be sure that any other forces caused by the environment have a negligible effect on the motion of the body.

As the agent that supplies the force, we choose a light spring. We observe that springs come in different stiffnesses, requiring different efforts to stretch them. We also observe that the effort needed to stretch a given spring increases as we stretch it through larger distances.

We place the standard body on a frictionless horizontal surface and attach the spring to it (Fig. 3-3*a*). By trial and error we stretch our spring until it gives an acceleration of exactly 1 m/s^2 to our standard body (Fig. 3-3*b*). We define this as *one unit* of force, and for future reference we record the amount of stretch ΔL in the spring that corresponds to this force. Now we repeat the experiment, stretching the spring by a greater amount until the standard body experiences an acceleration of 2 m/s^2. We define this as *two units* of force, and we again record the amount of stretch ΔL for the spring that gives this force. Continuing, we find the amount of stretch that gives an acceleration of 3 m/s^2, corresponding to *three units* of force. Eventually we have a complete calibration of our spring, giving the amount of stretch that gives our standard body any desired acceleration. Other springs of different stiffnesses could be calibrated in a similar way using the same standard body.

Based entirely on the acceleration given to the standard body, we now have a calibrated set of springs. Using these measuring devices, we can now proceed to measure unknown forces. For example, let us suspend an object vertically from a spring, as in Fig. 3-4*a*. Since the body is at rest, the net force on it must be zero. The magnitude of the upward force exerted by the spring must equal the magnitude of the downward force of gravity, so that the vector sum of the two forces is zero. By measuring the extension ΔL and checking our calibration for that spring to determine the corresponding force, we can determine the force of gravity on the body. In fact, calibrated spring scales are available for just this purpose, such as to weigh fruits and vegetables in grocery stores.

In a similar way we can measure frictional forces. By placing the body on a horizontal surface where it experiences a frictional force, we could attach a spring (as in Fig. 3-4*b*) and pull the body with just the right force so that it moves at constant velocity. In this case, the magnitude of the spring force equals that of the frictional force, so their vector sum is zero (because they have equal magnitudes and opposite directions). We can determine the magnitude of the force from the amount by which the spring is stretched. Once again, a calibrated spring scale can be used for this purpose.

Another way to measure force is to use a (commercially available) electronic *force probe,* which can be interfaced to a computer to read forces directly (see Fig. 3-5). A force applied to the probe causes a small deflection of a mechani-

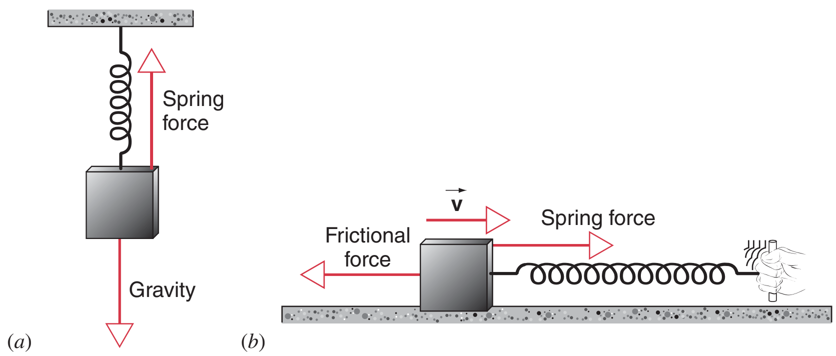


Figure 3-4. (*a*) A body is suspended at rest, acted upon by the spring force and gravity. (*b*) A body moves at constant velocity on a horizontal surface where a frictional force is exerted upon it.

FIGURE 3-5. A mass hangs from a spring attached to an electronic probe, which can be interfaced to a computer for measuring forces. Courtesy Vernier Software and Technology.

cal or electromagnetic device; the deflection can be read electronically and calibrated against a "standard" spring.

3-4 MASS

In the previous section we discussed a series of experiments that we used to calibrate a set of springs based on the accelerations given to a standard body when the springs were stretched to different lengths. Now we want to repeat those experiments to answer a different question: What effect will the *same force* have when applied to *different bodies?*

Everyday experience leads us to guess at a qualitative answer: it is much easier to accelerate a bicycle than a car by pushing it. Clearly the same force produces different accelerations when applied to different bodies. What makes these bodies differ in our ability to accelerate them by pushing is their *mass,* which is *the property of a body that determines its resistance to a change in its motion.*

Let us see how we can study the relationship between force and mass by accelerating bodies of different masses using our calibrated spring set. We start by obtaining a second identical standard body and attaching it to the first body. We then apply one unit of force (as previously determined from the single body) to this combined object, and we observe the acceleration to be 0.5 m/s^2. Applying two units of force, we find an acceleration of 1.0 m/s^2. We can repeat the experiment with three standard bodies joined together, then four, and so forth. Here is a possible set of outcomes of these experiments:

Applied Force	1 unit	2 units	3 units	4 units
Acceleration of one standard body	1.0 m/s^2	2.0 m/s^2	3.0 m/s^2	4.0 m/s^2
Acceleration of two standard bodies	0.5 m/s^2	1.0 m/s^2	1.5 m/s^2	2.0 m/s^2
Acceleration of three standard bodies	0.33 m/s^2	0.67 m/s^2	1.0 m/s^2	1.3 m/s^2

It is clear from the results of these measurements that, for each combination of bodies, the acceleration is directly proportional to the force (for example, in each case two units of force gives twice the acceleration as one unit of force). The proportionality constant between force and acceleration, however, is different for each horizontal row of the data table and is a characteristic property of the object being accelerated. This property of the object is its mass, which gives the proportionality between force and acceleration. Figure 3-6 illustrates these experiments, which show the relationships among force, mass, and acceleration.

From many experiments similar to these, we conclude that the greater the total mass of a body, the smaller the acceleration produced by a given force. That is, *the acceleration produced by a given force is inversely proportional to the mass being accelerated.* The mass of a body can thus be regarded as *a quantitative measure of the resistance of a body to acceleration by a given force.*

This observation gives us a direct way to compare the masses of different bodies: we apply the same force to both bodies and measure the resulting acceleration. The ratio of the masses of the two bodies is then the same as the *inverse* ratio of the accelerations. For example, suppose we apply a force F to the standard body (whose mass we take to be m_{std}) and measure acceleration a_{std}. We then observe that the same force F applied to body x of unknown mass m_x gives acceleration a_x. Forming the ratios, we then have

$$\frac{m_x}{m_{\text{std}}} = \frac{a_{\text{std}}}{a_x} \qquad \text{(same force } F \text{ acting).} \qquad (3\text{-}1)$$

This allows us to find the mass of the unknown body in terms of the mass of our chosen standard body. For example, if the acceleration of body x is $\frac{1}{3}$ of the acceleration of the standard body when the same force is applied to both, then the mass of body x is three times the mass of the standard body. Note that this remains true no matter how many units of force we choose to apply to both bodies, as you can see from the above table of measured values—for example, the accelerations of the triple body in the last line of the table are $\frac{1}{3}$ of the corresponding accelerations of the single

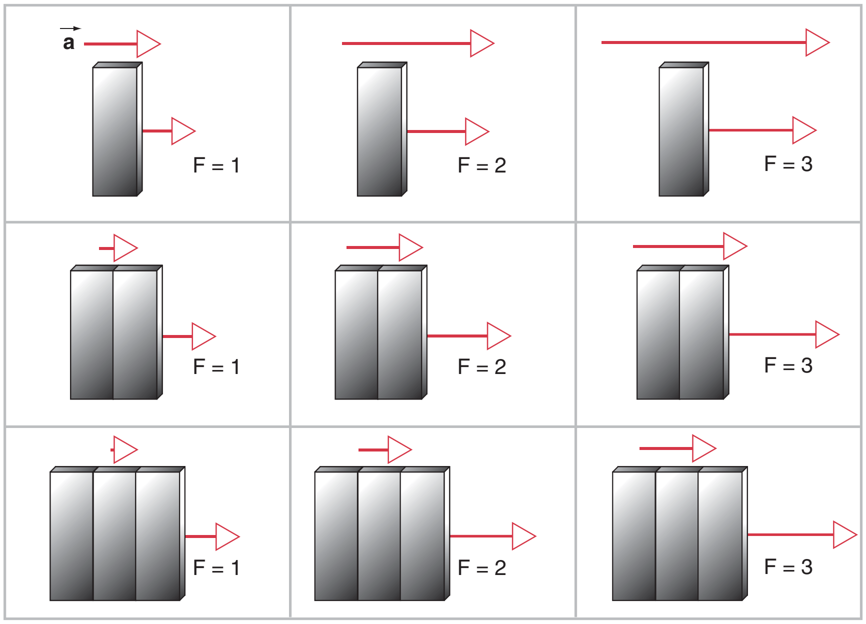


FIGURE 3-6. Experiments illustrating the relationship among force (given in arbitrary units), mass, and acceleration. Acceleration vectors are drawn to scale above the blocks. Reading across each row, we see that the acceleration is always proportional to the force, but the proportionality is different for different masses. Reading down each column, we see that when the same force acts, the acceleration is inversely proportional to the mass.

body for each value of the applied force. More specifically, if we apply a different force F' to both the standard body and body x, giving accelerations a'_{std} and a'_x, we then find that the ratio of the accelerations with force F' is the same as the ratio with force F:

$$\frac{m_x}{m_{\text{std}}} = \frac{a_{\text{std}}}{a_x} = \frac{a'_{\text{std}}}{a'_x}. \tag{3-2}$$

We obtain the same value for the unknown mass m_x, *no matter what the value of the common force.* The mass ratio m_x/m_{std} is independent of the force; the mass is a fundamental property of the object, unrelated to the value of the force used to compare the unknown mass to the standard mass.

By a simple extension of this procedure, we can compare the masses of any two bodies with each other, rather than comparing a single body with the standard. Consider two arbitrary objects, of masses m_1 and m_2. We apply a given force of magnitude F to m_1 and measure acceleration a_1. Applying the same force to m_2, we obtain acceleration a_2. The mass ratio is

$$\frac{m_2}{m_1} = \frac{a_1}{a_2} \quad \text{(same force acting)}, \tag{3-3}$$

which turns out to be identical with the ratio we would obtain by deducing the masses m_1 and m_2 separately by direct comparison with the standard, as in Eq. 3-1.

This procedure also shows that, when two masses m_1 and m_2 are fastened together, they behave mechanically like a single object of mass $m_1 + m_2$. This demonstrates that *masses add like (and are) scalar quantities.*

One practical example of the use of this technique—assigning masses by comparing the accelerations produced by a given force—is in the precise measurement of the masses of atoms. The force in this case is magnetic, and the acceleration is perpendicular to the velocity of the atom and so causes a deflection in its path, but the principle is exactly the same: the ratio of the masses of the two atoms is equal to the inverse ratio of their accelerations. Measuring the deflection permits precise mass ratios to be measured, and comparing with a standard mass (that of ^{12}C, defined to be exactly 12 u) permits precise values of masses, such as those shown in Table 1-6, to be obtained.

3-5 NEWTON'S SECOND LAW

We can now summarize all the previously described experiments and definitions in one equation, the fundamental equation of classical mechanics,

$$\sum \vec{\mathbf{F}} = m\vec{\mathbf{a}}. \tag{3-4}$$

In this equation $\Sigma \vec{\mathbf{F}}$ is the (vector) *sum* of *all* the forces acting *on* the body, m is the mass of the body, and $\vec{\mathbf{a}}$ is its (vector) acceleration. We shall usually refer to $\Sigma \vec{\mathbf{F}}$ as the *resultant* force or *net* force.

Equation 3-4 is a statement of *Newton's second law.* If we write it in the form $\vec{\mathbf{a}} = (\Sigma \vec{\mathbf{F}})/m$, we can easily see that the acceleration of the body is in magnitude directly proportional to the resultant force acting on it and in direction parallel to this force. We also see that the acceleration, for a given force, is inversely proportional to the mass of the body.

Note that the first law of motion appears to be contained in the second law as a special case, for if $\Sigma \vec{\mathbf{F}} = 0$, then $\vec{\mathbf{a}} = 0$. In other words, if the resultant force on a body is zero, the acceleration of the body is zero and the body moves with constant velocity, as stated by the first law. However, the first law has an independent and important role in defining inertial reference frames. Without that definition, we would not be able to choose the frames of reference in which to apply the second law. We therefore need *both laws* for a complete system of mechanics.

Equation 3-4 is a vector equation. As in the case of all vector equations, we can write this single vector equation as three one-dimensional equations,

$$\sum F_x = ma_x, \qquad \sum F_y = ma_y, \qquad \sum F_z = ma_z, \quad (3\text{-}5)$$

relating the x, y, and z components of the resultant force (ΣF_x, ΣF_y, and ΣF_z) to the x, y, and z compoonents of acceleration (a_x, a_y, and a_z) for the mass m. It should be emphasized that ΣF_x is the *algebraic* sum of the x components of *all* the forces, ΣF_y is the *algebraic* sum of the y components of *all* the forces, and ΣF_z is the *algebraic* sum of the z components of *all* the forces acting on m. In taking the algebraic sum, the signs of the components (that is, the relative directions of the forces) must be taken into account.

Like all equations, Newton's second law must be dimensionally consistent. On the right side, the dimensions are, recalling from Chapter 1 that [ ] denotes *the dimensions of,* $[m][a] = \mathrm{ML/T^2}$, and therefore these must also be the dimensions of force:

$$[F] = \mathrm{ML/T^2}.$$

No matter what the origin of the force—gravitational, electrical, nuclear, or whatever—and no matter how complicated the equation describing the force, these dimensions must hold for it.

In the SI system of units, the standard body has a mass of one kilogram (see Section 1-5), and we in effect measure the masses of objects by comparing them with the standard kilogram. To impart an acceleration of 1 m/s^2 to a mass of 1 kg requires a force of 1 kg·m/s^2. This combination of units is called the *newton* (abbreviated N):

$$1\ \mathrm{N} = 1\ \mathrm{kg \cdot m/s^2}.$$

If we measure the mass in kg and the acceleration in m/s^2, Newton's second law gives the force in N.

Two other systems of units in common use are the cgs (centimeter-gram-second) and the British systems. In the cgs system, mass is measured in grams and acceleration in cm/s^2. The force unit in this system is the *dyne* and is equivalent to the g·cm/s^2. Since 1 kg = 10^3 g and 1 m/s^2 = 100 cm/s^2, it follows that 1 N = 10^5 dyne. A dyne is a very small unit, roughly equal to the weight of a cubic millimeter of water. (A newton, on the other hand, is about the weight of a half-cup of water.)

In the British system, force is measured in pounds and acceleration in ft/s^2. In this system, the mass that is accelerated at 1 ft/s^2 by a force of 1 lb is called the *slug* (from the word *sluggish,* meaning slow or unresponsive).

Other variants on these basic systems are occasionally found, but these three are by far the most common. Table 3-2 summarizes these common force units; a more extensive listing can be found in Appendix G.

TABLE 3-2 Units in Newton's Second Law

System	*Force*	*Mass*	*Acceleration*
SI	newton (N)	kilogram (kg)	m/s^2
cgs	dyne	gram (g)	cm/s^2
British	pound (lb)	slug	ft/s^2

Dynamical Analysis Using Newton's Second Law

In analyzing problems using Newton's second law, there are several steps that you should follow:

1. Choose a suitable inertial reference frame. Select the orientation and the positive direction for the coordinate axes in that reference frame. Force components in the positive direction are taken to be positive, and those in the opposite direction are negative.
2. For each object in the problem, draw a *free-body diagram,* showing all of the forces acting on that body. The body is regarded as a particle in this diagram.
3. Label each force with two subscripts: the first one indicates the body on which the force acts, and the second indicates the body in the environment that causes the force. For example, F_{AB} indicates the force on body A due to body B, and F_{BA} indicates the force on body B due to body A. If there are several objects A, B, C . . . in the problem, then the forces on body A might include F_{AB}, F_{AC}, and so forth. This method of labeling the forces is very important, because it will help you avoid making the mistake of including a fictitious force that is not associated with a body in the environment.
4. For each body, find the vector sum of all the forces. In practice, this usually means separately adding (with proper attention to the signs) the x, y, and z components of the forces. Then use Eqs. 3-5 to find the acceleration components of that body.

The following examples illustrate the application of these procedures.

SAMPLE PROBLEM 3-1. A worker W pushes a loaded sled S whose mass m is 240 kg for a distance d of 2.3 m over the surface of a frozen lake. The sled moves with negligible friction on the ice. The worker exerts a constant horizontal force F_{SW} of 130 N (= 29 lb) as she does so; see Fig. 3-7*a*. If the sled starts from rest, what is its final velocity?

Solution As Fig. 3-7*b* shows, we lay out a horizontal x axis, we take the direction of increasing x to be to the right, and we treat

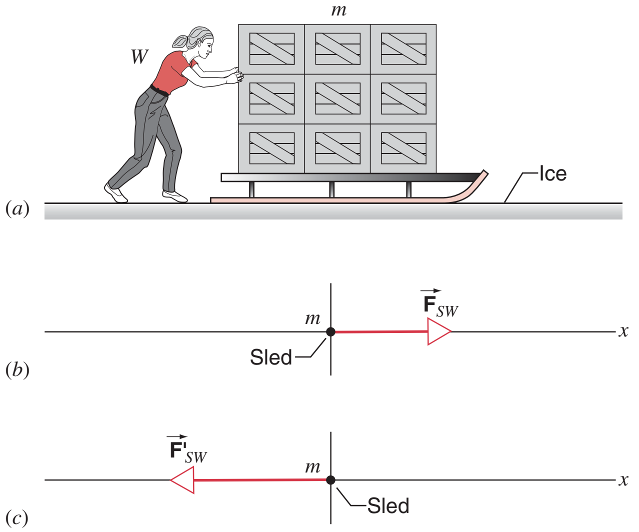


FIGURE 3-7. Sample Problems 3-1 and 3-2. (*a*) A worker pushing a loaded sled over a frictionless surface. (*b*) A free-body diagram, showing the sled as a "particle" and the force acting on it. (*c*) A second free-body diagram, showing the force acting when the worker pushes in the opposite direction.

the sled as a particle. Figure 3-7*b* is a *partial* free-body diagram. In drawing free-body diagrams, it is important always to include *all* forces that act on the particle, but here we have omitted two vertical forces that will be discussed later and that do not affect our solution. We assume that the force F_{SW} exerted by the worker is the only horizontal force acting on the sled, so that $\Sigma F_x = F_{SW}$. We can then find the acceleration of the sled from Newton's second law, or

$$a_x = \frac{\Sigma F_x}{m} = \frac{F_{SW}}{m} = \frac{130\text{ N}}{240\text{ kg}} = 0.54\text{ m/s}^2.$$

With this acceleration, we can find the time necessary to move a distance d using Eq. 2-28 ($x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$) with $x - x_0 = d$ and $v_{0x} = 0$). Solving, we obtain $t = \sqrt{2d/a_x} = 2.9$ s. Equation 2-26 (with $v_{0x} = 0$) now gives the final velocity

$$v_x = a_x t = (0.54\text{ m/s}^2)(2.9\text{ s}) = 1.6\text{ m/s}.$$

The force, acceleration, displacement, and final velocity of the sled are all positive, which means that they all point to the right in Fig. 3-7*b*.

Note that to continue applying the constant force, the worker would have to run faster and faster to keep up with the accelerating sled. Eventually, the velocity of the sled would exceed the fastest speed at which the worker could run, and thereafter she would no longer be able to apply a force to the sled. The sled would then continue (in the absence of friction) to coast at constant velocity.

SAMPLE PROBLEM 3-2. The worker in Sample Problem 3-1 wants to reverse the direction of the velocity of the sled in 4.5 s. With what constant force must she push on the sled to do so?

Solution If she exerts a constant force, then the acceleration of the sled will be constant. Let us find this constant acceleration, using Eq. 2-26 ($v_x = v_{0x} + a_x t$). Solving for a gives

$$a_x = \frac{v_x - v_{0x}}{t} = \frac{(-1.6\text{ m/s}) - (1.6\text{ m/s})}{4.5\text{ s}} = -0.71\text{ m/s}^2.$$

This is larger in magnitude than the acceleration in Sample Problem 3-1 (0.54 m/s^2), so it stands to reason that the worker must push harder this time. We find this (constant) force F'_{SW} from

$$\begin{aligned} F'_{SW} &= ma_x = (240\text{ kg})(-0.71\text{ m/s}^2) \\ &= -170\text{ N } (= -38\text{ lb}). \end{aligned}$$

The negative sign shows that the worker is pushing the sled in the direction of decreasing x—that is, to the left as shown in the free-body diagram of Fig. 3-7*c*.

SAMPLE PROBLEM 3-3. A crate whose mass m is 360 kg rests on the bed of a truck that is moving at a speed v_0 of 105 km/h, as in Fig. 3-8*a*. The driver applies the brakes and slows to a speed v of 62 km/h in 17 s. What force (assumed constant) acts on the crate during this time? Assume that the crate does not slide on the truck bed.

Solution We first find the (constant) acceleration of the crate. Solving Eq. 2-26 ($v_x = v_{0x} + a_x t$) for a_x yields

$$a_x = \frac{v_x - v_{0x}}{t} = \frac{(62\text{ km/h}) - (105\text{ km/h})}{(17\text{ s})(3600\text{ s/h})} = -0.70\text{ m/s}^2.$$

Because we have taken the positive sense of the horizontal direction to the right, the acceleration must point to the left.

The force F_{CT} exerted on the crate by the truck follows from Newton's second law:

$$\begin{aligned} F_{CT} &= ma_x \\ &= (360\text{ kg})(-0.70\text{ m/s}^2) = -250\text{ N}. \end{aligned}$$

This force acts in the same direction as the acceleration—namely, to the left in Fig. 3-8*b*. The force must be supplied by an external agent, such as the straps or other mechanical means used to secure the crate to the truck bed. If the crate is not secured, then friction between the crate and the truck bed must supply the required force. If there is not enough friction to provide a force of 250 N, the crate will slide on the truck bed because, as measured by a ground-based observer, it will slow down less rapidly than the truck.

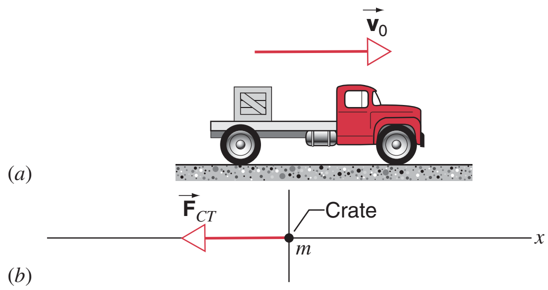


FIGURE 3-8. Sample Problem 3-3. (*a*) A crate on the truck that is slowing down. (*b*) The free-body diagram of the crate.

3-6 NEWTON'S THIRD LAW

Consider the Earth and the Moon. The Earth exerts a gravitational force on the Moon, and the Moon exerts a gravitational force on the Earth. All forces are part of such mutual interactions between two (or more) bodies—it is not possible to have only a single isolated force.

The forces that act on a body (let us call it body A) are due to the other bodies in its environment. Suppose that body B is one of these bodies in the environment of body A. Then among the forces acting on body A is $\vec{\mathbf{F}}_{AB}$, the force on body A due to body B. Alternatively, we might direct our attention to body B. Among the bodies in the environment of body B is body A, which exerts a force $\vec{\mathbf{F}}_{BA}$ on body B. Newton's third law concerns the relationship between $\vec{\mathbf{F}}_{AB}$ and $\vec{\mathbf{F}}_{BA}$.

We find by experiment that when one body exerts a force on a second body, then the second body always exerts a force on the first. Furthermore, we find the forces *always* to be equal in magnitude and opposite in direction. In the Earth–Moon system, the magnitude of the force on the Moon due to the Earth is equal to the magnitude of the force of the Earth due to the Moon. The forces are also opposite in direction—if we imagine a line connecting the Earth and the Moon, then the force on the Moon by the Earth acts along that line toward the Earth and the force on the Earth by the Moon acts along that same line toward the Moon.

Newton's third law summarizes these observations:

When one body exerts a force on another, the second exerts a force on the first. These two forces are always equal in magnitude and opposite in direction.

Formally (see Fig. 3-9), let body B exert a force $\vec{\mathbf{F}}_{AB}$ on body A; experiment then shows that body A exerts a force $\vec{\mathbf{F}}_{BA}$ on body B. These forces are related by

$$\vec{\mathbf{F}}_{AB} = -\vec{\mathbf{F}}_{BA}. \tag{3-6}$$

The negative sign reminds us that the forces act in opposite directions, as shown in Fig. 3-9.

It is customary to label the two forces $\vec{\mathbf{F}}_{AB}$ and $\vec{\mathbf{F}}_{BA}$ due to the mutal interaction of two bodies as the "action" and "reaction" forces. These labels are completely arbitrary; either force could be called the "action," and its partner would then be the "reaction." By using these common labels, we do not mean to imply that the "action" somehow causes the "reaction." Both forces exist due to the mutual interaction, and we simply pick one as the "action," which then leaves the other as the "reaction." This gives us a shorthand way of stating Newton's third law:

To every action there is an equal and opposite reaction.

This law requires that the reaction force must exist and it also specifies its magnitude and direction.

It is important to remember that the action and reaction forces *always* act on *different* bodies, as the subscripts remind us. Often you will encounter situations in which two equal and opposite forces act on the same body (as in Fig. 3-4). Equation 3-6 shows that these two forces *cannot* be an action–reaction pair, because they act on the same body. In a true action–reaction pair, one force acts on body A and the other on body B. If you have labeled your forces carefully by specifying the body on which the force acts and the body that causes the force, then you can identify the reaction force by simply interchanging the names of the two bodies. For example:

Action: Force on book due to table	*Reaction:* Force on table due to book
Action: Force on Moon due to Earth	*Reaction:* Force on Earth due to Moon
Action: Force on electron due to nucleus	*Reaction:* Force on nucleus due to electron
Action: Force on baseball by bat	*Reaction:* Force on bat by baseball

If our goal is to study the dynamics of one body (say, the book or the baseball) then only one member of the action–reaction pair would be considered (the force that acts on that body). The other member would be considered only if we were also studying the dynamics of the second body (the table or the bat).

The following examples illustrate applications of the third law.

1. *An orbiting satellite.* Figure 3-10 shows a satellite orbiting the Earth. The only force that acts on it is $\vec{\mathbf{F}}_{SE}$, the

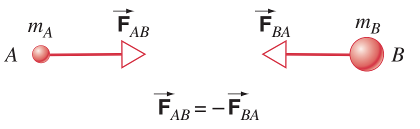


FIGURE 3-9. Newton's third law. Body A exerts a force $\vec{\mathbf{F}}_{BA}$ on body B. Body B must then exert a force $\vec{\mathbf{F}}_{AB}$ on body A, and $\vec{\mathbf{F}}_{AB} = -\vec{\mathbf{F}}_{BA}$.

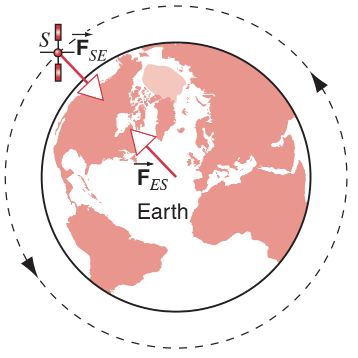


FIGURE 3-10. A satellite in Earth orbit. The forces shown are an action–reaction pair. Note that they act on different bodies.

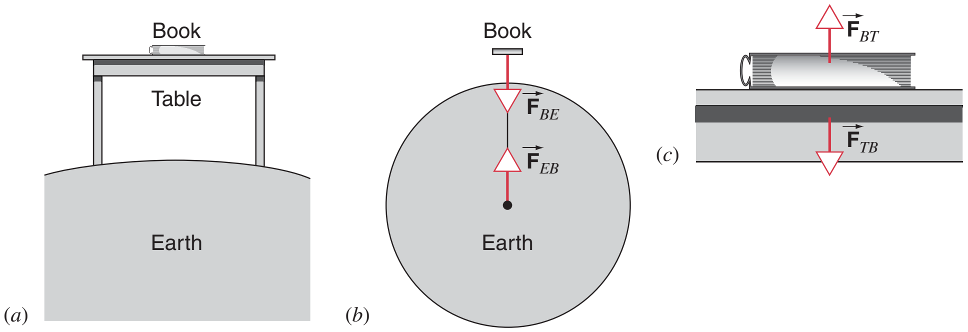


FIGURE 3-11. (*a*) A book rests on a table, which in turn rests on the Earth. (*b*) The book and the Earth exert gravitational forces on each other, forming an action–reaction pair. (*c*) The table and book exert action–reaction contact forces on each other.

force exerted *on* the satellite *by* the gravitational pull of the Earth. Where is the corresponding reaction force? It is $\vec{F}_{ES}$, the force acting on the Earth due to the gravitational pull of the satellite.

You may think that the tiny satellite cannot exert much of a gravitational pull on the Earth but it does, exactly as Newton's third law requires ($\vec{F}_{ES} = -\vec{F}_{SE}$). The force $\vec{F}_{ES}$ causes the Earth to accelerate, but, because of the Earth's large mass, its acceleration is so small that it cannot easily be detected.

2. *A book resting on a table.* Figure 3-11*a* shows a book resting on a table. The Earth pulls downward on the book with a force $\vec{F}_{BE}$. The book does not accelerate because the effect of this force is balanced by an equal and opposite contact force $\vec{F}_{BT}$ exerted on the book by the table.

Even though $\vec{F}_{BE}$ and $\vec{F}_{BT}$ are equal in magnitude and oppositely directed, they do *not* form an action–reaction pair. Why not? *Because they act on the same body—the book.* The two forces sum to zero and thus account for the fact that the book is not accelerating.

Each of these forces must then have a corresponding reaction force somewhere. Where are they? The reaction to $\vec{F}_{BE}$ is $\vec{F}_{EB}$, the (gravitational) force with which the book attracts the Earth. We show this action–reaction pair in Fig. 3-11*b*.

Figure 3-11*c* shows the reaction force to $\vec{F}_{BT}$. It is $\vec{F}_{TB}$, the contact force on the table due to the book. The action–reaction pairs involving the book in this problem, and the bodies on which they act, are

first pair: $\vec{F}_{BE} = -\vec{F}_{EB}$ (book and Earth)

and

second pair: $\vec{F}_{BT} = -\vec{F}_{TB}$ (book and table).

3. *Pushing a row of crates.* Figure 3-12 shows a worker *W* pushing two crates, each of which rests on a wheeled cart that can roll with negligible friction. The worker exerts a force $\vec{F}_{1W}$ on crate 1, which in turn pushes back on the worker with a reaction force $\vec{F}_{W1}$. Crate 1 pushes on crate 2 with a force $\vec{F}_{21}$, and crate 2 pushes back on crate 1 with a force $\vec{F}_{12}$. (Note that the worker exerts no force on crate 2 directly.) To move forward, the worker must push backward against the ground. The worker exerts a force $\vec{F}_{GW}$ on the ground, and the reaction force of the ground on the worker, $\vec{F}_{WG}$, pushes the worker forward. The figure shows three action–reaction pairs:

$$\vec{F}_{21} = -\vec{F}_{12} \qquad \text{(crate 1 and crate 2)},$$

$$\vec{F}_{1W} = -\vec{F}_{W1} \qquad \text{(worker and crate 1)},$$

$$\vec{F}_{WG} = -\vec{F}_{GW} \qquad \text{(worker and ground)}.$$

Note that in this example the worker is the active agent that is responsible for the motion, but it is the reaction force on the worker by the ground that makes this possible. If there were no friction between the worker's shoes and the ground, the worker could not move the system forward.

4. *Block hanging from a spring.* Figure 3-13*a* shows a block hanging at rest from a spring, the other end of which is fixed to the ceiling. The forces on the block, shown separately in Fig. 3-13*b*, are its weight $\vec{F}_{BE}$ (the gravitational force on the block by the Earth) and the force $\vec{F}_{BS}$ exerted on the block by the spring. The block is at rest under the influence of these forces, but they are *not* an action–reaction pair, because once again they act on the same body. The reaction force to the weight $\vec{F}_{BE}$ is the gravitational force $\vec{F}_{EB}$ that the block exerts on the Earth, which is not shown.

The reaction force to $\vec{F}_{BS}$ (the force exerted *on* the block *by* the spring) is the force $\vec{F}_{SB}$ exerted *on* the spring *by* the block. To show this force, we illustrate the forces acting on the spring in Fig. 3-13*c*. These forces include the reaction to $\vec{F}_{BS}$, which we show as a force $\vec{F}_{SB}$ ($= -\vec{F}_{BS}$) acting downward, the weight $\vec{F}_{SE}$ of the spring (usually negligible), and the upward pull $\vec{F}_{SC}$ on the spring by the ceiling. If the spring is at rest, the net force must be zero: $\vec{F}_{SC} + \vec{F}_{SE} + \vec{F}_{SB} = 0$.

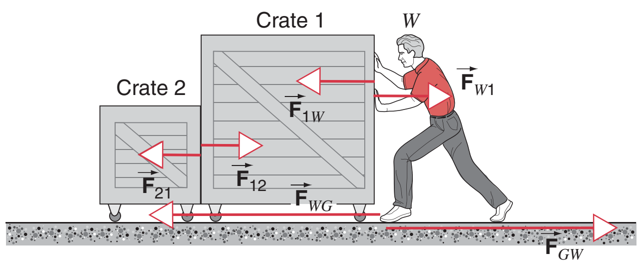


FIGURE 3-12. A worker pushes against crate 1, which in turn pushes on crate 2. The crates are on wheels that move freely, so there is no friction between the crates and the ground.

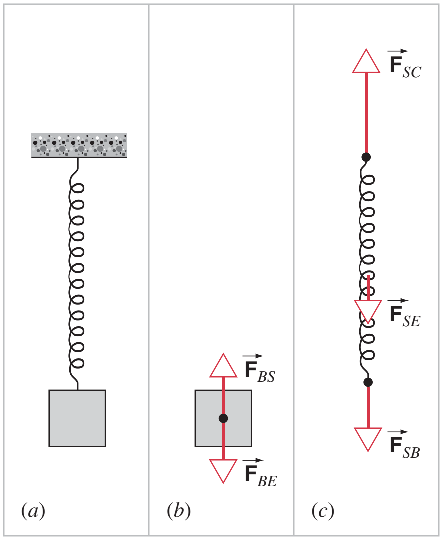


FIGURE 3-13. (*a*) A block hangs at rest supported by a stretched spring. (*b*) The forces on the block. (*c*) The forces on the spring.

The reaction force to $\vec{\mathbf{F}}_{SC}$ acts *on* the ceiling. Since we are not showing the ceiling as an independent body in this diagram, the reaction to $\vec{\mathbf{F}}_{SC}$ does not appear.

Verifying Newton's Third Law

We can easily verify Newton's third law by attaching electronic force probes (Fig. 3-5) to two carts that collide on a frictionless track. These force probes are connected to a computer, which plots the force instantaneously as the two carts collide.

Figure 3-14 shows the results of three different collisions between the carts. In Fig. 3-14*a*, cart 1 was originally at rest when cart 2 (of equal mass) collided with it. Note that at every instant of time, the force exerted on cart 1 by cart 2 is equal and opposite to that exerted on cart 2 by cart 1.

Figure 3-14*b* shows the results when the same two carts experience a head-on collision with both carts in motion, and Fig. 3-14*c* shows the results of a head-on collision between two carts when cart 2 has three times the mass of cart 1.

In every case, no matter which cart is in motion and no matter what the relative masses of the two carts are, the forces are exactly equal and opposite, just as Newton's third law requires.

3-7 WEIGHT AND MASS

Consider a body of mass m that is released from rest near the surface of the Earth. As we discussed in Section 2-6, the body will fall downward with the free-fall acceleration $\vec{\mathbf{g}}$ (if we neglect other forces that may be acting, such as air resistance). Here $\vec{\mathbf{g}}$ is a vector whose magnitude g is the free-fall acceleration and whose direction is vertically down (toward the center of the Earth). Assuming the Earth's surface to be an inertial frame so that we may apply Newton's second law, the net force on the object must equal $m\vec{\mathbf{a}}$, or $m\vec{\mathbf{g}}$ in our case since $\vec{\mathbf{a}} = \vec{\mathbf{g}}$. This force is due to the Earth's gravitational attraction for the body.

If we instead hold the body in our hand rather than releasing it, its acceleration is zero and hence the net vertical force must be zero (by Newton's second law). We have not "turned off" the Earth's gravitational attraction for the body; this force still acts on the body and can still be expressed as $m\vec{\mathbf{g}}$. The hand must therefore be exerting an upward force equal in magnitude to the downward force $m\vec{\mathbf{g}}$, so the net force is zero. (These two forces are *not* an action–reaction pair, because they act on the same body.) We sense the need to exert this upward force by the tension in our muscles; in this way we can "feel" the Earth's gravitational attraction for the body.

The downward force of the Earth's gravity acting on the body is called the *weight* of the body. The force of the Earth's gravity on the body is the same whether it is at rest or falling; the force has magnitude mg and a direction toward the center of the Earth. In terms of magnitudes, the weight W is

$$W = mg. \tag{3-7}$$

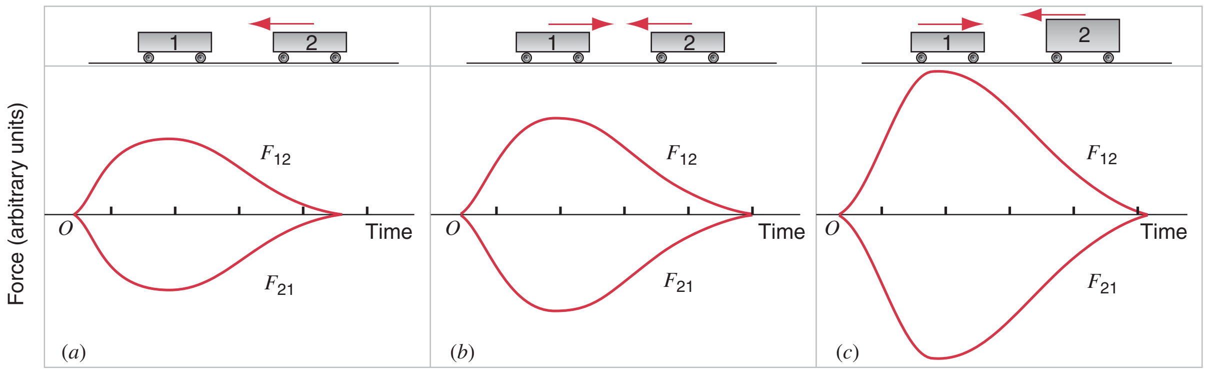


FIGURE 3-14. Electronic force probes of the type shown in Fig. 3-5 are mounted on two colliding carts. The probes simultaneously measure the force exerted on each cart by the other. The plots show the forces exerted on the two carts as functions of the time during the collisions. (*a*) Cart 2 is initially in motion and collides with cart 1, which is initially at rest. The two carts have equal masses. (*b*) A head-on collision between two carts of equal masses. (*c*) A head-on collision between two carts in which cart 2 has three times the mass of cart 1.

Weight is measured in force units, such as newtons or pounds.

We can measure the weight directly if we place the body on a platform scale (such as a bathroom scale) with a display that indicates the magnitude of the force that the platform is exerting on the body; if the body is at rest, then the net vertical force is zero, and so the upward force on the body due to the platform must equal the downward force on the body due to the Earth (the weight). We can also measure this force using a spring scale, such as might be found in the produce section of a supermarket; again, the net force on the body must be zero, and the spring exerts an upward force (which can be read on the scale) equal in magnitude to the downward force mg.

When you draw a free-body diagram for a body near the Earth's surface, you should include a force $m\vec{g}$ directed toward the center of the Earth. This represents the weight, the gravitational force on the body due to the Earth. The third-law reaction force to the weight is the gravitational force on the Earth due to the body; this force would appear only on a free-body diagram of the Earth (as in Figs. 3-10 or 3-11*b*).

In this discussion, we have assumed that the surface of the Earth is an inertial frame. This is only approximately true; because of the Earth's rotation its surface is not an inertial frame, but the error resulting from this assumption is very small—about 0.3% at the equator, where the effect is largest. That is, the weight indicated on a scale at the equator is 0.3% smaller than the force of gravity on the body. At the poles, the scale reading and the free-fall acceleration are not affected by the rotation. Neglecting this effect, we can regard the scale readings at the Earth's surface as a sufficiently precise measure of the weight of an object.

The Difference between Weight and Mass

As Eq. 3-7 shows, the weight depends on the mass—the greater the mass, the greater the weight. A second body with twice the mass of the first will have twice the weight at the same location. However, weight and mass are very different quantities. Our definition of mass in Section 3-4 and our operational procedure for measuring it make no reference to the gravitational force of attraction by the Earth. We could use the same procedure, and arrive at the same values of the mass in comparison with the standard kilogram, if we carried out the measurements on the Moon (where the free-fall acceleration is only $\frac{1}{6}$ the value on Earth) or even in empty space far from any planet or star (where the free-fall acceleration is zero). The weight of the body would be different in these locations, but the mass is the same; that is, we must apply the same amount of force at each location to produce a given acceleration.

The mass of a body has the same value in any location, but its weight will vary on the surface of the Earth where the free-fall acceleration varies with location. At a location on the Earth's surface where $g = 9.78\ \text{m/s}^2$ (near the equator, for example) a body of mass 1.00 kg has a weight of 9.78 N, while near the poles where $g = 9.83\ \text{m/s}^2$ (because the poles are closer to the center of the Earth than the equator) that same body has a weight of 9.83 N. Identical spring scales in the two locations would stretch by slightly different amounts to record these different weights. Unlike the mass, which is an *intrinsic* property of the body, the weight of a body depends on its location relative to the center of the Earth. (Variations in g on Earth are discussed in Section 14-4.)

Equation 3-7 shows that, for a given value of g, mass and weight are proportional to each other. Sometimes you may see an equation in which mass units are set equal to weight units—for example, 1 kg = 2.2 lb. This equation violates the rules for dimensional consistency we discussed in Section 1-7. Here the equal sign means "is equivalent to," and the equation should be taken to mean that at a location where g has a certain value, an object of mass 1 kg is equivalent to a weight of 2.2 lb. On the surface of the Moon, this equation would read 1 kg = 0.37 lb, but at the surface of Jupiter 1 kg = 5.1 lb. A "quarter-pounder" hamburger at a fast-food outlet on Jupiter would contain about 14 times as much meat as one on the Moon, but a 0.1-kg hamburger (about $\frac{1}{4}$ lb on Earth) would contain exactly the same amount of meat at all locations. When we colonize the planets, we should be sure to order supplies by mass, not by weight!

Weightlessness

Photographs of astronauts in orbiting space vehicles (such as Fig. 3-15) show them floating freely in a state that we call "weightless." We must be careful how we analyze the astronauts' motion, because in their rapidly orbiting craft they are not even approximately in an inertial frame. An object released from rest, according to the astronaut's noninertial reference frame, stays at the same location. Thus the free-fall acceleration appears to be zero in that frame. However, an object released from a nonorbiting craft at that same altitude (about 400 km in the case of the space shut-

Figure 3-15. Astronaut Dr. Mae C. Jemison in free-fall in the orbiting space shuttle *Endeavor* appears to float as if weightless.

FIGURE 3-16. Actors in simulated weightlessness during the filming of the movie *Apollo 13*. They are in free-fall in a KC-135 aircraft flying a parabolic trajectory. This aircraft, nicknamed the "Vomit Comet," is used by NASA for microgravity research.

tle) would fall toward the center of the Earth with a free-fall acceleration of about 8.7 m/s^2.

If we were to place a body on a platform scale or attach it to a spring scale in the astronaut's noninertial reference frame, the scale would read zero. In this reference frame, we cannot use the scale reading to determine the weight of the body. However, the body is certainly *not* "massless," nor is it "gravityless." With $g = 8.7\ m/s^2$ at that altitude, a 1-kg body would have a weight of 8.7 N, about 11% less than its weight on the Earth's surface.

Our perception or sensation of weight involves the force with which the floor pushes up on us. Floating in water, we are less aware of our *weight,* but we are fully aware of our *mass,* such as when we try to accelerate by swimming through the water. If we stand in an elevator that is accelerating upward, the floor exerts a force on us that is greater than the pull of gravity and so we feel as if our weight increases; when the elevator accelerates downward, we feel as if our weight has become smaller. If we stand on a platform scale in an accelerating elevator, the scale readings will confirm these perceptions (see Sample Problem 3-7). However, the magnitude of the weight remains mg, independent of any acceleration.

True weightlessness can be achieved only in deep space, far from any star or planet. In a spacecraft drifting with its engines off, the astronauts would float freely. If the engines fire, the resulting acceleration would cause the ship to be a noninertial frame; in the astronauts' frame of reference, the floor of the accelerating ship would exert an upward force that would be perceived by the astronauts as similar to weight. In a similar way, if the ship rotates, the outer wall of the ship would be the floor that provides the sensation of weight by pushing anything in contact with the floor toward the axis of rotation. This effect is sometimes referred to as "artificial gravity" and will be used in the International Space Station to provide the sensation of weight for biological specimens.

A body in free fall near the Earth's surface has no floor to push on it and therefore would feel weightless. If you were inside a chamber that is also in free fall (such as an elevator cab in which the cable breaks), the floor would not push on you and you would feel no sensation of weight. As we will learn in the next chapter, a projectile in free fall near the Earth's surface follows a parabolic trajectory; if an airplane flies a chosen parabolic trajectory, the passengers inside will be objects in free fall and will feel weightless because they are not in contact with the floor of the plane. This effect is used to train astronauts for working in the similar free fall of Earth orbit and has also been used in movies to simulate the effects of orbit (Fig. 3-16). Even though a body in free fall near the Earth's surface lacks the sensation of weight that ordinarily comes from the upward push of a floor, the weight remains at the value mg, indicating the strength of the Earth's gravitational attraction for the body.

3-8 APPLICATIONS OF NEWTON'S LAWS IN ONE DIMENSION

Although each problem to be solved with Newton's laws will require a unique approach, the general procedure that we gave in Section 3-5 forms the basis for the analysis of all such problems. The best way to learn the applications of the rules is to study the examples. Often in these problems there are two or more bodies to which Newton's laws must be separately applied.

In these problems we make some assumptions that simplify the analysis at the cost of some physical reality. Bodies are treated as particles, so that all forces can be considered to act at a single point. String are massless (no force is required to accelerate them) and inextensible (they do not stretch, so that bodies connected by taut strings have the same speed and acceleration). Despite these simplifications, the examples provide insight into the basic techniques of dynamical analysis. Later in the text we add new techniques that permit us to be more realistic in our analysis. For now, we ignore many admittedly important effects so that we can concentrate on the basic methods used to solve problems.

Sample Problem 3-4. A worker W is pushing a packing crate of mass $m_1 = 4.2$ kg. In front of the crate is a second crate of mass $m_2 = 1.4$ kg (Fig. 3-17*a*). Both crates slide across the floor without friction. The worker pushes on crate 1 with a force $P_{1W} = 3.0$ N. Find the accelerations of the crates and the force exerted by crate 1 on crate 2.

Solution We choose the positive x axis to be in the direction of motion of the crates, so that force and acceleration components to the right in Fig. 3-17 are positive. The worker pushes only on crate 1. The force $\vec{\mathbf{F}}_{21}$ pushing on crate 2 is exerted on it by crate 1. According to Newton's third law, crate 2 then exerts a force $\vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21}$ on crate 1. With F_{12} and F_{21} representing the magnitudes of these forces, the x component of $\vec{\mathbf{F}}_{12}$ is $-F_{12}$ and that of $\vec{\mathbf{F}}_{21}$ is F_{21}. Figures 3-17*b*, *c* show the free-body diagrams of crates 1 and 2. The net force on m_1 is $\Sigma F_x = P_{1W} - F_{12}$, and for crate 1 Newton's second law ($\Sigma F_x = ma_x$) then gives

$$\text{(crate 1)} \qquad P_{1W} - F_{12} = m_1 a_1,$$

where a_1 represents the x component of the acceleration of crate 1. The net force on crate 2 is $\Sigma F_x = F_{21}$, so Newton's second law gives

$$\text{(crate 2)} \qquad F_{21} = m_2 a_2.$$

If the two crates stay in contact, then $a_1 = a_2$. We call this common acceleration a. Adding these equations, we obtain

$$P_{1W} - F_{12} + F_{21} = m_1 a + m_2 a$$

or, using Newton's third law for the magnitudes of the contact forces ($F_{12} = F_{21}$) and solving for a,

$$a = \frac{P_{1W}}{m_1 + m_2} = \frac{3.0\ \text{N}}{4.2\ \text{kg} + 1.4\ \text{kg}} = 0.54\ \text{m/s}^2.$$

It should not be too surprising that the acceleration is determined by the total mass $m_1 + m_2$ of the system of two crates, because the force P_{1W} exerted by the worker is ultimately responsible for accelerating the entire system. To find the contact force exerted on crate 2 by crate 1, we have

$$F_{21} = m_2 a = (1.4\ \text{kg})(0.54\ \text{m/s}^2) = 0.76\ \text{N}.$$

Note that the force exerted on crate 2 by crate 1 (equal to 0.76 N) is smaller than the force exerted on crate 1 by the worker (3.0 N). This is reasonable, because F_{21} acts only to accelerate crate 2, but P_{1W} acts to give the same acceleration to both crates.

Sample Problem 3-5. A flat-bed cart of mass $m_C = 360$ kg rolls on frictionless wheels. Resting on the cart is a box of mass $m_B = 150$ kg (Fig. 3-18*a*). The box can slide on the cart, but each exerts a force (due to friction) on the other during the sliding. When a worker pulls on the box with a force $\vec{\mathbf{F}}_{BW}$, both the box and the cart move forward, but the box moves faster than the cart, because the frictional force is not strong enough to prevent the box from sliding forward on the cart. An observer measures the magnitudes of the accelerations to be 1.00 m/s² for the box and 0.167 m/s² for the cart. Find (*a*) the frictional force between the box and the cart and (*b*) the force that the worker is exerting on the box.

Solution (*a*) We choose the x axis so that its positive direction is to the right in Fig. 3-18. Force and acceleration components in that direction are positive. The worker exerts a force $\vec{\mathbf{F}}_{BW}$ on the box. It is customary to use a lower-case symbol $\vec{\mathbf{f}}$ to represent frictional forces, so the force on the box due to the cart is $\vec{\mathbf{f}}_{BC}$, which points to the left (opposing the motion of the box) and has x component $-f_{BC}$. By Newton's third law, there is an equal and opposite frictional force $\vec{\mathbf{f}}_{CB}$ on the cart exerted by the box. Figures 3-18*b*, *c* show the free-body diagrams of the cart and the box.

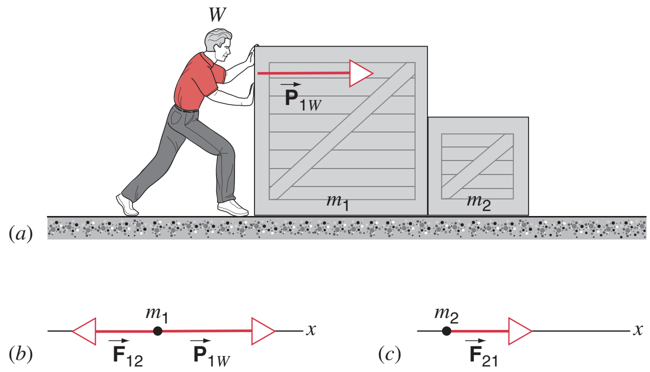


Figure 3-17. Sample Problem 3-4. (*a*) A worker pushes on a crate, which in turn pushes on another crate. (*b*) The free-body diagram of crate 1. (*c*) The free-body diagram of crate 2.

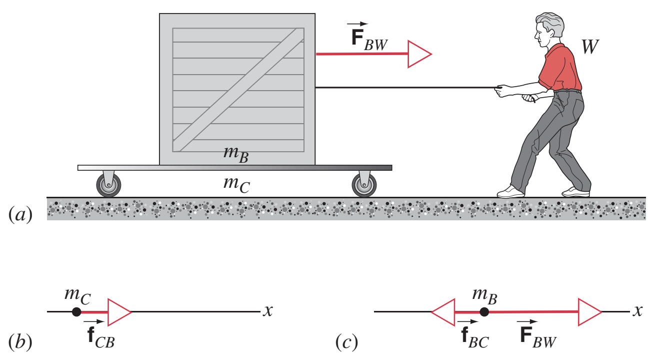


Figure 3-18. Sample Problems 3-5 and 3-6. (*a*) A worker pulls on a box that slides on a rolling cart. (*b*) The free-body diagram of the cart. (*c*) The free-body diagram of the box.

The net force on the cart is $\Sigma F_x = f_{CB}$, and applying Newton's second law ($\Sigma F_x = ma_x$) to the cart, we obtain

(cart) $\quad f_{CB} = m_C a_C = (360 \text{ kg})(0.167 \text{ m/s}^2) = 60 \text{ N}.$

(*b*) Similarly, the net force on the box is $\Sigma F_x = F_{BW} - f_{BC}$, so Newton's second law gives

(box) $\quad F_{BW} - f_{BC} = m_B a_B$

and solving for F_{BW} we obtain

$$F_{BW} = f_{BC} + m_B a_B = 60 \text{ N} + (150 \text{ kg})(1.00 \text{ m/s}^s) = 210 \text{ N},$$

where we have used $f_{BC} = f_{CB}$ for the magnitudes of the frictional forces, which form an action–reaction pair.

Friction with the box pulls the cart forward. Even though in this case friction produces the motion of the cart, friction between two objects always opposes their *relative* motion. If there were no friction in this problem, the cart would not move at all and there would be more relative motion between the box and the cart. If the frictional force were sufficiently large (see the next Sample Problem), the box and the cart could move together with *no* relative motion between them.

SAMPLE PROBLEM 3-6. Suppose in the previous Sample Problem that the frictional force were larger, so that the box does not slide on the cart (the box and the cart move together as a unit). If the force applied by the worker remains the same at 210 N, what is the frictional force of the box on the cart?

Solution The cart and the box move together, so they have the same acceleration a. Newton's second law gives

(cart) $\quad \Sigma F_x = f_{CB} = m_C a$

(box) $\quad \Sigma F_x = F_{BW} - f_{BC} = m_B a.$

Since we want to solve for the unknown $f_{CB} = f_{BC}$, we eliminate a from these two equations and obtain

$$f_{CB} = \frac{m_C F_{BW}}{m_C + m_B} = \frac{(360 \text{ kg})(210 \text{ N})}{360 \text{ kg} + 150 \text{ kg}} = 150 \text{ N}.$$

SAMPLE PROBLEM 3-7. A passenger P of mass $m =$ 72.2 kg is riding in an elevator while standing on a platform scale (Fig. 3-19*a*, which is essentially a calibrated spring scale that reads the upward force F_{PS} exerted on the passenger by the scale. (The situation would be exactly the same if the passenger were hanging from a spring scale.) What does the scale read when the elevator cab is (*a*) descending with constant velocity and (*b*) ascending with an acceleration of 3.20 m/s^2?

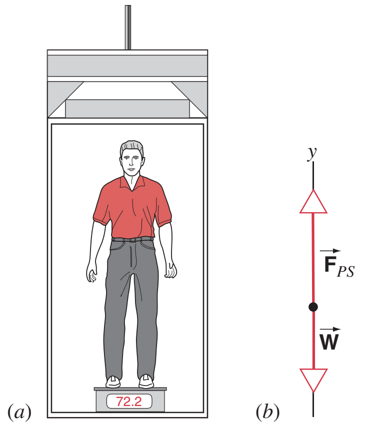


FIGURE 3-19. Sample Problem 3-7. (*a*) A passenger is riding in an elevator cab while standing on a scale. Like most such scales, this one reads in mass units (kilograms), rather than in the corresponding force units (newtons). (*b*) The free-body diagram of the passenger.

Solution (*a*) First we develop a general result that is valid for any acceleration a. We choose our inertial reference frame to be that of the building in which the elevator is located, because an accelerating elevator is not an inertial reference frame. Both g and a are measured by an observer in this inertial frame. We choose our coordinate system so that the y axis is vertical and positive upward. Figure 3-19*b* shows the free-body diagram of the passenger. Two forces act on the passenger: the upward force $\vec{F}_{PS}$ exerted by the scale and the downward weight $\vec{W}$ of magnitude $W = mg$ (the force of the Earth's gravity on the passenger).

The net force on the passenger is then $\Sigma F_y = F_{PS} - W$, and Newton's second law ($\Sigma F_y = ma_y$) gives

$$F_{PS} - W = ma_y$$

or

$$F_{PS} = W + ma_y = mg + ma_y.$$

When $a_y = 0$ (corresponding to motion with constant velocity) then

$$F_{PS} = mg = (72.2 \text{ kg})(9.80 \text{ m/s}^2) = 708 \text{ N } (= 159 \text{ lb}).$$

The scale reading does not depend on the velocity of the elevator, and the scale reads the same when the elevator moves with constant velocity as it does when the elevator is at rest.

(*b*) When $a_y = +3.20 \text{ m/s}^2$, we have

$$F_{PS} = m(g + a_y) = (72.2 \text{ kg})(9.80 \text{ m/s}^2 + 3.20 \text{ m/s}^2)$$
$$= 939 \text{ N } (= 211 \text{ lb}).$$

The scale reading increases when the elevator is accelerating upward and decreases when the elevator is accelerating downward. What does the scale read when the elevator is moving upward but accelerating downward (that is, it is slowing down)? What does it read when the cable breaks and the elevator is in free fall ($a_y = -g$)?

MULTIPLE CHOICE

3-1 Classical Mechanics

3-2 Newton's First Law

1. An interstellar spacecraft, far from the influence of any stars or planets, is moving at high speed under the influence of fusion rockets when the engines malfunction and stop. The spacecraft will

(A) immediately stop, throwing all of the occupants to the front of the craft.
(B) begin slowing down, eventually coming to a rest in the cold emptiness of space.
(C) keep moving at constant speed for a while, but then begin to slow down.
(D) keep moving forever at the same speed.

2. A small child is playing with a ball on a level surface. She gives the ball a push to get it rolling, then the ball rolls a short distance before coming to a stop. The ball slows to a stop because

(A) the child stopped pushing it.
(B) speed is proportional to force.
(C) there must have been some force on the ball opposing the direction of motion.
(D) the net force on the ball was zero, so it wanted to remain stationary.

3-3 Force

3. A student attaches a ruler to a block of wood sitting on a horizontal surface as shown in Fig. 3-20*a*. The surface exerts a large frictional force on the block. A nail is fastened at the 0-in. mark. The student pulls the rubber band tight at the 5-in. mark; he then pulls it tighter and when it reaches the 8-in. mark, the block of wood just starts to move (Fig. 3-20*b*). Before the block begins to move, the net force on the block when the rubber band is stretched 7 in. is

(A) greater than (B) equal to
(C) less than (D) unrelated to

the net force on the block when the rubber band is stretched 6 in.

4. The student in multiple-choice question 3 then pulls the rubber band so that it is at the 9-in. mark. The block of wood slides faster and faster with a constant acceleration, and as it moves, the student moves his hand so that the rubber band is always pulled to the 9-in. mark. The net force on the block when the rubber band is stretched 9 in. is

(A) greater than (B) equal to
(C) less than (D) unrelated to

the net force on the block when the rubber band was stretched 7 in.

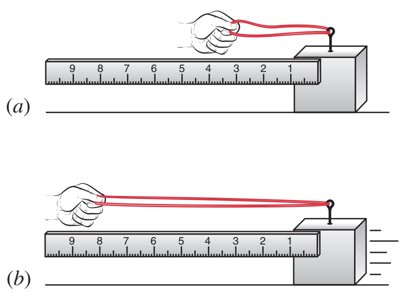


FIGURE 3-20. Multiple-choice questions 3 and 4.

3-4 Mass

5. Two objects with masses M and m ($M > m$) are on a frictionless surface. A force F will accelerate the smaller object with an acceleration a. If that same force is applied to the larger object then it will

(A) move with a greater acceleration.
(B) move with the same acceleration.
(C) move but with a smaller acceleration.
(D) move only if the force F is greater than some minimum value.

3-5 Newton's Second Law

6. An object is moving north. From only this information one can conclude

(A) that there is a single force on the object directed north.
(B) that there is a net force on the object directed north.
(C) that there may be several forces on the object, but the largest must be directed north.
(D) nothing about the forces on the object.

7. An object is moving north with an increasing speed. From only this information one can conclude

(A) that there is a single force on the object directed north.
(B) that there is a net force on the object directed north.
(C) that there may be several forces on the object, but the largest must be directed north.
(D) nothing about the forces on the object.

8. Which of the following objects is not experiencing a net force directed north?

(A) An object moving south at a decreasing speed.
(B) An object moving north at an increasing speed.
(C) An object instantaneously at rest that then begins to move north.
(D) An object moving north at a constant speed.

3-6 Newton's Third Law

9. A rock rests on a level surface. The magnitude of the force on the surface due to the rock is F_{SR}, and the magnitude of the force on the rock due to the surface is F_{RS}. If we compare these forces, we find

(A) $F_{SR} < F_{RS}$. (B) $F_{SR} = F_{RS}$. (C) $F_{SR} > F_{RS}$.
(D) There is not enough information to compare the two forces.

10. A rock is on an inclined surface. The rock is originally at rest, but starts to slide down the incline. The magnitude of the force on the surface due to the rock is F_{SR}, and the magnitude of the force on the rock due to the surface is F_{RS}. If we compare these forces, we find

(A) $F_{SR} < F_{RS}$ always.
(B) $F_{SR} = F_{RS}$ when the rock is at rest but then $F_{SR} > F_{RS}$.
(C) $F_{SR} = F_{RS}$ always.
(D) $F_{SR} > F_{RS}$ always.

11. A piano is rolling down a frictionless slope at an ever increasing speed. The piano tuner sees it, runs up to it and pushes on it, slowing it down to a constant speed. The magnitude of the force on the man by the piano is F_{MP}; the magnitude of the force on the piano by the man is F_{PM}. If we compare these forces, we find

(A) $F_{PM} > F_{MP}$ always.

(B) $F_{PM} > F_{MP}$ while the piano slows down but $F_{PM} = F_{MP}$ when the piano is moving at constant speed.
(C) $F_{PM} = F_{MP}$ always.
(D) $F_{PM} = F_{MP}$ while the piano slows down but $F_{PM} < F_{MP}$ when the piano is moving at constant speed.

3-7 Weight and Mass

12. A large rock *falls* on your toe. Which of the concepts is most important in determining how much it hurts?
(A) The mass of the rock.
(B) The weight of the rock.
(C) Both the mass and the weight of the rock are important.
(D) Either the mass or the weight, as the two are related by a single multiplicative constant g.

13. A large rock *sits* on your toe. Which of the concepts is most important in determining how much it hurts?
(A) The mass of the rock.
(B) The weight of the rock.
(C) Both the mass and the weight of the rock are important.
(D) Either the mass or the weight, as the two are related by a single multiplicative constant g.

3-8 Applications of Newton's Laws in One Dimension

14. An object is free to move on a table, except that there is a constant frictional force f that opposes the motion of the object when it moves. If a force of 10.0 N pulls the object, the acceleration is 2.0 m/s^2. If a force of 20.0 N pulls the object, the acceleration is 6.0 m/s^2.
(*a*) What is the force of friction f?
(A) 1.0 N. (B) 3.33 N. (C) 5.0 N. (D) 10.0 N.
(*b*) What is the mass of the object?
(A) 0.40 kg. (B) 2.5 kg. (C) 3.33 kg. (D) 5.0 kg.

15. A parachutist is in free fall before opening her chute. The net force on her has magnitude F and is directed down; this net force is somewhat less than her weight W because of air friction. Then she opens her chute. At the instant after her chute fully inflates the net force on her is
(A) greater than F and still directed down.
(B) less than F and still directed down.
(C) zero.
(D) directed upward, but could be more or less than F.

16. (*a*) You stand on a spring-loaded bathroom scale in a bathroom. The scale "reads" your mass. What is the scale actually *measuring?* (*b*) You stand on a spring-loaded bathroom scale in an elevator that is accelerating upward at 2.0 m/s^2. The scale "reads" your mass. What is the scale *measuring?*
(A) Your mass.
(B) Your weight.
(C) The force of the scale pushing up on your feet.
(D) The force of your feet pushing down on the scale.

QUESTIONS

1. Of the objects listed in Table 3-1, which might be considered as a particle for the motion described? For those that behave like particles, can you describe a type of motion in which they could *not* be considered as particles?

2. Why do you fall forward when a moving bus decelerates to a stop and fall backward when it accelerates from rest? Subway standees often find it convenient to face the side of the car when the train is starting or stopping and to face the front or rear when it is running at constant speed. Why?

3. Why did we specify the use of a "light" spring for the experiments described in Section 3-3? What would be the difference if we used a "heavy" spring?

4. A block with mass m is supported by a cord C from the ceiling, and a similar cord D is attached to the bottom of the block (Fig. 3-21). Explain this: If you give a sudden jerk to D, it will break, but if you pull on D steadily, C will break.

5. Criticize the statement, often made, that the mass of a body is a measure of the "quantity of matter" in it.

6. Using force, length, and time as fundamental quantities, what are the dimensions of mass?

7. How many slugs are in one kilogram?

8. A car moving at constant speed is suddenly braked. The occupants, all wearing seat belts, are thrown forward. The instant the car stops, however, the occupants are all jerked backward. Why? Is it possible to stop an automobile without this "jerk"?

9. Can Newton's first law be considered merely a special case of the second law with $\vec{a} = 0$? If so, is the first law really needed? Discuss.

10. What is the relation—if any—between the force acting on an object and the direction in which the object is moving?

11. Suppose a body that is acted on by exactly two forces is accelerated. Does it then follow that (*a*) the body cannot move with constant speed; (*b*) the velocity can never be zero; (*c*) the sum of the two forces cannot be zero; (*d*) the two forces must act in the same line?

12. A horse is urged to pull a wagon. The horse refused to try, citing Newton's third law as a defense: the pull of the horse on the wagon is equal but opposite to the pull of the wagon on the horse. "If I can never exert a greater force on the wagon than it exerts on me, how can I ever start the wagon moving?" asks the horse. How would you reply?

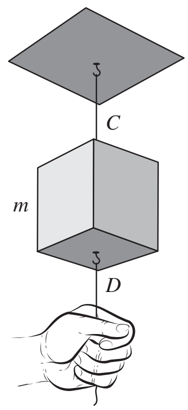


FIGURE 3-21. Question 4.

13. Comment on whether the following pairs of forces are examples of action–reaction: (*a*) The Earth attracts a brick; the brick attracts the Earth. (*b*) A propellered airplane pushes air toward the tail; the air pushes the plane forward. (*c*) A horse pulls forward on a cart, moving it; the cart pulls backward on the horse. (*d*) A horse pulls forward on a cart without moving it; the cart pulls back on the horse. (*e*) A horse pulls forward on a cart without moving it; the Earth exerts an equal and opposite force on the cart. (*f*) The Earth pulls down on the cart; the ground pushes up on the cart with an equal and opposite force.

14. The following statement is true; explain it. Two teams are having a tug of war; the team that pushes harder (horizontally) against the ground wins.

15. Two students try to break a rope. First they pull against each other and fail. Then they tie one end to a wall and pull together. Is this procedure better than the first? Explain your answer.

16. What is your mass in slugs? Your weight in newtons?

17. A French citizen, filling out a form, writes "78 kg" in the space marked Poids (weight). However, weight is a force and the kilogram is a mass unit. What do the French (among others) have in mind when they use a mass unit to report their weight? Why don't they report their weight in newtons? How many newtons does this person weigh? How many pounds?

18. Comment on the following statements about mass and weight taken from examination papers. (*a*) Mass and weight are the same physical quantities expressed in different units. (*b*) Mass is a property of one object alone, whereas weight results from the interaction of two objects. (*c*) The weight of an object is proportional to its mass. (*d*) The mass of a body varies with changes in its local weight.

19. A horizontal force acts on a body that is free to move. Can it produce an acceleration if the force is less than the weight of that body?

20. Why does the acceleration of a freely falling object not depend on the weight of the object?

21. Describe several ways in which you could, even briefly, experience weightlessness.

22. Under what circumstances would your weight be zero? Does your answer depend on the choice of a reference system?

23. The "mechanical arm" on the space shuttle can handle a 2200-kg satellite when extended to 12 m; see Fig. 3-22. Yet, on the ground, this remote manipulator system (RMS) cannot support its own weight. In the "weightlessness" of an orbiting shuttle, why does the RMS have to be able to exert any force at all?

24. In November 1984, astronauts Joe Allen and Dale Gardner salvaged a Westar-6 communications satellite from a faulty orbit and placed it into the cargo bay of the space shuttle *Discovery;* see Fig. 3-22. Describing the experience, Joe Allen said of the satellite, "It's not heavy; it's massive." What did he mean?

25. The owner's manual of a car suggests that your seat belt should be adjusted "to fit snugly" and that the front seat head rest should not be adjusted so that it fits comfortably at the back of your neck but so that "the top of the head rest is level with the top of your ears." How do Newton's laws support these good recommendations?

FIGURE 3-22. Questions 23 and 24.

26. Is it possible to derive $\Sigma\,\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ from some other principle? Is $\Sigma\,\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ an experimental conclusion?

27. Observers in two different inertial frames will measure the same acceleration of a moving object. Will they measure the same velocity of a moving object? Will they measure the same force on the moving object?

28. You are an astronaut in the lounge of an orbiting space station and you remove the cover from a long thin jar containing a single olive. Describe several ways—all taking advantage of the inertia of either the olive or the jar—to remove the olive from the jar.

29. In Fig. 3-23, a needle has been placed in each end of a broomstick, the tips of the needles resting on the edges of filled wine glasses. The experimenter strikes the broomstick a swift and sturdy blow with a stout rod. The broomstick breaks and falls to the floor but the wine glasses remain in place and no wine is spilled. This impressive parlor stunt was popular at the end of the nineteenth century. What is the physics behind it? (If you try it, practice first with empty soft drink cans. Come to think of it, you might ask your physics instructor to do it, as a lecture demonstration!)

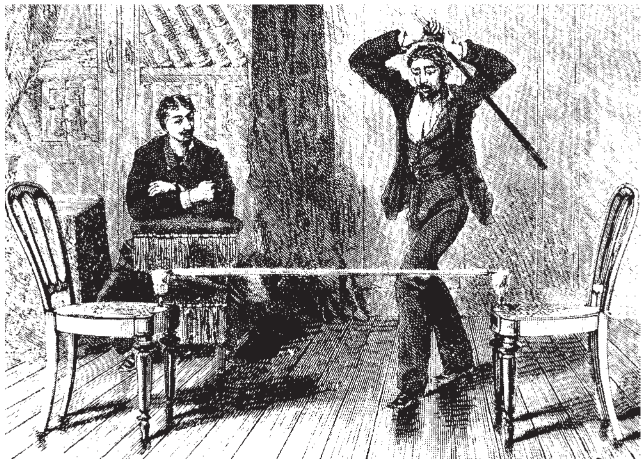


FIGURE 3-23. Question 29.

30. An elevator is supported by a single cable. There is no counterweight. The elevator receives passengers at the ground floor and takes them to the top floor, where they disembark.

New passengers enter and are taken down to the ground floor. During this round trip, when is the force exerted by the cable on the elevator equal to the weight of the elevator plus passengers? Greater? Less?

31. You are on the flight deck of the orbiting space shuttle *Discovery* and someone hands you two wooden balls, outwardly identical. One, however, has a lead core but the other does not. Describe several ways of telling them apart.

32. You stand on the large platform of a spring scale and note your weight. You then take a step on this platform and note that the scale reads less than your weight at the beginning of the step and more than your weight at the end of the step. Explain.

33. Could you weigh yourself on a scale whose maximum reading is less than your weight? If so, how?

34. A weight hangs by a spring scale from the ceiling of an elevator. In which of the following cases will the reading of the spring scale be greatest: (*a*) elevator at rest; (*b*) elevator rising with uniform speed; (*c*) elevator descending with decreasing speed; (*d*) elevator descending with increasing speed? In which will it be the least?

35. A woman stands on a spring scale in an elevator. In which of the following cases will the scale record the minimum reading: (*a*) elevator stationary; (*b*) elevator cable breaks, free fall; (*c*) elevator accelerating upward; (*d*) elevator accelerating downward; (*e*) elevator moving at constant velocity? In which will it record the maximum reading?

36. Figure 3-24 shows comet Kohoutek as it appeared in 1973. Like all comets, it moves around the Sun under the influence of the gravitational pull that the Sun exerts on it. The nucleus of the comet is a relatively massive core at a position indicated by *P*. The tail of a comet is produced by the action of the solar wind, which consists of charged particles streaming outward from the Sun. By inspection, what, if anything, can you say about the direction of the force that acts on the nucleus of the comet? What about the direction in which the nucleus is being accelerated? What about the direction in which the comet is moving?

37. In general (see Fig. 3-24), comets have a dust tail, consisting of dust particles pushed away from the Sun by the pressure of sunlight. Why is this tail often curved?

FIGURE 3-24. Questions 36 and 37.

38. Can you think of physical phenomena involving the Earth in which the Earth cannot be treated as a particle?

39. Consider a jump off a high dive. While waiting to get the courage to jump, your acceleration is zero, and you "feel" the force of gravity. When you jump you accelerate toward the water, but during this "free fall" you feel weightless, as if there was no force of gravity. Does this contradict Newton's laws of motion? How could you explain this to a non-physics student?

EXERCISES

3-1 Classical Mechanics
3-2 Newton's First Law
3-3 Force
3-4 Mass
3-5 Newton's Second Law

1. Suppose that the Sun's gravitational force was suddenly turned off, so that Earth became a free object rather than being confined to orbit the Sun. How long would it take for Earth to reach a distance from the Sun equal to Pluto's present orbital radius? (Hint: You will find some of the data you need in Appendix C.)

2. A 5.5-kg block is initially at rest on a frictionless horizontal surface. It is pulled with a constant horizontal force of 3.8 N. (*a*) What is its acceleration? (*b*) How long must it be pulled before its speed is 5.2 m/s? (*c*) How far does it move in this time?

3. An electron travels in a straight line from the cathode of a vacuum tube to its anode, which is 1.5 cm away. It starts with zero speed and reaches the anode with a speed of 5.8×10^6 m/s. Assume constant acceleration and compute the force on the electron. This force is electrical in origin. The electron's mass is 9.11×10^{-31} kg.

4. A neutron travels at a speed of 1.4×10^7 m/s. Nuclear forces are of very short range, being essentially zero outside a nucleus but very strong inside. If the neutron is captured and brought to rest by a nucleus whose diameter is 1.0×10^{-14} m, what is the minimum magnitude of the force, presumed to be constant, that acts on this neutron? The neutron's mass is 1.67×10^{-27} kg.

5. In a modified tug-of-war game, two people pull in opposite directions, not on a rope, but on a 25-kg sled resting on an icy road. If the participants exert forces of 90 N and 92 N, what is the acceleration of the sled?

6. A car traveling at 53 km/h hits a bridge abutment. A passenger in the car moves forward a distance of 65 cm (with respect to the road) while being brought to rest by an inflated air bag. What force (assumed constant) acts on the passenger's upper torso, which has a mass of 39 kg?

7. An electron is projected horizontally at a speed of 1.2×10^7 m/s into an electric field that exerts a constant vertical force of 4.5×10^{-16} N on it. The mass of the electron is

9.11×10^{-31} kg. Determine the vertical distance the electron is deflected during the time it has moved forward 33 mm horizontally.

8. The Sun yacht *Diana,* designed to navigate in the solar system using the pressure of sunlight, has a sail area of 3.1 km^2 and a mass of 930 kg. Near Earth's orbit, the Sun could exert a radiation force of 29 N on its sail. (*a*) What acceleration would such a force impart to the craft? (*b*) A small acceleration can produce large effects if it acts steadily for a long enough time. Starting from rest then, how far would the craft have moved after 1 day under these conditions? (*c*) What would then be its speed? (See "The Wind from the Sun," a fascinating science fiction account by Arthur C. Clarke of a Sun yacht race.)

9. A certain force gives object m_1 an acceleration of 12.0 m/s^2. The same force gives object m_2 an acceleration of 3.30 m/s^2. What acceleration would the force give to an object whose mass is (*a*) the difference between m_1 and m_2 and (*b*) the sum of m_1 and m_2?

10. (*a*) Neglecting gravitational forces, what force would be required to accelerate a 1200-metric-ton spaceship from rest to one-tenth the speed of light in 3 days? In 2 months? (One metric ton = 1000 kg.) (*b*) Assuming that the engines are shut down when this speed is reached, what would be the time required to complete a 5-light-month journey for each of these two cases? (Use 1 month = 30 days.)

3-6 Newton's Third Law

11. Two blocks, with masses $m_1 = 4.6$ kg and $m_2 = 3.8$ kg, are connected by a light spring on a horizontal frictionless table. At a certain instant, when m_2 has an acceleration $a_2 = 2.6$ m/s^2, (*a*) what is the force on m_2 and (*b*) what is the acceleration of m_1?

3-7 Weight and Mass

12. What are the weight in newtons and the mass in kilograms of (*a*) a 5.00-lb bag of sugar, (*b*) a 240-lb fullback, and (*c*) a 1.80-ton car? (1 ton = 2000 lb.)

13. What are the mass and weight of (*a*) a 1420-lb snowmobile and (*b*) a 412-kg heat pump?

14. A space traveler whose mass is 75.0 kg leaves Earth. Compute his weight (*a*) on Earth, (*b*) on Mars, where $g = 3.72$ m/s^2, and (*c*) in interplanetary space. (*d*) What is his mass at each of these locations?

15. A certain particle has a weight of 26.0 N at a point where the acceleration due to gravity is 9.80 m/s^2. (*a*) What are the weight and mass of the particle at a point where the acceleration due to gravity is 4.60 m/s^2? (*b*) What are the weight and mass of the particle if it is moved to a point in space where the gravitational force is zero?

16. A 12,000-kg airplane is in level flight at a speed of 870 km/h. What is the upward-directed lift force exerted by the air on the airplane?

17. What is the net force acting on a 3900-lb automobile accelerating at 13 ft/s^2?

18. A 523-kg experimental rocket sled can be accelerated from rest to 1620 km/h in 1.82 s. What net force is required?

19. A jet plane starts from rest on the runway and accelerates for takeoff at 2.30 m/s^2 (= 7.55 ft/s^2). It has two jet engines, each of which exerts a thrust of 1.40×10^5 N (= 15.7 tons). What is the weight of the plane?

3-8 Applications of Newton's Laws

20. (*a*) Two 10-lb weights are attached to a spring scale as shown in Fig. 3-25*a*. What is the reading of the scale? (*b*) A single 10-lb weight is attached to a spring scale which itself is attached to a wall, as shown in Fig. 3-25*b*. What is the reading of the scale? (Ignore the weight of the scale.)

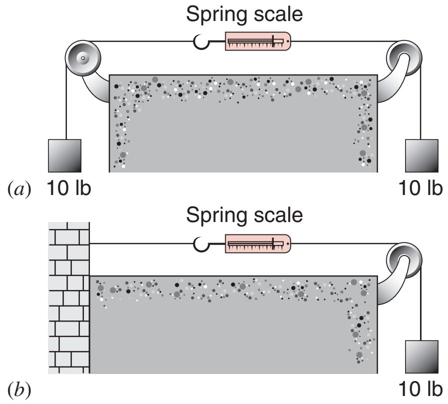


FIGURE 3-25. Exercise 20.

21. A car moving initially at a speed of 50 mi/h (≈80 km/h) and weighing 3000 lb (≈13,000 N) is brought to a stop in a distance of 200 ft (≈61 m). Find (*a*) the braking force and (*b*) the time required to stop. Assuming the same braking force, find (*c*) the distance and (*d*) the time required to stop if the car were going 25 mi/h (≈40 km/h) initially.

22. A meteor of mass 0.25 kg is falling vertically through Earth's atmosphere with an acceleration of 9.2 m/s^2. In addition to gravity, a vertical retarding force (due to the frictional drag of the atmosphere) acts on the meteor. What is the magnitude of this retarding force? See Fig. 3-26.

FIGURE 3-26. Exercise 22.

23. A man of mass 83 kg (weight 180 lb) jumps down to a concrete patio from a window ledge only 0.48 m (= 1.6 ft) above the ground. He neglects to bend his knees on landing, so that his motion is arrested in a distance of about 2.2 cm (= 0.87 in.). (*a*) What is the average acceleration of the man from the time his feet first touch the patio to the time he is brought fully to rest? (*b*) With what average force does this jump jar his bone structure?

24. What strength fishing line is needed to stop a 19-lb salmon swimming at 9.2 ft/s in a distance of 4.5 in.?

25. How could a 100-lb object be lowered from a roof using a cord with a breaking strength of 87 lb without breaking the cord?

26. An object is hung from a spring scale attached to the ceiling of an elevator. The scale reads 65 N when the elevator is standing still. (*a*) What is the reading when the elevator is moving upward with a constant speed of 7.6 m/s? (*b*) What is the reading of the scale when the elevator is moving upward with a speed of 7.6 m/s and decelerating at 2.4 m/s^2?

27. Workers are loading equipment into a freight elevator at the top floor of a building. However, they overload the elevator and the worn cable snaps. The mass of the loaded elevator at the time of the accident is 1600 kg. As the elevator falls, the guide rails exert a constant retarding force of 3700 N on the elevator. At what speed does the elevator hit the bottom of the shaft 72 m below?

28. A 26-ton Navy jet (Fig. 3-27) requires an air speed of 280 ft/s for lift-off. Its own engine develops a thrust of 24,000 lb. The jet is to take off from an aircraft carrier with a 300-ft flight deck. What force must be exerted by the catapult of the carrier? Assume that the catapult and the jet's engine each exert a constant force over the 300-ft takeoff distance.

29. A rocket and its payload have a total mass of 51,000 kg. How large is the thrust of the rocket engine when (*a*) the rocket is "hovering" over the launch pad, just after ignition, and (*b*) when the rocket is accelerating upward at 18 m/s^2?

30. A 77-kg person is parachuting and experiencing a downward acceleration of 2.5 m/s^2 shortly after opening the parachute. The mass of the parachute is 5.2 kg. (*a*) Find the upward force exerted on the parachute by the air. (*b*) Calculate the downward force exerted by the person on the parachute.

31. A 15,000-kg helicopter is lifting a 4500-kg car with an upward acceleration of 1.4 m/s^2. Calculate (*a*) the vertical force the air exerts on the helicopter blades and (*b*) the tension in the upper supporting cable; see Fig. 3-28.

FIGURE 3-27. Exercise 28.

FIGURE 3-28. Exercise 31.

PROBLEMS

1. A light beam from a satellite-carried laser strikes an object ejected from an accidentally launched ballistic missile; see Fig. 3-29. The beam exerts a force of 2.7×10^{-5} N on the target. If the "dwell time" of the beam on the target is 2.4 s, by how much is the object displaced if it is (*a*) a 280-kg warhead and (*b*) a 2.1-kg decoy? (These displacements can be measured by observing the reflected beam.)

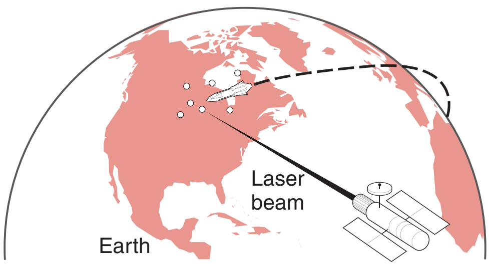


FIGURE 3-29. Problem 1.

2. A 40-kg girl and an 8.4-kg sled are on the surface of a frozen lake, 15 m apart. By means of a rope the girl exerts a 5.2-N force on the sled, pulling it toward her. (*a*) What is the acceleration of the sled? (*b*) What is the acceleration of the girl? (*c*) How far from the girl's initial position do they meet, presuming the force to remain constant? Assume that no frictional forces act.

3. A block is released from rest at the top of a frictionless inclined plane 16 m long. It reaches the bottom 4.2 s later. A second block is projected up the plane from the bottom at the instant the first block is released in such a way that it returns to the bottom simultaneously with the first block. (*a*) Find the acceleration of each block on the incline. (*b*) What is the initial velocity of the second block? (*c*) How far up the incline does it travel? You can assume that both blocks experience the same acceleration.

4. A 1400-kg jet engine is fastened to the fuselage of a passenger jet by just three bolts (this is the usual practice). Assume that each bolt supports one-third of the load. (*a*) Calculate the force on each bolt as the plane waits in line for clearance to take off. (*b*) During flight, the plane encounters turbulence, which suddenly imparts an upward vertical acceleration of 2.60 m/s^2 to the plane. Calculate the force on each bolt now. Why are only three bolts used? See Fig. 3-30.

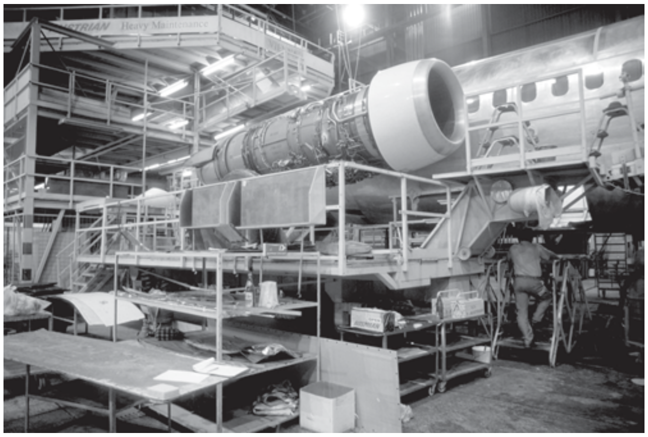


FIGURE 3-30. Problem 4.

5. A landing craft approaches the surface of Callisto, one of the satellites (moons) of the planet Jupiter (Fig. 3-31). If an upward thrust of 3260 N is supplied by the rocket engine, the craft descends with constant speed. Callisto has no atmosphere. If the upward thrust is 2200 N, the craft accelerates downward at 0.390 m/s^2. (*a*) What is the weight of the landing craft in the vicinity of Callisto's surface? (*b*) What is the mass of the craft? (*c*) What is the acceleration due to gravity near the surface of Callisto?

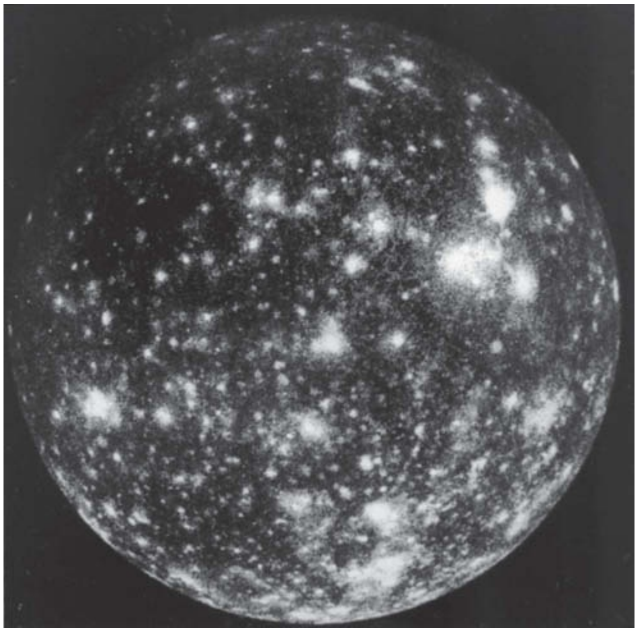


FIGURE 3-31. Problem 5.

6. A research balloon of total mass M is descending vertically with downward acceleration a (see Fig. 3-32). How much ballast must be thrown from the car to give the balloon an upward acceleration a, assuming that the upward lift of the air on the balloon does not change?

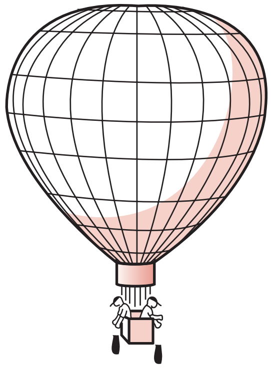


FIGURE 3-32. Problem 6.

7. A child's toy consists of three cars that are pulled in tandem on small frictionless rollers (Fig. 3-33). The cars have masses $m_1 = 3.1$ kg, $m_2 = 2.4$ kg, and $m_3 = 1.2$ kg. If they are pulled to the right with a horizontal force $P = 6.5$ N, find (*a*) the acceleration of the system, (*b*) the force exerted by the second car on the third car, and (*c*) the force exerted by the first car on the second car.

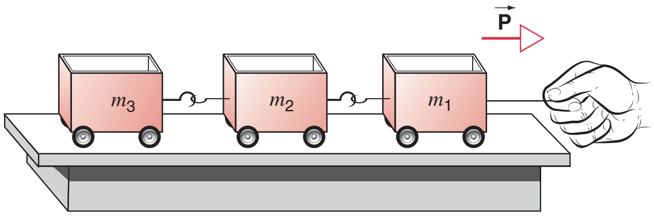


FIGURE 3-33. Problem 7.

8. Figure 3-34 shows three crates with masses $m_1 = 45.2$ kg, $m_2 = 22.8$ kg, and $m_3 = 34.3$ kg on a horizontal frictionless surface. (*a*) What horizontal force F is needed to push the crates to the right, as one unit, with an acceleration of 1.32 m/s^2? (*b*) Find the force exerted by m_2 on m_3. (*c*) By m_1 on m_2.

9. A chain consisting of five links, each with mass 100 g, is lifted vertically with a constant acceleration of 2.50 m/s^2, as shown in Fig. 3-35. Find (*a*) the forces acting between adja-

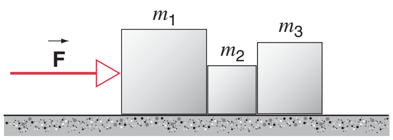


FIGURE 3-34. Problem 8.

cent links, (*b*) the force *F* exerted on the top link by the agent lifting the chain, and (*c*) the net force on each link.

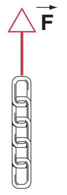


FIGURE 3-35. Problem 9.

10. Two blocks are in contact on a frictionless table. A horizontal force is applied to one block, as shown in Fig. 3-36. (*a*) If $m_1 = 2.3$ kg, $m_2 = 1.2$ kg, and $F = 3.2$ N, find the force of contact between the two blocks. (*b*) Show that if the same force *F* is applied to m_2 rather than to m_1, the force of contact between the blocks is 2.1 N, which is not the same value derived in (a). Explain.

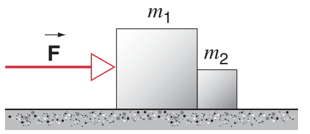


FIGURE 3-36. Problem 10.

11. A block of mass *M* is pulled along a horizontal frictionless surface by a rope of mass *m*, as shown in Fig. 3-37. A horizontal force $\vec{\mathbf{P}}$ is applied to one end of the rope. Assuming that the sag in the rope is negligible, find (*a*) the accelerationof rope and block, and (*b*) the force that the rope exerts on the block.

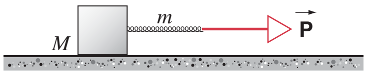


FIGURE 3-37. Problem 11.

COMPUTER PROBLEM

1. A 10.0 kg object is launched vertically into the air with an initial velocity of 50.0 m/s. In addition to the force of gravity there is a frictional force which is proportional to velocity according to $f_y = -bv_y$; note that this frictional force is negative (down) when the object is moving up, but positive (up) when the object is moving down.

(*a*) Numerically generate distance-time graphs for the object, using $b = 0$, but use several different step sizes for Δt, such as 1.0 s, 0.1 s, 0.01 s, and 0.001 s. Show the results on a single graph. How does the highest point vary with the step size?

(*b*) Numerically generate distance-time graphs for the object, using a step size of $\Delta t = 0.01$ s. Now, however, try non-zero values for *b*, such as 0.1 N·s/m, 0.5 N·s/m, 1.0 N·s/m, 5.0 N·s/m, and 10.0 N·s/m. How does the highest point vary with *b*? What do you notice about the shape of the graphs as *b* increases?

CHAPTER 4

MOTION IN TWO AND THREE DIMENSIONS

In this chapter we consider an extension of the concepts presented in Chapters 2 and 3. In those chapters we introduced kinematics and dynamics in terms of vectors, but we considered only applications in one dimension. In this chapter we broaden the discussion to include two- and three-dimensional applications. Keeping track of the separate x, y, and z components of the motion is greatly simplified if we rely on vectors to describe the particle's position, velocity, and acceleration, as well as the forces that may act on the particle. To illustrate the vector techniques, we discuss two examples: a projectile launched with both horizontal and vertical velocity components in the Earth's gravity, and an object moving in a circular path.

4-1 MOTION IN THREE DIMENSIONS WITH CONSTANT ACCELERATION

In Section 2-5 we developed a procedure for analyzing the position, velocity, and acceleration of a particle that moves in one dimension with constant acceleration. Knowing the acceleration, we can find the velocity at all times according to Eq. 2-26 ($v_x = v_{0x} + a_x t$) and the position at all times from Eq. 2-28 ($x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$).

Now we consider the possibility that the particle moves in three dimensions with constant acceleration. That is, as the particle moves, the acceleration does not vary in either magnitude or direction. Equivalently, we can represent the acceleration as a vector $\vec{\mathbf{a}}$ with three components (a_x, a_y, a_z), each of which is constant. In general the particle moves in a curved path. As is the case in one-dimensional motion, we would like to know the particle's velocity $\vec{\mathbf{v}}$ (a vector with components v_x, v_y, v_z) and its position $\vec{\mathbf{r}}$ (a vector with components x, y, z) at all times.

We can obtain the general equations for motion with constant $\vec{\mathbf{a}}$ by setting

$$a_x = \text{constant}, \quad a_y = \text{constant}, \quad \text{and} \quad a_z = \text{constant}.$$

The particle starts at $t = 0$ with initial position $\vec{\mathbf{r}}_0 = x_0\hat{\mathbf{i}} + y_0\hat{\mathbf{j}} + z_0\hat{\mathbf{k}}$ and an initial velocity $\vec{\mathbf{v}}_0 = v_{0x}\hat{\mathbf{i}} + v_{0y}\hat{\mathbf{j}} + v_{0z}\hat{\mathbf{k}}$. We now proceed as we did in Section 2-5 and develop, in analogy with Eq. 2-26, *three* one-dimensional equations: $v_x = v_{0x} + a_x t$, $v_y = v_{0y} + a_y t$, and $v_z = v_{0z} + a_z t$, which we write as the single three-dimensional vector equation

$$\vec{\mathbf{v}} = \vec{\mathbf{v}}_0 + \vec{\mathbf{a}}t. \qquad (4\text{-}1)$$

When using this or any other vector equation, remember that it represents three independent one-dimensional equations. That is, a vector equality such as $\vec{\mathbf{A}} = \vec{\mathbf{B}}$ means that three conditions *must* be fulfilled: $A_x = B_x$, $A_y = B_y$, and $A_z = B_z$. In this way it is clear how Eq. 4-1 represents the three one-dimensional equations for the components.

By referring to the three component equations as "independent," we mean that the velocity components vary independently of one another—for example, a_x affects only v_x and not v_y or v_z. If $a_y = a_z = 0$ but $a_x \neq 0$, then v_y and v_z would remain constant but v_x would vary with time.

The second term on the right side of Eq. 4-1 involves the multiplication of the vector $\vec{\mathbf{a}}$ by the scalar t. As discussed in Appendix H, this gives a vector of length at that points in the same direction as the original vector $\vec{\mathbf{a}}$.

In a similar way, we can write the three equations for the components of the position vector, as in Eq. 2-28: $x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$, $y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2$, $z = z_0 + v_{0z}t + \frac{1}{2}a_z t^2$. These three one-dimensional equations can be combined into a single three-dimensional vector equation:

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \tfrac{1}{2}\vec{a}t^2, \qquad (4\text{-}2)$$

which contains within it the three one-dimensional equations for the components.

SAMPLE PROBLEM 4-1. Starship *Enterprise* is coasting through space (where gravity is negligible) at a speed of 15.0 km/s relative to a particular inertial reference frame. Suddenly, the ship is gripped by a tractor beam, which pulls it in a direction perpendicular to its original velocity and gives it an acceleration of 4.2 km/s^2 in that direction. After the tractor beam has acted for 4.0 s, the *Enterprise* fires its engines, giving the ship a constant acceleration of 18.0 km/s^2 in the direction parallel to its original motion. After an additional 3.0 s, both the engines and the tractor beam stop operating, and the ship is again coasting. Find the ship's velocity at that time and its position relative to its location when the tractor beam first appeared.

Solution We set up our coordinate system with the positive x axis in the direction of the ship's original motion and the positive y axis in the direction of the tractor beam's pull, and we choose the origin ($x = 0$, $y = 0$) at the location where the tractor beam began to act. (With this choice of coordinate system, there is no motion in the z direction.) The problem breaks into two parts, which must be analyzed separately: (1) from $t = 0$ to $t = 4.0$ s the ship moves with $a_x = 0$, $a_y = +4.2$ km/s^2, and (2) from $t = 4.0$ s to $t = 7.0$ s it moves with $a_x = +18.0$ km/s^2, $a_y = +4.2$ km/s^2.

We analyze each part in turn. For the first part, with $v_{0x} = +15.0$ km/s and $v_{0y} = 0$, the x and y components of Eqs. 4-1 and 4-2 become

$$v_x = v_{0x} + a_x t = 15.0\ \text{km/s} + 0 = 15.0\ \text{km/s}$$

$$v_y = v_{0y} + a_y t = 0 + (4.2\ \text{km/s}^2)(4.0\ \text{s}) = 16.8\ \text{km/s}$$

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2 = 0 + (15.0\ \text{km/s})(4.0\ \text{s}) + 0 = 60.0\ \text{km}$$

$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2 = 0 + 0 + \tfrac{1}{2}(4.2\ \text{km/s}^2)(4.0\ \text{s})^2 = 33.6\ \text{km}.$$

For the 3.0-s interval from $t = 4.0$ s to $t = 7.0$ s, we write a similar set of equations using a new variable t' that ranges from 0 to 3.0 s (but we keep the origin of the coordinate system in the original location). For this interval, the starting velocities and locations are the values found above for $t = 4.0$ s ($v_{0x} = 15.0$ km/s, $v_{0y} = 16.8$ km/s, $x_0 = 60.0$ km, $y_0 = 33.6$ km), and so

$$v_x = v_{0x} + a_x t' = 15.0\ \text{km/s} + (18.0\ \text{km/s}^2)(3.0\ \text{s}) = 69.0\ \text{km/s}$$

$$v_y = v_{0y} + a_y t' = 16.8\ \text{km/s} + (4.2\ \text{km/s}^2)(3.0\ \text{s}) = 29.4\ \text{km/s}$$

$$x = x_0 + v_{0x}t' + \tfrac{1}{2}a_x t'^2 = 60.0\ \text{km} + (15.0\ \text{km/s})(3.0\ \text{s}) + \tfrac{1}{2}(18.0\ \text{km/s}^2)(3.0\ \text{s})^2 = 186\ \text{km}$$

$$y = y_0 + v_{0y}t' + \tfrac{1}{2}a_y t'^2 = 33.6\ \text{km} + (16.8\ \text{km/s})(3.0\ \text{s}) + \tfrac{1}{2}(4.2\ \text{km/s}^2)(3.0\ \text{s})^2 = 103\ \text{km}.$$

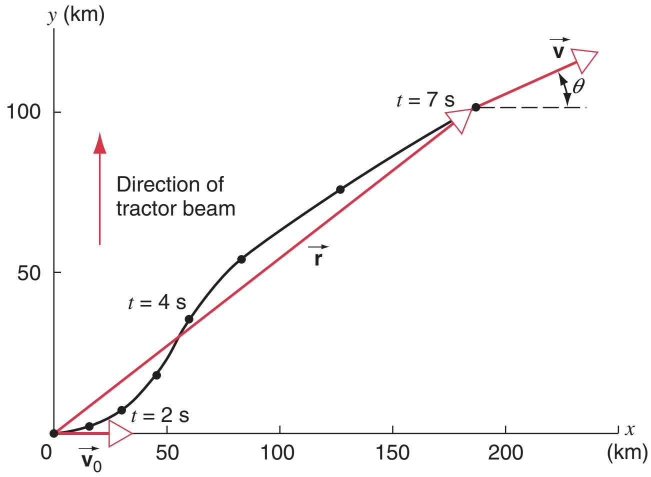


FIGURE 4-1. Sample Problem 4-1. The dots show the position of the starship at successive 1-second intervals from $t = 0$ to $t = 7$ s. The vectors $\vec{r}$ and $\vec{v}$ show the position and velocity at $t = 7$ s. Note that $\vec{v}_0$ is tangent to the path at $t = 0$, and $\vec{v}$ is tangent to the path at $t = 7$ s.

At $t = 7.0$ s, the ship is at $x = 186$ km, $y = 103$ km (or, a distance $r = \sqrt{x^2 + y^2} = 213$ km from the initial reference point). The components of its velocity are $v_x = 69$ m/s, $v_y = 29$ m/s, corresponding to a speed $v = \sqrt{v_x^2 + v_y^2} = 75$ km/s and a direction given by the angle $\theta = \tan^{-1} v_y/v_x = 23°$ relative to the x axis. Figure 4-1 shows the path of the starship and its position at various times. In general, as shown at $t = 7$ s, the directions of the position and velocity vectors are different.

Note in this problem how using the components of vectors helps us to separate the x and y motions. That is, the solutions to the equations for the x components do not depend on the y motion. If the tractor beam did not exist and the *Enterprise* fired its engines in the same way from 4.0 s to 7.0 s, it would still be at $x = 186$ km moving with velocity $v_x = 69$ km/s at 7.0 s. If the tractor beam were present but the engines did not fire, the *Enterprise* would still be at $y = 103$ km moving with $v_y = 29$ km/s at 7.0 s.

4-2 NEWTON'S LAWS IN THREE-DIMENSIONAL VECTOR FORM

Before we can write Newton's laws in their three-dimensional vector form, we first must verify that force, as we have defined it, is a vector quantity. We have seen in Chapter 3 that, even in one dimension, we must carefully account for the direction of each force. Since force has both magnitude and direction, we suspect that it may be a vector quantity. However, to be a vector it is not enough for a quantity to have both magnitude and direction; it also must obey the laws of vector addition described in Section 2-2. Only from experiment can we learn whether forces, as we defined them, obey these laws.

Let us arrange to exert a force of 4 N along the x axis and a force of 3 N along the y axis. We apply these forces first separately and then simultaneously to the standard 1-kg body placed, as before, on a horizontal, frictionless sur-

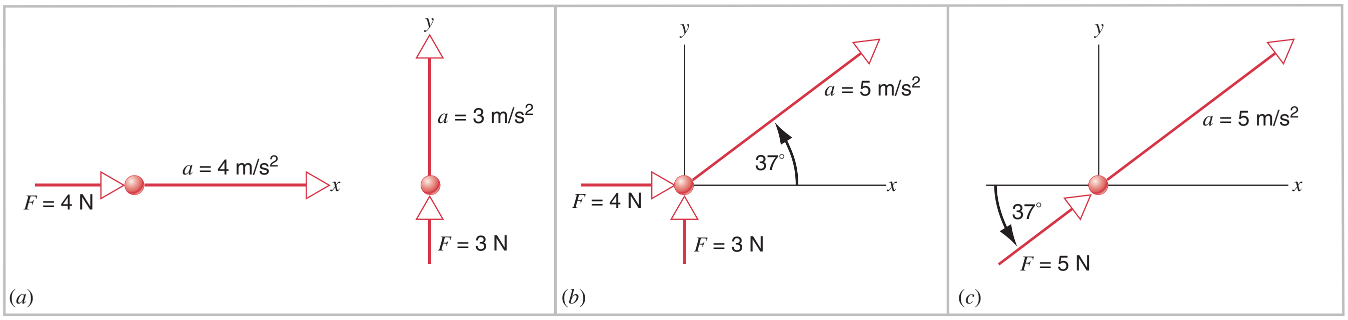


Figure 4-2. (*a*) A 4-N force in the x direction gives an acceleration of 4 m/s^2 in the x direction, and a 3-N force in the y direction gives an acceleration of 3 m/s^2 in the y direction. (*b*) When the forces are applied simultaneously, the resultant acceleration is 5 m/s^2 in the direction shown. (*c*) The same acceleration can be produced by a single 5-N force in the direction shown.

face. What will be the acceleration of the standard body? We would find by experiment that the 4-N force in the x direction acting alone produced an acceleration of 4 m/s^2 in the x direction, and that the 3-N force in the y direction acting alone produced an acceleration of 3 m/s^2 in the y direction (Fig. 4-2*a*). When the forces are applied simultaneously, as shown in Fig. 4-2*b*, we find that the acceleration is 5 m/s^2 directed along a line that makes an angle of 37° with the x axis. This is the same acceleration that would be produced if the standard body were experiencing a force of 5 N in that direction. This same result can be obtained if we first add the 4-N and 3-N forces vectorially (Fig. 4-2*c*) to a 5-N resultant directed at 37° from the x axis, and then apply that single 5-N net force to the body. Experiments of this kind show conclusively that forces are vectors; they have magnitude and direction, *and* they add according to the vector addition law.

Now that we are convinced that force is a vector, we are justified in writing Newton's second law in vector form, as we already have done in Chapter 3:

$$\sum \vec{\mathbf{F}} = m\vec{\mathbf{a}} \tag{4-3}$$

which includes the three component equations

$$\sum F_x = ma_x, \qquad \sum F_y = ma_y, \qquad \sum F_z = ma_z. \tag{4-4}$$

That is, all three equations must simultaneously be satisfied when we apply Newton's second law.

Equation 4-3 suggests that we can find the direction of the acceleration by taking the vector sum of all the forces acting on the particle. Because the mass m is a scalar, the direction of $\vec{\mathbf{a}}$ is the same as the direction of $\Sigma \vec{\mathbf{F}}$. Once we have the resultant force we can also find the magnitude of $\vec{\mathbf{a}}$ from Eq. 4-3. However, as we shall see in the examples in this chapter, it is often easier to use Eqs. 4-4 to solve problems by first resolving each force into its components and then finding the sum of each of the force components to obtain each component of the acceleration.

If the forces are constant, then the acceleration is constant and we can use the equations of Section 4-1 to find the position and velocity of the particle at all times. If the forces are not constant, then it is not possible to use the equations for constant acceleration; an example of a nonconstant force, the drag on a projectile, is discussed later in this chapter.

Newton's third law is also a vector equation:

$$\vec{\mathbf{F}}_{AB} = -\vec{\mathbf{F}}_{BA}, \tag{4-5}$$

which tells us that, no matter what the direction in three-dimensional space of the vector $\vec{\mathbf{F}}_{AB}$ that represents the force exerted on A by B, the vector $\vec{\mathbf{F}}_{BA}$ that represents the force on B by A has the same magnitude and acts in the opposite direction.

Sample Problem 4-2. A crate of mass $m = 62$ kg is sliding without friction with an initial velocity of $v_0 = 6.4$ m/s along the floor. In an attempt to move it in a different direction, Tom pushes opposite to its initial motion with a constant force of magnitude $F_{CT} = 81$ N, while Jane pushes in a perpendicular direction with a constant force of magnitude $F_{CJ} = 105$ N (Fig. 4-3*a*). If they each push for 3.0 s, in what direction is the crate moving when they stop pushing?

Solution Let us take the positive x direction to be that of the initial motion of the crate (so that Tom's force $\vec{\mathbf{F}}_{CT}$ is in the negative x direction) and the positive y direction to be that of Jane's force $\vec{\mathbf{F}}_{CJ}$. Figure 4-3*b* shows the free-body diagram of the crate. The only force in the x direction is that exerted by Tom, so $\Sigma F_x = -F_{CT}$; similarly, $\Sigma F_y = F_{CJ}$. Using Newton's second law in its component form (Eqs. 4-4) we can write the equations of motion of the crate as

$$x \text{ direction } \left(\sum F_x = ma_x\right): \qquad -F_{CT} = ma_x$$

$$y \text{ direction } \left(\sum F_y = ma_y\right): \qquad -F_{CJ} = ma_y$$

Solving, we find

$$a_x = -\frac{F_{CT}}{m} = \frac{-81 \text{ N}}{62 \text{ kg}} = -1.31 \text{ m/s}^2$$

$$a_y = \frac{F_{CJ}}{m} = \frac{105 \text{ N}}{62 \text{ kg}} = 1.69 \text{ m/s}^2.$$

Using Eq. 4-1 in its component form, we can find the velocity components at $t = 3.0$ s:

$$v_x = v_{0x} + a_x t = 6.4 \text{ m/s} + (-1.31 \text{ m/s}^2)(3.0 \text{ s}) = 2.5 \text{ m/s}$$

$$v_y = v_{0y} + a_y t = 0 + (1.69 \text{ m/s}^2)(3.0 \text{ s}) = 5.1 \text{ m/s}.$$

A graph of the path of the crate is shown in Fig. 4-3*c*, which also shows the velocity components at $t = 3.0$ s. To find the magnitude

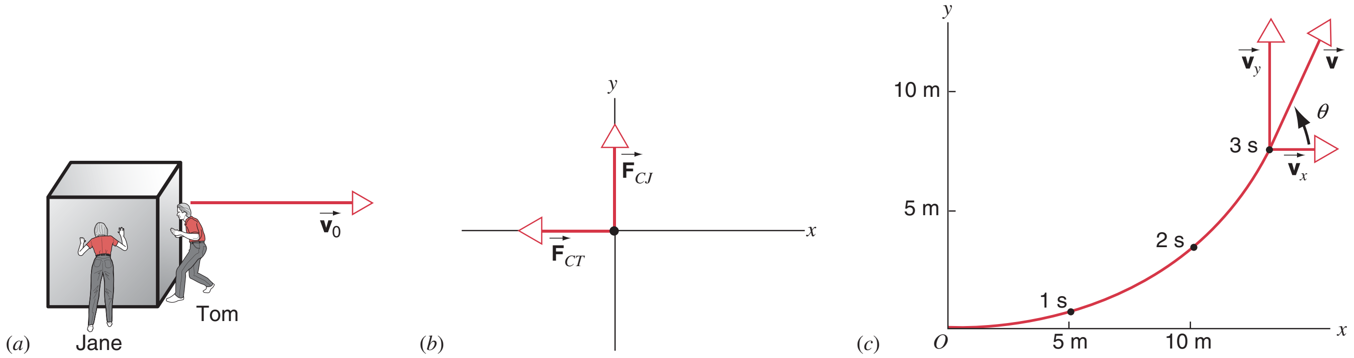


FIGURE 4-3. Sample Problem 4-2. (*a*) Tom pushes opposite to the initial motion of the crate and Jane pushes in a perpendicular direction. (*b*) The free-body diagram of the crate. (*c*) The tangent to the path gives the direction of the crate's motion. The velocity components at $t = 3.0$ s are shown.

of the velocity and its direction, we use Eqs. 2-2:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(2.5\text{ m/s})^2 + (5.1\text{ m/s})^2} = 5.7\text{ m/s}$$

$$\theta = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\frac{5.1\text{ m/s}}{2.5\text{ m/s}} = 64°.$$

Note that $\vec{v}$ is in the direction of motion (tangent to the curve that represents the path of the crate).

Can you find the location of the crate at $t = 3.0$ s?

4-3 PROJECTILE MOTION

A common example of motion in two dimensions is projectile motion near the Earth's surface, in which a projectile, such as a golf ball or a baseball, is launched in an arbitrary direction. For the time being we neglect air resistance (a nonconstant force), which simplifies the calculation.

Figure 4-4*a* shows the initial motion of the projectile at the instant of launch. Its initial velocity is $\vec{v}_0$ directed at an angle ϕ_0 from the horizontal. We choose our coordinate system with the *x* axis horizontal, its positive direction corresponding to the horizontal component of the initial velocity. The *y* axis is vertical, with its positive direction upward. We also place the origin of the coordinate system at the location of launch, so that $x_0 = 0$ and $y_0 = 0$. The components of the initial velocity are

$$v_{0x} = v_0 \cos\phi_0, \qquad v_{0y} = v_0 \sin\phi_0. \tag{4-6}$$

The free-body diagram of the projectile (of mass *m*) is shown in Fig. 4-4*b*. Gravity is the only force that acts. It is

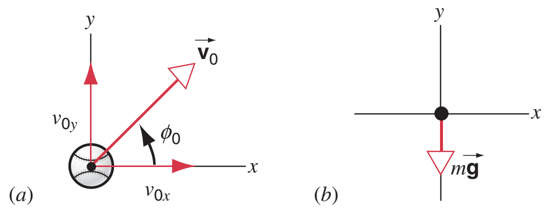


FIGURE 4-4. (*a*) A projectile is launched with initial velocity $\vec{v}_0$. (*b*) The free-body diagram of the projectile.

a constant force, having a downward direction and the same magnitude *mg* everywhere on the path of the projectile, no matter what its location or direction of motion. With our choice of coordinate system, the components of the net force are

$$\sum F_x = 0 \qquad \text{and} \qquad \sum F_y = -mg \tag{4-7}$$

and the vector components of Newton's second law (Eqs. 4-4) then give

$$a_x = \frac{\sum F_x}{m} = 0 \qquad \text{and} \qquad a_y = \frac{\sum F_y}{m} = -g. \tag{4-8}$$

The horizontal component of the acceleration is zero everywhere along the path, and the vertical component of the acceleration is $-g$ everywhere.

Applying the formulas for constant acceleration from Section 4-1, we obtain

velocity components:

$$(a)\ v_x = v_{0x}, \qquad (b)\ v_y = v_{0y} - gt \tag{4-9}$$

position components:

$$(a)\ x = v_{0x}t, \qquad (b)\ y = v_{0y}t - \tfrac{1}{2}gt^2 \tag{4-10}$$

Note that the horizontal component of the velocity remains constant (and equal to its initial value) throughout the flight. We know from Eqs. 4-4 that the *x* component of the acceleration can be affected *only* by a net force with an *x* component. In this case $\sum F_x = 0$, so v_x remains constant. The force in the *y* direction affects only *y* and v_y, not *x* or v_x.

The equations for the vertical motion (Eqs. 4-9*b* and 4-10*b*) are exactly those of free fall (Eqs. 2-29 and 2-30). In fact, if we observed the motion from a car traveling along the ground at velocity v_{0x} in the direction of the projectile, the motion would appear to be that of a projectile thrown vertically upward with initial speed v_{0y}.

Figure 4-5 shows the motion. At any point, the magnitude of the velocity vector is

$$v = \sqrt{v_x^2 + v_y^2} \tag{4-11}$$

and its direction is given by

$$\tan\phi = \frac{v_y}{v_x}, \tag{4-12}$$

where ϕ is the angle that the velocity vector makes with the horizontal. At every point of the motion, the velocity vector is tangent to the path of the projectile.

Equations 4-10 give us x and y as functions of the common parameter t. By combining and eliminating t from them, we obtain

$$y = (\tan \phi_0)x - \frac{g}{2(v_0 \cos \phi_0)^2} x^2, \qquad (4\text{-}13)$$

which relates y to x and is the equation of the *trajectory* of the projectile. Since v_0, ϕ_0, and g are constants, this equation has the form

$$y = bx - cx^2,$$

the equation of a parabola. Hence the trajectory of a projectile is parabolic, as shown in Fig. 4-5.

The *horizontal range R* of the projectile, as shown in Fig. 4-5, is defined as the distance along the horizontal where the projectile returns to the level from which it was launched. We can find the range by putting $y = 0$ into Eq. 4-13. One solution immediately arises at $x = 0$; the other gives the range:

$$R = \frac{2v_0^2}{g} \sin \phi_0 \cos \phi_0 = \frac{v_0^2}{g} \sin 2\phi_0, \qquad (4\text{-}14)$$

using the trigonometric identity $\sin 2\theta = 2 \sin \theta \cos \theta$. Note that, for a given initial speed, we get the maximum range for $\phi_0 = 45°$, such that $\sin 2\phi_0 = 1$.

Figure 4-6 shows a strobe photo of the path of a projectile that is not severely affected by air resistance. The path certainly appears parabolic in its shape. Figure 4-7 compares the motions of a projectile dropped from rest to one simultaneously fired horizontally. Here you can see directly the predictions of Eqs. 4-10 when $\phi_0 = 0$. Note that (1) the horizontal motion of ball 2 does indeed follow Eq. 4-10: its x coordinate increases by equal amounts in equal intervals of time, independent of the y motion; and (2) the y motions of the two projectiles are identical: the vertical increments of the position of the two projectiles are the same, independent of the horizontal motion of one of them.

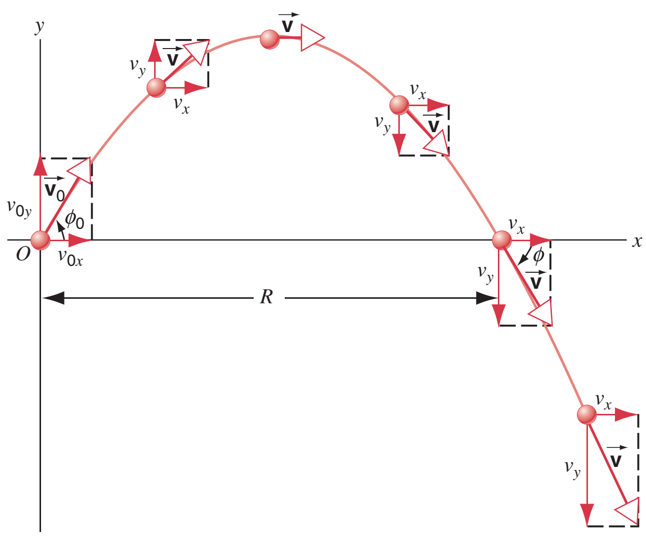


FIGURE 4-5. The trajectory of a projectile, showing the initial velocity $\vec{v}_0$ and its components and also the velocity $\vec{v}$ and its components at five later times. Note that $v_x = v_{0x}$ throughout the flight. The horizontal distance R is the range of the projectile.

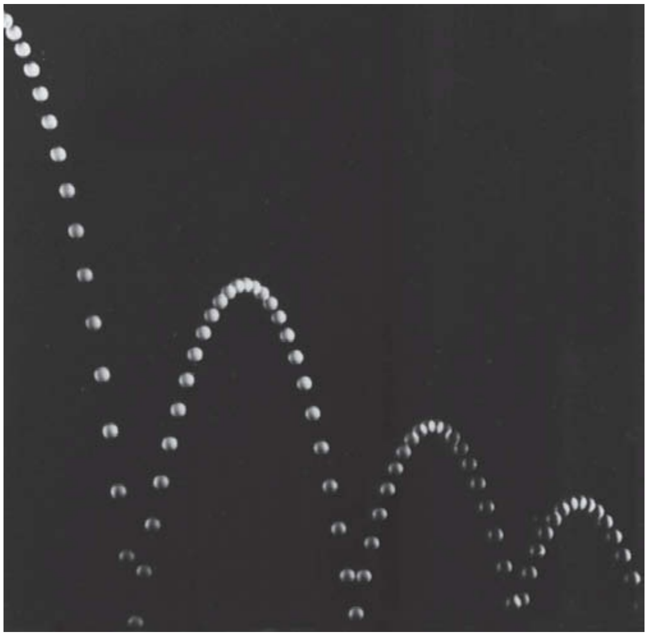


FIGURE 4-6. A strobe photo of a golf ball (which enters the photo from the left) bouncing off a hard surface. Between impacts, the ball shows the parabolic path characteristic of projectile motion. Why do you suppose the height of successive bounces is decreasing? (Chapter 6 may provide the answer.)

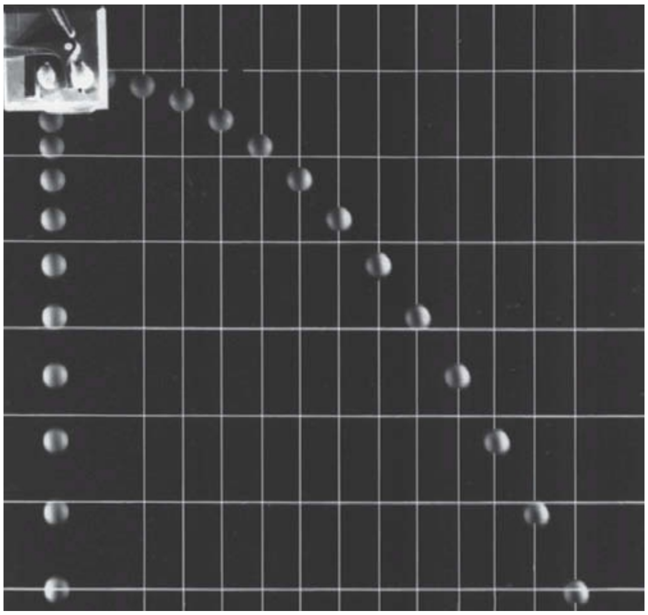


FIGURE 4-7. One ball is released from rest at the same instant that a second ball is fired to the right. Note that both balls fall at exactly the same rate; the horizontal motion of ball 2 does not affect its vertical rate of fall. The exposures in this strobe photo were taken at intervals of 1/30 s. Does the horizontal velocity of the second ball appear to be constant?

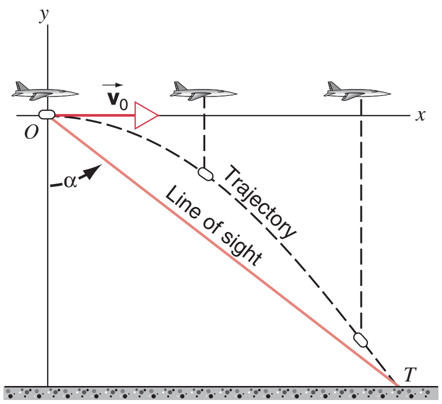


FIGURE 4-8. Sample Problem 4-3.

SAMPLE PROBLEM 4-3. In a contest to drop a package on a target, one contestant's plane is flying at a constant horizontal velocity of 155 km/h at an elevation of 225 m toward a point directly above the target. At what angle of sight α should the package be released to strike the target (Fig. 4-8)?

Solution We choose a reference frame fixed with respect to the Earth, its origin O being the release point. The motion of the package at the moment of release is the same as that of the plane. Hence the initial package velocity $\vec{v}_0$ is horizontal and its magnitude is 155 km/h. The angle of projection ϕ_0 is zero.

We find the time of fall from Eq. 4-10*b*. With $v_{0y} = 0$ and $y = -225$ m at the ground this gives

$$t = \sqrt{-\frac{2y}{g}} = \sqrt{-\frac{(2)(-225\text{ m})}{9.8\text{ m/s}^2}} = 6.78\text{ s}.$$

Note that the time of fall does not depend on the speed of the plane for a horizontal projection.

The horizontal distance traveled by the package in this time is given by Eq. 4-10*a*:

$$x = v_{0x}t = (155\text{ km/h})(1\text{ h/3600 s})(6.78\text{ s})$$
$$= 0.292\text{ km} = 292\text{ m}$$

so that the angle of sight (Fig. 4-8) should be

$$\alpha = \tan^{-1}\frac{x}{|y|} = \tan^{-1}\frac{292\text{ m}}{225\text{ m}} = 52°.$$

Does the motion of the package appear to be parabolic when viewed from a reference frame fixed with respect to the plane? (Can you recall having seen films of bombs dropping from a plane, taken by a camera either on that plane or on another plane flying a parallel course at the same speed?)

SAMPLE PROBLEM 4-4. A soccer player kicks a ball at an angle of 36° from the horizontal with an initial speed of 15.5 m/s. Assuming that the ball moves in a vertical plane, find (*a*) the time t_1 at which the ball reaches the highest point of its trajectory, (*b*) its maximum height, (*c*) its time of flight and range, and (*d*) its velocity when it strikes the ground.

Solution (*a*) The vertical component of the initial velocity is $v_{0y} = v_0 \sin\phi_0 = (15.5\text{ m/s})\sin 36° = 9.1$ m/s. At the top of its trajectory, $v_y = 0$. Solving Eq. 4-9*b* for the time and substituting the numerical values we obtain:

$$t_1 = \frac{v_{0y} - v_y}{g} = \frac{9.1\text{ m/s} - 0}{9.8\text{ m/s}^2} = 0.93\text{ s}.$$

(*b*) The maximum height y_{max} is reached at $t_1 = 0.93$ s. Using Eq. 4-10*b*, we have

$$y_{max} = v_{0y}t_1 - \tfrac{1}{2}gt_1^2$$
$$= (9.1\text{ m/s})(0.93\text{ s}) - \tfrac{1}{2}(9.8\text{ m/s}^2)(0.93\text{ s})^2 = 4.2\text{ m}.$$

(*c*) To find the total time of flight t_2, we set $y = 0$ in Eq. 4-10*b* and, after eliminating the solution $t = 0$ (which reminds us that the ball did indeed start at $y = 0$ when $t = 0$) we solve to find the other time when the ball is at $y = 0$:

$$t_2 = \frac{2v_{0y}}{g} = \frac{2(9.1\text{ m/s})}{9.8\text{ m/s}^2} = 1.86\text{ s}.$$

Note that $t_2 = 2t_1$, which must occur because the time required for the ball to go up (reach its maximum height from the ground) is the same as the time required for it to come down (reach the ground from its maximum height). The range is the horizontal distance traveled during the time t_2:

$$x = v_{0x}t_2 = (v_0\cos\phi_0)t_2 = (15.5\text{ m/s})(\cos 36°)(1.86\text{ s}) = 23.3\text{ m}.$$

(*d*) To find the velocity of the ball when it strikes the ground, we use Eq. 4-9*a* to obtain v_x, which remains constant throughout the flight:

$$v_x = v_{0x} = v_0\cos\phi_0 = (15.5\text{ m/s})(\cos 36°) = 12.5\text{ m/s},$$

and from Eq. 4-9*b* we obtain v_y for $t = t_2$,

$$v_y = v_{0y} - gt = 9.1\text{ m/s} - (9.8\text{ m/s}^2)(1.86\text{ s})$$
$$= -9.1\text{ m/s}.$$

Hence, the velocity has magnitude given by

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(12.5\text{ m/s})^2 + (-9.1\text{ m/s})^2} = 15.5\text{ m/s},$$

and direction given by

$$\tan\phi = v_y/v_x = (-9.1\text{ m/s})/(12.5\text{ m/s}),$$

so that $\phi = -36°$, or 36° clockwise from the x axis. Note that $\phi = -\phi_0$, as we expect from symmetry (Fig. 4-5).

The final speed turned out to be equal to the initial speed. Can you explain this? Is it a coincidence?

Shooting a Falling Target

In a favorite lecture demonstration an air gun is sighted at an elevated target, which is released in free fall by a trip mechanism as the "bullet" leaves the muzzle. No matter what the initial speed of the bullet, it always hits the falling target.

To understand this surprising outcome, consider that if there were no acceleration due to gravity, the target would not fall and the bullet would move along the line of sight directly into the target (Fig. 4-9). The effect of gravity is to cause each body to accelerate down at the same rate from the position it would otherwise have had. Therefore, in the time t, the bullet will fall a distance $\frac{1}{2}gt^2$ from the position it would have had

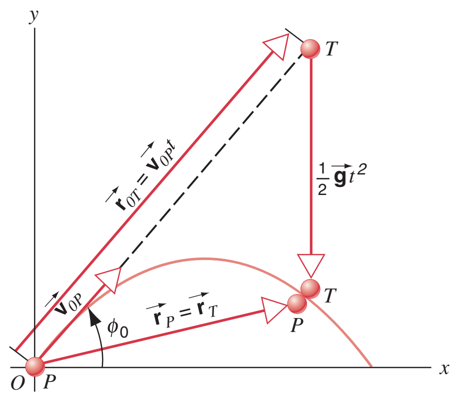


Figure 4-9. In the motion of a projectile, its displacement from the origin at any time t can be thought of as the sum of two vectors: $\vec{v}_{0P}t$, directed along $\vec{v}_{0P}$, and $\frac{1}{2}\vec{g}t^2$, directed down.

along the line of sight and the target will fall the same distance from its starting point. When the bullet reaches the line of fall of the target, it will be the same distance below the target's initial position as the target is and hence the collision. If the bullet moves faster than shown in the figure (v_0 larger), it will have a greater range and will cross the line of fall at a higher point; but since it gets there sooner, the target will fall a correspondingly smaller distance in the same time and collide with it. A similar argument holds for slower speeds.

For an equivalent analysis, let us use Eq. 4-2

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \tfrac{1}{2}\vec{a}t^2$$

to describe the positions of the projectile and the target at any time t. For the projectile P, $\vec{r}_0 = 0$ and $\vec{a} = \vec{g}$, and we have

$$\vec{r}_P = \vec{v}_{0P}t + \tfrac{1}{2}\vec{g}t^2. \tag{4-15}$$

For the target T, $\vec{r}_0 = \vec{r}_{0T}$, $\vec{v}_0 = 0$, and $\vec{a} = \vec{g}$, leading to

$$\vec{r}_T = \vec{r}_{0T} + \tfrac{1}{2}\vec{g}t^2. \tag{4-16}$$

If there is a collision, we must have $\vec{r}_P = \vec{r}_T$. Inspection shows that this will always occur at a time t given by $\vec{r}_{0T} = \vec{v}_{0P}t$—that is, in the time $t\ (= r_{0T}/v_{0P})$ that it would take for an unaccelerated projectile to travel to the target position along the line of sight. Because multiplying a vector by a positive scalar gives another vector in the same direction, the equation $\vec{r}_{0T} = \vec{v}_{0P}t$ tells us that $\vec{r}_{0T}$ and $\vec{v}_{0P}$ must be in the same direction. That is, the gun must be aimed at the initial position of the target.

4-4 DRAG FORCES AND THE MOTION OF PROJECTILES (Optional)

Raindrops fall from clouds whose height h above the ground is about 2 km. Using our equations for freely falling bodies (Eqs. 2-29 and 2-30), we expect the raindrop to strike the ground with a speed of $v \approx 200$ m/s, or about 440 mi/h. The impact of a projectile, even a raindrop, at that speed would be lethal; since raindrops move at much slower speeds, we have obviously made an error in the analysis.

The error occurs when we neglect the effect of the frictional force exerted by the air on the falling raindrop. This frictional force is an example of a *drag force,* experienced by any object that moves through a fluid medium, such as air or water. Drag forces have important effects on a variety of objects, such as baseballs, which deviate considerably from the ideal drag-free trajectory, and downhill skiers, who try to streamline their bodies and skiing position to reduce the drag. Drag forces must be taken into account in the design of aircraft and seacraft. From the standpoint of falling bodies, from raindrops to skydivers, drag forces prevent the velocity from increasing without limit and they impose a maximum or *terminal* speed that can be attained by a falling body.

One particular characteristic of drag forces is that they depend on the speed: the faster an object moves, the greater is the drag force. We therefore cannot use our formulas for constant acceleration to analyze motion under drag forces.

We illustrate the technique for handling problems with nonconstant forces by considering a body of mass m that is dropped from rest. We assume that the magnitude of the drag force D depends linearly on the speed:

$$D = bv \tag{4-17}$$

and always acts in a direction opposite to the motion. The constant b depends on properties of the falling object (its size and shape, for instance) and on the properties of the fluid (air, in this case) through which the object falls. Our goal is to find the velocity of the falling object as a function of the time.

Figure 4-10 shows the free-body diagram, which changes with time as the object falls. When the object is released, $D = 0$ (because $v_y = 0$), and D increases as the object falls. As D continues to increase, at some point it will equal the weight of the object, and at that point there is no net force acting on the object; its acceleration is zero, so its velocity remains constant, as does the drag force. From that time on, the object falls with constant velocity, which is the terminal velocity.

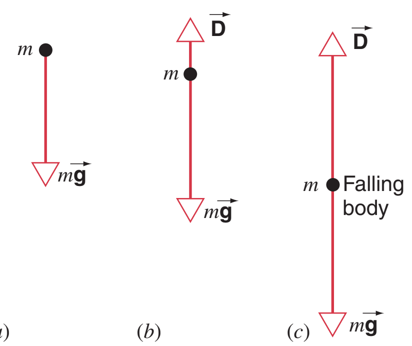


Figure 4-10. Forces acting on a body falling in air. (*a*) At the instant of release, $v_y = 0$ and there is no drag force. (*b*) The drag force increases as the body gains speed. (*c*) Eventually the drag force equals the weight; for all later times it remains equal to the weight and the body falls at its constant terminal speed.

We choose the y axis to be vertical and the positive direction to be downward. (The choice of direction is arbitrary, and here it is convenient to work with positive velocity and acceleration components.) With the weight mg acting downward and the drag force bv_y acting upward, the net force is then $\sum F_y = mg - bv_y$, so that Newton's second law $\sum F_y = ma_y$ gives

$$mg - bv_y = ma_y \tag{4-18}$$

or

$$a_y = g - \frac{bv_y}{m}. \tag{4-19}$$

Our goal is to find the velocity as a function of the time. We begin by substituting $a_y = dv_y/dt$ in Eq. 4-19, which gives

$$\frac{dv_y}{g - bv_y/m} = dt. \tag{4-20}$$

With $v_y = 0$ at time $t = 0$, we seek the velocity v_y at time t. We can therefore integrate the left side of Eq. 4-20 from velocity 0 to v_y and the right side from time 0 to t. (See Eq. 5 of Appendix I.) The result is

$$-\frac{m}{b}\ln\left(\frac{mg - bv_y}{mg}\right) = t \tag{4-21}$$

and solving for v_y we obtain

$$v_y(t) = \frac{mg}{b}(1 - e^{-bt/m}). \tag{4-22}$$

This is the expression for the velocity as a function of time.

It is interesting to examine this result in the two limiting cases of small and large values of t. The velocity starts with $v_y = 0$ at $t = 0$. Just after $t = 0$, near the beginning of the projectile's fall, we can find the velocity by approximating the exponential function using $e^{-x} \approx 1 - x$ for small x ($x \ll 1$). This gives

$$v_y(t) \approx \frac{mg}{b}\left[1 - \left(1 - \frac{bt}{m}\right)\right] = gt \qquad \text{(small } t\text{)}, \tag{4-23}$$

which agrees with Eq. 2-29 when $v_{0y} = 0$ (recalling that here we chose the positive y direction to be downward). Early in the motion, when the velocity is small and the drag force has not increased significantly, the object is approximately in free fall.

For large t, the exponential approaches zero ($e^{-x} \rightarrow 0$ as $x \rightarrow \infty$) and the magnitude of the velocity approaches the *terminal speed* given by

$$v_T = \frac{mg}{b}. \tag{4-24}$$

We can also find the terminal speed directly from Eq. 4-19—when the speed increases to the point at which the drag force and the weight are equal, $a_y = 0$ and Eq. 4-19 then gives Eq. 4-24.

We see that, just as we expect, the larger is the drag force coefficient b, the smaller is the terminal speed. The terminal speed of a pebble falling in water is less than that of the same pebble falling in air, because the drag coefficient is much larger in water.

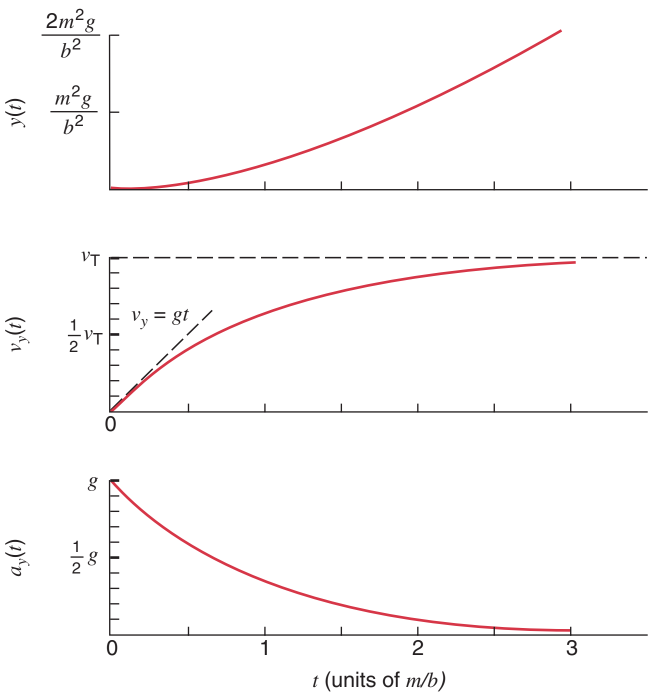


FIGURE 4-11. Position, velocity, and acceleration for a falling body subject to a drag force. Note that the acceleration starts at g and falls to zero; the velocity starts at zero and approaches v_T. Note also that $y(t)$ becomes nearly linear at large t, as we expect for motion with constant velocity.

Now that we have an expression for $v_y(t)$, we can differentiate it to find $a_y(t)$ or integrate it to find $y(t)$. (See Problem 17.) Figure 4-11 shows the time dependence of y, v_y, and a_y.

A drag force proportional to v is representative of *viscous drag,* which is the force that might be experienced by a small particle falling through a thick fluid. Large objects

TABLE 4-1 Some Terminal Speeds in Air

Object	*Terminal Speed* (m/s)	*95% Distance*[a] (m)
16-lb shot	145	2500
Skydiver (typical)	60	430
Baseball	42	210
Tennis ball	31	115
Basketball	20	47
Ping-Pong ball	9	10
Raindrop (radius = 1.5 mm)	7	6
Parachutist (typical)	5	3

[a]This is the distance through which the body must fall from rest to reach 95% of its terminal speed.

Source: Adapted from Peter J. Brancazio, *Sport Science* (Simon & Schuster, 1984).

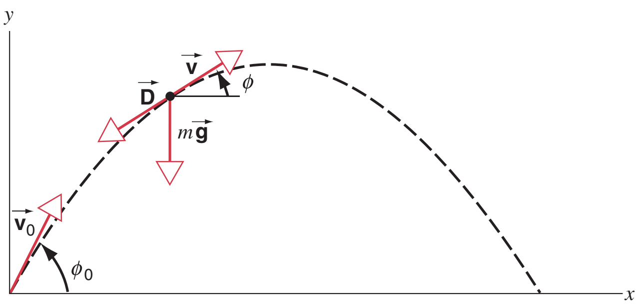


Figure 4-12. A projectile in motion. It is launched with velocity v_0 at an angle ϕ_0 with the horizontal. At a certain time later its velocity is $\vec{v}$ at the angle ϕ. The weight and the drag force (which always points in a direction opposite to $\vec{v}$) are shown at that time.

in air experience *aerodynamic drag,* in which D is proportional to v^2. This case is more complicated mathematically, but it also yields a terminal speed (different from the terminal speed calculated for $D \propto v$).

Table 4-1 shows typical measured values of terminal speeds of different objects in air.

Projectile Motion with Air Resistance

Drag calculations are also important for two-dimensional projectile motion. A baseball, for example, leaves the bat with a speed of about 100 mi/h or 45 m/s. This is already greater than its terminal speed in air when dropped from rest (Table 4-1). The magnitude of the drag force $D = bv$ can be estimated from our previous calculation. From Eq. 4-24 we see that the constant b is the weight mg of the baseball (about 1.4 N, corresponding to a mass of 0.14 kg) divided by its terminal speed, 42 m/s. Thus $b = 0.033$ N/(m/s). If the ball travels at 45 m/s, it experiences a drag force bv with a magnitude of about 1.5 N, which is greater than its weight and therefore has a substantial effect on its motion.

Figure 4-12 shows the free-body diagram at a particular point in the baseball's trajectory. Like all frictional forces, $\vec{D}$ is in a direction opposite to $\vec{v}$, and we assume no wind is blowing. If we take $\vec{D} = -b\vec{v}$, we can use Newton's laws to find an analytic solution for the trajectory, an example of which is illustrated in Fig. 4-13. When air resistance is taken into account, the range is reduced from 179 m to 72 m and the maximum height from 78 m to 48 m. Note also that the trajectory is no longer symmetric about the maximum; the descending motion is much steeper than the ascending motion. For $\phi_0 = 60°$, the projectile strikes the ground at an angle of $-79°$, while in the absence of drag it would strike the ground at an angle equal to $-\phi_0 = -60°$.

For other (and more realistic) choices for the drag force $\vec{D}$, the calculation must be done numerically.* ■

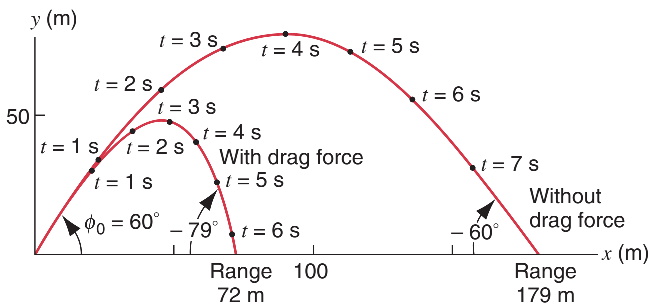


Figure 4-13. Projectile motion with and without a drag force, calculated for $v_0 = 45$ m/s and $\phi_0 = 60°$.

4-5 UNIFORM CIRCULAR MOTION

In projectile motion in the absence of air resistance, the acceleration is constant in both magnitude and direction, but the velocity changes in both magnitude and direction. We now examine a different case of motion in two dimensions in which a particle moves at constant *speed* in a circular path. As we shall see, both the velocity and acceleration are constant in magnitude, but both change their directions continuously. This situation is called *uniform circular motion.* Examples of this kind of motion may include Earth satellites and points on spinning rotors such as fans or merry-go-rounds. In fact, to the extent that we can regard ourselves as particles, we are in uniform circular motion because of the rotation of the Earth.

As an example, imagine you are swinging a ball on a string in a horizontal plane, as in Fig. 4-14. (We neglect the drag force and the force of gravity for the time being.) As you swing the ball, your fingers are exerting a force on the string (and the string in turn exerts a force on the ball). If you were to loosen your grip on the string slightly, the string would slide between your fingers and the ball would move away from the center of the circle, so to prevent this

* You can find more information about this calculation in "Trajectory of a Fly Ball," by Peter J. Brancazio, *The Physics Teacher,* January 1985, p. 20. For an interesting collection of articles about similar problems, see *The Physics of Sports,* edited by Angelo Armenti, Jr. (American Institute of Physics, 1992). See http://www.physics.uoguelph.ca/fun/JAVA/trajplot/trajplot.html for an interesting program that allows you to display the trajectories of a projectile for various choices of launch angle and air resistance.

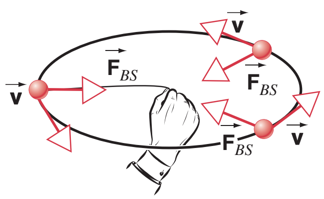


Figure 4-14. A ball on a string is whirled in a horizontal circle. Vectors representing the velocity and the force of the string on the ball are shown at three different instants.

from happening your fingers must be exerting an inward force on the string.

A similar example occurs in planetary motion. As the Moon moves in its orbit about the Earth, the Earth exerts a gravitational force that always points toward the center of the Earth (Fig. 4-15).

In both of these cases, the force is constant in magnitude but varies in direction as the object revolves in its circular path. Because the force always points toward the center of the circle, it is known as a *centripetal* ("seeking the center") force. Since no other force acts, the acceleration must also point toward the center of the circle (centripetal acceleration). From the geometry of the circular motion, we can obtain an expression for the centripetal acceleration.

It is important to note that, in uniform circular motion, the magnitude of the velocity stays constant but the particle is still accelerating because the *direction* of its velocity is changing. Even though we usually associate an acceleration with a change in the magnitude of $\vec{\mathbf{v}}$, there must also be an acceleration present to change the direction of $\vec{\mathbf{v}}$.

To find the relationship between this acceleration and the constant magnitude of the velocity, consider the geometry of Fig. 4-16. A particle is moving in a circle of radius r. We set up an xy coordinate system with its origin at the center of the circle, and we examine the motion of the particle at two locations: at P_1, where its velocity is $\vec{\mathbf{v}}_1$, and at P_2, where its velocity is $\vec{\mathbf{v}}_2$. The points P_1 and P_2 are located symmetrically with respect to the y axis, with the radius to each location making an angle θ with the y axis.

The magnitudes of $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ are equal, but they have different directions, each being tangent to the circle at the location of the particle. The velocity components are:

$$\begin{aligned} v_{1x} &= +v\cos\theta \qquad & v_{1y} &= +v\sin\theta \\ v_{2x} &= +v\cos\theta \qquad & v_{2y} &= -v\sin\theta \end{aligned} \tag{4-25}$$

where we have used v to represent the common magnitude of $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$.

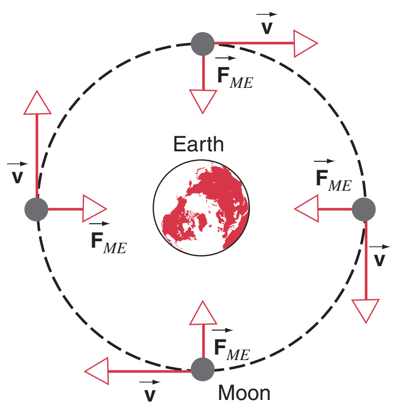


FIGURE 4-15. The Moon moves in its orbit around the Earth. The velocity and force vectors are shown at four different instants. The velocity is always tangent to the circular path, and the force on the Moon due to the Earth always points toward the center of the circle.

As the particle moves along the arc from P_1 to P_2, it covers a distance of $2r\theta$ (where θ is measured in radians), and if it does so in a time interval Δt then its speed v is $2r\theta/\Delta t$. We can therefore express the time interval as

$$\Delta t = \frac{2r\theta}{v}. \tag{4-26}$$

Now we can find the components of the average acceleration. We use the definition of average acceleration from Eq. 2-14, $\vec{\mathbf{a}}_{\text{av}} = \Delta\vec{\mathbf{v}}/\Delta t$ where $\Delta\vec{\mathbf{v}}$ means $\vec{\mathbf{v}}_2 - \vec{\mathbf{v}}_1$. The x component of the average acceleration is then

$$a_{\text{av},x} = \frac{v_{2x} - v_{1x}}{\Delta t} = \frac{v\cos\theta - v\cos\theta}{\Delta t} = 0. \tag{4-27}$$

As shown in Fig. 4-16, the x components of the velocity are the same at P_1 and P_2, so it is not surprising that the x component of the average acceleration is zero in that interval. The y component of the average acceleration is

$$\begin{aligned} a_{\text{av},y} &= \frac{v_{2y} - v_{1y}}{\Delta t} = \frac{-v\sin\theta - v\sin\theta}{\Delta t} \\ &= \frac{-2v\sin\theta}{2r\theta/v} = -\left(\frac{v^2}{r}\right)\left(\frac{\sin\theta}{\theta}\right). \end{aligned} \tag{4-28}$$

We can find the instantaneous acceleration from this result by taking the limit as the time interval approaches zero. Equivalently, we can let the angle θ go to zero, so that P_1 and P_2 both approach P, which gives

$$a_y = \lim_{\theta\to 0}\left[-\left(\frac{v^2}{r}\right)\left(\frac{\sin\theta}{\theta}\right)\right] = -\left(\frac{v^2}{r}\right)\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right).$$

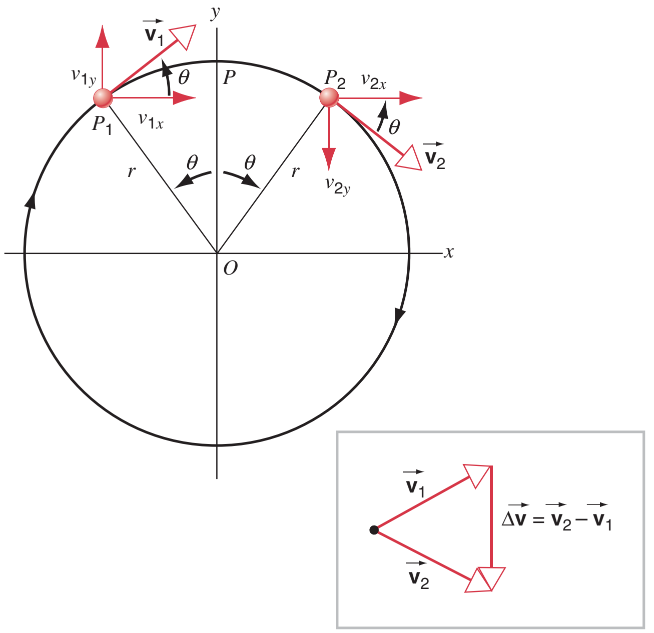


FIGURE 4-16. A particle moves at constant speed in a circle of radius r. It is shown at locations P_1 and P_2, where the radius makes equal angles θ on opposite sides of the y axis. The inset shows the vector $\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}}_2 - \vec{\mathbf{v}}_1$; this vector always points toward the center of the circle, no matter where we choose points P_1 and P_2.

For small angles, sin $\theta \approx \theta$ (in radians), so the limit approaches the value 1. The y component of the instantaneous acceleration at P is then $a_y = -v^2/r$, the minus sign indicating that the acceleration at P points in the negative y direction—that is, toward the center of the circle.

Point P is an arbitrary point on the circle. We could have repeated the above calculation for any point on the circle and we would have obtained the same result: *the acceleration points toward the center of the circle and has magnitude v^2/r*. This is a general result for any particle that moves in a circle at constant speed; the centripetal acceleration is

$$a_c = \frac{v^2}{r}. \tag{4-29}$$

The centripetal acceleration is sometimes also called the *radial* acceleration, since its direction is always along a radius of the circle. In Fig. 4-16 you can see that the direction of $\vec{\mathbf{a}}$ is the same as the direction of $\Delta\vec{\mathbf{v}}$, just as the vector relationship of Eq. 2-14 requires.

Both in free fall and in projectile motion $\vec{\mathbf{a}}$ is constant in direction and magnitude, and we can use the equations developed for constant acceleration. We cannot use these equations for uniform circular motion because $\vec{\mathbf{a}}$ varies in direction and is therefore not constant.

The units of centripetal acceleration are the same as those of an acceleration resulting from a change in the magnitude of a velocity. Dimensionally, we have

$$[a] = \frac{[v^2]}{[r]} = \frac{(\mathrm{L/T})^2}{\mathrm{L}} = \frac{\mathrm{L}}{\mathrm{T}^2},$$

which are the usual dimensions of acceleration. The units therefore may be m/s^2, km/h^2, or similar units of dimension L/T^2.

The acceleration resulting from a change in direction of a velocity is just as real and just as much an acceleration in every sense as that arising from a change in magnitude of a velocity. By definition, acceleration is the time rate of change of velocity, and velocity, being a vector, can change in direction as well as magnitude. If a physical quantity is a vector, its directional aspects cannot be ignored, for their effects will prove to be every bit as important and real as those produced by changes in magnitude.

According to Newton's second law in its vector form ($\sum \vec{\mathbf{F}} = m\vec{\mathbf{a}}$), the acceleration and the net force must have the same direction. In the case of circular motion at constant speed, the net force must thus point toward the center of the circle. For now we will write this result in terms of magnitudes: $|\sum \vec{\mathbf{F}}| = ma$. For uniform circular motion, $a = a_c = v^2/r$ and so

$$\left|\sum \vec{\mathbf{F}}\right| = \frac{mv^2}{r}. \tag{4-30}$$

The quantity on the left side of Eq. 4-30 is sometimes called the "centripetal force." The centripetal force is not a new kind of force. When a particle moves in a circular path at constant speed, several forces may act on it. The resultant of all those forces must point toward the center of the circle, and we call that resultant the centripetal force. Newton's second law then gives the magnitude and direction of the acceleration.

In Fig. 4-14, the string provides the centripetal force that acts on the ball; in Fig. 4-15, the gravitational force of the Earth provides the centripetal force that acts on the Moon. To label a force as "centripetal" simply means that it acts toward the center of the circle, but that label tells us nothing about the nature of the force or the body that is exerting it. All forces, including those that act centripetally, must always be associated with a specific body in the environment. The centripetal force can be any type of force and might, for example, be provided by the action of gravity, strings, springs, or electric charges. As indicated in Eq. 4-30, it can also be a combination of two or more forces, as long as the direction of the resultant force is toward the center of the circle.

In this section we have discussed uniform circular motion as an example of a case in which vector laws are essential to understand motion in two dimensions. More general vector techniques can be used to describe the case in which the acceleration has both radial and tangential components. These techniques are described in Chapter 8.

Sample Problem 4-5. The Moon revolves about the Earth, making a complete revolution in 27.3 days. Assume that the orbit is circular and has a radius $r = 238{,}000$ miles. What is the magnitude of the gravitational force exerted on the Moon by the Earth?

Solution We have $r = 238{,}000\ \text{mi} = 3.82 \times 10^8$ m. From Appendix C, we find the mass of the Moon is $m = 7.36 \times 10^{22}$ kg. The time for one complete revolution, called the period, is $T = 27.3\ \text{d} = 2.36 \times 10^6$ s. The speed of the Moon (assumed constant) is therefore

$$v = \frac{2\pi r}{T} = \frac{2\pi(3.82 \times 10^8\ \text{m})}{2.36 \times 10^6\ \text{s}} = 1018\ \text{m/s}.$$

The centripetal force is provided by the gravitational force on the Moon by the Earth:

$$F_{ME} = \frac{mv^2}{r} = \frac{(7.36 \times 10^{22}\ \text{kg})(1018\ \text{m/s})^2}{3.82 \times 10^8\ \text{m}}$$
$$= 2.00 \times 10^{20}\ \text{N}.$$

Sample Problem 4-6. A satellite of mass 1250 kg is to be placed in a circular orbit at a height $h = 210$ km above the Earth's surface, where $g = 9.2$ m/s^2. (*a*) What is the weight of the satellite at this altitude? (*b*) With what tangential speed must it be inserted into its orbit? The Earth's radius is $R = 6370$ km.

Solution (*a*) The weight of the satellite is

$$W = mg = (1250\ \text{kg})(9.2\ \text{m/s}^2) = 1.15 \times 10^4\ \text{N}.$$

(*b*) The weight is the force of gravity F_{SE} exerted on the satellite by the Earth. Since this is the only force that acts on the satellite, it must provide the centripetal force. Solving Eq. 4-30 for the tangential speed v, we obtain (with $r = R + h$):

$$v = \sqrt{\frac{F_{SE}r}{m}} = \sqrt{\frac{(1.15 \times 10^4 \text{ N})(6370 \text{ km} + 210 \text{ km})}{1250 \text{ kg}}}$$

$$= 7780 \text{ m/s} = 17{,}400 \text{ mi/h}.$$

At this speed, the satellite completes one orbit every 1.48 h.

4-6 RELATIVE MOTION

In Section 3-2 we discussed inertial frames and how observers in motion relative to one another will deduce identical accelerations if they are both in inertial frames. These observers will thus agree on the application of Newton's second law.

In this section, we expand the comparison of observations from different inertial frames using vector considerations. As before, we consider the description of the motion of a single particle by two observers who are in uniform motion relative to one another. The two observers might be, for example, a person in a car moving at constant velocity along a long, straight road and another person standing at rest on the ground. The particle they are both observing might be a ball tossed in the air or another moving car.

We call the two observers S and S'. Each has a corresponding reference frame to which is attached a Cartesian coordinate system. For convenience, we assume the observers to be located at the origins of their respective coordinate systems. We make only one restriction on this situation: *the relative velocity between S and S' must be a constant.* Here we mean constant in both magnitude and direction. Note that this restriction does not include the motion of the particle being observed by S and S'. The particle need not necessarily be moving with constant velocity, and indeed the particle may well be accelerating.

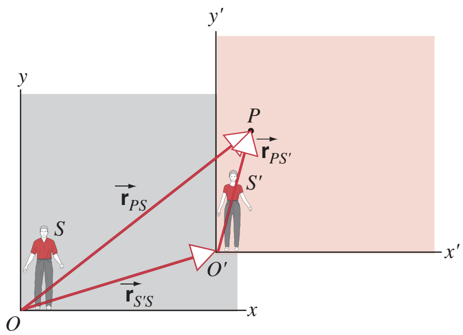


FIGURE 4-17. Observers S and S', who are moving with respect to each other, observe the same moving particle P. At the time shown, they measure the position of the particle with respect to the origins of their coordinate systems to be $\vec{r}_{PS}$ and $\vec{r}_{PS'}$, respectively. At this same instant, observer S measures the position of S' with respect to the origin O to be $\vec{r}_{S'S}$.

Figure 4-17 shows, at a particular time t, the two coordinate systems belonging to S and S'. For simplicity, we consider motion in only two dimensions, the common xy and $x'y'$ planes shown in Fig. 4-17. The origin of the S' system is located with respect to the origin of the S system by the vector $\vec{r}_{S'S}$. Note in particular the order of the subscripts we use to label the vector: the first subscript gives the system being located (in this case, the coordinate system of S') and the second subscript gives the system with respect to which we are doing the locating (in this case, the coordinate system of S). The vector $\vec{r}_{S'S}$ would then be read as "the position of S' with respect to S."

Figure 4-17 also shows a particle P in the common xy and $x'y'$ planes. Both S and S' locate the particle P with respect to their coordinate systems. According to S, the particle P is at the position indicated by the vector $\vec{r}_{PS}$, whereas according to S' the particle P is at $\vec{r}_{PS'}$. From Fig. 4-17 we can deduce the following relationship among the three vectors:

$$\vec{r}_{PS} = \vec{r}_{S'S} + \vec{r}_{PS'} = \vec{r}_{PS'} + \vec{r}_{S'S}, \qquad (4\text{-}31)$$

where we have used the commutative law of vector addition to exchange the order of the two vectors. Once again, pay careful attention to the order of the subscripts. In words, Eq. 4-31 tells us: "the position of P as measured by S is equal to the position of P as measured by S' plus the position of S' as measured by S."

Suppose the particle P is moving with velocity $\vec{v}_{PS'}$ according to S'. What velocity will S measure for the particle? To answer this question, we need only take the derivative with respect to time of Eq. 4-31, which gives

$$\frac{d\vec{r}_{PS}}{dt} = \frac{d\vec{r}_{PS'}}{dt} + \frac{d\vec{r}_{S'S}}{dt}.$$

The rate of change of each position vector gives the corresponding velocity, so that

$$\vec{v}_{PS} = \vec{v}_{PS'} + \vec{v}_{S'S}. \qquad (4\text{-}32)$$

Thus, at any instant, the velocity of P as measured by S is equal to the velocity of P as measured by S' plus the relative velocity of S' with respect to S. Although we have illustrated Eqs. 4-31 and 4-32 for motion in two dimensions, they hold equally well in three dimensions.

Equation 4-32 is a law of the *transformation of velocities.* It permits us to transform a measurement of velocity made by an observer in one frame of reference—say, S'—to another frame of reference—say, S—as long as we know the relative velocity between the two reference frames. It is a law firmly grounded both in the common sense of everyday experience and in the concepts of space and time that are essential to the classical physics of Galileo and Newton. In fact, Eq. 4-32 is often called the *Galilean form of the law of transformation of velocities.*

We consider here only the very important special case in which the two reference frames are moving at constant velocity with respect to one another. That is, $\vec{v}_{S'S}$ is constant both in magnitude and direction. The velocities $\vec{\mathbf{v}}_{PS}$ and $\vec{\mathbf{v}}_{PS'}$ that S and S' measure for the particle P may not be constant, and of course they will in general not be equal to one another. If, however, one of the observers—say, S'—measures a velocity that is constant in time, then both terms on the right-hand side of Eq. 4-32 are independent of time and therefore the left side of Eq. 4-32 must also be independent of time. Thus, if one observer concludes that the particle moves with constant velocity, then all other observers conclude the same, as long as the other observers are in frames of reference that move at constant velocity with respect to the frame of the first observer.

We can see this in a more formal way by differentiating Eq. 4-32:

$$\frac{d\vec{\mathbf{v}}_{PS}}{dt} = \frac{d\vec{\mathbf{v}}_{PS'}}{dt} + \frac{d\vec{\mathbf{v}}_{S'S}}{dt}. \tag{4-33}$$

The last term of Eq. 4-33 vanishes, because we assume that the relative velocity of the two reference frames is a constant. Thus

$$\frac{d\vec{\mathbf{v}}_{PS}}{dt} = \frac{d\vec{\mathbf{v}}_{PS'}}{dt}.$$

Replacing these two derivatives of velocity with the corresponding accelerations, we obtain

$$\vec{\mathbf{a}}_{PS} = \vec{\mathbf{a}}_{PS'}. \tag{4-34}$$

The accelerations of P measured by the two observers are identical!

Equation 4-34 indicates directly why Newton's laws can be equally well applied by observers in any inertial frame. If our observers deduce identical accelerations for the moving particle, then they will agree on the results of applying $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$. If observer S successfully tests to determine that Newton's laws are valid, then all other observers whose reference frames move relative to S with a velocity that is constant in both magnitude and direction will likewise find Newton's laws to be valid.

SAMPLE PROBLEM 4-7. The compass in an airplane indicates that it is headed due east; its air speed indicator reads 215 km/h. A steady wind of 65 km/h is blowing due north. (*a*) What is the velocity of the plane with respect to the ground? (*b*) If the pilot wishes to fly due east, what must be the heading? That is, what must the compass read?

Solution (*a*) The moving "particle" in this problem is the plane P. There are two reference frames, the ground (G) and the air (A). We let the ground be our S system and the air be the S' system, and by a simple change of notation, we can rewrite Eq. 4-32 as

$$\vec{\mathbf{v}}_{PG} = \vec{\mathbf{v}}_{PA} + \vec{\mathbf{v}}_{AG}.$$

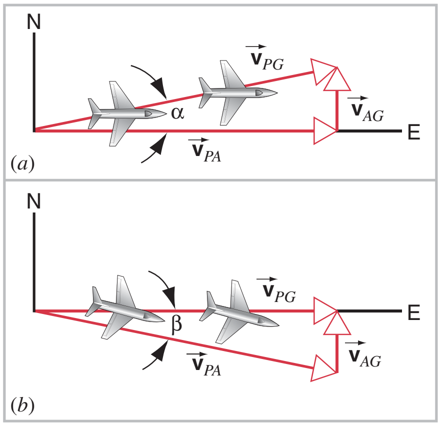


FIGURE 4-18. Sample Problem 4-7. (*a*) A plane, heading due east, is blown to the north. (*b*) To travel due east, the plane must head into the wind.

Figure 4-18*a* shows these vectors, which form a right triangle. The terms are, in sequence, the velocity of the plane with respect to the ground, the velocity of the plane with respect to the air, and the velocity of the air with respect to the ground (that is, the wind velocity). Note the orientation of the plane, which is consistent with a due east reading on its compass.

The magnitude of the ground velocity (the ground speed) is found from

$$v_{PG} = \sqrt{v_{PA}^2 + v_{AG}^2} = \sqrt{(215\text{ km/h})^2 + (65\text{ km/h})^2} = 225\text{ km/h}.$$

The angle α in Fig. 4-18*a* follows from

$$\alpha = \tan^{-1}\frac{v_{AG}}{v_{PA}} = \tan^{-1}\frac{65\text{ km/h}}{215\text{ km/h}} = 16.8°.$$

Thus, with respect to the ground, the plane is flying at 225 km/h in a direction 16.8° north of east. Note that its ground speed is greater than its air speed.

(*b*) In this case the pilot must head into the wind so that the velocity of the plane with respect to the ground points east. The wind remains unchanged and the vector diagram representing Eq. 4-32 is as shown in Fig. 4-18*b*. Note that the three vectors still form a right triangle, as they did in Fig. 4-18*a*, but in this case the hypotenuse is v_{PA} rather than v_{PG}.

The pilot's ground speed is now

$$v_{PG} = \sqrt{v_{PA}^2 - v_{AG}^2} = \sqrt{(215\text{ km/h})^2 - (65\text{ km/h})^2} = 205\text{ km/h}.$$

As the orientation of the plane in Fig. 4-18*b* indicates, the pilot must head into the wind by an angle β given by

$$\beta = \sin^{-1}\frac{v_{AG}}{v_{PA}} = \sin^{-1}\frac{65\text{ km/h}}{215\text{ km/h}} = 17.6°.$$

Note that, by heading into the wind as the pilot has done, the ground speed is now less than the air speed.

MULTIPLE CHOICE

4-1 Motion in Three Dimensions with Constant Acceleration

1. An object moves in the xy plane with an acceleration that has a positive x component. At time $t = 0$ the object has a velocity given by $\vec{\mathbf{v}} = 3\hat{\mathbf{i}} + 0\hat{\mathbf{j}}$.
(*a*) What can be concluded about the y component of the acceleration?
   (A) The y component must be positive and constant.
   (B) The y component must be negative and constant.
   (C) The y component must be zero.
   (D) Nothing at all can be concluded about the y component.

(*b*) What can be concluded about the y component of the velocity?
   (A) The y component must be increasing.
   (B) The y component must be constant.
   (C) The y component must be decreasing.
   (D) Nothing at all can be concluded about the variation of the y component.

(*c*) What can be concluded about magnitude of the velocity?
   (A) The magnitude of the velocity must be increasing.
   (B) The magnitude of the velocity must be constant.
   (C) The magnitude of the velocity component must be decreasing.
   (D) Nothing at all can be concluded about the magnitude of the velocity.

2. An object moves with a constant acceleration $\vec{\mathbf{a}}$. Which of the following expressions are also constant?
   (A) $d|\vec{\mathbf{v}}|/dt$ (B) $|d\vec{\mathbf{v}}/dt|$
   (C) $d(v^2)/dt$ (D) $d(\vec{\mathbf{v}}/|\vec{\mathbf{v}}|)/dt$

4-2 Newton's Laws in Three-Dimensional Vector Form

3. Suppose the net force $\vec{\mathbf{F}}$ on an object is a nonzero constant. Which of the following could also be constant?
   (A) Position. (B) Speed.
   (C) Velocity. (D) Acceleration.

4. Two forces of magnitude F_1 and F_2 are acting on an object. The magnitude of the net force F_{net} on the object will be in the range
   (A) $F_1 \le F_{net} \le F_2$.
   (B) $(F_1 - F_2)/2 \le F_{net} \le (F_1 + F_2)/2$.
   (C) $|F_1 - F_2| \le F_{net} \le |F_1 + F_2|$.
   (D) $F_1^2 - F_2^2 \le (F_{net})^2 \le F_1^2 + F_2^2$.

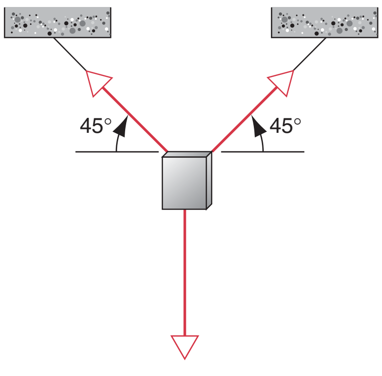


FIGURE 4-19. Multiple-choice questions 5 and 6.

5. A small 2.0-kg object is suspended at rest from two strings as shown in Fig. 4-19. The magnitude of the force exerted by each string on the object is 13.9 N; the magnitude of the force of gravity is 19.6 N. The magnitude of the net force on the object is
   (A) 47.4 N. (B) 33.5 N. (C) 13.9 N.
   (D) 8.2 N. (E) 0 N.

6. The string on the left in Fig. 4-19 suddenly snaps. At the instant when the string snaps, the magnitude of the net force on the object is
   (A) 47.4 N. (B) 33.5 N. (C) 13.9 N.
   (D) 8.2 N. (E) 0 N.

4-3 Projectile Motion

7. A projectile is launched with initial velocity $\vec{\mathbf{v}}_0$ at an angle ϕ_0 with the horizontal. Neglect air resistance. (*a*) Where in the motion is the net force on the projectile equal to zero? (*b*) Where in the motion is the acceleration of the projectile equal to zero?
   (A) Somewhere before reaching its maximum height.
   (B) At its highest point.
   (C) Somewhere after reaching its maximum height.
   (D) Nowhere in the trajectory.

8. An object is launched into the air with an initial velocity given by $\vec{\mathbf{v}}_0 = (4.9\hat{\mathbf{i}} + 9.8\hat{\mathbf{j}})$ m/s. Ignore air resistance.
(*a*) At the highest point the magnitude of the velocity is
   (A) 0. (B) $\sqrt{4.9^2}$ m/s.
   (C) $\sqrt{9.8^2}$ m/s. (D) $\sqrt{4.9^2 + 9.8^2}$ m/s.

(*b*) At $t = 0.5$ s the magnitude of the velocity is
   (A) $\sqrt{(4.9 + 9.8/2)^2}$ m/s. (B) $\sqrt{4.9^2 + (9.8/2)^2}$ m/s.
   (C) $\sqrt{(4.9/2)^2 + 9.8^2}$ m/s. (D) $\sqrt{(4.9/2)^2 + (9.8/2)^2}$ m/s.

9. During the Battle of Tarawa in World War II, battleships fired ballistic projectiles at the Japanese garrisons on Betio from as far as 40 miles out at sea. Assuming no air resistance, and assuming the trajectories were chosen to give the optimum range,
(*a*) the projectiles would have risen to a maximum altitude in the range of
   (A) 0 to $\frac{1}{2}$ mi. (B) $\frac{1}{2}$ to 2 mi. (C) 2 to 5 mi.
   (D) 5 to 8 mi. (E) 8 to 12 mi.

before returning to the ground.
(*b*) What would be the approximate muzzle velocity of the projectiles?
   (A) 25,000 ft/s. (B) 2,500 ft/s.
   (C) 250 ft/s. (D) 25 ft/s.

10. A projectile fired vertically up from a cannon rises 200 meters before returning to the ground. If the same cannon were to fire the same projectile at an angle, then the maximum range would be approximately
   (A) 200 m. (B) 400 m.
   (C) 800 m. (D) 1600 m.
(Assume that air resistance is negligible.)

4-4 Drag Forces and the Motion of Projectiles

11. Which graph in Fig. 4-20 best shows the velocity–time graph for an object launched vertically into the air when air resistance is given by $D = bv$? The dashed line shows the velocity graph if there were no air resistance.

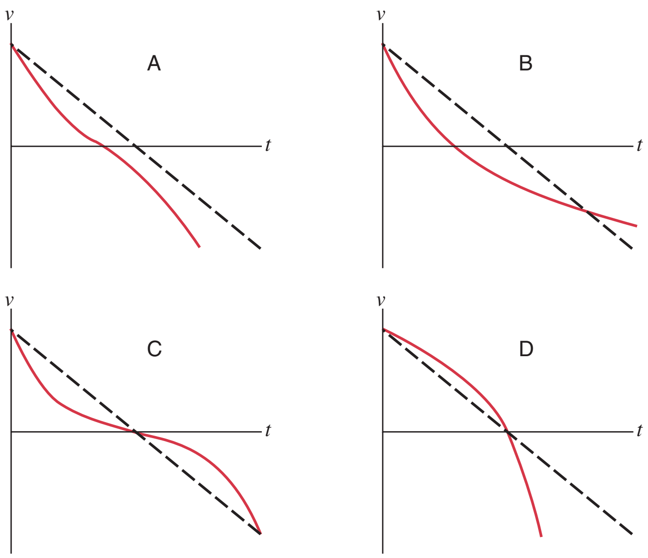


FIGURE 4-20. Multiple-choice question 11.

12. You calculate that to throw an object vertically to a height h it needs to be launched with an initial upward velocity v_0, assuming no air resistance. The dashed lines in Fig. 4-21 show the motion according to this calculation. Which of the velocity–time graphs shows the motion of an object tossed with initial upward velocity v_0' that will also rise to height h, but this time with air resistance?

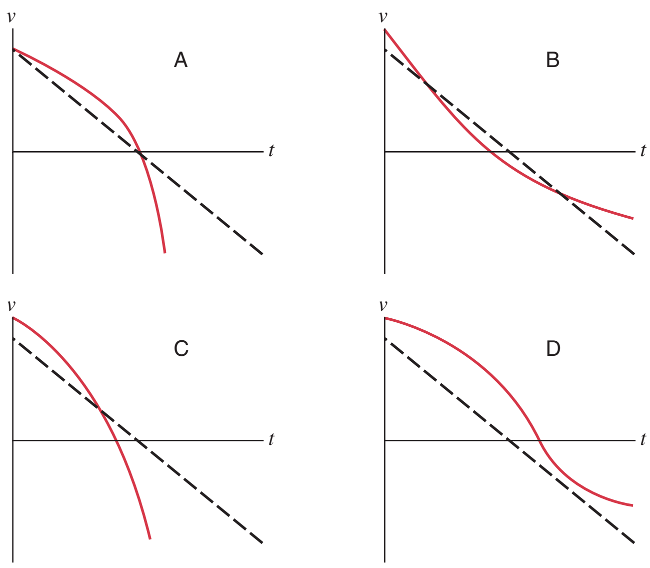


FIGURE 4-21. Multiple-choice question 12.

13. A parachutist jumps out of a plane. He falls freely for some time, and then opens his chute. Shortly after his chute inflates the parachutist
- (A) keeps falling but quickly slows down.
- (B) momentarily stops, then starts falling again, but more slowly.
- (C) suddenly shoots upward, and then starts falling again, but more slowly.
- (D) suddenly shoots upward, and then starts falling again, eventually acquiring the same speed as before the chute opened.

4-5 Uniform Circular Motion

14. Which statement is most correct?
- (A) Uniform circular motion causes a constant force toward the center.
- (B) Uniform circular motion is caused by a constant force toward the center.
- (C) Uniform circular motion is caused by a constant magnitude *net* force toward the center.
- (D) Uniform circular motion is caused by a constant magnitude *net* force away from the center.

15. A puck is moving in a circle of radius r_0 with a constant speed v_0 on a level frictionless table. A string is attached to the puck, which holds it in the circle; the string passes through a frictionless hole and is attached on the other end to a hanging object of mass M. (See Fig. 4-22.)

(*a*) The puck is now made to move with a speed $v' = 2v_0$, but still in a circle. The mass of the hanging object is left unchanged. The acceleration a' of the puck and the radius r' of the circle are now given by

(A) $a' = 4a_0$ and $r' = r_0$. (B) $a' = 2a_0$ and $r' = r_0$.
(C) $a' = 2a_0$ and $r' = 2r_0$. (D) $a' = a_0$ and $r' = 4r_0$.

(*b*) The puck continues to move at speed $v' = 2v_0$ in a circle, but now the mass of the hanging object is doubled. The acceleration a' of the puck and the radius r' of the circle are now given by

(A) $a' = 4a_0$ and $r' = r_0$. (B) $a' = 2a_0$ and $r' = r_0$.
(C) $a' = 2a_0$ and $r' = 2r_0$. (D) $a' = a_0$ and $r' = 4r_0$.

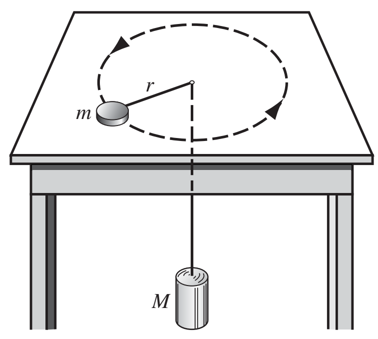


FIGURE 4-22. Multiple-choice question 15.

4-6 Relative Motion

16. An object has velocity $\vec{\mathbf{v}}_1$ relative to the ground. An observer moving past with a constant velocity $\vec{\mathbf{v}}_0$ relative to the ground measures the velocity of the object to be $\vec{\mathbf{v}}_2$ (relative to the observer). The magnitudes of these velocities are related by

(A) $v_0 \leq v_1 + v_2$. (B) $v_1 \leq v_2 + v_0$.
(C) $v_2 \leq v_0 + v_1$. (D) All the above are true.

17. (*a*) A boy sitting in a railroad car moving at constant velocity throws a ball straight up into the air, according to the person sitting next to him. Where will the ball fall?

(A) Behind him (B) In front of him
(C) Into his hands (D) Beside him

(*b*) Where would the ball fall if the train accelerates forward while the ball is in the air? If it rounds a curve?

(A) Behind him (B) In front of him
(C) Into his hands (D) Beside him

QUESTIONS

1. A particle moves in three-dimensional space with a constant acceleration. Can the z component of the acceleration affect the x component of its location? Can the z component of the acceleration affect the y component of the velocity?

2. Describe a physical situation in which an object that moves in the xy plane might have an acceleration with a constant positive x component and a constant negative y component.

3. Can the acceleration of a body change its direction without its velocity changing direction?

4. Let $\vec{v}$ and $\vec{a}$ represent the velocity and acceleration, respectively, of an automobile. Describe circumstances in which (*a*) $\vec{v}$ and $\vec{a}$ are parallel; (*b*) $\vec{v}$ and $\vec{a}$ are antiparallel; (*c*) $\vec{v}$ and $\vec{a}$ are perpendicular to one another; (*d*) $\vec{v}$ is zero but $\vec{a}$ is not zero; (*e*) $\vec{a}$ is zero but $\vec{v}$ is not zero.

5. In Fig. 4-23, we show four forces that are equal in magnitude. What combination of three of these, acting together on the same particle, might keep that particle at rest?

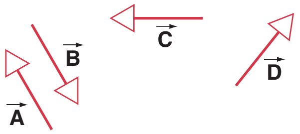


FIGURE 4-23. Question 5.

6. You shoot an arrow into the air and keep your eye on it as it follows a parabolic flight path to the ground. You note that the arrow turns in flight so that it is always tangent to its flight path. What makes it do that?

7. In a tug-of-war, three men pull on a rope to the left at A and three men pull to the right at B with forces of equal magnitude. Now a 5-lb weight is hung vertically from the center of the rope. (*a*) Can the men get the rope AB to be horizontal? (*b*) If not, explain. If so, determine the magnitude of the forces required at A and B to do this.

8. A tube in the shape of a rectangle with rounded corners is placed in a vertical plane, as shown in Fig. 4-24. You introduce two ball bearings at the upper right-hand corner. One travels by path AB and the other by path CD. Which will arrive first at the lower left-hand corner?

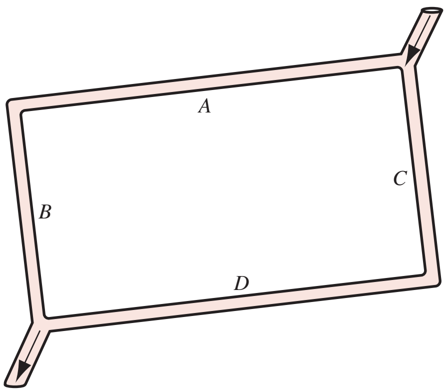


FIGURE 4-24. Question 8.

9. In broad jumping, sometimes called long jumping, does it matter how high you jump? What factors determine the span of the jump?

10. Why doesn't the electron in the beam from an electron gun fall as much because of gravity as a water molecule in the stream from a hose? Assume horizontal motion initially in each case.

11. At what point or points in its path does a projectile have its minimum speed? Its maximum?

12. Figure 4-25 shows the path followed by a NASA airplane in a run designed to simulate low-gravity conditions for a short period of time. Make an argument to show that, if the plane follows a particular parabolic path, the passengers will experience weightlessness.

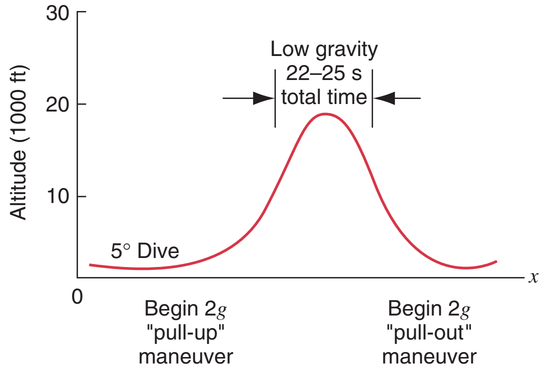


FIGURE 4-25. Question 12.

13. A shot-putter heaves a shot from above ground level. The launch angle that will produce the longest range is less than 45°; that is, a flatter trajectory has a longer range. Explain why.

14. Consider a projectile at the top of its trajectory. (*a*) What is its speed in terms of v_0 and ϕ_0? (*b*) What is its acceleration? (*c*) How is the direction of its acceleration related to that of its velocity?

15. Trajectories are shown in Fig. 4-26 for three kicked footballs. Pick the trajectory for which (*a*) the time of flight is least, (*b*) the vertical velocity component at launch is greatest, (*c*) the horizontal velocity component at launch is greatest, and (*d*) the launch speed is least. Ignore air resistance.

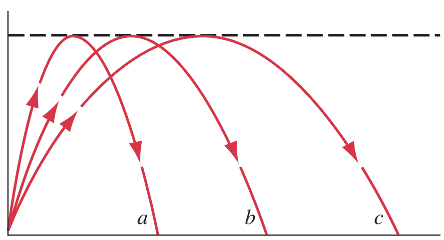


FIGURE 4-26. Question 15.

16. A rifle is bore-sighted with its barrel horizontal. Show that, at the same range, it will shoot too high when shooting either uphill or downhill. (See "A Puzzle in Elementary Ballistics," by Ole Anton Haugland, *The Physics Teacher,* April 1983, p. 246.)

17. In his book, *Sport Science,* Peter Brancazio, with such projectiles as baseballs and golf balls in mind, writes: "Everything else being equal, a projectile will travel farther on a hot day than on a cold day, farther at high altitude than at sea level, farther in humid than in dry air." How can you explain these claims?

18. A graph of height versus time for an object thrown vertically upward is a parabola. The path of a projectile, thrown upward but not vertically upward, is also a parabola. Is this a coincidence? Justify your answer.

19. Long-range artillery pieces are not set at the "maximum range" angle of 45° but at larger elevation angles, in the range of 55° to 65°. What is wrong with 45°?

20. In projectile motion when air resistance is negligible, is it ever necessary to consider three-dimensional motion rather than two-dimensional?

21. Under what conditions would it be necessary to consider three-dimensional motion of a projectile?

22. Discuss how the choice of angle for maximum range of a projectile would be affected by the resistance of the air to motion of the projectile through it.

23. Which raindrops, if either, fall faster: small ones or large ones?

24. The terminal speed of a baseball is 95 mi/h. However, the measured speeds of pitched balls often exceed this, topping 100 mi/h. How can this be?

25. Describe the motion of an object that is fired vertically downward with an initial speed that is greater than its terminal speed.

26. A log is floating downstream. How would you calculate the drag force acting on it?

27. You drop two objects of different masses simultaneously from the top of a tower. Show that, if you assume the air resistance to have the same constant value for each object, the one with the larger mass will strike the ground first. How good is this assumption?

28. Why does Table 4-1 list the "95% distance" and not the "100% distance"?

29. Is it possible to be accelerating if you are traveling at constant speed? Is it possible to round a curve with zero acceleration? With constant acceleration?

30. Describe qualitatively the acceleration acting on a bead that, sliding along a frictionless wire, moves inward with constant speed along a flat spiral.

31. Show that, taking the Earth's rotation and revolution into account, a book resting on your table moves faster at night than it does during the daytime. In what reference frame is this statement true?

32. An aviator, pulling out of a dive, follows the arc of a circle and is said to have "pulled $3g$'s" in pulling out of the dive. Explain what this statement means.

33. Could the acceleration of a projectile be represented in terms of a radial and a tangential component at each point of the motion? If so, is there any advantage to this representation?

34. If the acceleration of a body is constant in a given reference frame, is it necessarily constant in all other reference frames?

35. A woman on the rear platform of a train moving with constant velocity drops a coin while leaning over the rail. Describe the path of the coin as seen by (*a*) the woman on the train, (*b*) a person standing on the ground near the track, and (*c*) a person in a second train moving in the opposite direction to the first train on a parallel track.

36. An elevator is descending at a constant speed. A passenger drops a coin to the floor. What accelerations would (*a*) the passenger and (*b*) a person at rest with respect to the elevator shaft observe for the falling coin?

37. Water is collecting in a bucket during a steady downpour. Will the rate at which the bucket is filling change if a steady horizontal wind starts to blow?

38. A bus with a vertical windshield moves along in a rainstorm at speed v_b. The raindrops fall vertically with a terminal speed v_r. At what angle do the raindrops strike the windshield?

39. Drops are falling vertically in a steady rain. In order to go through the rain from one place to another in such a way as to encounter the least number of raindrops, should you move with the greatest possible speed, the least possible speed, or some intermediate speed? (See "An Optimal Speed for Traversing a Constant Rain," by S. A. Stem, *American Journal of Physics,* September 1983, p. 815.)

40. What is wrong with Fig. 4-27? The boat is sailing with the wind.

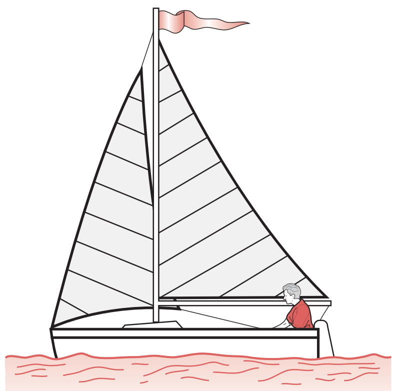


FIGURE 4-27. Question 40.

41. The Galilean velocity transformation, Eq. 4-32, is so instinctively familiar from everyday experience that it is sometimes claimed to be "obviously correct, requiring no proof." Many so-called refutations of relativity theory turn out to be based on this claim. How would you refute someone who made this claim?

EXERCISES

4-1 Motion in Three-Dimensions with Constant Acceleration

1. In a cathode-ray tube, a beam of electrons is projected horizontally with a speed of 9.6×10^8 cm/s into the region between a pair of horizontal plates 2.3 cm long. An electric field between the plates causes a constant downward acceleration of the electrons of magnitude 9.4×10^{16} cm/s^2. Find (*a*) the time required for the electrons to pass through the plates, (*b*) the vertical displacement of the beam in passing through the plates, and (*c*) the horizontal and vertical components of the velocity of the beam as it emerges from the plates.

2. An iceboat sails across the surface of a frozen lake with constant acceleration produced by the wind. At a certain instant its velocity is $6.30\hat{\mathbf{i}} - 8.42\hat{\mathbf{j}}$ in m/s. Three seconds later the boat is instantaneously at rest. What is its acceleration during this interval?

3. A particle moves so that its position as a function of time is

$$\vec{\mathbf{r}}(t) = A\hat{\mathbf{i}} + Bt^2\hat{\mathbf{j}} + Ct\hat{\mathbf{k}},$$

where $A = 1.0$ m, $B = 4.0$ m/s^2, and $C = 1.0$ m/s. Write expressions for (*a*) its velocity and (*b*) its acceleration as functions of time. (*c*) What is the shape of the particle's trajectory?

4. A particle leaves the origin at $t = 0$ with an initial velocity $\vec{\mathbf{v}}_0 = (3.6\text{ m/s})\hat{\mathbf{i}}$. It experiences a constant acceleration $\vec{\mathbf{a}} = -(1.2\text{ m/s}^2)\hat{\mathbf{i}} - (1.4\text{ m/s}^2)\hat{\mathbf{j}}$. (*a*) At what time does the particle reach its maximum x coordinate? (*b*) What is the velocity of the particle at this time? (*c*) Where is the particle at this time?

4.2 Newton's Laws in Three-Dimensional Form

5. A body with mass m is acted on by two forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$, as shown in Fig. 4-28. If $m = 5.2$ kg, $F_1 = 3.7$ N, and $F_2 = 4.3$ N, find the vector acceleration of the body.

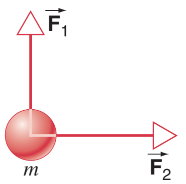


FIGURE 4-28. Exercise 5.

6. An 8.5-kg object passes through the origin with a velocity of 42 m/s parallel to the x axis. It experiences a constant 19-N force in the direction of the positive y axis. Calculate (*a*) the velocity and (*b*) the position of the particle after 15 s have elapsed.

7. A 5.1-kg block is pulled along a frictionless floor by a cord that exerts a force $P = 12$ N at an angle $\theta = 25°$ above the horizontal, as shown in Fig. 4-29. (*a*) What is the acceleration of the block? (*b*) The force P is slowly increased. What is the value of P just before the block is lifted off the floor? (*c*) What is the acceleration of the block just before it is lifted off the floor?

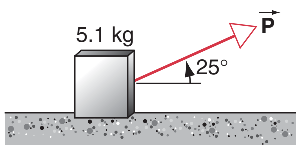


FIGURE 4-29. Exercise 7.

8. A worker drags a crate across a factory floor by pulling on a rope tied to the crate. The rope, which is inclined at 38.0° above the horizontal, exerts a force of 450 N on the crate. The floor exerts a horizontal resistive force of 125 N, as shown in Fig. 4-30. Calculate the acceleration of the crate (*a*) if its mass is 96.0 kg, and (*b*) if its weight is 96.0 N.

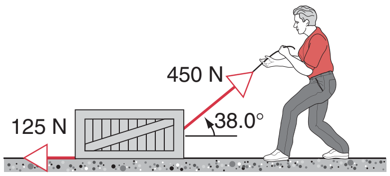


FIGURE 4-30. Exercise 8.

9. A 1200-kg car is being towed up an 18° incline by means of a rope attached to the rear of a truck. The rope makes an angle of 27° with the incline. What is the greatest distance that the car can be towed in the first 7.5 s starting from rest if the rope has a breaking strength of 4.6 kN? Ignore all resistive forces on the car. See Fig. 4-31.

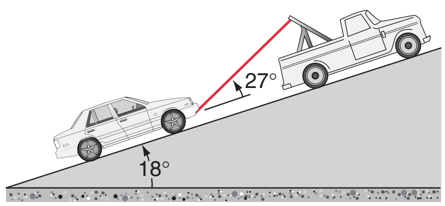


FIGURE 4-31. Exercise 9.

10. A 110-kg crate is pushed at constant speed up the frictionless 34° ramp shown in Fig. 4-32. What horizontal force F is required? (Hint: Resolve forces into components parallel to the ramp.)

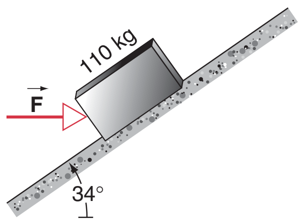


FIGURE 4-32. Exercise 10.

11. In earlier days, horses pulled barges down canals in the manner shown in Fig. 4-33. Suppose that the horse pulls a rope that exerts a horizontal force of 7900 N at an angle of 18° to the direction of motion of the barge, which is headed straight along the canal. The mass of the barge is 9500 kg and its acceleration is 0.12 m/s^2. Calculate the horizontal force exerted by the water on the barge.

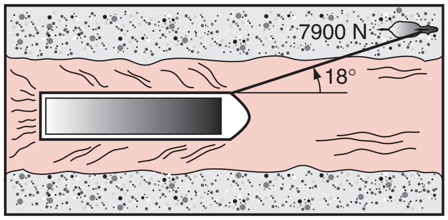


FIGURE 4-33. Exercise 11.

12. A jet fighter takes off at an angle of 27.0° with the horizontal, accelerating at 2.62 m/s^2. The weight of the plane is 79,300 N. Find (*a*) the thrust T of the engine on the plane and (*b*) the lift force L exerted by the air perpendicular to the wings; see Fig. 4-34. Ignore air resistance.

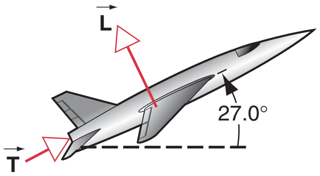


FIGURE 4-34. Exercise 12.

4-3 Projectile Motion

13. A ball rolls off the edge of a horizontal tabletop, 4.23 ft high. It strikes the floor at a point 5.11 ft horizontally away from the edge of the table. (*a*) For how long was the ball in the air? (*b*) What was its speed at the instant it left the table?

14. Electrons, like all forms of matter, fall under the influence of gravity. If an electron is projected horizontally with a speed of 3.0×10^7 m/s (one-tenth the speed of light), how far will it fall in traversing 1.0 m of horizontal distance?

15. A dart is thrown horizontally toward the bull's eye, point P on the dart board, with an initial speed of 10 m/s. It hits at point Q on the rim, vertically below P, 0.19 s later; see Fig. 4-35. (*a*) What is the distance PQ? (*b*) How far away from the dart board did the player stand?

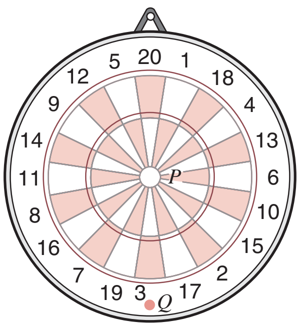


FIGURE 4-35. Exercise 15.

16. You throw a ball from a cliff with an initial velocity of 15 m/s at an angle of 20° below the horizontal. Find (*a*) its horizontal displacement and (*b*) its vertical displacement 2.3 s later.

17. Show that the maximum height reached by a projectile is

$$y_{\max} = (v_0 \sin \phi_0)^2/2g.$$

18. A ball rolls off the top of a stairway with a horizontal velocity of magnitude 5.0 ft/s. The steps are 8.0 in. high and 8.0 in. wide. Which step will the ball hit first?

19. A ball is thrown from the ground into the air. At a height of 9.1 m, the velocity is observed to be $\vec{\mathbf{v}} = (7.6\ \text{m/s})\hat{\mathbf{i}} + (6.1\ \text{m/s})\hat{\mathbf{j}}$ (x axis horizontal, y axis vertical and up). (*a*) To what maximum height will the ball rise? (*b*) What will be the total horizontal distance traveled by the ball? (*c*) What is the velocity of the ball (magnitude and direction) the instant before it hits the ground?

20. If the pitcher's mound is 1.25 ft above the baseball field, can a pitcher release a fast ball horizontally at 92.0 mi/h and still get it into the strike zone over the plate 60.5 ft away? Assume that, for a strike, the ball must fall at least 1.30 ft but no more than 3.60 ft.

21. According to Eq. 4-14, the range of a projectile depends not only on v_0 and ϕ_0 but also on the value g of the gravitational acceleration, which varies from place to place. In 1936, Jesse Owens established a world's running broad jump record of 8.09 m at the Olympic Games in Berlin (g = 9.8128 m/s^2). Assuming the same values of v_0 and ϕ_0, by how much would his record have differed if he had competed instead in 1956 at Melbourne (g = 9.7999 m/s^2)? (In this connection see "The Earth's Gravity," by Weikko A. Heiskanen, *Scientific American,* September 1955, p. 164.)

22. At what initial speed must the basketball player throw the ball, at 55° above the horizontal, to make the foul shot, as shown in Fig. 4-36? The basket rim is 18 in. in diameter. Obtain other data from Fig. 4-36.

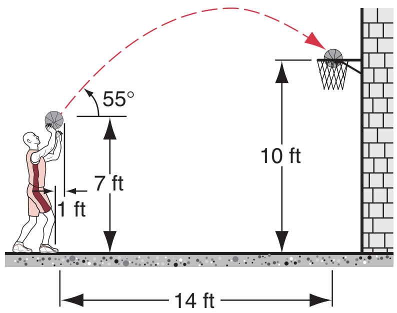


FIGURE 4-36. Exercise 22.

23. A football player punts the football so that it will have a "hang time" (time of flight) of 4.50 s and land 50 yd (= 45.7 m) away. If the ball leaves the player's foot 5.0 ft (= 1.52 m) above the ground, what is its initial velocity (magnitude and direction)?

24. A certain airplane has a speed of 180 mi/h and is diving at an angle of 27° below the horizontal when a radar decoy is released. The horizontal distance between the release point and the point where the decoy strikes the ground is 2300 ft. (*a*) How long was the decoy in the air? (*b*) How high was the plane when the decoy was released? See Fig. 4-37.

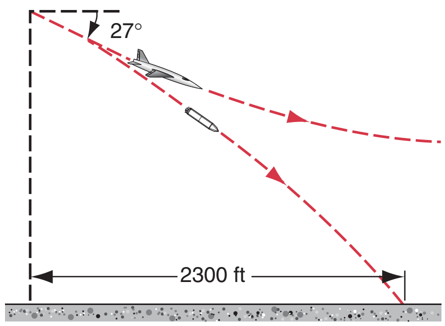


FIGURE 4-37. Exercise 24.

25. (*a*) During a tennis match, a player serves at 23.6 m/s (as recorded by radar gun), with the ball leaving the racquet 2.37 m above the court surface, horizontally. By how much does the ball clear the net, which is 12 m away and 0.90 m high? (*b*) Suppose the player serves the ball as before except that the ball leaves the racquet at 5.0° below the horizontal. Does the ball clear the net now?

26. A batter hits a pitched ball at a height 4.0 ft above the ground so that its angle of projection is 45° and its horizontal range is 350 ft. The ball travels down the left field line where a 24-ft-high fence is located 320 ft from home plate. Will the ball clear the fence? If so, by how much?

27. In a baseball game, a batter hits the ball at a height of 4.60 ft above the ground so that its angle of projection is 52.0° to the horizontal. The ball lands in the grandstand, 39.0 ft up from the bottom; see Fig. 4-38. The grandstand seats slope upward at 28.0° with the bottom seats 358 ft from home plate. Calculate the speed with which the ball left the bat. (Ignore air resistance.)

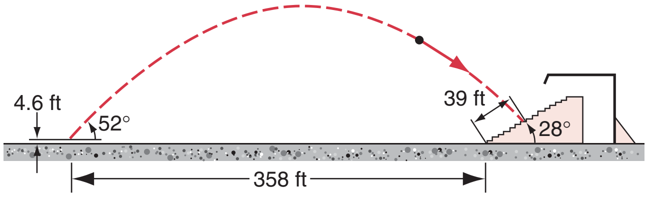


FIGURE 4-38. Exercise 27.

28. What is the maximum vertical height to which a baseball player can throw a ball if he can throw it a maximum distance of 60.0 m? Assume that the ball is released at a height of 1.60 m with the same speed in both cases.

4-4 Drag Forces and the Motion of Projectiles

29. A small 150-g pebble is 3.4 km deep in the ocean and is falling with a constant terminal speed of 25 m/s. What force does the water exert on the falling pebble?

30. An object is dropped from rest. Find the terminal speed assuming that the drag force is given by $D = bv^2$.

31. How long does it take for the object described by Eq. 4-22 to reach one-half of its terminal speed?

32. From Table 4-1, calculate the value of b for the raindrop, assuming that the drag force is given by $D = bv$. The density of water is 1.0 g/cm^3.

33. A locomotive accelerates a 23-car train along a level track. Each car has a mass of 48.6 metric tons and is subject to a drag force $f = 243v$, where v is the speed in m/s and the force f is in N. At the instant when the speed of the train is 34.5 km/h, the acceleration is 0.182 m/s^2. (*a*) Calculate the force exerted by the locomotive on the first car. (*b*) Suppose that the force found in part (*a*) is the greatest force the locomotive can exert on the train. What, then, is the steepest grade up which the locomotive can pull the train at 34.5 km/h? (1 metric ton = 1000 kg.)

4-5 Uniform Circular Motion

34. In Bohr's model of the hydrogen atom, an electron revolves around a proton in a circular orbit of radius 5.29×10^{-11} m with a speed of 2.18×10^6 m/s. (*a*) What is the acceleration of the electron in this model of the hydrogen atom? (*b*) What is the magnitude and direction of the net force that acts on the electron?

35. An astronaut is rotated in a centrifuge of radius 5.2 m. (*a*) What is the speed if the acceleration is $6.8g$? (*b*) How many revolutions per minute are required to produce this acceleration?

36. A carnival Ferris wheel has a 15-m radius and completes five turns about its horizontal axis every minute. (*a*) What is the acceleration, magnitude and direction, of a passenger at the highest point? (*b*) What is the acceleration at the lowest point? (*c*) What force (magnitude and direction) must the Ferris wheel exert on a 75-kg person at the highest point and at the lowest point?

37. Certain neutron stars (extremely dense stars) are believed to be rotating at about 1 rev/s. If such a star has a radius of 20 km (a typical value), (*a*) what is the speed of a point on the equator of the star, and (*b*) what is the centripetal acceleration of this point?

38. (*a*) What is the centripetal acceleration of an object on the Earth's equator due to the rotation of the Earth? (*b*) A 25.0-kg object hangs from a spring scale at the equator. If the free-fall acceleration due only to the Earth's gravity is 9.80 m/s^2, what is the reading on the spring scale?

4-6 Relative Motion

39. A person walks up a stalled 15-m-long escalator in 90 s. When standing on the same escalator, now moving, the person is carried up in 60 s. How much time would it take that person to walk up the moving escalator? Does the answer depend on the length of the escalator?

40. The airport terminal in Geneva, Switzerland has a "moving sidewalk" to speed passengers through a long corridor. Peter, who walks through the corridor but does not use the moving sidewalk, takes 150 s to do so. Paul, who simply stands on the moving sidewalk, covers the same distance in 70 s. Mary not only uses the sidewalk but walks along it. How long does Mary take? Assume that Peter and Mary walk at the same speed.

41. A transcontinental flight at 2700 mi is scheduled to take 50 min longer westward than eastward. The air speed of the jet is 600 mi/h. What assumptions about the jet-stream wind velocity, presumed to be east or west, are made in preparing the schedule?

42. A train travels due south at 28 m/s (relative to the ground) in a rain that is blown to the south by the wind. The path of each

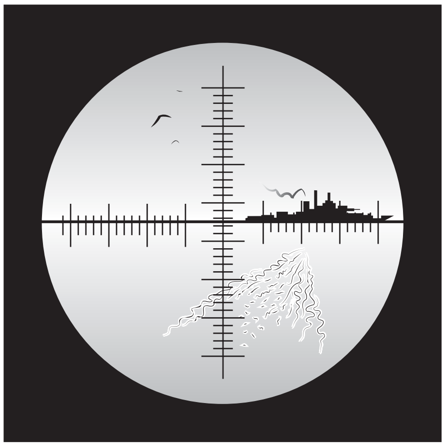


FIGURE 4-39. Exercise 45.

raindrop makes an angle of 64° with the vertical, as measured by an observer stationary on the Earth. An observer on the train, however, sees perfectly vertical tracks of rain on the windowpane. Determine the speed of the drops relative to the Earth.

43. An elevator ascends with an upward acceleration of 4.0-ft/s^2. At the instant its upward speed is 8.0 ft/s, a loose bolt drops from the ceiling of the elevator 9.0 ft from the floor. Calculate (*a*) the time of flight of the bolt from ceiling to floor and (*b*) the distance it has fallen relative to the elevator shaft.

44. A light plane attains an air speed of 480 km/h. The pilot sets out for a destination 810 km to the north but discovers that the plane must be headed 21° east of north to fly there directly. The plane arrives in 1.9 h. What was the vector wind velocity?

45. A battleship steams due east at 24 km/h. A submarine 4.0 km away fires a torpedo that has a speed of 50 km/h; see Fig. 4-39. If the bearing of the ship as seen from the submarine is 20° east of north, (*a*) in what direction should the torpedo be fired to hit the ship, and (*b*) what will be the running time for the torpedo to reach the battleship?

PROBLEMS

1. A particle A moves along the line $y = d$ (30 m) with a constant velocity $\vec{v}$ ($v = 3.0$ m/s) directed parallel to the positive x axis (Fig. 4-40). A second particle B starts at the origin with zero speed and constant acceleration $\vec{a}$ ($a = 0.40$ m/s^2) at the same instant that particle A passes the y axis. What angle θ between $\vec{a}$ and the positive y axis would result in a collision between these two particles?

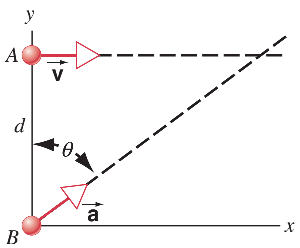


FIGURE 4-40. Problem 1.

2. A ball is dropped from a height of 39.0 m. The wind is blowing horizontally and imparts a constant acceleration of 1.20 m/s^2 to the ball. (*a*) Show that the path of the ball is a straight line and find the values of R and θ in Fig. 4-41. (*b*) How long does it take for the ball to reach the ground? (*c*) With what speed does the ball hit the ground?

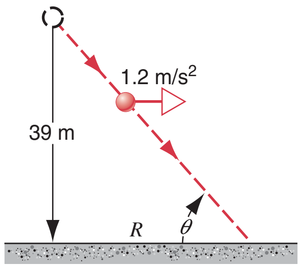


FIGURE 4-41. Problem 2.

3. A rocket with mass 3030 kg is fired from rest from the ground at an elevation angle of 58.0°. The motor exerts a thrust of 61.2 kN at a constant angle of 58.0° with the horizontal for 48.0 s and then cuts out. Ignore the mass of fuel consumed and neglect aerodynamic drag. Calculate (*a*) the altitude of the rocket at motor cut-out and (*b*) the total distance from firing point to impact.

4. A baseball leaves the pitcher's hand horizontally at a speed of 92.0 mi/h. The distance to the batter is 60.0 ft. (*a*) How long does it take for the ball to travel the first 30.0 ft horizontally? The second 30.0 ft? (*b*) How far does the ball fall under gravity during the first 30 ft of its horizontal travel? (*c*) During the second 30.0 ft? (*d*) Why are these quantities not equal? Ignore the effects of air resistance.

5. You throw a ball with a speed of 25.3 m/s at an angle of 42.0° above the horizontal directly toward a wall as shown in Fig. 4-42. The wall is 21.8 m from the release point of the ball. (*a*) How long is the ball in the air before it hits the wall? (*b*) How far above the release point does the ball hit the wall? (*c*) What are the horizontal and vertical components of its velocity as it hits the wall? (*d*) Has it passed the highest point on its trajectory when it hits?

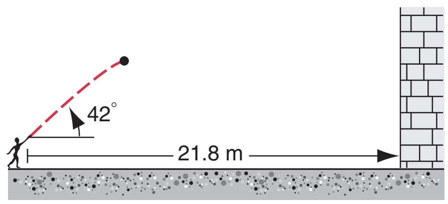


FIGURE 4-42. Problem 5.

6. A projectile is fired from the surface of level ground at an angle ϕ_0 above the horizontal. (*a*) Show that the elevation angle θ of the highest point as seen from the launch point is related to ϕ_0 by $\tan\theta = \frac{1}{2}\tan\phi_0$. See Fig. 4-43. (*b*) Calculate θ for $\phi_0 = 45°$.

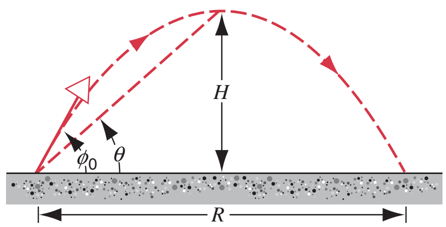


FIGURE 4-43. Problem 6.

7. A stone is projected at an initial speed of 120 ft/s directed 62° above the horizontal, at a cliff of height h, as shown in Fig. 4-44. The stone strikes the ground at A 5.5 s after launching. Find (a) the height h of the cliff, (b) the speed of the stone just before impact at A, and (c) the maximum height H reached above the ground.

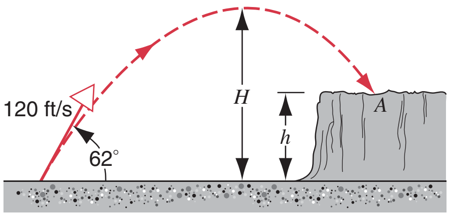


FIGURE 4-44. Problem 7.

8. (a) In Galileo's Two New Sciences, the author states that "for elevations [angles of projection] which exceed or fall short of 45° by equal amounts, the ranges are equal." Prove this statement. See Fig. 4-45. (b) For an initial speed of 30.0 m/s and a range of 20.0 m, find the two possible elevation angles of projection.

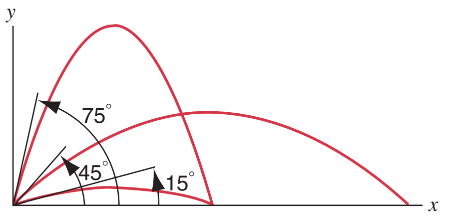


FIGURE 4-45. Problem 8.

9. The kicker on a football team can give the ball an initial speed of 25 m/s. Within what angular range must he kick the ball if he is to just score a field goal from a point 50 m in front of the goalposts whose horizontal bar is 3.44 m above the ground?

10. A radar observer on the ground is "watching" an approaching projectile. At a certain instant she has the following information: the projectile is at maximum altitude and is moving horizontally with speed v; the straight-line distance to the projectile is L; the line of sight to the projectile is at an angle θ above the horizontal. (a) Find the distance D between the observer and the point of impact on the projectile. D is to be expressed in terms of the observed quantities v, L, θ, and the known value of g. Assume a flat Earth; assume also that the observer lies in the plane of the projectile's trajectory. (b) How can you tell whether the projectile will pass over the observer's head or strike the ground before reaching her?

11. Show that for a projectile $d^2(v^2)/dt^2 = 2g^2$.

12. A projectile is launched from the origin at an angle ϕ_0 with the horizontal; the subsequent position is given by $\vec{\mathbf{r}}(t)$. For small enough angles the distance from the origin $r = |\vec{\mathbf{r}}|$ always increases. However, if a projectile is launched nearly straight up it rises to a highest point, and then moves back toward the origin, so that the distance to the origin first increases then decreases. What initial launch angle ϕ_c divides the two types of motion? (See "Projectiles: Are They Coming or Going?," by James S. Walker, *The Physics Teacher,* May 1995, p. 282.)

13. A balloon is descending through still air at a constant speed of 1.88 m/s. The total weight of the balloon, including payload, is 10.8 kN. A constant upward buoyant force of 10.3 kN is exerted on the balloon. The air also exerts a drag force given by $D = bv^2$, where v is the speed of the balloon and b is a constant. The crew drops 26.5 kg of ballast. What will be the eventual constant downward speed of the balloon?

14. Repeat Problem 13, but this time assume that the drag force is given by $D = bv$. Note that the constant b must be reevaluated.

15. A body of mass m falls from rest through the air. A drag force $D = bv^2$ opposes the motion of the body. (a) What is the initial downward acceleration of the body? (b) After some time the speed of the body approaches a constant value. What is this terminal speed v_T? (c) What is the downward acceleration of the body when $v = v_T/2$?

16. A canal barge of mass m is traveling at speed v_i when it shuts off its engines. The drag force D with the water is given by $D = bv$. (a) Find an expression for the time required for the barge to reduce its speed to v_f. (b) Evaluate the time numerically for a 970-kg barge traveling initially at 32 km/h to reduce its speed to 8.3 km/h; the value of b is 68 N · s/m.

17. Consider the falling object in Section 4-4. (a) Find the acceleration as a function of time. What is the acceleration at small t; at large t? (b) Find the distance the object falls, as a function of time.

18. (a) Assuming that the drag force D is given by $D = bv$, show that the distance y_{95} through which an object must fall from rest to reach 95% of its terminal speed is given by

$$y_{95} = (v_T^2/g)(\ln 20 - \tfrac{19}{20}),$$

where v_T is the terminal speed. (Hint: Use the result for $y(t)$ obtained in Problem 17.) (b) Using the terminal speed of 42 m/s for the baseball given in Table 4-1, calculate the 95% distance. Why does the result not agree with the value listed in Table 4-1?

19. The fast train known as the TGV Atlantique (Train Grande Vitesse) that runs south from Paris to Le Mans in France has a top speed of 310 km/h. (a) If the train goes around a curve at this speed and the acceleration experienced by the passengers is to be limited to $0.05g$, what is the smallest radius of curvature for the track that can be tolerated? (b) If there is a curve with a 0.94-km radius, to what speed must the train be slowed?

20. A particle P travels with constant speed on a circle of radius 3.0 m and completes one revolution in 20 s (Fig. 4-46). The particle passes through O at $t = 0$. With respect to the origin O, find (a) the magnitude and direction of the vectors describing its position 5.0, 7.5, and 10 s later, (b) the magnitude and direction of the displacement in the 5.0-s interval from the fifth to the tenth second, (c) the average velocity vector in this interval, (d) the instantaneous velocity vector at the beginning

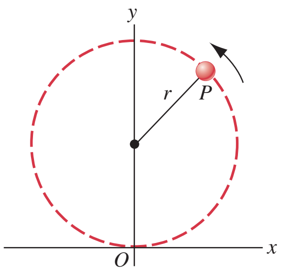


FIGURE 4-46. Problem 20.

and at the end of this interval, and (e) the instantaneous acceleration vector at the beginning and at the end of this interval. Measure angles counterclockwise from the x axis.

21. A child whirls a stone in a horizontal circle 1.9 m above the ground by means of a string 1.4 m long. The string breaks, and the stone flies off horizontally, striking the ground 11 m away. What was the centripetal acceleration of the stone while in circular motion?

22. A woman 1.6 m tall stands upright at latitude 50° for 24 h. (a) During this interval, how much farther does the top of her head move than the soles of her feet? (b) How much greater is the acceleration of the top of her head than the acceleration of the soles of her feet? Consider only effects associated with the rotation of the Earth.

23. A particle moves in a plane according to

$$x = R \sin \omega t + \omega R t$$

$$y = R \cos \omega t + R,$$

where ω and R are constants. This curve, called a cycloid, is the path traced out by a point on the rim of a wheel that rolls without slipping along the x axis. (a) Sketch the path. (b) Calculate the instantaneous velocity and acceleration when the particle is at its maximum and minimum value of y.

24. Snow is falling vertically at a constant speed of 7.8 m/s. (a) At what angle from the vertical and (b) with what speed do the snowflakes appear to be falling as viewed by the driver of a car traveling on a straight road with a speed of 55 km/h?

25. One of the early attempts to measure the speed of light was to measure the position of a star located at right angles to the path of the Earth in its orbit (Fig. 4-47). (a) If the measured angle θ is found to be between 89°59′39.3″ and 89°59′39.4″, then what would be the range of values for the speed of light? (b) Describe a reasonable method for measuring this angle to the above accuracy. *The answer to this question might not be as straightforward as you think!*

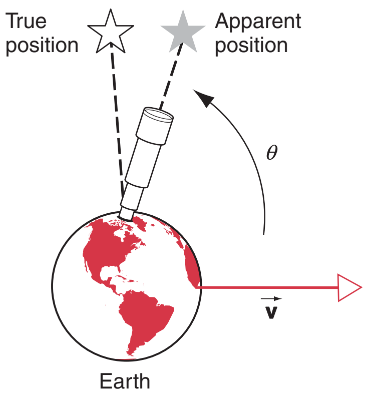


FIGURE 4-47. Problem 25.

26. A pilot is supposed to fly due east from A to B and then back again to A due west. The velocity of the plane in air is $\vec{v}$ and the velocity of the air with respect to the ground is $\vec{u}$. The distance between A and B is l and the plane's air speed is constant. (a) If $u = 0$ (still air), show that the time for the round trip is $t_0 = 2l/v$. (b) Suppose that the air velocity is due east (or west). Show that the time for a round trip is then

$$t_E = \frac{t_0}{1 - u^2/v^2}.$$

(c) Suppose that the air velocity is due north (or south). Show that the time for a round trip is then

$$t_N = \frac{t_0}{\sqrt{1 - u^2/v^2}}.$$

(d) In parts (b) and (c) one must assume that $u < v$. Why?

27. Two highways intersect, as shown in Fig. 4-48. At the instant shown, a police car P is 41 m from the intersection and moving at 76 km/h. Motorist M is 57 m from the intersection and moving at 62 km/h. At this moment, what is the velocity (magnitude and angle with the line of sight) of the motorist with respect to the police car?

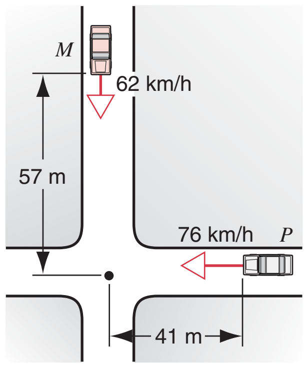


FIGURE 4-48. Problem 27.

28. The New Hampshire State Police use aircraft to enforce highway speed limits. Suppose that one of the airplanes has a speed of 135 mi/h in still air. It is flying straight north so that it is at all times directly above a north-south highway. A ground observer tells the pilot by radio that a 70 mi/h wind is blowing but neglects to give the wind direction. The pilot observes that in spite of the wind the plane can travel 135 mi along the highway in 1 h. In other words, the ground speed is the same as if there were no wind. (a) What is the direction of the wind? (b) What is the heading of the plane—that is, the angle between its axis and the highway?

COMPUTER PROBLEMS

1. The force on a 5.0-kg particle is given by $F_x = -(20.0\text{ N/m})x$ and $F_y = -(20.0\text{ N/m})y$. Plot the motion of the particle on an xy graph. Use an initial position of $x_0 = 2.0$ m, $y_0 = 0$ and an initial velocity of $v_{0x} = 0$, $v_{0y} = 4.0$ m/s. Try different step sizes for Δt until you find one for which the trajectory takes the object back to within 1.0 cm of the initial position. What is the shape of the motion? How long did it take to return to the starting point? What happens to the trajectory if you use $v_{0y} = 3.0$ m/s instead?

2. The acceleration of a particle is given by $a_x = -(10.0\ \text{m}^2/\text{s}^2)x/|x|^3$ and $a_y = -(10.0\ \text{m}^2/\text{s}^2)y/|y|^3$. The initial position of the particle is at $\vec{\mathbf{r}}_0 = 5\hat{\mathbf{i}}$ m and the initial velocity is in the y direction only. (*a*) Using a step size of $\Delta t = 0.1$ s, choose an initial value for v_y so that the numerical solution of the trajectory is a circle. Compare your result to the theoretical value. (*b*) Repeat, except now look for an initial value of v_y that results in a trajectory which is an ellipse that is twice as long as it is wide. There are actually two answers to part (*b*); find both.

3. A 150-g ball is thrown straight upward from the edge of a cliff with an initial speed of 25 m/s. On the way down it misses the cliff edge and continues to fall to the ground 300 m below. In addition to the force of gravity it is subjected to a force of air resistance given by $D = bv$ with $b = 0.0150$ kg/s. (*a*) How long is the ball in flight? (*b*) What is its speed just before it hits the ground? (*c*) What is the ratio of this speed to its terminal speed? (Try using the Euler method with a time interval of $\Delta t = 0.001$ s.)

4. The velocity of a projectile subject to air resistance approaches a terminal velocity. Suppose the net force is $m\vec{\mathbf{g}} - b\vec{\mathbf{v}}$, where b is the drag coefficient and the y axis is chosen to be positive in the upward direction. At terminal velocity v_T the net force vanishes, so $v_\text{T} = -(mg/b)$. Notice it has no horizontal component. The projectile eventually falls straight down.

Use a computer program or spreadsheet to "watch" a projectile approach terminal velocity. Consider a 2.5-kg projectile launched with an initial speed of 150 m/s, at an angle of 40° above the horizontal. Take the drag coefficient to be $b = 0.50$ kg/s. Numerically integrate Newton's second law and display results for every 0.5 s from $t = 0$ (the time of launch) to the time the y component of the velocity is 90% of v_T. Plot $v_x(t)$ and $v_y(t)$ on the same graph. Notice that v_x approaches 0 as v_y approaches v_T.

CHAPTER 5

APPLICATIONS OF NEWTON'S LAWS

In Chapter 3 we introduced Newton's laws and gave some examples of their applications. Those examples were deliberately oversimplified, so that the use of the laws could be illustrated. In the process of oversimplification, we lost some of the physical insight.

In this chapter we continue with further applications of Newton's laws, particularly to friction and other contact forces, to circular motion, and to nonconstant forces. Finally, we show how using a noninertial reference frame produces effects that can be analyzed by introducing inertial forces or pseudoforces that, in contrast to real forces, are not caused by specific objects in the environment.

5-1 FORCE LAWS

Before we return to applications of Newton's laws, we should briefly discuss the nature of the forces themselves. We have used the equations of motion to analyze and calculate the *effects* of forces, but they tell us nothing about the *causes* of the forces. To understand what causes a force we must have a detailed microscopic understanding of the interactions of objects with their environment. On the most fundamental level, nature appears to operate through a small number of fundamental forces. Physicists have traditionally identified four basic forces: (1) *the gravitational force,* which originates with the presence of matter; (2) *the electromagnetic force,* which includes basic electric and magnetic interactions and is responsible for the binding of atoms and the structure of solids; (3) *the weak nuclear force,* which causes certain radioactive decay processes and certain reactions among the fundamental particles; and (4) *the strong force,* which operates among the fundamental particles and is responsible for binding the nucleus together.

On the most microscopic scale—for example, two protons in a typical nucleus—the relative strengths of these forces would be: strong (relative strength = 1); electromagnetic (10^{-2}); weak (10^{-9}); gravitational (10^{-38}). On the fundamental scale, gravity is exceedingly weak and has negligible effects. You can get some appreciation for the weakness of gravity from some common experiments—for example, lifting a few bits of paper with an electrostatically charged comb or lifting a few nails or paper clips with a magnet. The magnetic force of a small magnet is sufficient to overcome the gravitational force exerted by the entire Earth on these objects!

The search for ever more simplification has led physicists to try to reduce the number of forces even below four. In 1967, a theory was proposed according to which the weak and electromagnetic forces could be regarded as parts of a single force, called the *electroweak* force. The combination or *unification* of these two forces is similar to the 19th-century unification of the separate electric and magnetic forces into a single electromagnetic force. Other new theories, called *grand unification theories,* have been proposed that combine the strong and electroweak forces into a single framework, and there are even "theories of everything," which attempt to include gravity as well.

Fortunately, our analysis of mechanical systems need not invoke such theories. In fact, everything we study about ordinary mechanical systems involves only two forces: gravity and electromagnetism. The gravitational force is apparent in the Earth's attraction for objects, which gives them their weight. The much weaker gravitational attraction of one laboratory object for another is almost always negligible.

All the other forces we normally consider are ultimately electromagnetic in origin: contact forces, such as the normal force exerted when one object pushes on another and the frictional force produced when one surface rubs against another; viscous forces, such as air resistance; tensile forces, such as in a stretched rope or string; elastic forces, as in a spring; and many others. Microscopically, these forces originate with the forces exerted by one atom on another. Fortunately, when we deal with ordinary mechanical systems we can ignore the microscopic basis and replace the complicated substructure with a single effective force of a specified magnitude and direction.

5-2 TENSION AND NORMAL FORCES*

Figure 5-1*a* shows a worker pulling with a force $\vec{\mathbf{P}}$ on a string that is attached to a crate, accelerating it over the surface that we assume to be frictionless. The force on the crate is exerted not directly by the worker, but rather by the string. We call this force a *tension* $\vec{\mathbf{T}}$.

Partial free-body diagrams (including only horizontal forces) for the string and crate are shown in Fig. 5-1*b*. The string pulls on the crate with tension $\vec{\mathbf{T}}$, and therefore by Newton's third law the crate must pull on the string with a force equal in magnitude to $\vec{\mathbf{T}}$ but in the opposite direction. We assume that the string is very thin, so the tension force is always along the direction of the string. Furthermore, we assume that the string is of negligible mass.

Choosing the x axis to be horizontal with its positive direction to the right in Fig. 5-1, we find the net force on the string in the x direction to be $\sum F_x = P - T$. (Here P and T represent respectively the magnitudes of the forces $\vec{\mathbf{P}}$ and $\vec{\mathbf{T}}$.) Newton's second law in the form $\sum F_x = ma_x$ then gives $P - T = m_{\text{string}} a_x = 0$, because we have assumed that the mass of the string is zero. From this we conclude that $P = T$.

The net force on the crate in the x direction is $\sum F_x = T$, and Newton's second law gives $T = m_{\text{crate}} a_x$. Thus $a_x = T/m_{\text{crate}} = P/m_{\text{crate}}$. The thin, massless string simply transmits the applied force from one end to the other with no change in direction or magnitude—that is, the force $\vec{\mathbf{P}}$ that the worker applies to the string is equal to the force that the string applies to the crate.

The ideal string likewise does not stretch. Suppose we add another crate to our system, creating the configuration shown in Fig. 5-2*a*. As before, the magnitude of the tension $\vec{\mathbf{T}}_1$ in the first string is equal to P. Again choosing the x axis to be horizontal with positive to the right, we can find the x component of the net force on crate 1 to be $\sum F_x = T_1 - T_2 = P - T_2$ and, similarly for crate 2, $\sum F_x = T_2$. Applying Newton's second law gives:

*To simplify the notation in this chapter, we will no longer label each force with subscripts indicating the body that the force acts on and the body that causes the force. However, as you study the examples in this chapter and solve the problems, you should continue to identify these two bodies for each force that acts.

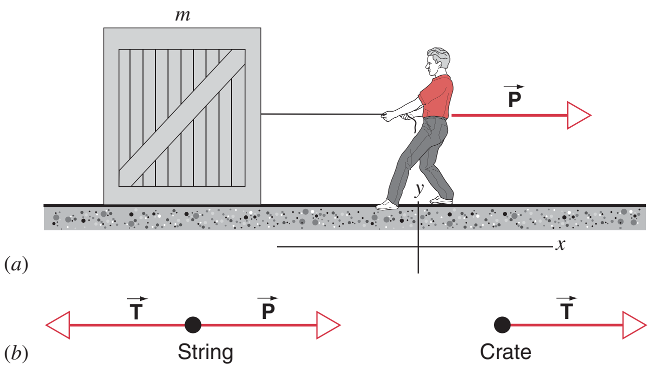


FIGURE 5-1. (*a*) A worker pulls with force P on a string attached to a crate. (*b*) Partial free-body diagrams of the string and the crate, showing only the horizontal forces.

$$\text{crate 1:} \qquad P - T_2 = m_1 a_{1x} \tag{5-1}$$

$$\text{crate 2:} \qquad T_2 = m_2 a_{2x} \tag{5-2}$$

If the second string (which connects m_1 and m_2) does not stretch, then m_1 and m_2 move with the same velocity and acceleration. Putting $a_{1x} = a_{2x} = a_x$, we can combine Eqs. 5-1 and 5-2 to find

$$a_x = \frac{P}{m_1 + m_2} \qquad \text{and} \qquad T_2 = \frac{m_2}{m_1 + m_2} P. \tag{5-3}$$

That is, the two crates accelerate just like a single system of mass $m_1 + m_2$ to which the force $\vec{\mathbf{P}}$ is applied. Considering only the effort exerted by the worker, we could replace the two crates by a single crate of mass $m_1 + m_2$.

The tension force arises because each small element of the string pulls on the element next to it (and is in turn pulled by that element, according to Newton's third law). In this way, a force pulling on one end of the string is transmitted to an object at the other end. This force is due to the force between atoms and is ultimately electromagnetic in origin.

As shown in Fig. 5-3*a*, any particular element i of the string experiences a tension $\vec{\mathbf{T}}$ acting in one direction due to element $i - 1$, and an equal tension acting in the opposite direction due to element $i + 1$. If we were to cut the string at any point and tie a spring scale between the cut ends, the

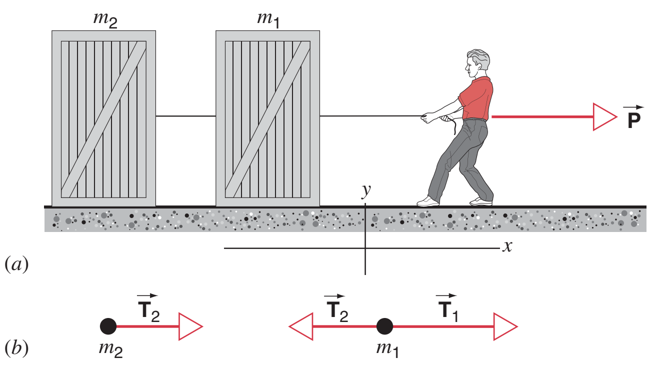


FIGURE 5-2. (*a*) A worker pulls on a string attached to a row of two crates. (*b*) Partial free-body diagrams of the two crates, showing only the horizontal forces.

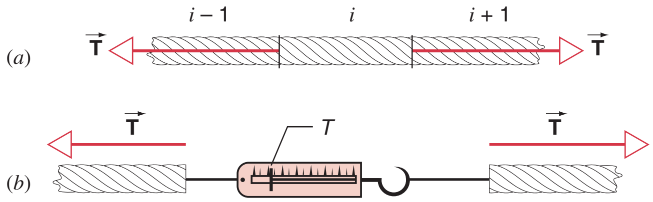


Figure 5-3. (*a*) Three small elements of a stretched string, labeled $i-1$, i, and $i+1$. The forces acting on element i are shown. (*b*) If the string is cut so that element i is replaced by a spring scale (the rest of the string being undisturbed), the scale reads the tension T.

spring scale would read the magnitude of the tension T directly (Fig. 5-3*b*).

Note that the spring scale does not read $2T$, even though a tension $\vec{\mathbf{T}}$ pulls in each direction on the scale. In the same way, when we hang an object of weight W from a spring scale, the scale reads W and not $2W$, even though there is a downward force W on the spring scale from the weight of the object and also an upward force equal to W on the top end of the scale due to whatever is supporting the scale.

Sample Problem 5-1. Figure 5-4*a* shows a block of mass $m = 15.0$ kg hanging from three strings. What are the tensions in the three strings?

Solution We first consider the knot at the junction of the three strings to be the "body." Figure 5-4*b* shows the free-body diagram of the knot, which is at rest under the action of the three forces $\vec{\mathbf{T}}_A$, $\vec{\mathbf{T}}_B$, and $\vec{\mathbf{T}}_C$, which are due to the tensions in the strings. Choosing the x and y axes as shown, we can resolve the forces into their x and y components, as shown in Fig. 5-4*c*. The acceleration components are zero, so Newton's second law applied to the knot gives

$$x \text{ component: } \sum F_x = -T_A \cos 30° + T_B \cos 45° = ma_x = 0$$

$$y \text{ component: } \sum F_y = T_A \sin 30° + T_B \sin 45° - T_C = ma_y = 0.$$

Figure 5-4*d* shows the free-body diagram of the block. The forces have only y components, and once again the acceleration is zero, so

$$\sum F_y = T_C - mg = ma_y = 0.$$

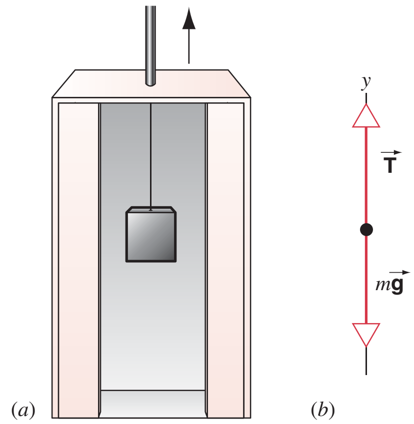


Figure 5-5. Sample Problem 5-2. (*a*) A package hangs from a string in an ascending elevator. (*b*) The free-body diagram of the package.

Solving for T_C, we find

$$T_C = mg = (15.0 \text{ kg})(9.80 \text{ m/s}^2) = 147 \text{ N}.$$

Substituting this result, we can solve the two equations for the forces on the knot simultaneously to find

$$T_A = 108 \text{ N} \quad \text{and} \quad T_B = 132 \text{ N}.$$

Check these results to see whether the vector sum of the three forces acting on the knot is indeed zero.

Sample Problem 5-2. A package (mass 2.4 kg) tied to a string hangs from the ceiling of an elevator (Fig. 5-5*a*). What is the tension in the string when the elevator is (*a*) descending with constant velocity, and (*b*) ascending with an acceleration of 3.2 m/s²?

Solution (*a*) The free-body diagram of the package is shown in Fig. 5-5*b*. Two forces act on the package: the upward force due to the tension in the string and the downward force of the Earth's gravity. We choose the y axis to be vertical and positive upward. The net force on the package is $\Sigma F_y = T - mg$. Newton's second law ($\Sigma F_y = ma_y$) then gives $T - mg = ma_y$ or, solving for the tension T,

$$T = m(g + a_y).$$

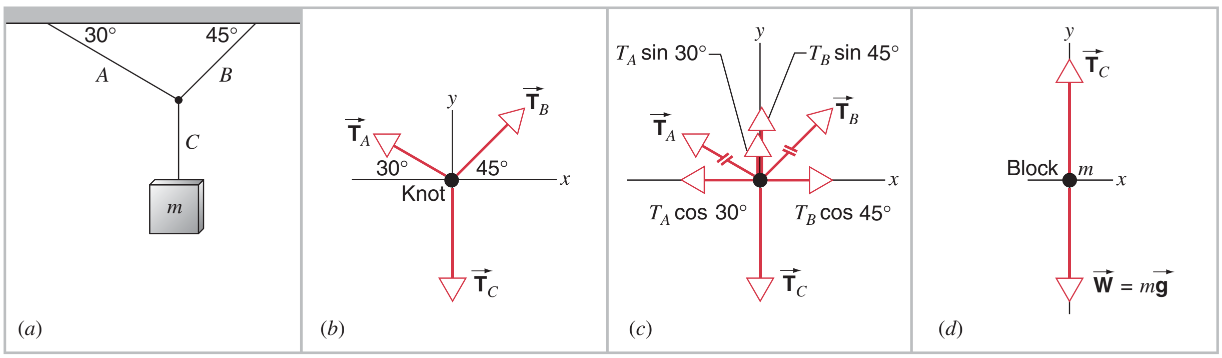


Figure 5-4. Sample Problem 5-1. (*a*) A block hangs from three strings A, B, and C. (*b*) The free-body diagram of the knot that joins the strings. (*c*) The free-body diagram of the knot, with $\vec{\mathbf{T}}_A$ and $\vec{\mathbf{T}}_B$ resolved into their x and y vector components. The double lines on a vector remind us that we have replaced the vector by its vector components. (*d*) The free-body diagram of the block.

When the elevator moves with constant velocity, $a_y = 0$ and so

$$T = mg = (2.4 \text{ kg})(9.8 \text{ m/s}^2) = 24 \text{ N}.$$

(*b*) When the elevator moves with $a_y = +3.2 \text{ m/s}^2$, the tension is

$$T = m(g + a_y) = (2.4 \text{ kg})(9.8 \text{ m/s}^2 + 3.2 \text{ m/s}^2) = 31 \text{ N}.$$

In this case the elevator is moving upward and increasing its speed. Would you expect the tension to be greater than it is when the elevator is at rest or moving with constant velocity? Suppose the elevator were moving downward and braking, so that its acceleration is upward and again equal to $+3.2 \text{ m/s}^2$. Would the tension in the string have the same value? Is this reasonable? What would be the tension if the elevator were in free fall?

Compare this problem with Sample Problem 3-7 and account for any similarities or differences.

The Normal Force

Consider a book resting on a table, as shown in Fig. 5-6*a*. Gravity exerts a downward force on the book, but the book has no vertical acceleration. The net vertical force on the book must therefore be zero, so there must be an additional upward force acting on it. This force is the *normal* force exerted on the book by the table. In this sense, the word "normal" means "perpendicular"—the normal force exerted by a surface is *always* perpendicular to (or normal to) the surface.

Even though the normal force shown in the free-body diagram of Fig. 5-6*b* is equal and opposite to the weight, it is *not* the reaction force to the weight. The weight is the force of the Earth on the book, and its reaction force is the force exerted by the book on the Earth. The reaction force to the normal force is the downward force exerted by the book on the table; it would appear in a free-body diagram of the table. Remember that the action–reaction pairs of Newton's third law *never* act on the same body, so the forces N and W that act on the book cannot be an action–reaction pair.

If someone placed a hand on top of the book and pushed downward with a force P, the book would remain at rest. For an acceleration of zero, the net force on the book must be zero and so the total downward force $W + P$ must equal the total upward force N. The normal force must therefore increase as P increases, since $N = W + P$. Eventually, P could become large enough to exceed the ability of the table to provide the upward normal force, and the book would break through the tabletop.

Tension and normal forces are examples of *contact* forces, in which one body exerts a force on another because of the contact between them. These forces originate with the atoms of each body—each atom exerts a force on its neighbor (which may be an atom of another body). A contact force can be maintained only if it does not exceed the interatomic forces within either of the bodies; otherwise the binding between atoms can be overcome and the string or the surface will split into pieces.

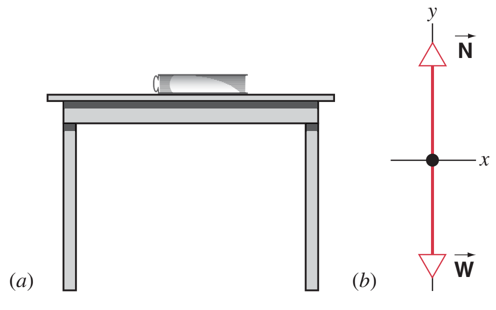


FIGURE 5-6. (*a*) A book resting on a table. (*b*) The free-body diagram of the book.

SAMPLE PROBLEM 5-3. A sled of mass $m = 7.5$ kg is pulled along a frictionless horizontal surface by a cord (Fig. 5-7*a*). A constant force of $P = 21.0$ N is applied to the cord. Analyze the motion if (*a*) the cord is horizontal and (*b*) the cord makes an angle of $\theta = 15°$ with the horizontal.

Solution (*a*) The free-body diagram with the cord horizontal is shown in Fig. 5-7*b*. The surface exerts a force N, the normal force, on the sled. The components of the net force acting on the sled are $\Sigma F_x = P$ and $\Sigma F_y = N - mg$, and applying Newton's second law we obtain

x component ($\Sigma F_x = ma_x$): $\quad P = ma_x$

y component ($\Sigma F_y = ma_y$): $\quad N - mg = ma_y$

If there is to be no vertical motion, the sled remains on the surface and $a_y = 0$. Thus

$$N = mg = (7.5 \text{ kg})(9.80 \text{ m/s}^2) = 74 \text{ N}.$$

The horizontal acceleration is

$$a_x = \frac{P}{m} = \frac{21.0 \text{ N}}{7.5 \text{ kg}} = 2.80 \text{ m/s}^2.$$

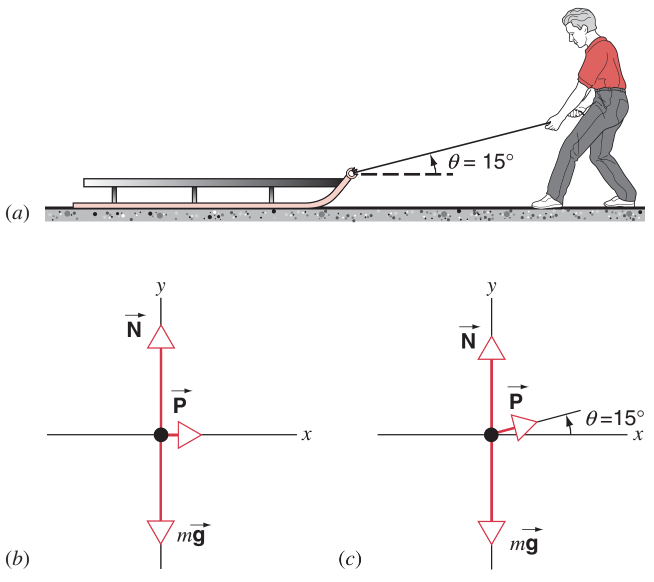


FIGURE 5-7. Sample Problem 5-3. (*a*) A sled is pulled along a frictionless horizontal surface. (*b*) The free-body diagram of the sled when $\theta = 0°$. (*c*) The free-body diagram of the sled when $\theta = 15°$.

Note that, if the surface is truly frictionless, as we have assumed, the person would find it difficult to continue to exert this force on the sled for very long. After 30 s at this acceleration, the sled would be moving at 84 m/s or 188 mi/h!

(*b*) When the pulling force is not horizontal, the free-body diagram is shown in Fig. 5-7*c*, and the components of the net force are $\Sigma F_x = P\cos\theta$ and $\Sigma F_y = N + P\sin\theta - mg$. Newton's second law then gives

$$x \text{ component } (\Sigma F_x = ma_x): \qquad P\cos\theta = ma_x$$

$$y \text{ component } (\Sigma F_y = ma_y): \qquad N + P\sin\theta - mg = ma_y$$

Let us assume for the moment that the sled stays on the surface; that is, $a_y = 0$. Then

$$N = mg - P\sin\theta = 74\text{ N} - (21.0\text{ N})(\sin 15°) = 69\text{ N},$$

$$a_x = \frac{P\cos\theta}{m} = \frac{(21.0\text{ N})(\cos 15°)}{7.5\text{ kg}} = 2.70\text{ m/s}^2.$$

A normal force is always perpendicular to the surface in contact; with the coordinates chosen as in Fig. 5-7*b*, N must be positive. If we increase $P\sin\theta$, N would decrease and at some point would become zero. At that point the sled would leave the surface, under the influence of the upward component of P, and we would need to analyze its vertical motion. With the values of P and θ we have used, the sled remains on the surface and $a_y = 0$.

Note that a_x is smaller in part (*b*) than in part (*a*). Can you explain this?

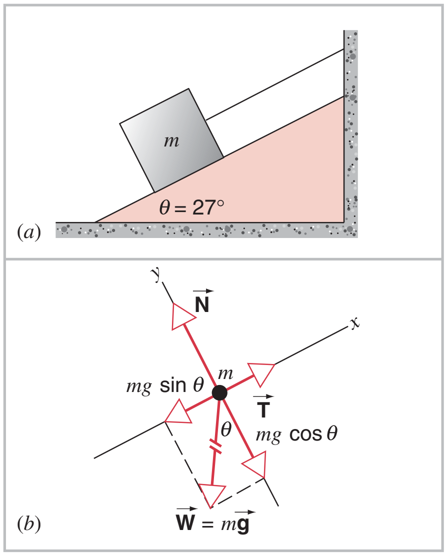


Figure 5-8. Sample Problem 5-4. (*a*) A mass m is supported at rest by a string on a frictionless inclined plane. (*b*) The free-body diagram of m. Note that the xy coordinate system is tilted so that the x axis is parallel to the plane. The weight $m\vec{g}$ has been resolved into its vector components; the double line through the vector $m\vec{g}$ reminds us that this vector has been replaced by its components.

Sample Problem 5-4. A block of mass $m = 18.0$ kg is held in place by a string on a frictionless plane inclined at an angle of 27° (see Fig. 5-8*a*). (*a*) Find the tension in the string and the normal force exerted on the block by the plane. (*b*) Analyze the subsequent motion after the string is cut.

Solution (*a*) The free-body diagram of the block is shown in Fig. 5-8*b*. The block is acted on by the normal force $\vec{N}$, its weight $\vec{W} = m\vec{g}$ and a force due to the tension $\vec{T}$ of the string. We choose a coordinate system with the x axis along the plane and the y axis perpendicular to it. With this choice, two of the forces ($\vec{T}$ and $\vec{N}$) are already resolved into components, and the motion that will eventually occur along the plane has only one component as well. The weight is resolved into its x component $-mg\sin\theta$ and its y component $-mg\cos\theta$. The net force in the x direction is then $\Sigma F_x = T - mg\sin\theta$ and the net force in the y direction is $\Sigma F_y = N - mg\cos\theta$.

In the static case, $a_x = 0$ and $a_y = 0$. Newton's second law then gives $\Sigma F_x = ma_x = 0$ and $\Sigma F_y = ma_y = 0$, so

$$T - mg\sin\theta = 0 \qquad \text{and} \qquad N - mg\cos\theta = 0.$$

Examine these equations. Are they reasonable? What happens in the limit $\theta = 0°$? It looks like the tension would be zero. Would you expect the tension to be zero if the block were resting on a horizontal surface? What happens to the normal force when $\theta = 0°$? Is this reasonable? What happens to T and N in the limit of $\theta = 90°$? You should form the habit of asking questions like these before starting on the algebra to find the solution. If there is an error, now is the best time to find and correct it.

Solving the equations,

$$T = mg\sin\theta = (18.0\text{ kg})(9.80\text{ m/s}^2)(\sin 27°) = 80\text{ N},$$

$$N = mg\cos\theta = (18.0\text{ kg})(9.80\text{ m/s}^2)(\cos 27°) = 157\text{ N}.$$

(*b*) When the string is cut, the tension disappears from the equations and the block is no longer in equilibrium. The components of the net force are now $\Sigma F_x = -mg\sin\theta$ and $\Sigma F_y = N - mg\cos\theta$. Newton's second law for the x and y components now gives

$$-mg\sin\theta = ma_x \qquad \text{and} \qquad N - mg\cos\theta = ma_y.$$

Cutting the string doesn't change the motion in the y direction (the block doesn't jump off the plane!), so $a_y = 0$ as before and the normal force still equals $mg\cos\theta$, or 157 N. In the x direction

$$a_x = -g\sin\theta = -(9.80\text{ m/s}^2)(\sin 27°) = -4.45\text{ m/s}^2.$$

The negative sign shows that the block accelerates in the negative x direction—that is, down the plane. Check the limits $\theta = 0°$ and $\theta = 90°$. Are they consistent with your expectations?

Additional Applications

Here we consider some additional applications of Newton's laws. These examples involve two objects that must be analyzed separately but not quite independently, because the motion of one object is constrained by the motion of another, such as when they are attached to one another by a string of fixed length. Study these examples, and note the independent choices of coordinate systems used for the separate objects.

Sample Problem 5-5. Two blocks with unequal masses m_1 and m_2 are connected by a string that passes over an ideal pulley (whose mass is negligible and that rotates with negligible friction), as shown in Fig. 5-9. (The arrangement is also known as an

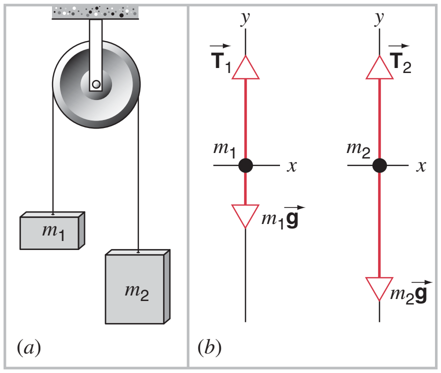


FIGURE 5-9. Sample Problem 5-5. (*a*) Diagram of Atwood's machine, consisting of two suspended blocks connected by a string that passes over a pulley. (*b*) Free-body diagrams of m_1 and m_2.

*Atwood's machine.**) Let m_2 be greater than m_1. Find the tension in the string and the acceleration of the blocks.

Solution We choose our coordinate system with the positive y axis upward; only y components of forces and accelerations need be considered. The free-body diagrams are shown in Fig. 5-9*b*. For m_1 the net force is $\Sigma F_y = T_1 - m_1 g$; for m_2, $\Sigma F_y = T_2 - m_2 g$. Applying Newton's second law in the y direction for each block gives:

block 1: $$T_1 - m_1 g = m_1 a_{1y}$$

block 2: $$T_2 - m_2 g = m_2 a_{2y}$$

where a_{1y} and a_{2y} are the respective accelerations of m_1 and m_2. If the string is massless and doesn't stretch, and if the pulley is massless and frictionless, then the tension has the same magnitude everywhere in the string and the magnitudes of the accelerations of the blocks are equal. (This ideal pulley doesn't change the magnitude of the tension or the acceleration from one side of the string to the other; its only function is to change their directions.) We set $T_1 = T_2 = T$, the common value of the tension in the string. If we let a represent the common magnitude of the accelerations, then $a_{1y} = a$ (a positive number, because the less massive block 1 moves upward) and $a_{2y} = -a$ (a negative number, because the more massive block 2 moves downward). Making these substitutions and solving the two equations simultaneously, we find

$$a = \frac{m_2 - m_1}{m_2 + m_1} g \quad \text{and} \quad T = \frac{2m_1 m_2}{m_1 + m_2} g. \qquad (5\text{-}4)$$

Consider what happens in the limiting cases $m_1 = 0$, $m_2 = 0$, $g = 0$, and $m_1 = m_2$. Note that $m_1 g < T < m_2 g$, and be sure you understand why this must be true. If the system is accelerating, then the tension in the string is *not* equal to the weight of either block.

*George Atwood (1745–1807) was an English mathematician who developed this device in 1784 for demonstrating the laws of accelerated motion and measuring g. By making the difference between m_1 and m_2 small, he was able to "slow down" the effect of free fall and time the motion of the falling weight with a pendulum clock, the most precise way to measure time intervals in his day.

SAMPLE PROBLEM 5-6. Figure 5-10*a* shows a block of mass m_1 on a frictionless horizontal surface. The block is pulled by a string of negligible mass that is attached to a hanging block of mass m_2. The string passes over a pulley whose mass is negligible and whose axle rotates with negligible friction. Find the tension in the string and the acceleration of each block.

Solution Figures 5-10*b* and 5-10*c* show the free-body diagrams for the two blocks.

Block 1 is acted on by a normal force due to the surface, by gravity, and by a force due to the tension in the string. The components of the net force on block 1 are $\Sigma F_x = T_1$ and $\Sigma F_y = N - m_1 g$, and Newton's second law for block 1 then gives:

$$T_1 = m_1 a_{1x} \quad \text{and} \quad N - m_1 g = m_1 a_{1y}.$$

We expect that block 1 does not move in the y direction, so that $a_{1y} = 0$.

For block 2, there are no forces in the x direction. The net force in the y direction is $\Sigma F_y = T_2 - m_2 g$, and Newton's second law gives

$$T_2 - m_2 g = m_2 a_{2y}.$$

If the string is of negligible mass and the pulley is ideal (frictionless and of negligible mass), then the magnitudes of the tension forces T_1 and T_2 are equal; we let T represent the common value of the tension. If the string doesn't stretch, then the accelerations of the blocks are equal; letting a be the common value of the acceleration, we set $a_{1x} = a$ and $a_{2y} = -a$. We now have two equations:

$$T = m_1 a \quad \text{and} \quad T - m_2 g = m_2(-a).$$

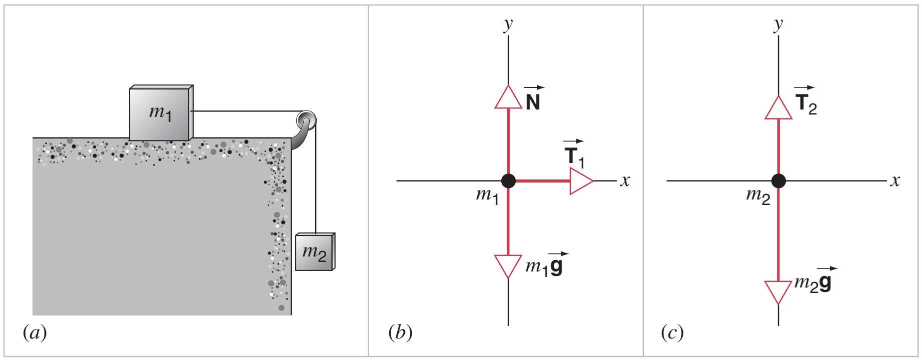


FIGURE 5-10. Sample Problem 5-6. (*a*) Block m_1 is pulled along a smooth horizontal surface by a string that passes over a pulley and is attached to block m_2. (*b*) The free-body diagram of block m_1. (*c*) The free-body diagram of block m_2.

Solving these simultaneously, we obtain

$$a = \frac{m_2}{m_1 + m_2} g \quad \text{and} \quad T = \frac{m_1 m_2}{m_1 + m_2} g. \qquad (5\text{-}5)$$

It is helpful to consider the limiting cases of these results. What happens when m_1 is zero? We would expect the string to be slack ($T = 0$) and m_2 to be in free fall ($a = g$). The equations correctly predict these limits. When $m_2 = 0$, there is no horizontal force on block 1 and it does not accelerate; again, the equations give the correct prediction.

Note that $a < g$, as we should expect. Also, note that T is less than mg, as we should expect when the block is accelerating downward (see Sample Problem 5-2).

Do Eqs. 5-5 behave properly in the limit $g = 0$?

SAMPLE PROBLEM 5-7. In the system shown in Fig. 5-11*a*, a block (of mass $m_1 = 9.5$ kg) slides on a frictionless plane inclined at an angle $\theta = 34°$. The block is attached by a string to a second block (of mass $m_2 = 2.6$ kg). The system is released from rest. Find the acceleration of the blocks and the tension in the string.

Solution The free-body diagrams for blocks 1 and 2 are shown in Figs. 5-11*b* and 5-11*c*. We choose coordinate systems as shown, so that one coordinate axis is parallel to the anticipated acceleration of each body. As in the previous examples, we expect that the tension has a common value and that the vertical motion of m_2 and the motion along the plane of m_1 can be described by accelerations of the same magnitude. We arbitrarily assume that m_1 moves in the positive x direction (if our assumption is wrong, a will come out negative). The components of the net force on m_1 are $\Sigma F_x = T - m_1 g \sin\theta$ and $\Sigma F_y = N - m_1 g \cos\theta$, and Newton's second law gives (with $a_{1x} = a$ and $a_{1y} = 0$):

$$T - m_1 g \sin\theta = m_1 a \quad \text{and} \quad N - m_1 g \cos\theta = 0.$$

For m_2 the y component of the net force is $\Sigma F_y = T - m_2 g$, and Newton's second law is (with $a_{2y} = -a$)

$$T - m_2 g = m_2(-a).$$

Solving simultaneously gives

$$a = \frac{m_2 - m_1 \sin\theta}{m_1 + m_2} g \qquad (5\text{-}6a)$$

and

$$T = \frac{m_1 m_2 g}{m_1 + m_2}(1 + \sin\theta). \qquad (5\text{-}6b)$$

Note that these results duplicate Eqs. 5-5 of Sample Problem 5-6 if we put $\theta = 0$ (so that block 1 moves horizontally) and Eqs. 5-4 of Sample Problem 5-5 if we put $\theta = 90°$ (so that block 1 moves vertically).

Putting in the numbers, we have

$$a = \frac{2.6\text{ kg} - (9.5\text{ kg})(\sin 34°)}{9.5\text{ kg} + 2.6\text{ kg}}(9.80\text{ m/s}^2) = -2.2\text{ m/s}^2.$$

The acceleration comes out to be negative, which means our initial guess about the direction of motion was wrong. Block 1 slides down the plane, and block 2 moves upward. Because the dynamical equations do not involve forces that depend on the direction of motion, this incorrect initial guess has no effect on the equations and we can accept the final value as correct. In general, this will not be the case when we consider frictional forces that act opposite to the direction of motion.

For the tension in the string, we find

$$T = \frac{(9.5\text{ kg})(2.6\text{ kg})(9.80\text{ m/s}^2)}{9.5\text{ kg} + 2.6\text{ kg}}(1 + \sin 34°) = 31\text{ N}.$$

This value is greater than the weight of m_2 ($m_2 g = 26$ N), which is consistent with the acceleration of m_2 being upward.

5-3 FRICTIONAL FORCES*

A block of mass m moving with initial velocity $\vec{v}_0$ along a long horizontal table will eventually come to rest. This means that, while it is moving, it experiences an average acceleration $\vec{a}_{av}$ that points in the direction opposite to its motion. If (in an inertial frame) we see that a body is accelerated, we always associate a force, defined from Newton's second law, with the motion. In this case we declare that the table exerts a force of *friction,* whose average value is $m\vec{a}_{av}$, on the sliding block. We generally take friction to mean a contact interaction between solids. Frictionlike effects caused by liquids and gases are described by other terms (see Section 4-4).

Actually, whenever the surface of one body slides over that of another, each body exerts a frictional force on the other. The frictional force on each body is in a direction

*See "Friction at the Atomic Scale" by Jacqueline Krim, *Scientific American,* October 1996, p. 74.

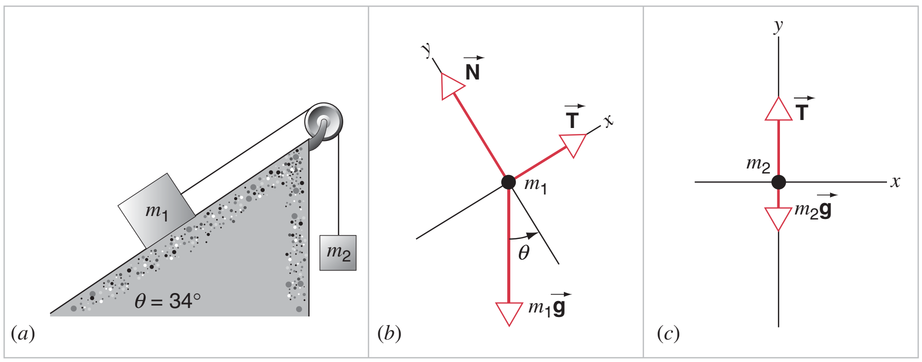


FIGURE 5-11. Sample Problem 5-7. (*a*) Block m_1 slides on a frictionless inclined plane. Block m_2 hangs from a string attached to m_1. (*b*) Free-body diagram of m_1. (*c*) Free-body diagram of m_2.

opposite to its motion relative to the other body. Frictional forces automatically oppose this relative motion and never aid it. Even when there is no relative motion, frictional forces may exist between surfaces.

Although we have ignored its effects up to now, friction is very important in our daily lives. Left to act alone it brings every rotating shaft to a halt. In an automobile, about 20% of the engine power is used to counteract frictional forces. Friction causes wear and seizing of moving parts, and much engineering effort is devoted to reducing it. On the other hand, without friction we could not walk; we could not hold a pencil, and if we could it would not write; wheeled transport as we know it would not be possible.

We want to express frictional forces in terms of the properties of the body and its environment; that is, we want to know the force law for frictional forces. In what follows we consider the sliding (not rolling) of one dry (unlubricated) surface over another. As we shall see later, friction, viewed at the microscopic level, is a very complicated phenomenon. The force laws for dry, sliding friction are empirical in character and approximate in their predictions. They do not have the elegant simplicity and accuracy that we find for the gravitational force law (Chapter 14) or for the electrostatic force law (Chapter 25). It is remarkable, however, considering the enormous diversity of surfaces one encounters, that many aspects of frictional behavior can be understood qualitatively on the basis of a few simple mechanisms.

Consider a block at rest on a horizontal table as in Fig. 5-12*a*. Attach a spring to it to measure the horizontal force $\vec{\mathbf{F}}$ required to set the block in motion. We find that the block will not move even though we apply a small force (Fig. 5-12*b*). We say that our applied force is balanced by an opposite frictional force $\vec{\mathbf{f}}$ exerted on the block by the table, acting along the surface of contact. As we increase the applied force (Fig. 5-12*c, d*) we find some definite force at which the block will "break away" from the surface and begin to accelerate (Fig. 5-12*e*). By reducing the force once motion has started, we find that it is possible to keep the block in uniform motion without acceleration (Fig. 5-12*f*). Figure 5-12*g* shows the results of an experiment to measure the frictional force. An increasing force F is applied starting at about $t = 2$ s, after which the frictional force increases with the applied force and the object remains at rest. At $t = 4$ s, the object suddenly begins to move and the frictional force becomes constant, independent of the applied force.

The frictional forces acting between surfaces at rest with respect to each other are called forces of *static friction.* The maximum force of static friction (corresponding to the peak at $t = 4$ s in Fig. 5-12*g*) will be the same as the smallest applied force necessary to start motion. Once motion is started, the frictional forces acting between the surfaces usually decrease so that a smaller force is necessary to maintain uniform motion (corresponding to the nearly constant force at $t > 4$ s in Fig. 5-12*g*). The forces acting between surfaces in relative motion are called forces of *kinetic friction.*

The maximum force of static friction between any pair of dry unlubricated surfaces follows these two empirical

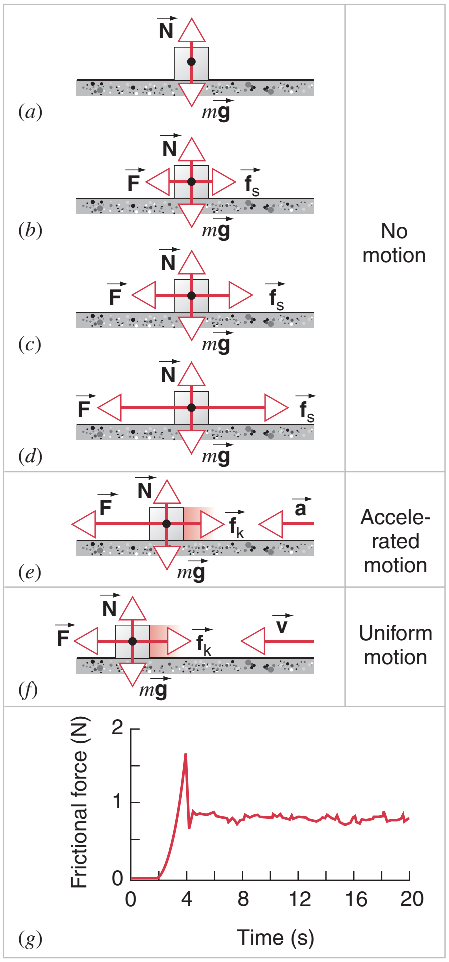


FIGURE 5-12. (*a–d*) An external force $\vec{\mathbf{F}}$, applied to a resting block, is counterbalanced by an equal but opposite frictional force $\vec{\mathbf{f}}$. As $\vec{\mathbf{F}}$ is increased, $\vec{\mathbf{f}}$ also increases, until $\vec{\mathbf{f}}$ reaches a certain maximum value. (*e*) The block then "breaks away," accelerating to the left. (*f*) If the block is to move with constant velocity, the applied force $\vec{\mathbf{F}}$ must be reduced from the maximum value it had just before the block began to move. (*g*) Experimental results; here the applied force $\vec{\mathbf{F}}$ is increased from zero starting at about $t = 2$ s, and the motion suddenly begins at about $t = 4$ s. For details of the experiment, see "Undergraduate Computer-Interfacing Projects," by Joseph Priest and John Snyder, *The Physics Teacher,* May 1987, p. 303.

laws: (1) It is approximately independent of the area of contact over wide limits and (2) it is proportional to the normal force.*

The ratio of the magnitude of the *maximum* force of static friction to the magnitude of the normal force is called the *coefficient of static friction* for the surfaces involved. If

*The two laws of friction were first discovered experimentally by Leonardo da Vinci (1452–1519). Leonardo's statement of the two laws was remarkable, coming as it did two centuries before Newton developed the concept of force. The mathematical expressions of the laws of friction and the concept of the coefficient of friction were developed by Charles Augustin Coulomb (1736–1806), who is best known for his studies of electrostatics (see Chapter 25).

Table 5-1 Coefficients of Friction[a]

Surfaces	μ_s	μ_k
Wood on wood	0.25–0.5	0.2
Glass on glass	0.9–1.0	0.4
Steel on steel, clean surfaces	0.6	0.6
Steel on steel, lubricated	0.09	0.05
Rubber on dry concrete	1.0	0.8
Waxed wood ski on dry snow	0.04	0.04
Teflon on Teflon	0.04	0.04

[a] Values are approximate and are intended only as estimates. The actual coefficients of friction for any pair of surfaces depend on such conditions as the cleanliness of the surfaces, the temperature, and the humidity.

Figure 5-13. A magnified section of a highly polished steel surface. The vertical scale of the surface irregularities is several thousand atomic diameters. The section has been cut at an angle so that the vertical scale is exaggerated with respect to the horizontal scale by a factor of 10.

f_s represents the magnitude of the force of static friction, we can write

$$f_s \leq \mu_s N, \tag{5-7}$$

where μ_s is the coefficient of static friction and N is the magnitude of the normal force. The equality sign holds only when f_s has its maximum value.

The force of kinetic friction f_k between dry, unlubricated surfaces follows the same two laws as those of static friction. (1) It is approximately independent of the area of contact over wide limits and (2) it is proportional to the normal force. The force of kinetic friction is also reasonably independent of the relative speed with which the surfaces move over each other.

The ratio of the magnitude of the force of kinetic friction to the magnitude of the normal force is called the *coefficient of kinetic friction.* If f_k represents the magnitude of the force of kinetic friction, then

$$f_k = \mu_k N, \tag{5-8}$$

where μ_k is the coefficient of kinetic friction.

Both μ_s and μ_k are dimensionless constants, each being the ratio of (the magnitudes of) two forces. Usually, for a given pair of surfaces $\mu_s > \mu_k$. The actual values of μ_s and μ_k depend on the nature of both the surfaces in contact. In most cases we can regard them as being constants (for a given pair of surfaces) over the range of forces and velocities we commonly encounter. Both μ_s and μ_k can exceed unity, although commonly they are less than 1. Table 5-1 shows some representative values of μ_s and μ_k.

Note that Eqs. 5-7 and 5-8 are relations between the *magnitudes only* of the normal and frictional forces. These forces are always directed perpendicularly to one another.

The Microscopic Basis of Friction

On the atomic scale even the most finely polished surface is far from flat. Figure 5-13, for example, shows an actual profile, highly magnified, of a steel surface that would be considered to be highly polished. One can readily believe that when two bodies are placed in contact, the actual microscopic area of contact is much less than the true area of the surface; in a particular case these areas can easily be in the ratio of $1:10^4$.

The actual (microscopic) area of contact is proportional to the normal force, because the contact points deform plastically under the great stresses that develop at these points. Many contact points actually become "cold-welded" together. This phenomenon, *surface adhesion,* occurs because at the contact points the molecules on opposite sides of the surface are so close together that they exert strong intermolecular forces on each other.

When one body (a metal, say) is pulled across another, the frictional resistance is associated with the rupturing of these thousands of tiny welds, which continually reform as new chance contacts are made (see Fig. 5-14). Radioactive tracer experiments have shown that, in the rupturing process, small fragments of one metallic surface may be sheared off and adhere to the other surface. If the relative speed of the two surfaces is great enough, there may be local melting at certain contact areas even though the surface as a whole may feel only moderately warm. The "stick and

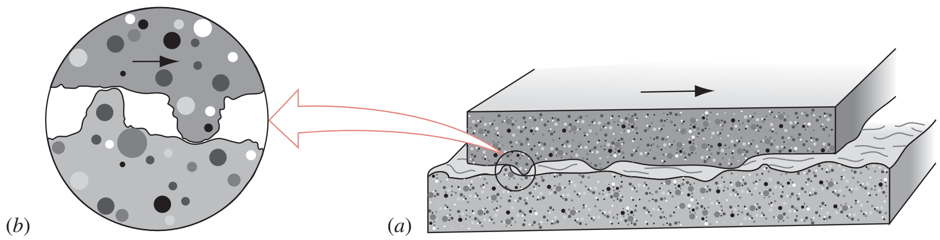


Figure 5-14. The mechanism of sliding friction. (*a*) The upper surface is sliding to the right over the lower surface in this enlarged view. (*b*) A detail, showing two spots where cold welding has occurred. Force is required to break these welds and maintain the motion. If the normal force increases, the surfaces are pushed together so that more welds form and the frictional force increases.

slip" events are responsible for the noises that dry surfaces make when sliding across one another as, for example, the squealing chalk on the blackboard.

The coefficient of friction depends on many variables, such as the nature of the materials, surface finish, surface films, temperature, and extent of contamination. For example, if two carefully cleaned metal surfaces are placed in a highly evacuated chamber so that surface oxide films do not form, the coefficient of friction rises to enormous values and the surfaces actually become firmly "welded" together. The admission of a small amount of air to the chamber so that oxide films may form on the opposing surfaces reduces the coefficient of friction to its "normal" value.

The frictional force that opposes one body *rolling* over another is much less than that for sliding motion; this gives the advantage to the rolling wheel over the sliding sledge. This reduced friction is due in large part to the fact that, in rolling, the microscopic contact welds are "peeled" apart rather than "sheared" apart as in sliding friction. This reduces the frictional force by a large factor.

Frictional resistance in dry, sliding friction can be reduced considerably by lubrication. This technique was used in ancient Egypt to move the blocks from which the pyramids were built. A still more effective technique is to introduce a layer of gas between the sliding surfaces; a laboratory air track and the gas-supported bearing are two examples. Friction can be reduced still further by suspending an object by means of magnetic forces. Magnetically levitated trains now under development have the potential for high-speed, nearly frictionless travel.

SAMPLE PROBLEM 5-8. A block is at rest on an inclined plane making an angle θ with the horizontal, as in Fig. 5-15*a*. As the angle of incline is raised, it is found that slipping just begins at an angle of inclination $\theta_s = 15°$. What is the coefficient of static friction between block and incline?

Solution The forces acting on the block, considered to be a particle, are shown in Fig. 5-15*b*. The weight of the block is $m\vec{\mathbf{g}}$, the normal force exerted on the block by the inclined surface is $\vec{\mathbf{N}}$, and the force of friction exerted by the inclined surface on the block is $\vec{\mathbf{f}}_s$. Note that the resultant force exerted by the inclined surface on the block, $\vec{\mathbf{N}} + \vec{\mathbf{f}}_s$, is no longer perpendicular to the surface of contact, as was true for frictionless surfaces ($\vec{\mathbf{f}}_s = 0$). The block is at rest, so that Newton's second law gives $\sum \vec{\mathbf{F}} = 0$.

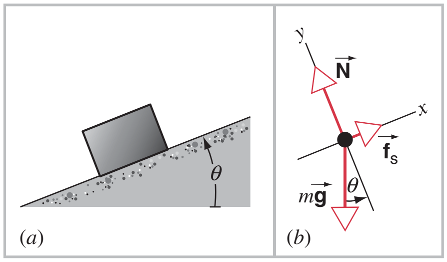


FIGURE 5-15. Sample Problem 5-8. (*a*) A block at rest on an inclined plane. (*b*) A free-body diagram of the block.

Resolving the weight into its x and y components (see Fig. 5-8), we can find the components of the net force to be $\sum F_x = f_s - mg \sin\theta$ and $\sum F_y = N - mg\cos\theta$. If the block is at rest, then $a_x = 0$ and $a_y = 0$, and Newton's second law gives

$$f_s - mg\sin\theta = 0 \qquad \text{and} \qquad N - mg\cos\theta = 0.$$

At the angle θ_s where slipping just begins, f_s has its maximum value of $\mu_s N$. Evaluating f_s and N from the above equations, we then have

$$\mu_s = \frac{f_s}{N} = \frac{mg\sin\theta_s}{mg\cos\theta_s} = \tan\theta_s = \tan 15° = 0.27.$$

Hence measurement of the angle of inclination at which slipping just starts provides a simple experimental method for determining the coefficient of static friction between two surfaces. Note that this determination is independent of the weight of the object.

You can use similar arguments to show that the angle of inclination θ_k required to maintain a *constant speed* for the block as it slides down the plane, once it has been started by a gentle tap, is given by

$$\mu_k = \tan\theta_k,$$

where $\theta_k < \theta_s$. With the aid of a ruler to measure the tangent of the angle of inclination, you can now determine μ_s and μ_k for a coin sliding down your textbook.

SAMPLE PROBLEM 5-9. Consider an automobile moving along a straight horizontal road with a speed v_0. The driver applies the brakes and brings the car to a halt without skidding. If the coefficient of static friction between the tires and the road is μ_s, what is the shortest distance in which the automobile can be stopped?

Solution The forces acting on the automobile are shown in Fig. 5-16. The car is assumed to be moving in the positive x direction. If we assume that f_s is a constant force, we have uniformly decelerated motion.

Our plan is to use Newton's laws to find the acceleration of the automobile, and then to use the equations of kinematics from Chapter 2 to find the stopping distance. From the free-body diagram of Fig. 5-16*b*, we obtain the component equations for the net force, $\sum F_x = -f_s$ and $\sum F_y = N - mg$, and thus for Newton's second law

$$-f_s = ma_x \qquad \text{and} \qquad N - mg = ma_y = 0,$$

where we have put $a_y = 0$ because the car does not move in the vertical direction. From these equations and the force of static friction ($f_s = \mu_s N$) we obtain

$$a_x = -\frac{f_s}{m} = -\frac{\mu_s N}{m} = -\frac{\mu_s(mg)}{m} = -\mu_s g.$$

If the car starts with velocity v_{0x} and ends with velocity $v_x = 0$, we can use Eq. 2-26 ($v_x = v_{0x} + a_x t$) to find the stopping time $t = -v_{0x}/a_x = v_0/\mu_s g$. The stopping distance $d = x - x_0$ can then be found from Eq. 2-28 ($x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$) using this value for t:

$$d = v_{0x}t + \tfrac{1}{2}a_x t^2 = v_0\left(\frac{v_0}{\mu_s g}\right) + \tfrac{1}{2}(-\mu_s g)\left(\frac{v_0}{\mu_s g}\right)^2 = \frac{v_0^2}{2\mu_s g}.$$

The greater is the initial speed, the longer is the distance required to come to a stop; in fact, this distance varies as the square

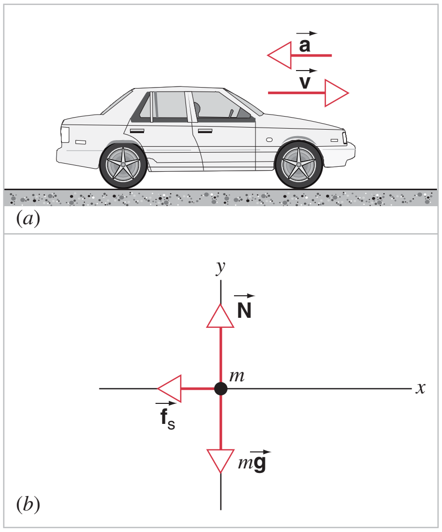


Figure 5-16. Sample Problem 5-9. (*a*) A decelerating automobile. (*b*) A free-body diagram of the decelerating automobile, considered to be a particle. For convenience, all forces are taken to act at a common point. In reality, the forces $\vec{N}$ and $\vec{f}_s$ are sums of the individual forces exerted by the road on each of the four tires.

of the initial velocity. Also, the greater is the coefficient of static friction between the surfaces, the less is the distance required to come to a stop.

We have used the coefficient of static friction in this problem, rather than the coefficient of kinetic friction, because we assume there is no sliding between the tires and the road. Furthermore, we have assumed that the maximum force of static friction ($f_s = \mu_s N$) operates because the problem seeks the shortest distance for stopping. With a smaller static frictional force the distance for stopping would obviously be greater. The correct braking technique required here is to keep the car just on the verge of skidding. (Cars equipped with anti-lock braking systems maintain this condition automatically). If the surface is smooth and the brakes are fully applied, sliding may occur. In this case μ_k replaces μ_s, and the distance required to stop would increase because μ_k is smaller than μ_s.

As a specific example, if $v_0 = 60$ mi/h $= 27$ m/s, and $\mu_s = 0.60$ (a typical value), we obtain

$$d = \frac{v_0^2}{2\mu_s g} = \frac{(27\ \text{m/s})^2}{2(0.60)(9.8\ \text{m/s}^2)} = 62\ \text{m}.$$

Note that this result is independent of the mass of the car. On rear-wheel drive cars, with the engine in front, it is a common practice to "weigh down" the trunk in order to increase safety when driving on icy roads. How can this practice be consistent with our result that the stopping distance is independent of the mass of the car? (*Hint:* See Exercise 10.)

Sample Problem 5-10. Repeat Sample Problem 5-7, taking into account a frictional force between block 1 and the plane. Use the values $\mu_s = 0.24$ and $\mu_k = 0.15$.

Solution As in Sample Problem 5-7, we assume that block 1 moves down the plane, and so the frictional force acts up the plane. Figure 5-17*a* shows the free-body diagram for m_1. Resolving the forces on m_1 into their components, we have $\Sigma F_x = T + f - m_1 g \sin\theta$ and $\Sigma F_y = N - m_1 g \cos\theta$, and Newton's second law gives

$$T + f - m_1 g \sin\theta = m_1 a \qquad \text{and} \qquad N - m_1 g \cos\theta = 0,$$

where we have put $a_{1x} = a$ and $a_{1y} = 0$, as in Sample Problem 5-7. The net force on block 2 is $\Sigma F_y = T - m_2 g$, and with $a_{2y} = -a$ Newton's second law gives

$$T - m_2 g = m_2(-a).$$

Putting $f = \mu_k N = \mu_k m_1 g \cos\theta$ and solving the remaining two equations for a and T, we obtain

$$a = \frac{m_2 - m_1(\sin\theta - \mu_k\cos\theta)}{m_1 + m_2}\, g, \qquad (5\text{-}9a)$$

$$T = \frac{m_1 m_2 g}{m_1 + m_2}(1 + \sin\theta - \mu_k\cos\theta). \qquad (5\text{-}9b)$$

Note that, in the limit of $\mu_k \to 0$, Eqs. 5-9 reduce to Eqs. 5-6 of Sample Problem 5-7.

Let us now find the numerical values of a and T:

$$a = \frac{2.6\ \text{kg} - 9.5\ \text{kg}(\sin 34° - 0.15\cos 34°)}{2.6\ \text{kg} + 9.5\ \text{kg}}(9.80\ \text{m/s}^2)$$

$$= -1.2\ \text{m/s}^2,$$

$$T = \frac{(9.5\ \text{kg})(2.6\ \text{kg})(9.80\ \text{m/s}^2)}{9.5\ \text{kg} + 2.6\ \text{kg}}(1 + \sin 34° - 0.15\cos 34°)$$

$$= 29\ \text{N}.$$

The negative value of a is consistent with the way we set up our equations; the block moves down the plane, as it did in Sample Problem 5-7, but with less acceleration than it did in the frictionless case (2.2 m/s^2).

The tension in the string is less than it was in the frictionless case (31 N). Block 1 accelerates less rapidly down the plane when there is friction, so it doesn't pull as strongly on the string attached to block 2.

One additional question that must be answered is whether the system will move at all. That is, is there enough force down the plane to exceed the static friction and start the motion? When the system is initially at rest, the tension in the string is equal to the weight of m_2, or (2.6 kg) (9.8 m/s^2) $= 26$ N. The maximum static friction, which opposes the tendency to move down the plane, is $\mu_s N = \mu_s m_1 g \cos\theta = 19$ N. The component of the weight of m_1 acting down the plane is $m_1 g \sin\theta = 52$ N. Thus

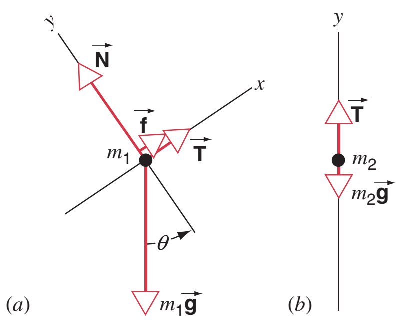


Figure 5-17. Sample Problem 5-10. The free-body diagrams of Fig. 5-11, in the case of friction along the plane.

there is more than enough weight acting down the plane (52 N) to overcome the total of the tension and the static frictional force (26 N + 19 N = 45 N), and the system does indeed move. You should be able to show that if the static coefficient of friction is greater than 0.34 then there will be no motion.

5-4 THE DYNAMICS OF UNIFORM CIRCULAR MOTION

As we discussed in Section 4-5, when an object of mass m moves in a circle of radius r at a uniform speed v, it experiences a radial or centripetal acceleration of magnitude v^2/r. In this section we consider the circular motion resulting from several forces acting on the object. Newton's second law must hold in this case in vector form: $\sum \vec{\mathbf{F}} = m\vec{\mathbf{a}}$. Since $\vec{\mathbf{a}}$ is always in the radial direction, the net force must also be radial. Its magnitude must be given by

$$\left|\sum \vec{\mathbf{F}}\right| = ma = \frac{mv^2}{r}. \tag{5-10}$$

Whatever the nature or origin of the forces that act on the object in uniform circular motion, the resultant of all the forces must be (1) in the radial direction, and (2) of magnitude mv^2/r. Even though the magnitude of the object's velocity remains constant, there is an acceleration and thus a net force, because the *direction* of the velocity is changing.

The following examples illustrate applications of Newton's laws to uniform circular motion.

The Conical Pendulum

Figure 5-18 shows a small body of mass m revolving in a horizontal circle with constant speed v at the end of a string of length L. As the body swings around, the string sweeps over the surface of an imaginary cone. This device is called a *conical pendulum.* Let us find the time required for one complete revolution of the body.

If the string makes an angle θ with the vertical, the radius of the circular path is $R = L \sin \theta$. The forces acting on the body of mass m are its weight $m\vec{\mathbf{g}}$ and the tension $\vec{\mathbf{T}}$ of the string, as shown in Fig. 5-18*b*. We can resolve $\vec{\mathbf{T}}$ at any instant into a radial and a vertical component

$$T_r = -T \sin \theta \qquad \text{and} \qquad T_z = T \cos \theta. \tag{5-11}$$

The radial component is negative if we define the radial direction to be positive outward from the central axis.

Using the coordinate system shown in Fig. 5-18*b*, we can write the components of the net force on the body as $\sum F_r = T_r = -T \sin \theta$ and $\sum F_z = T \cos \theta - mg$. Since the body has no vertical acceleration, we can write the z component of Newton's second law as

$$T \cos \theta - mg = 0. \tag{5-12}$$

The radial component of Newton's second law is $\sum F_r = ma_r$. The radial acceleration is $a_r = -v^2/R$, negative because it acts radially inward and we have chosen the outward radial direction to be positive. For this case Newton's second law gives

$$-T \sin \theta = ma_r = m\left(\frac{-v^2}{R}\right). \tag{5-13}$$

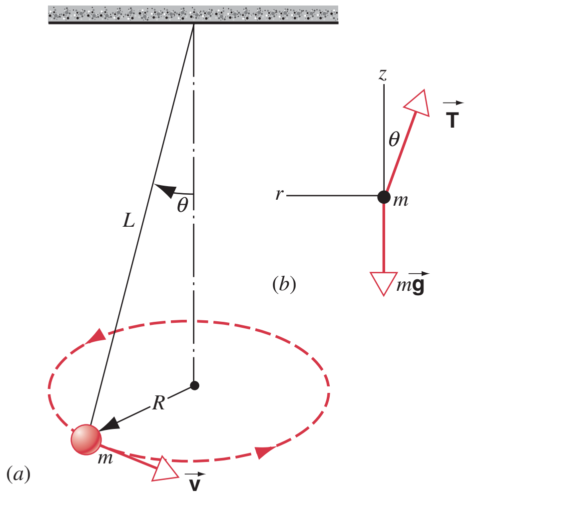


FIGURE 5-18. The conical pendulum. (*a*) A body of mass m suspended from a string of length L moves in a circle; the string describes a right circular cone of semiangle θ. (*b*) A free-body diagram of the body.

Eliminating T between these two equations, we can solve for the speed of the body:

$$v = \sqrt{Rg \tan \theta}. \tag{5-14}$$

If we let t represent the time for one complete revolution of the body, then

$$v = \frac{2\pi R}{t}$$

or

$$t = \frac{2\pi R}{v} = \frac{2\pi R}{\sqrt{Rg \tan \theta}} = 2\pi\sqrt{\frac{R}{g \tan \theta}}.$$

However, $R = L \sin \theta$, so that

$$t = 2\pi\sqrt{\frac{L \cos \theta}{g}}. \tag{5-15}$$

This equation gives the relation between t, L, and θ. Note that t, called the *period* of motion, does not depend on m.

If $L = 1.2$ m and $\theta = 25°$, what is the period of the motion? We have

$$t = 2\pi\sqrt{\frac{(1.2\text{ m})(\cos 25°)}{9.8\text{ m/s}^2}} = 2.1\text{ s}.$$

The Rotor

In many amusement parks we find a device often called the rotor. The rotor is a hollow cylindrical room that can be set rotating about the central vertical axis of the cylinder. A person enters the rotor, closes the door, and stands up against the wall. The rotor gradually increases its rotational speed from rest until, at a predetermined speed, the floor below the person is opened downward, revealing a deep pit. The person does not fall but remains "pinned up" against the wall of the rotor. What minimum rotational speed is necessary to prevent falling?

The forces acting on the person are shown in Fig. 5-19. The person's weight is $m\vec{g}$, the force of static friction between person and rotor wall is $\vec{f}_s$, and $\vec{N}$ is the normal force exerted by the wall on the person (which, as we shall see, provides the needed centripetal force). As we did in the previous calculation, we resolve the forces into radial and vertical components, with the positive radial direction outward from the axis of rotation and the positive z axis upward. The components of the net force on the person are then $\Sigma F_r = -N$ and $\Sigma F_z = f_s - mg$. Note that $\vec{N}$ provides the centripetal force in this case.

The radial acceleration is $a_r = -v^2/R$ and the vertical acceleration is $a_z = 0$. The radial and vertical components of Newton's second law then give

$$-N = ma_r = m\left(\frac{-v^2}{R}\right) \quad \text{and} \quad f_s - mg = ma_z = 0.$$

Writing $f_s = \mu_s N$ and substituting $N = mv^2/R$ from the first equation and $f_s = mg$ from the second, we can solve for v to find

$$v = \sqrt{\frac{gR}{\mu_s}}. \tag{5-16}$$

This equation relates the coefficient of friction necessary to prevent slipping to the tangential speed of an object on the wall. Note that the result does not depend on the person's weight.

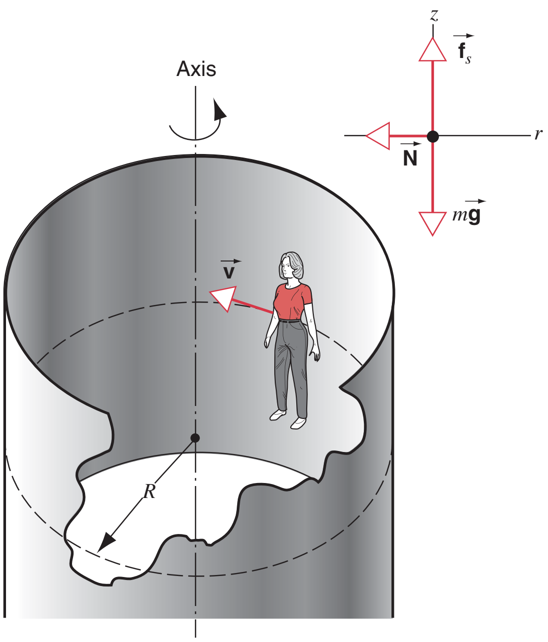


FIGURE 5-19. The rotor. Forces acting on the person are shown.

As a practical matter the coefficient of friction between the textile material of clothing and a typical rotor wall (canvas) is about 0.40. For a typical rotor the radius is 2.0 m, so that v must be about 7.0 m/s or more. The circumference of the circular path is $2\pi R = 12.6$ m, and at 7.0 m/s it takes a time of $t = 12.6$ m/(7.0 m/s) $= 1.80$ s to complete each revolution. The rotor must therefore turn at a rate of at least 1 revolution/1.80 s $= 0.56$ revolution/s or about 33 rpm, the same rate of rotation as a phonograph turntable.

The Banked Curve

Let the block in Fig. 5-20*a* represent an automobile or railway car moving at constant speed v on a *level* roadbed around a curve having a radius of curvature R. In addition to two vertical forces—namely, the weight $m\vec{g}$ and a normal force $\vec{N}$—a horizontal force $\vec{P}$ must act on the car. The force $\vec{P}$ provides the centripetal force necessary for motion in a circle. In the case of the automobile this force is supplied by a sidewise frictional force exerted by the road on the tires; in the case of the railway car the force is supplied by the rails exerting a sidewise force on the inner rims of the car's wheels. Neither of these sidewise forces can safely be relied on to be large enough at all times, and both cause

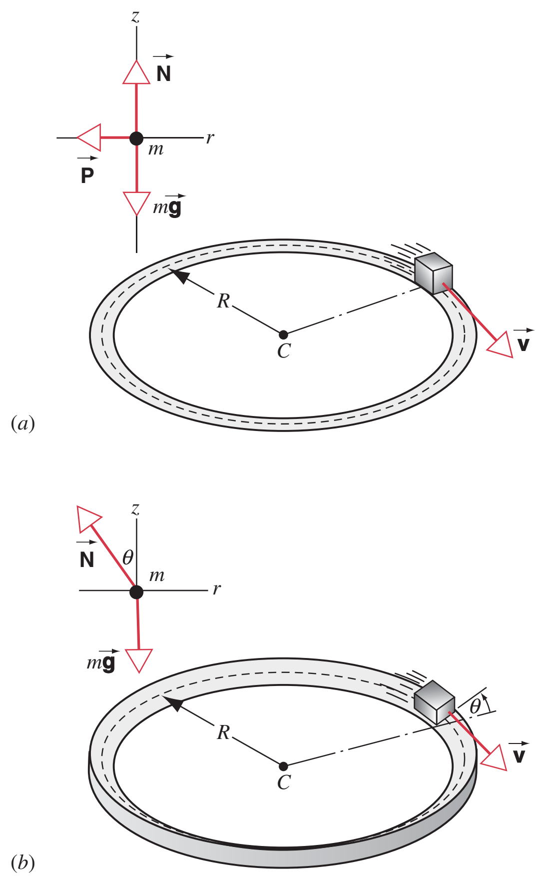


FIGURE 5-20. (*a*) A level roadbed. A free-body diagram of the moving body is shown at left. The centripetal force must be supplied by friction between tires and road. (*b*) A banked roadbed. No friction is necessary to round the curve safely.

unnecessary wear. Hence, the roadbed is *banked* on curves, as shown in Fig. 5-20*b*. In this case, the normal force $\vec{\mathbf{N}}$ has not only a vertical component, as before, but also a horizontal component that supplies the centripetal force necessary for uniform circular motion. Thus no additional sidewise forces are needed with a roadbed that is properly banked for vehicles of a particular speed.

The correct angle θ of banking in the absence of friction can be obtained as follows. We begin, as usual, with Newton's second law, and we refer to the free-body diagram shown in Fig. 5-20*b*. The radial and vertical components of the net force on the moving body are $\Sigma F_r = -N \sin\theta$ and $\Sigma F_z = N\cos\theta - mg$. As before, the radial acceleration is $a_r = -v^2/R$ and the vertical acceleration is $a_z = 0$, so we can write the components of Newton's second law as

$$-N\sin\theta = ma_r = m\left(\frac{-v^2}{R}\right)$$

and

$$N\cos\theta - mg = ma_z = 0.$$

Solving these two equations for $\sin\theta$ and $\cos\theta$ and dividing the resulting expressions, we obtain

$$\tan\theta = \frac{v^2}{Rg}. \tag{5-17}$$

Note that the proper angle of banking depends on the speed of the car and the curvature of the road. It does not depend on the mass of the car; for a given banking angle, all cars will be able to travel safely. For a given curvature, the road is banked at an angle corresponding to an expected average speed. Curves are often marked by signs giving the proper speed for which the road was banked. If vehicles exceed that speed, the friction between tires and road must supply the additional centripetal force needed to travel the curve safely.

Check the banking formula for the limiting cases $v = 0$, $R \to \infty$, v large, and R small. Also, note that Eq. 5-17, if solved for v, gives the same result that we derived for the speed of the bob of a conical pendulum. Compare Figs. 5-18*b* and 5-20*b*, noting their similarities.

5-5 TIME-DEPENDENT FORCES (Optional)*

In Chapter 2, we analyzed the braking of an automobile by assuming the acceleration to be constant. In practice, this is seldom the case. Under many circumstances, especially at high speed, we usually apply the brakes slowly at first and then more strongly as the car slows. The braking force therefore depends on the time during the interval over which the car is slowing, and so the acceleration $a(t)$ is a function of how we apply the brakes.

Even though the force is not constant, we can still use Newton's laws to analyze the motion, but we cannot use the equations of Chapter 2 to find the position and velocity as functions of the time, because those equations were derived for constant acceleration. We assume here for simplicity that the forces and the motion are in one dimension, which we take to be the x direction. We can find the x component of the net force $F_x(t)$ in the usual way from a free-body diagram, and then we continue by writing $a_x = dv_x/dt$ and using Newton's second law:

$$a_x(t) = \frac{dv_x}{dt} = \frac{F_x(t)}{m}$$

or

$$dv_x = \frac{F_x(t)}{m}\,dt. \tag{5-18}$$

Suppose the object begins its motion at $t = 0$ with initial velocity v_{0x}. What is its velocity v_x at time t? We integrate Eq. 5-18 on the left between v_{0x} and v_x, and on the right between 0 and t:

$$\int_{v_{0x}}^{v_x} dv_x = \int_0^t \frac{F_x(t)}{m}\,dt.$$

$$v_x - v_{0x} = \frac{1}{m}\int_0^t F_x(t)\,dt$$

so

$$v_x(t) = v_{0x} + \frac{1}{m}\int_0^t F_x(t)\,dt. \tag{5-19}$$

Note that this reduces to Eq. 2-26 if F_x is a constant $(= ma_x)$ and so can be taken out of the integral.

Continuing in the same way with $v_x = dx/dt$, we can find the position as a function of the time

$$x(t) = x_0 + \int_0^t v_x(t)\,dt. \tag{5-20}$$

This reduces to Eq. 2-28 when F_x is a constant, in which case $v_x(t) = v_{0x} + a_x t$.

When we have a force that depends on the time, we can use Eqs. 5-19 and 5-20 to find analytical expressions for $v_x(t)$ and $x(t)$. We have already seen in Section 4-4 how this can be done in a similar way for a force that depends on the velocity. More often, especially when there is no analytical expression for the integrals, we find it is necessary or convenient to use numerical or computational methods.

SAMPLE PROBLEM 5-11. A car of mass $m = 1260$ kg is moving at 105 km/h (about 65 mi/h or 29.2 m/s). The driver begins to apply the brakes so that the magnitude of the braking force increases linearly with time at the rate of 3360 N/s. (*a*) How much time passes before the car comes to rest? (*b*) How far does the car travel in the process?

*This section involves integral calculus and can be postponed until the student is more familiar with integration methods.

Solution (*a*) If we choose the direction of the car's velocity as the positive x direction, then we can represent the braking force as $F_x = -ct$ where $c = 3360$ N/s. (The negative sign indicates that the direction of the braking force is opposite that of the velocity). Using Eq. 5-19, we then have

$$v_x(t) = v_{0x} + \frac{1}{m}\int_0^t (-ct)\,dt = v_{0x} - \frac{ct^2}{2m}.$$

To find the time t_1 when the car comes to rest, we set this expression for $v_x(t)$ equal to zero and solve for t:

$$t_1 = \sqrt{\frac{2v_{0x}m}{c}} = \sqrt{\frac{2(29.2\text{ m/s})(1260\text{ kg})}{3360\text{ N/s}}} = 4.68\text{ s}.$$

(*b*) To find how far the car travels in this time, we need an expression for $x(t)$, for which we must integrate $v_x(t)$ according to Eq. 5-20:

$$x(t) = x_0 + \int_0^t \left(v_{0x} - \frac{ct^2}{2m}\right) dt = x_0 + v_{0x}t - \frac{ct^3}{6m}.$$

Evaluating this expression at $t = t_1$ (setting x_0 to 0), we obtain

$$x(t_1) = 0 + (29.2\text{ m/s})(4.68\text{ s}) - \frac{(3360\text{ N/s})(4.68\text{ s})^3}{6(1260\text{ kg})} = 91.1\text{ m}.$$

Figure 5-21 shows the time dependence of x, v_x, and a_x. In contrast with the case of constant acceleration, $v_x(t)$ is not a straight line.

With this method of braking, most of the change in velocity occurs near the end of the motion. The change in velocity in the first second after the brakes are applied is only 1.3 m/s (about 3 mi/h); in the last second, however, the change is 11.2 m/s (about 25 mi/h). (Recall that in the case of constant acceleration, the change in velocity is the same in equal time intervals.) Can you think of an advantage to braking in this manner? Are there also disadvantages?

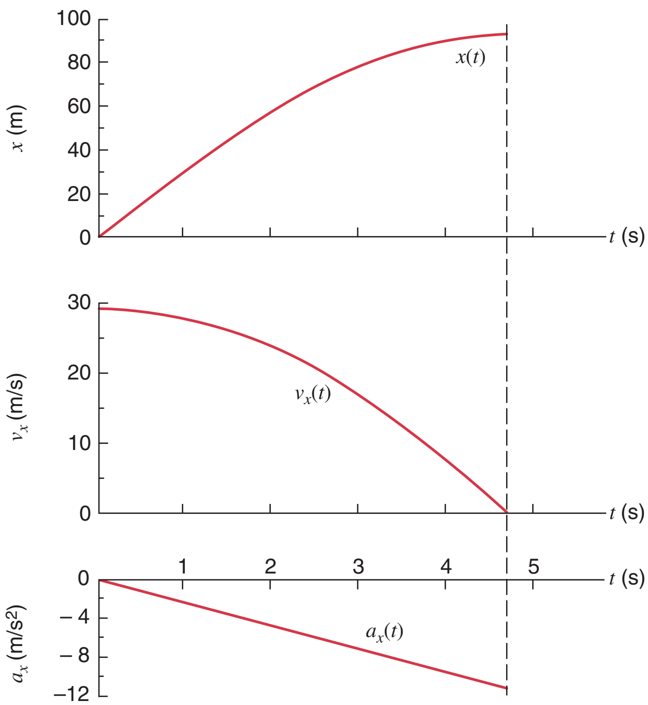


FIGURE 5-21. Sample Problem 5-11. The deduced position $x(t)$ and velocity $v_x(t)$ are shown corresponding to $a_x(t)$, which varies linearly with time. The dashed line marks the instant ($t = 4.68$ s) at which the car comes to rest.

5-6 NONINERTIAL FRAMES AND PSEUDOFORCES (Optional)

In our treatment of classical mechanics thus far, we have assumed that measurements and observations were made from an inertial reference frame. This is one of the set of reference frames defined by Newton's first law—namely, that set of frames in which a body will not be accelerated ($\vec{\mathbf{a}} = 0$) if there are no force-producing bodies in its environment ($\Sigma\,\vec{\mathbf{F}} = 0$). The choice of a reference frame is always ours to make, so that if we choose to select only inertial frames, we do not restrict in any way our ability to apply classical mechanics to natural phenomena.

Nevertheless, if we find it convenient, we can apply classical mechanics from the point of view of an observer in a *noninertial frame*—that is, a frame attached to a body that is accelerating as viewed from an inertial frame. The frames of an accelerating car or a rotating merry-go-round are examples of noninertial frames.

To apply classical mechanics in noninertial frames we must introduce additional forces known as *pseudoforces* (sometimes called inertial forces). Unlike the forces we have examined thus far, we cannot associate pseudoforces with any particular object in the environment of the body on which they act, and we cannot classify them into any of the categories discussed in Section 5-1. Moreover, if we view the body from an inertial frame, the pseudoforces disappear. Pseudoforces are simply devices that permit us to apply classical mechanics in the normal way to events if we insist on viewing the events from a noninertial reference frame.

As an example, consider an observer S' riding in a van that is moving at constant velocity. The van contains a long airtrack with a frictionless 0.25-kg glider resting at one end (Fig. 5-22*a*). The driver of the van applies the brakes, and the van begins to decelerate. An observer S on the ground measures the constant acceleration of the van to be -2.8 m/s². The observer S' riding in the van is therefore in a noninertial frame of reference when the van begins to decelerate. S' observes the glider to move down the track with an acceleration of $+2.8$ m/s² (Fig. 5-22*b*). How might each observer use Newton's second law to account for the motion of the glider?

For ground observer S, who is in an inertial reference frame, the analysis is straightforward. The glider, which had been moving forward at constant velocity before the van started to brake, simply continues to do so. According to S, the glider has no acceleration and therefore no horizontal force need be acting on it.

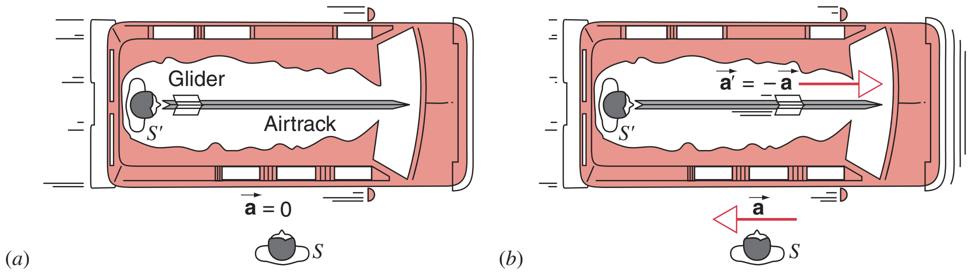


FIGURE 5-22. (*a*) Ground-based observer *S* watches observer *S*′ traveling in a van at constant velocity. The van is traveling to the right, which we take to be the positive *x* direction. Both observers are in inertial reference frames. (*b*) The van brakes with constant acceleration *a* according to observer *S*. Observer *S*′, now in a noninertial reference frame, sees the glider move forward on its airtrack with constant acceleration $\vec{\mathbf{a}}' = -\vec{\mathbf{a}}$. Observer *S*′ accounts for this motion in terms of a pseudoforce.

Observer *S*′, however, sees the glider accelerate and can find no object in the environment of the glider that exerted a force on it to provide its observed forward acceleration. To preserve the applicability of Newton's second law, *S*′ must assume that a force (in this case a pseudoforce) acts on the glider. According to *S*′, this force $\vec{\mathbf{F}}'$ must equal $m\vec{\mathbf{a}}'$, where $\vec{\mathbf{a}}'$ $(=-\vec{\mathbf{a}})$ is the acceleration of the glider measured by *S*′. The *x* component of this pseudoforce is

$$F'_x = ma'_x = (0.25\text{ kg})(2.8\text{ m/s}^2) = 0.70\text{ N},$$

and its direction is the same as $\vec{\mathbf{a}}'$—that is, toward the front of the van. This force, which is very real from the point of view of *S*′, is not apparent to ground-based observer *S*, who has no need to introduce it to account for the motion of the glider.

One indication that pseudoforces are non-Newtonian is that they violate Newton's third law. To apply Newton's third law, *S*′ must find a reaction force exerted *by* the glider *on* some other body. No such reaction force can be found, and so Newton's third law is violated.

Pseudoforces are very real to those that experience them. Imagine yourself riding in a car that is rounding a curve to the left. To a ground observer, the car is experiencing a centripetal acceleration and therefore constitutes a noninertial reference frame. If the car has smooth vinyl seats, you will find yourself sliding across the seat to the right. To the ground observer, who is in an inertial frame, this is quite natural; your body is simply trying to obey Newton's first law and move in a straight line, and it is the car that is sliding to the left under you. From your point of view in the noninertial reference frame of the car, you must ascribe your sliding motion to a pseudoforce pulling you to the right. This type of pseudoforce is called a *centrifugal force,* meaning a force directed *away from* the center.

Riding on a merry-go-round, you are again in an accelerated and therefore noninertial reference frame in which objects will apparently move outward from the axis of rotation under the influence of the centrifugal force. If you hold a ball in your hand, it seems to you that the net horizontal force on the ball is zero, the outward centrifugal force being balanced by the inward force exerted on the ball by your hand. To a ground observer, who is in an inertial reference frame, the ball is moving in a circle, accelerating toward the center under the influence of the *centripetal* force you exert on it with your hand. To the ground observer, there is no centrifugal force because the net force on the ball is *not* zero: it is accelerating radially inward.

Pseudoforces can be used as the basis of practical devices. Consider the centrifuge, one of the most useful of laboratory instruments. As a mixture of substances moves rapidly in a circle, the more massive substances experience a larger centrifugal force mv^2/r and move farther away from the axis of rotation. The centrifuge thus uses a pseudoforce to separate substances by mass, just as the mass spectrometer (Section 3-4) uses an electromagnetic force to separate atoms by mass.

Another pseudoforce is called a *Coriolis* force. Suppose that you roll a ball inward with constant speed along a radial line painted on the floor of a rotating merry-go-round. At the instant you release it at the radius *r*, it has just the right tangential velocity (the same as yours) to be in circular motion. As it moves inward it would take a smaller tangential speed to maintain its circular motion at the same rate as its immediate surroundings. Because it has no way to lose tangential speed (we assume little friction between the ball and the floor), it moves a bit ahead of the painted line representing a uniform rotational speed. That is, in your rotating noninertial reference frame you would suggest that a sideways pseudoforce—a Coriolis force—causes the ball to veer steadily away from the line as it rolls inward. To a ground observer in an inertial frame, there is no Coriolis force: the ball moves in a straight line at a speed determined by the components of its velocity at the instant of release.

Perhaps the most familiar example of the effects of the Coriolis force is in the motion of the atmosphere around centers of low or high pressure. Figure 5-23 shows a diagram of a low-pressure center in the northern hemisphere.

Because the pressure is lower than the surroundings, air flows radially inward in all directions. As the Earth rotates (making it a noninertial frame), the effect is similar to that of the ball on the merry-go-round: air rushing inward from the south moves a bit ahead of an imaginary line drawn on the rotating Earth, while air from the north (like a ball rolled outward on the merry-go-round) lags a bit behind the line. The total effect is that the air rotates in a counterclockwise direction around the low-pressure center. This Coriolis effect is thus responsible for the circulation of winds in a cyclone or hurricane. In the southern hemisphere the effects are reversed.

It is necessary to correct for the Coriolis effect of the rotating Earth in the motion of long-range artillery shells. For a typical shell of range 10 km, the Coriolis effect may cause a deflection as large as 20 m. Such corrections are built into the computer programs used to control the aiming and firing of long-range weapons. Things can go wrong, however, as the British Navy discovered in a World War I battle near the Falkland Islands. Its fire control manuals were written for the northern hemisphere, and the Falklands are in the southern hemisphere where the Coriolis correction would be in the opposite direction. The British shells were landing about 100 m from their targets, because the correction for the Coriolis effect was being made in the wrong direction!

In mechanical problems, then, we have two choices: (1) select an *inertial* reference frame and consider only "real" forces—that is, forces that we can associate with definite bodies in the environment; or (2) select a *noninertial* reference frame and consider not only the "real" forces but suitably defined pseudoforces. Although we usually choose the first alternative, we sometimes choose the second; both are completely equivalent and the choice is a matter of convenience. ■

5-7 LIMITATIONS OF NEWTON'S LAWS (Optional)

In the first five chapters, we have described a system for analyzing mechanical behavior with a seemingly vast range of applications. With little more than the equations of Newton's laws, you can design great skyscrapers and suspension bridges, or even plan the trajectory of an interplanetary spacecraft (Fig. 5-24). Newtonian mechanics, which provided these computational tools, was the first truly revolutionary development in theoretical physics.

Here is an example of our faith in Newton's laws. Galaxies and clusters of galaxies are often observed to rotate, and by observation we can deduce the speed of rotation. From this we can calculate the amount of matter that must be present in the galaxy or cluster for gravity to supply the centripetal force corresponding to the observed rotation. Yet the amount of matter that we actually observe with telescopes is far less than we expect. Therefore, it has been proposed that there is additional "dark matter" that we can-

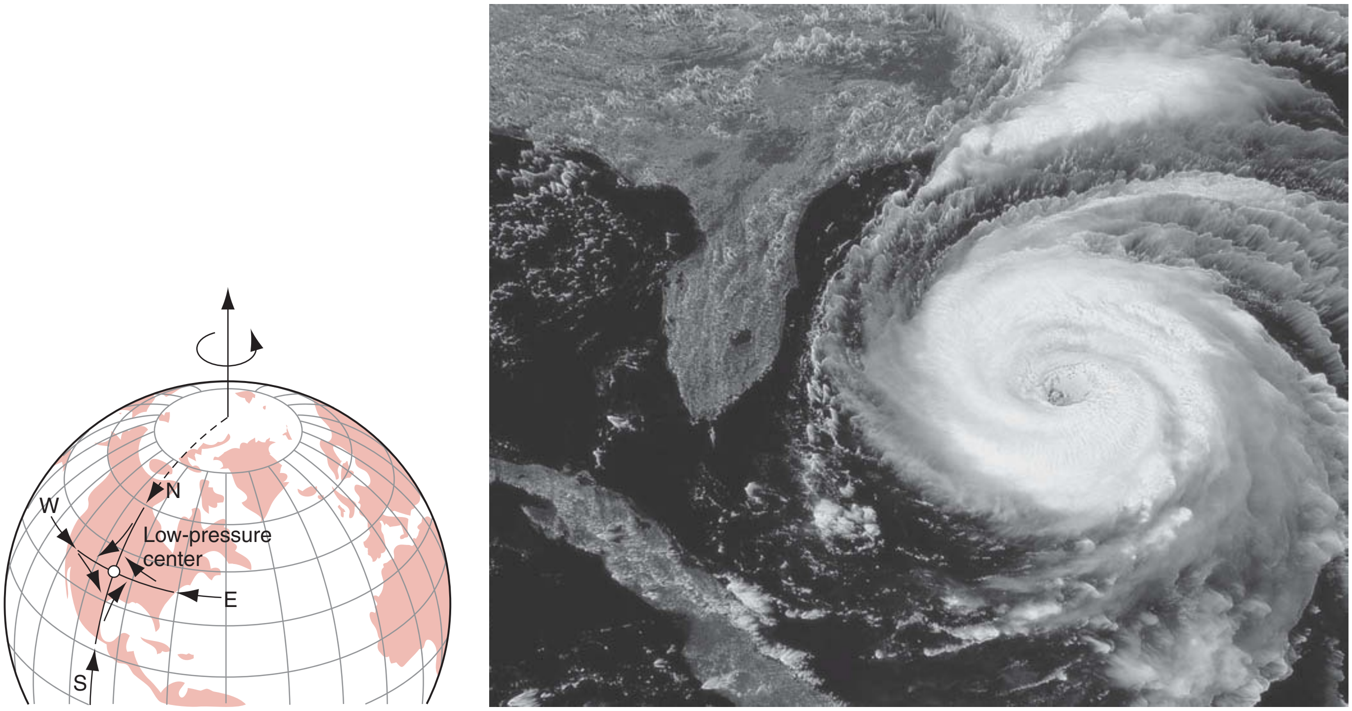


FIGURE 5-23. A low-pressure center on the rotating Earth. As the air flows inward, it appears to rotate counterclockwise to noninertial observers in the northern hemisphere of the rotating Earth. A hurricane (photo) is such a low-pressure center.

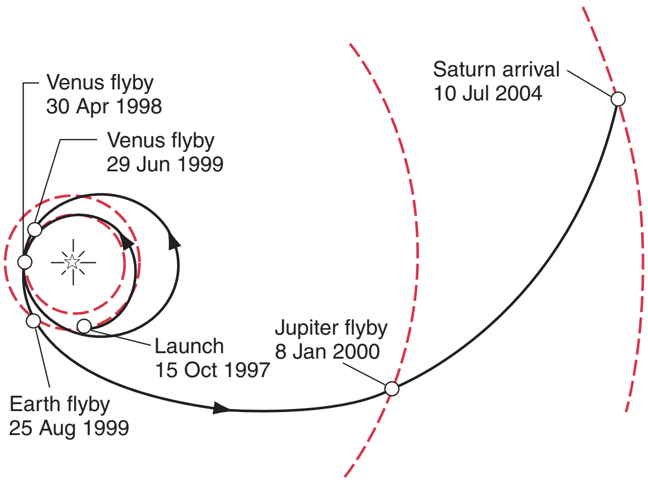


FIGURE 5-24. The trajectory of the Cassini mission to Saturn, launched from Earth on October 15, 1997. The ability to calculate such trajectories with pinpoint accuracy is a triumph of the methods of classical mechanics. The four planetary flybys are used to provide "gravity assists" that increase the speed of the spacecraft (see Section 6-1) and enable it to reach Saturn. To learn more about this mission, see the Web site at http://www.jpl.nasa.gov/cassini.

not see with telescopes but that must be present to provide the needed gravitational force. There is as yet no convincing candidate for the type or nature of this dark matter, and so other explanations have been proposed for the apparent inconsistency between the amount of matter actually observed in the galaxies and the amount we think is needed to satisfy Newton's laws. One proposed explanation is that our calculations are incorrect because Newton's laws do not hold in the conditions that we find on the very large scale—that is, when the accelerations are very small (below a few times 10^{-10} m/s^2). In particular, it has been proposed that for these very small accelerations, the force is proportional to a^2 instead of a.

Figure 5-25 shows the results of an experiment testing this supposition. If force depended on the acceleration to some power other than 1, the data would not fall on a straight line. From this extremely precise experiment, we conclude that down to accelerations of about 10^{-10} m/s^2, force is proportional to acceleration and Newton's second law holds.

In the 20th century, we have experienced three other revolutionary developments: Einstein's special theory of relativity (1905), his general theory of relativity (1915), and quantum mechanics (in about 1925). Special relativity teaches us that we cannot extrapolate the use of Newton's laws to particles moving at speeds comparable to the speed of light. General relativity shows that we cannot use Newton's laws in the vicinity of extremely massive objects. Quantum mechanics teaches us that we cannot extrapolate Newton's laws to objects as small as atoms.

Special relativity, which involves a distinctly non-Newtonian view of space and time, can be applied under all circumstances, at both high speeds and low speeds. In the limit of low speeds, it can be shown that the dynamics of special relativity reduces directly to Newton's laws. Similarly, general relativity can be applied to weak as well as strong gravitational forces, but its equations reduce to Newton's laws for weak forces. Quantum mechanics can be applied to individual atoms, where a certain randomness of behavior is predicted, or to ordinary objects containing a huge number of atoms, in which case the randomness averages out to give Newton's laws once again.

Within the past two decades, another apparently revolutionary development has emerged. This new development concerns mechanical systems whose behavior is described as *chaotic*. One of the hallmarks of Newton's laws is their ability to predict the future behavior of a system, if we know the forces that act and the initial motion. For example, from the initial position and velocity of a space probe that experiences known gravitational forces from the Sun and the planets, we can calculate its exact trajectory. On the other hand, consider a twig floating in a turbulent stream. Even though it is acted on at all times by forces governed by Newtonian mechanics, its path downstream is totally unpredictable. If two twigs are released side-by-side in the stream, they may be found very far apart downstream. One particular theme of chaotic dynamics is that tiny changes in the initial conditions of a problem can be greatly amplified and can cause substantial differences in the predicted outcomes. Chaotic dynamics is often invoked in weather forecasting, and it has been said that the fluttering of a butterfly over Japan could be related to the subsequent development of a hurricane over the Gulf of Mexico.

Such chaotic motions occur not only in complex systems like a turbulent stream but also in simple physical systems as well, such as a pendulum, a slowly dripping faucet, or an oscillating electrical circuit. In the 1960s it was discovered that the seemingly chaotic behavior of these sys-

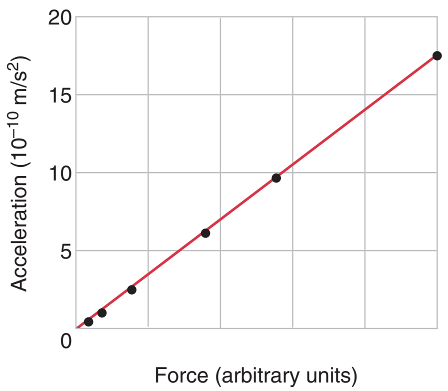


FIGURE 5-25. Results of an experiment to test whether Newton's second law holds for small accelerations below 10^{-9} m/s^2. The straight line shows that acceleration is proportional to the applied force down to 10^{-10} m/s^2, and so Newton's law remains valid even at such small accelerations.

tems conceals a hidden order and regularity, the study of which has formed the core of a new branch of science, *chaos*.* Applications of the laws of chaos have been found not only in physical systems but in biological systems as well. Even areas of social science, such as economics and population dynamics, show chaotic behavior.

Calculations combining the Newtonian mechanics of particles with chaos theory have shown that the orbit of the planet Pluto is chaotic on a time scale of tens of millions of years (a short time compared with the age of the solar system, about 4.5 billion years, but a long time compared with Pluto's orbital period around the Sun, about 250 years). Chaos theory has also been used to explain two properties of the asteroid belt (which lies between the orbits of Mars and Jupiter) that could not be understood within the framework of conventional Newtonian mechanics: (1) many asteroids stray from what should be stable orbits, some of them becoming meteorites that rain steadily down on Earth, and (2) within the asteroid belt are several empty gaps where the number of orbiting asteroids is small or zero. It has been only within the past decade that high-speed computers have permitted detailed calculations of the dynamics of such systems to be followed for the time scales necessary to observe this unusual behavior, and as additional calculations are done, new applications of this exciting field continue to be discovered. ■

*See *Chaos—Making a New Science,* by James Gleick (Penguin Books, 1987).

MULTIPLE CHOICE

5-1 Force Laws

5-2 Tension and Normal Forces

1. A spring balance is attached on both ends to strings; the strings hang over frictionless pulleys and are each connected to 20 N weights as shown in Fig. 5-26. The reading on the scale will be closest to

(A) 0 N (B) 10 N (C) 20 N (D) 40 N

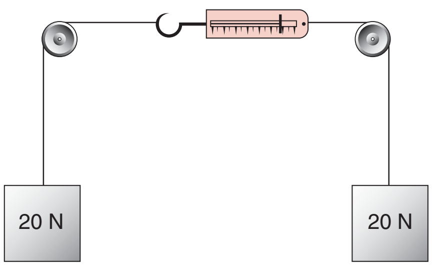


FIGURE 5-26. Multiple-choice question 1.

2. Which of the following statements is most correct?
- (A) The normal force is the same thing as the weight.
- (B) The normal force is different from the weight, but always has the same magnitude.
- (C) The normal force is different from the weight, but the two form an action–reaction pair according to Newton's third law.
- (D) The normal force is different from the weight, but the two may have the same magnitude in certain cases.

3. A wooden box is sitting on a table. The normal force on the box from the table is 75 N. A second identical box is placed on top of the first box. The normal force on the first box by the table will

(A) decrease. (B) remain at 75 N.
(C) increase to 150 N. (D) increase to 300 N.

4. A woman can stand in running shoes or stiletto high heels on a level surface. Assuming her total mass is the same regardless of the shoes she wears, then the normal force on her shoes by the ground is
- (A) greater in the case of the running shoes, because of the larger area of contact with the ground.
- (B) the same for either pair of shoes.
- (C) greater for the stiletto heels, because of the smaller area of contact with the ground.
- (D) solely dependent on whether she is standing with her knees bent.

5. A *real* rope is hanging by one end from the ceiling. The other end dangles freely. If the mass of the rope is 100 g, then the tension is
- (A) 0.98 N along the entire length of the rope.
- (B) 0.49 N along the entire length of the rope.
- (C) 0.98 N at the bottom of the rope, and varies linearly to zero at the top of the rope.
- (D) 0.98 N at the top of the rope, and varies linearly to zero at the bottom of the rope.

6. A bird of weight W is resting at the center of a stretched wire of negligible mass. Each half of the wire makes a small angle with the horizontal. What can be concluded about the tension T in the wire?

(A) $T < W/2$ (B) $W/2 \leq T \leq W$ (C) $T > W$
(D) More information is needed to answer the question.

5-3 Frictional Forces

7. Which statement is correct about the weight of an object and the force of kinetic friction on that object?
- (A) The weight is always greater than the frictional force.
- (B) The weight is always equal to the frictional force.
- (C) The weight is less than the frictional force for sufficiently light objects.
- (D) The weight can be more or less than the frictional force.

8. A 2.0-kg block of wood is on a level surface where $\mu_s = 0.80$ and $\mu_k = 0.60$. A 13.7-N force is being applied to the block parallel to the surface.

(*a*) If the block was originally at rest, then

(A) it will remain at rest, and the force of friction will be about 15.7 N.
(B) it will remain at rest, and the force of friction will be about 13.7 N.
(C) it will remain at rest, and the force of friction will be 11.8 N.
(D) it will begin to slide with a net force of about 1.9 N acting on the block.

(*b*) If the block was originally in motion, and the 13.7-N applied force is in the direction of motion, then

(A) it will accelerate under a net force of about 1.9 N.
(B) it will move at constant speed.
(C) it will decelerate under a net force of about 1.9 N.
(D) it will decelerate under a net force of 11.8 N.

9. Two similar wooden blocks are tied one behind the other and pulled across a level surface. Friction is not negligible. The force required to pull them at constant speed is F. If one block is stacked upon the other then the new force required to pull them at constant speed will be approximately

(A) $F/2$. (B) F. (C) $\sqrt{2}F$. (D) $2F$.

10. Automatic braking systems (ABS) on automobiles prevent the tires from locking by sensing when the tires stop spinning and then reducing the braking force until they begin to spin again. Knowing that $\mu_s > \mu_k > \mu_{\text{rolling}}$, an automobile equipped with ABS will

(A) always stop in a shorter distance
(B) stop in shorter distance on dry pavement but not on wet pavement
(C) stop in about the same distance
(D) always stop in a longer distance

than an automobile that stops by locking the tires.

11. A 1.0-kg block of wood sits on top of an identical block of wood, which sits on top of a flat level table made of plastic. The coefficient of static friction between the wood surfaces is μ_1, and the coefficient of static friction between the wood and plastic is μ_2.

(*a*) A horizontal force F is applied to the top block only, and this force is increased until the top block starts to slide. The bottom block will slide with the top block if and only if

(A) $\mu_1 < \frac{1}{2}\mu_2$. (B) $\frac{1}{2}\mu_2 < \mu_1 < \mu_2$.
(C) $\mu_2 < \mu_1$. (D) $2\mu_2 < \mu_1$.

(*b*) Instead a horizontal force F is applied to the bottom block only, and this force is increased until the bottom block just starts to slide. Under what conditions will the top block slide with the bottom block?

(A) If $\mu_1 > 0$ the top block will slide regardless of μ_2.
(B) $\frac{1}{2}\mu_2 < \mu_1 < \mu_2$ (C) $\mu_2 < \mu_1$ (D) $2\mu_2 < \mu_1$

5-4 The Dynamics of Uniform Circular Motion

12. A motorcycle moves around a vertical circle with a constant speed under the influence of the force of gravity $\vec{\mathbf{W}}$, the force of friction between the wheels and the track $\vec{\mathbf{f}}$, and the normal force between the wheels and the track $\vec{\mathbf{N}}$.

(*a*) Which of the following quantities has a constant magnitude?

(A) $\vec{\mathbf{N}}$ (B) $\vec{\mathbf{N}} + \vec{\mathbf{f}}$ (C) $\vec{\mathbf{f}} + \vec{\mathbf{W}}$ (D) $\vec{\mathbf{N}} + \vec{\mathbf{W}} + \vec{\mathbf{f}}$

(*b*) Which of the following quantities, when nonzero, is always directed toward the center of the circle?

(A) $\vec{\mathbf{f}}$ (B) $\vec{\mathbf{W}}$ (C) $\vec{\mathbf{f}} + \vec{\mathbf{W}}$ (D) $\vec{\mathbf{N}} + \vec{\mathbf{f}}$

13. An automobile drives over some hilly terrain. The motion of the automobile at the top of a hill is instantaneously similar to circular motion with the center of curvature beneath the road. The motion of the automobile at the bottom of a dip is instantaneously similar to circular motion with the center of curvature above the road. At any time there are three forces on the car: weight $\vec{\mathbf{W}}$, normal $\vec{\mathbf{N}}$, and friction $\vec{\mathbf{f}}$ of the tires with the road. The magnitudes of these three forces are given respectively by W, N, and f.

(*a*) When the car is at the top of a hill, which of the following gives the magnitude of the centripetal force?

(A) N (B) $W + N$ (C) $W - N$ (D) $N - W$

(*b*) When the car is at the bottom of a dip, which of the following gives the magnitude of the centripetal force?

(A) N (B) $W + N$ (C) $W - N$ (D) $N - W$

5-5 Time-Dependent Forces

5-6 Noninertial Frames and Pseudoforces

5-7 Limitations of Newton's Laws

QUESTIONS

1. You can pull a wagon with a rope, but you can't push it with a rope. Is there such a thing as a "negative" tension?

2. There is a limit beyond which further polishing of a surface increases rather than decreases frictional resistance. Explain why.

3. A crate, heavier than you are, rests on a rough floor. The coefficient of static friction between the crate and the floor is the same as that between the soles of your shoes and the floor. Can you push the crate across the floor? See Fig. 5-27.

4. In baseball, a base runner can usually get to a base quicker by running than by sliding. Explain why this is so. Why slide then?

5. How could a person who is at rest on completely frictionless ice covering a pond reach shore? Could she do this by walking, rolling, swinging her arms, or kicking her feet? How could a person be placed in such a position in the first place?

6. Why do tires grip the road better on level ground than they do when going uphill or downhill?

FIGURE 5-27. Question 3.

7. What is the purpose of curved surfaces, called spoilers, placed on the rear of sports cars? They are designed so that air flowing past exerts a downward force.

8. Two surfaces are in contact but are at rest relative to each other. Nevertheless, each exerts a force of friction on the other. Explain.

9. Your car skids across the centerline on an icy highway. Should you turn the front wheels in the direction of the skid or in the opposite direction (*a*) when you want to avoid a collision with an oncoming car and (*b*) when no other car is near but you want to regain control of the steering? Assume rear-wheel drive, then front-wheel drive.

10. Why is it that racing drivers actually speed up when traversing a curve?

11. You are flying a plane at constant altitude and you wish to make a 90° turn. Why do you bank in order to do so?

12. When a wet dog shakes itself, people standing nearby tend to get wet. Why does the water fly outward from the dog in this way?

13. You must have noticed (Einstein did) that when you stir a cup of tea, the floating tea leaves collect at the center of the cup rather than at the outer rim. Can you explain this? (Einstein could.)

14. Suppose that you need to measure whether a tabletop in a train is truly horizontal. If you use a spirit level, can you determine this when the train is moving down or up a grade? When the train is moving along a curve? (Hint: There are two horizontal components.)

15. In the conical pendulum, what happens to the period and the speed when $\theta = 90°$? Why is this angle not achievable physically? Discuss the case for $\theta = 0°$.

16. A coin is put on a phonograph turntable. The motor is started but, before the final speed of rotation is reached, the coin flies off. Explain why.

17. A car is riding on a country road that resembles a roller coaster track. If the car travels with uniform speed, compare the force it exerts on a horizontal section of the road to the force it exerts on the road at the top of a hill and at the bottom of a hill. Explain.

18. You are driving a station wagon at uniform speed along a straight highway. A beach-ball rests at the center of the wagon bed and a helium-filled balloon floats above it, touching the roof of the wagon. What happens to each if you (*a*) turn a corner at constant speed or (*b*) apply the brakes?

19. How does the Earth's rotation affect the measurement of the weight of an object at the equator?

20. Explain why a plumb bob will not hang exactly in the direction of the Earth's gravitational attraction (toward the center of the Earth) at most latitudes.

21. Astronauts in the orbiting space shuttle want to keep a daily record of their weight. Can you think how they might do it, considering that they are "weightless"?

22. Explain how the question "What is the linear velocity of a point on the equator?" requires an assumption about the reference frame used. Show how the answer changes as you change reference frames.

23. What is the distinction between inertial reference frames and those differing only by a translation or rotation of the axes?

24. A passenger in the front seat of a car finds himself sliding toward the door as the driver makes a sudden left turn. Describe the forces on the passenger and on the car at this instant if the motion is viewed from a reference frame (*a*) attached to the Earth and (*b*) attached to the car.

25. Do you have to be concerned with the Coriolis effect when playing tennis or golf? If not, why not?

26. Suppose that you are standing on the balcony of a tall tower, facing east. You drop an object so that it falls to the ground below; see Fig. 5-28. Suppose also that you can locate the impact point very precisely. Will the object strike the ground at *a*, vertically below the release point, at *b* to the east, or at *c* to the west? The object was released from rest; the Earth rotates from west to east.

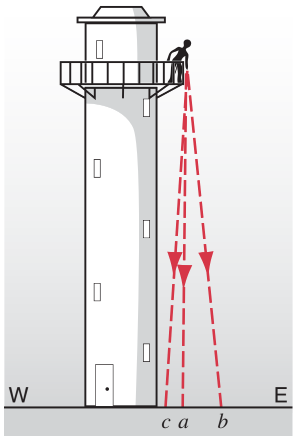


FIGURE 5-28. Question 26.

27. Show by a qualitative argument that, because of the rotation of the Earth, a wind in the northern hemisphere blowing from north to south will be deflected to the right. What about a wind that is blowing from south to north? What is the situation in the southern hemisphere?

EXERCISES

5-1 Force Laws

5-2 Tension and Normal Forces

1. A charged sphere of mass 2.8×10^{-4} kg is suspended from a string. An electric force acts horizontally on the sphere so that the string makes an angle of 33° with the vertical when at rest. Find (*a*) the magnitude of the electric force and (*b*) the tension in the string.

2. An elevator weighing 6200 lb is pulled upward by a cable with an acceleration of 3.8 ft/s^2. (*a*) What is the tension in the cable? (*b*) What is the tension when the elevator is accelerating downward at 3.8 ft/s^2 but is still moving upward?

3. A lamp hangs vertically from a cord in a descending elevator. The elevator has a deceleration of 2.4 m/s^2 before coming to a stop. (*a*) If the tension in the cord is 89 N, what is the mass of

the lamp? (*b*) What is the tension in the cord when the elevator ascends with an upward acceleration of $2.4\ m/s^2$?

4. An elevator and its load have a combined mass of 1600 kg. Find the tension in the supporting cable when the elevator, originally moving downward at 12.0 m/s, is brought to rest with constant acceleration in a distance of 42.0 m.

5. A 110-kg man lowers himself to the ground from a height of 12 m by holding on to a rope passed over a frictionless pulley and attached to a 74-kg sandbag. (*a*) With what speed does the man hit the ground? (*b*) Is there anything he could do to reduce the speed with which he hits the ground?

6. An 11-kg monkey is climbing a massless rope attached to a 15-kg log over a frictionless tree limb. (*a*) With what minimum acceleration must the monkey climb up the rope so that it can raise the 15-kg log off the ground? If, after the log has been raised off the ground, the monkey stops climbing and hangs on to the rope, what will now be (*b*) the monkey's acceleration and (*c*) the tension in the rope?

7. Figure 5-29 shows a section of an alpine cable-car system. The maximum permitted mass of each car with occupants is 2800 kg. The cars, riding on a support cable, are pulled by a second cable attached to each pylon. What is the difference in tension between adjacent sections of pull cable if the cars are accelerated up to 35° incline at $0.81\ m/s^2$?

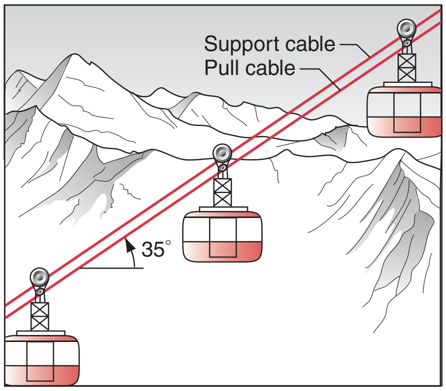


FIGURE 5-29. Exercise 7.

8. The man in Fig. 5-30 weighs 180 lb; the platform and attached frictionless pulley weigh a total of 43 lb. Ignore the weight of the rope. With what force must the man pull up on the rope in order to lift himself and the platform upward at $1.2\ ft/s^2$?

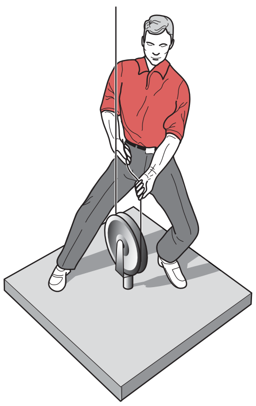


FIGURE 5-30. Exercise 8.

5-3 Frictional Forces

9. The coefficient of static friction between Teflon and scrambled eggs is about 0.04. What is the smallest angle from the horizontal that will cause the eggs to slide across the bottom of a Teflon-coated skillet?

10. Suppose that only the rear wheels of an automobile can accelerate it, and that half the total weight of the automobile is supported by those wheels. (*a*) What is the maximum acceleration attainable if the coefficient of static friction between tires and road is μ_s? (*b*) Take $\mu_s = 0.56$ and get a numerical value for this acceleration.

11. What is the greatest acceleration that can be generated by a runner if the coefficient of static friction between shoes and road is 0.95?

12. A baseball player (Fig. 5-31) with mass 79 kg, sliding into a base, is slowed by a force of friction of 470 N. What is the coefficient of kinetic friction between the player and the ground?

FIGURE 5-31. Exercise 12.

13. A horizontal bar is used to support a 75-kg object between two walls, as shown in Fig. 5-32. The equal forces F exerted by the bar against the walls can be varied by adjusting the length of the bar. Only friction between the ends of the bar and the walls supports the system. The coefficient of static friction between bar and walls is 0.41. Find the minimum value of the forces F for the system to remain at rest.

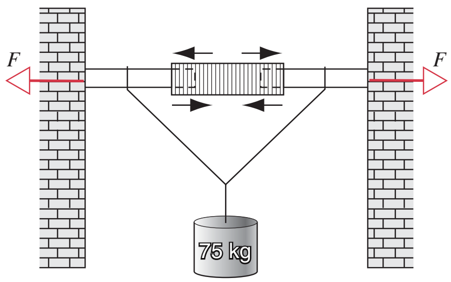


FIGURE 5-32. Exercise 13.

14. A 53-lb (= 240-N) trunk rests on the floor. The coefficient of static friction between the trunk and the floor is 0.41, while the coefficient of kinetic friction is 0.32. (*a*) What is the minimum horizontal force with which a person must push on the trunk to start it moving? (*b*) Once moving, what horizontal force must the person apply to keep the trunk moving with constant velocity? (*c*) If, instead, the person continued to push with the force used to start the motion, what would be the acceleration of the trunk?

15. The coefficient of static friction between the tires of a car and a dry road is 0.62. The mass of the car is 1500 kg. What maximum braking force is obtainable (*a*) on a level road and (*b*) on an 8.6° downgrade?

16. A house is built on the top of a hill with a 42° slope. Subsequent slumping of material on the slope surface indicates that the slope gradient should be reduced. If the coefficient of friction of soil on soil is 0.55, through what additional angle ϕ (see Fig. 5-33) should the slope surface be regraded?

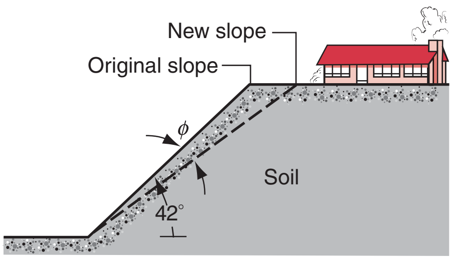


FIGURE 5-33. Exercise 16.

17. A 136-kg crate is at rest on the floor. A worker attempts to push it across the floor by applying a 412-N force horizontally. (*a*) Take the coefficient of static friction between the crate and floor to be 0.37 and show that the crate does not move. (*b*) A second worker helps by pulling up on the crate. What minimum vertical force must this worker apply so that the crate starts to move across the floor? (*c*) If the second worker applies a horizontal rather than a vertical force, what minimum force, in addition to the original 412-N force, must be exerted to get the crate started?

18. A student wants to determine the coefficients of static friction and kinetic friction between a box and a plank. She places the box on the plank and gradually raises one end of the plank. When the angle of inclination with the horizontal reaches 28.0°, the box starts to slip and slides 2.53 m down the plank in 3.92 s. Find the coefficients of friction.

19. Frictional heat generated by the moving ski is the chief factor promoting sliding in skiing. The ski sticks at the start, but once in motion will melt the snow beneath it. Waxing the ski makes it water repellent and reduces friction with the film of water. A magazine reports that a new type of plastic ski is even more water repellent and that, on a gentle 203-m slope in the Alps, a skier reduced his time from 61 to 42 s with the new skis. Assuming a 3.0° slope, compute the coefficient of kinetic friction for each case.

20. A block slides down an inclined plane of slope angle θ with constant velocity. It is then projected up the same plane with an initial speed v_0. (*a*) How far up the incline will it move before coming to rest? (*b*) Will it slide down again?

21. A piece of ice slides from rest down a rough 33.0° incline in twice the time it takes to slide down a frictionless 33.0° incline of the same length. Find the coefficient of kinetic friction between the ice and the rough incline.

22. In Fig. 5-34, *A* is a 4.4-kg block and *B* is a 2.6-kg block. The coefficients of static and kinetic friction between *A* and the table are 0.18 and 0.15. (*a*) Determine the minimum mass of the block *C* that must be placed on *A* to keep it from sliding. (*b*) Block *C* is suddenly lifted off *A*. What is the acceleration of block *A*?

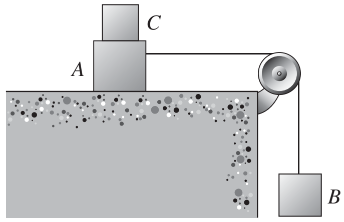


FIGURE 5-34. Exercise 22.

23. A 4.8-kg block on a 39° inclined plane is acted on by a horizontal force of 46 N (see Fig. 5-35). The coefficient of kinetic friction between block and plane is 0.33. (*a*) What is the acceleration of the block if it is moving up the plane? (*b*) With the horizontal force still acting, how far up the plane will the block go if it has an initial upward speed of 4.3 m/s? (*c*) What happens to the block after it reaches the highest point?

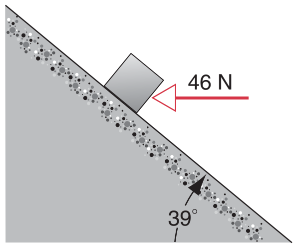


FIGURE 5-35. Exercise 23.

24. A 12-kg block of steel is at rest on a horizontal table. The coefficient of static friction between block and table is 0.52. (*a*) What is the magnitude of the horizontal force that will just start the block moving? (*b*) What is the magnitude of a force acting upward 62° from the horizontal that will just start the block moving? (*c*) If the force acts down at 62° from the horizontal, how large can it be without causing the block to move?

25. A worker drags a 150-lb crate across a floor by pulling on a rope inclined 17° above the horizontal. The coefficient of static friction is 0.52 and the coefficient of kinetic friction is 0.35. (*a*) What tension in the rope is required to start the crate moving? (*b*) What is the initial acceleration of the crate?

26. A wire will break under tensions exceeding 1.22 kN. If the wire, not necessarily horizontal, is used to drag a box across the floor, what is the greatest weight that can be moved if the coefficient of static friction is 0.35?

27. Block B in Fig. 5-36 weighs 712 N. The coefficient of static friction between block B and the table is 0.25. Find the maximum weight of block A for which block B will remain at rest.

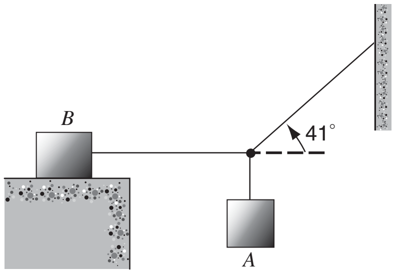


FIGURE 5-36. Exercise 27.

28. Block m_1 in Fig. 5-37 has a mass of 4.20 kg and block m_2 has a mass of 2.30 kg. The coefficient of kinetic friction between m_2 and the horizontal plane is 0.47. The inclined plane is frictionless. Find (*a*) the acceleration of the blocks and (*b*) the tension in the string.

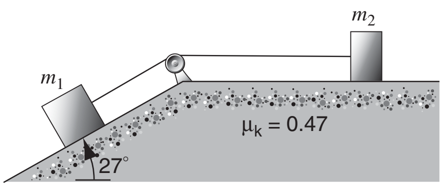


FIGURE 5-37. Exercise 28.

29. In Fig. 5-38, object B weighs 94.0 lb and object A weighs 29.0 lb. Between object B and the plane the coefficient of static friction is 0.56 and the coefficient of kinetic friction is 0.25. (*a*) Find the acceleration of the system if B is initially at rest. (*b*) Find the acceleration if B is moving up the plane. (*c*) What is the acceleration if B is moving down the plane? The plane is inclined by 42.0°.

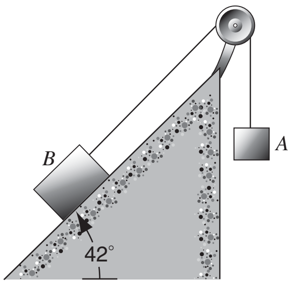


FIGURE 5-38. Exercise 29.

30. A crate slides down an inclined right-angled trough as in Fig. 5-39. The coefficient of kinetic friction between the crate and the material composing the trough is μ_k. Find the acceleration of the crate.

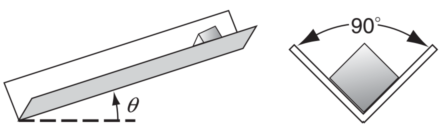


FIGURE 5-39. Exercise 30.

31. A 42-kg slab rests on a frictionless floor. A 9.7-kg block rests on top of the slab, as in Fig. 5-40. The coefficient of static friction between the block and the slab is 0.53, while the coefficient of kinetic friction is 0.38. The 9.7-kg block is acted on by a horizontal force of 110 N. What are the resulting accelerations of (*a*) the block and (*b*) the slab?

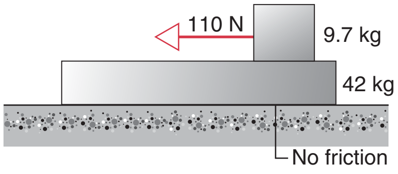


FIGURE 5-40. Exercise 31.

5-4 The Dynamics of Uniform Circular Motion

32. During an Olympic bobsled run, a European team takes a turn of radius 25 ft at a speed of 60 mi/h. What acceleration do the riders experience (*a*) in ft/s^2 and (*b*) in units of g?

33. A 2400-lb (= 10.7-kN) car traveling at 30 mi/h (= 13.4 m/s) attempts to round an unbanked curve with a radius of 200 ft (= 61.0 m). (*a*) What force of friction is required to keep the car on its circular path? (*b*) What minimum coefficient of static friction between the tires and road is required?

34. A circular curve of highway is designed for traffic moving at 60 km/h (= 37 mi/h). (*a*) If the radius of the curve is 150 m (= 490 ft), what is the correct angle of banking of the road? (*b*) If the curve were not banked, what would be the minimum coefficient of friction between tires and road that would keep traffic from skidding at this speed?

35. A conical pendulum is formed by attaching a 53-g pebble to a 1.4-m string. The pebble swings around in a circle of radius 25 cm. (*a*) What is the speed of the pebble? (*b*) What is its acceleration? (*c*) What is the tension in the string?

36. A bicyclist (Fig. 5-41) travels in a circle of radius 25 m at a constant speed of 8.7 m/s. The combined mass of the bicycle and rider is 85 kg. Calculate the force—magnitude and angle with the vertical—exerted by the road on the bicycle.

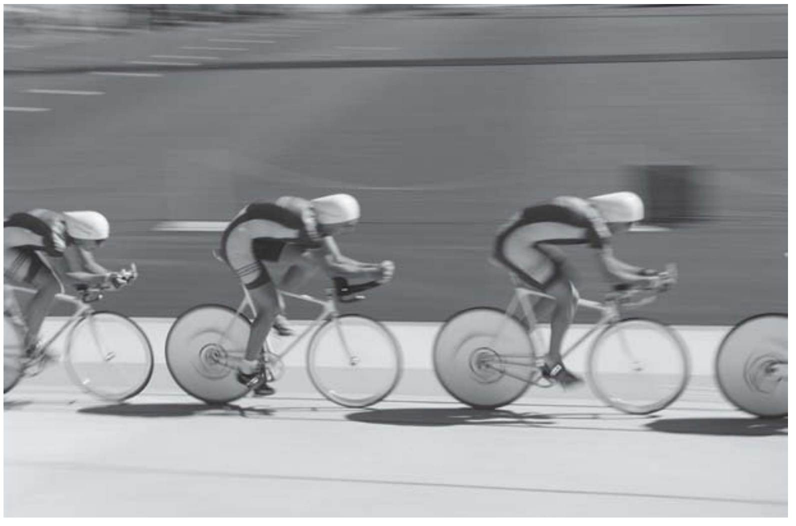


FIGURE 5-41. Exercise 36.

37. In the Bohr model of the hydrogen atom, the electron revolves in a circular orbit around the nucleus. If the radius is 5.3×10^{-11} m and the electron makes 6.6×10^{15} rev/s, find (*a*) the speed of the electron, (*b*) the acceleration of the electron, and (*c*) the force acting on the electron. (This force is the result of the attraction between the positively charged nucleus and the negatively charged electron.)

38. A child places a picnic basket on the outer rim of a merry-go-round that has a radius of 4.6 m and revolves once every 24 s. How large must the coefficient of static friction be for the basket to stay on the merry-go-round?

39. A disk of mass m on a frictionless table is attached to a hanging cylinder of mass M by a cord through a hole in the table (see Fig. 5-42). Find the speed with which the disk must move in a circle of radius r for the cylinder to stay at rest.

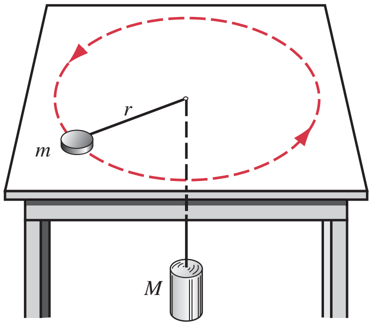


FIGURE 5-42. Exercise 39.

40. A driver's manual states that a driver traveling at 48 km/h and desiring to stop as quickly as possible travels 10 m before the foot reaches the brake. The car travels an additional 21 m before coming to rest. (*a*) What coefficient of friction is assumed in these calculations? (*b*) What is the minimum radius for turning a corner at 48 km/h without skidding?

41. A banked circular highway curve is designed for traffic moving at 95 km/h. The radius of the curve is 210 m. Traffic is moving along the highway at 52 km/h on a stormy day. (*a*) What is the minimum coefficient of friction between tires and road that will allow cars to negotiate the turn without sliding? (*b*) With this value of the coefficient of friction, what is the greatest speed at which the curve can be negotiated without sliding?

42. A 150-lb student on a steadily rotating Ferris wheel is sitting on a scale that reads 125 lb at the highest point. (*a*) What is the scale reading at the lowest point? (*b*) What would be the scale reading at the highest point if the speed of the Ferris wheel were doubled?

43. A small object is placed 13.0 cm from the center of a phonograph turntable. It is observed to remain on the table when it rotates at $33\frac{1}{3}$ rev/min but slides off when it rotates at 45.0 rev/min. Between what limits must the coefficient of static friction between the object and the surface of the turntable lie?

44. An airplane is flying in a horizontal circle at a speed of 482 km/h. The wings of the plane are tilted at 38.2° to the horizontal; see Fig. 5-43. Find the radius of the circle in which the plane is flying. Assume that the centripetal force is provided entirely by the lift force perpendicular to the wing surface.

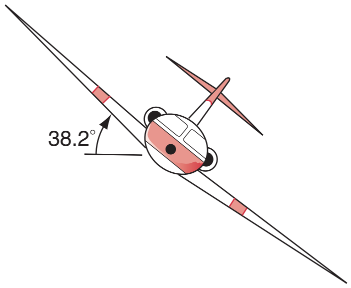


FIGURE 5-43. Exercise 44.

45. A frigate bird is soaring in a horizontal circular path. Its bank angle is estimated to be 25° and it takes 13 s for the bird to complete one circle. (*a*) How fast is the bird flying? (*b*) What is the radius of the circle? (See "The Amateur Scientist" by Jearl Walker, *Scientific American,* March 1985, p. 122.)

46. A model airplane of mass 0.75 kg is flying at constant speed in a horizontal circle at one end of a 33-m cord and at a height of 18 m. The other end of the cord is tethered to the ground. The airplane makes 4.4 rev/min and the lift is perpendicular to the unbanked wings. (*a*) What is the acceleration of the plane? (*b*) What is the tension in the cord? (*c*) What is the lift produced by the plane's wings?

47. Assume that the standard kilogram would weigh exactly 9.80 N at sea level on the equator if the Earth did not rotate. Then take into account the fact that the Earth does rotate, so that this object moves in a circle of radius 6370 km (the Earth's radius) in one day. (*a*) Determine the centripetal force needed to keep the standard kilogram moving in its circular path. (*b*) Find the force exerted by the standard kilogram on a spring balance from which it is suspended at the equator (its apparent weight).

5-5 Time-Dependent Forces

48. The position of a particle of mass 2.17 kg traveling in a straight line is given by

$$x = (0.179\ \text{m/s}^4)t^4 - (2.08\ \text{m/s}^2)t^2 + 17.1\ \text{m}.$$

Find the (*a*) velocity, (*b*) acceleration, and (*c*) force on the particle at time $t = 7.18$ s.

49. A particle of mass m is subjected to a net force $\vec{\mathbf{F}}(t)$ given by $\vec{\mathbf{F}}(t) = F_0(1 - t/T)\hat{\mathbf{i}}$; that is, $F(t)$ equals F_0 at $t = 0$ and decreases linearly to zero in time T. The particle passes the origin $x = 0$ with velocity $v_0\hat{\mathbf{i}}$. Show that at the instant $t = T$ that $F(t)$ vanishes, the speed v and distance x traveled are given by $v(T) = v_0 + a_0T/2$, and $x(T) = v_0T + a_0T^2/3$, where $a_0 = F_0/m$ is the initial acceleration. Compare these results with Eqs. 2-26 and 2-28.

5-6 Noninertial Frames and Pseudoforces

5-7 Limitations of Newton's Laws

PROBLEMS

1. A block of mass m_1 on a frictionless inclined plane making an angle θ_1 with the horizontal is connected by a cord over a small frictionless massless pulley to a second block of mass m_2 on a frictionless plane at an angle θ_2 (see Fig. 5-44). (*a*) Show that the acceleration of each block is

$$a = \frac{m_1 \sin\theta_1 - m_2 \sin\theta_2}{m_1 + m_2} g$$

and that the tension in the cord is

$$T = \frac{m_1 m_2 g}{m_1 + m_2}(\sin\theta_1 + \sin\theta_2).$$

(*b*) Evaluate the acceleration and tension for $m_1 = 3.70$ kg and $m_2 = 4.86$ kg when $\theta_1 = 28°$ and $\theta_2 = 42°$. What direction does m_1 move along the plane? (*c*) Using the values of m_1, θ_1, and θ_2 given above, for what values of m_2 does m_1 accelerate up the plane? Accelerate down the plane? Not accelerate?

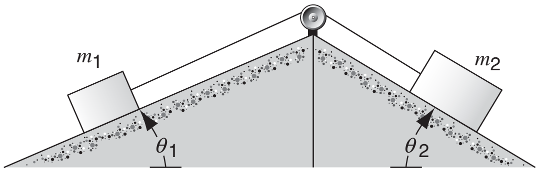


FIGURE 5-44. Problem 1.

2. Someone exerts a force F directly up on the axle of the pulley shown in Fig. 5-45. Consider the pulley and string to be massless and the bearing frictionless. Two objects, m_1 of mass 1.2 kg and m_2 of mass 1.9 kg, are attached as shown to the opposite ends of the string, which passes over the pulley. The object m_2 is in contact with the floor. (*a*) What is the largest value the force $\vec{\mathbf{F}}$ may have so that m_2 will remain at rest on the floor? (*b*) What is the tension in the string if the upward force F is 110 N? (*c*) With the tension determined in part (*b*), what is the acceleration of m_1?

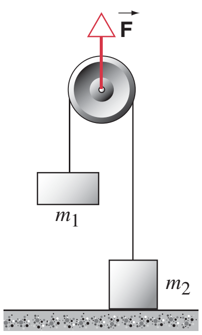


FIGURE 5-45. Problem 2.

3. Two particles, each of mass m, are connected by a light string of length $2L$, as shown in Fig. 5-46. A steady force $\vec{\mathbf{F}}$ is applied at the midpoint of the string ($x = 0$) at a right angle to the initial position of the string. Show that the acceleration of each mass in the direction at 90° to $\vec{\mathbf{F}}$ is given by

$$a_x = \frac{F}{2m}\frac{x}{(L^2 - x^2)^{1/2}}$$

in which x is the perpendicular distance of one of the particles from the line of action of $\vec{\mathbf{F}}$. Discuss the situation when $x = L$.

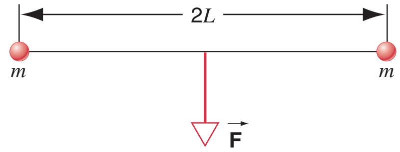


FIGURE 5-46. Problem 3.

4. A horizontal force F of 12 lb pushes a block weighing 5.0 lb against a vertical wall (Fig. 5-47). The coefficient of static friction between the wall and the block is 0.60 and the coefficient of kinetic friction is 0.40. Assume the block is not moving initially. (*a*) Will the block start moving? (*b*) What is the force exerted on the block by the wall?

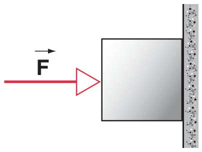


FIGURE 5-47. Problem 4.

5. A 7.96-kg block rests on a plane inclined at 22.0° to the horizontal, as shown in Fig. 5-48. The coefficient of static friction is 0.25, while the coefficient of kinetic friction is 0.15. (*a*) What is the minimum force F, parallel to the plane, that will prevent the block from slipping down the plane? (*b*) What is the minimum force F that will start the block moving up the plane? (*c*) What force F is required to move the block up the plane at constant velocity?

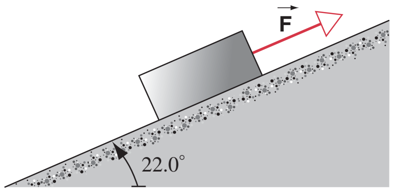


FIGURE 5-48. Problem 5.

6. A worker wishes to pile sand onto a circular area in his yard. The radius of the circle is R. No sand is to spill onto the surrounding area; see Fig. 5-49. Show that the greatest volume of sand that can be stored in this manner is $\pi\mu_s R^3/3$, where μ_s is the coefficient of static friction of sand on sand. (The volume of a cone is $Ah/3$, where A is the base area and h is the height.)

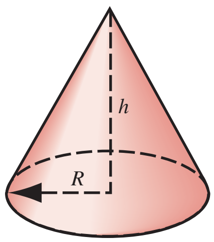


FIGURE 5-49. Problem 6.

7. The handle of a floor mop of mass m makes an angle θ with the vertical direction; see Fig. 5-50. Let μ_k be the coefficient of kinetic friction between mop and floor and μ_s the coefficient of static friction between mop and floor. Neglect the mass of the handle. (*a*) Find the magnitude of the force F directed along the handle required to slide the mop with uniform velocity across the floor. (*b*) Show that if θ is smaller than a certain angle θ_0 the mop cannot be made to slide across the floor no matter how great a force is directed along the handle. What is the angle θ_0?

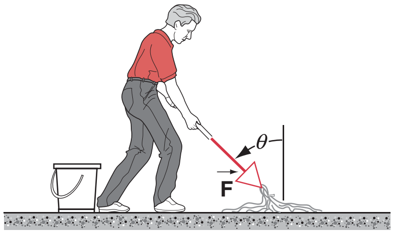


FIGURE 5-50. Problem 7.

8. Figure 5-51 shows the cross section of a road cut into the side of a mountain. The solid line AA' represents a weak bedding plane along which sliding is possible. The block B directly above the highway is separated from uphill rock by a large crack (called a *joint*), so that only the force of friction between the block and the likely surface of failure prevent sliding. The mass of the block is 1.8×10^7 kg, the dip angle of the failure plane is 24°, and the coefficient of static friction between block and plane is 0.63. (*a*) Show that the block will not slide. (*b*) Water seeps into the joint, exerting a hydrostatic force F parallel to the incline on the block. What minimum value of F will trigger a slide?

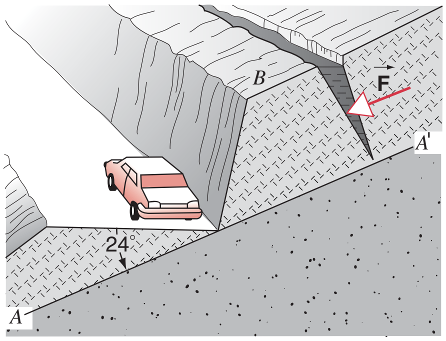


FIGURE 5-51. Problem 8.

9. The two blocks, $m = 16$ kg and $M = 88$ kg, shown in Fig. 5-52 are free to move. The coefficient of static friction between the blocks is $\mu_s = 0.38$, but the surface beneath M is frictionless. What is the minimum horizontal force F required to hold m against M?

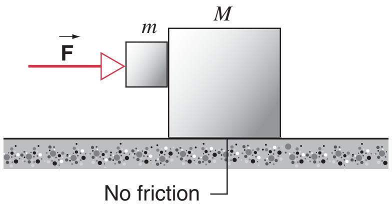


FIGURE 5-52. Problem 9.

10. Two objects, with masses $m_1 = 1.65$ kg and $m_2 = 3.22$ kg, attached by a massless rod parallel to the incline on which both slide, as shown in Fig. 5-53, travel down the plane with m_1 trailing m_2. The angle of the incline is $\theta = 29.5°$. The coefficient of kinetic friction between m_1 and the incline is $\mu_1 = 0.226$; between m_2 and the incline the corresponding coefficient is $\mu_2 = 0.127$. Compute (*a*) the common acceleration of the two objects and (*b*) the tension in the rod. (*c*) What are the answers to (*a*) and (*b*) if m_2 trails m_1?

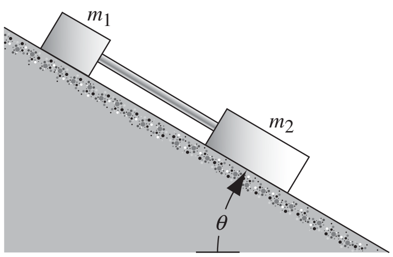


FIGURE 5-53. Problem 10.

11. A massless rope is tossed over a wooden dowel of radius r in order to lift a heavy object of weight W off of the floor, as shown in Fig. 5-54. The coefficient of sliding friction between the rope and the dowel is μ. Show that the minimum downward pull on the rope necessary to lift the object is

$$F_{\text{down}} = We^{\pi\mu}.$$

(Hint: This problem requires techniques from integral calculus.)

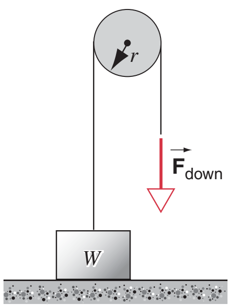


FIGURE 5-54. Problem 11.

12. A 4.40-kg block is put on top of a 5.50-kg block. In order to cause the top block to slip on the bottom one, held fixed, a horizontal force of 12.0 N must be applied to the top block. The assembly of blocks is now placed on a horizontal, frictionless table; see Fig. 5-55. Find (*a*) the maximum horizontal force F that can be applied to the lower block so that the blocks will move together, (*b*) the resulting acceleration of the blocks, and (*c*) the coefficient of static friction between the blocks.

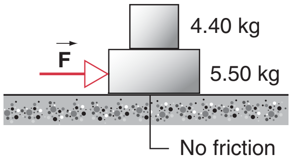


FIGURE 5-55. Problem 12.

13. You are driving a car at a speed of 85 km/h when you notice a barrier across the road 62 m ahead. (*a*) What is the minimum coefficient of static friction between tires and road that will allow you to stop without striking the barrier? (*b*) Suppose that you are driving at 85 km/h on a large empty parking lot. What is the minimum coefficient of static friction that would allow you to turn the car in a 62-m radius circle and, in this way, avoid collision with a wall 62 m ahead?

14. A car moves at a constant speed on a straight but hilly road. One section has a crest and dip of the same 250-m radius; see Fig. 5-56. (*a*) As the car passes over the crest, the normal force on the car is one-half the 16-kN weight of the car. What will be the normal force on the car as it passes through the bottom of the dip? (*b*) What is the greatest speed at which the car can move without leaving the road at the top of the hill? (*c*) Moving at the speed found in (*b*), what will be the normal force on the car as it moves through the bottom of the dip?

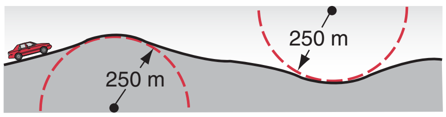


FIGURE 5-56. Problem 14.

15. A small coin is placed on a flat, horizontal turntable. The turntable is observed to make exactly three revolutions in 3.3 s. (*a*) What is the speed of the coin when it rides without slipping at a distance of 5.2 cm from the center of the turntable? (*b*) What is the acceleration (magnitude and direction) of the coin in part (*a*)? (*c*) What is the force of friction acting on the coin in part (*a*) if the coin has a mass of 1.7 g? (*d*) What is the coefficient of static friction between the coin and the turntable if the coin is observed to slide off the turntable when it is more than 12 cm from the center of the turntable?

16. A certain string can withstand a maximum tension of 9.2 lb without breaking. A child ties a 0.82-lb stone to one end and, holding the other end, whirls the stone in a vertical circle of radius 2.9 ft, slowly increasing the speed until the string breaks. (*a*) Where is the stone on its path when the string breaks? (*b*) What is the speed of the stone as the string breaks?

17. A 1.34-kg ball is attached to a rigid vertical rod by means of two massless strings each 1.70 m long. The strings are attached to the rod at points 1.70 m apart. The system is rotating about the axis of the rod, both strings being taut and forming an equilateral triangle with the rod, as shown in Fig. 5-57. The tension in the upper string is 35.0 N. (*a*) Find the tension in the lower string. (*b*) Calculate the net force on the ball at the instant shown in the figure. (*c*) What is the speed of the ball?

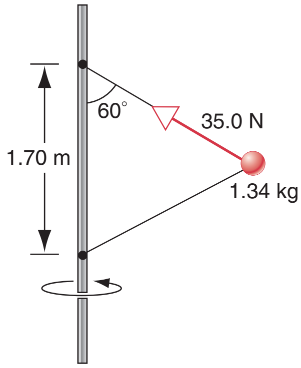


FIGURE 5-57. Problem 17.

18. A very small cube of mass m is placed on the inside of a funnel (see Fig. 5-58) rotating about a vertical axis at a constant rate of ω revolutions per second. The wall of the funnel makes an angle θ with the horizontal. The coefficient of static friction between cube and funnel is μ_s and the center of the cube is at a distance r from the axis of rotation. Find the (*a*) largest and (*b*) smallest values of ω for which the cube will not move with respect to the funnel.

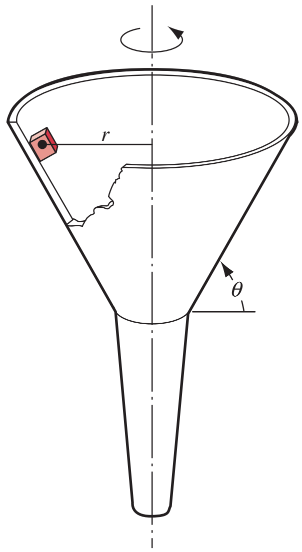


FIGURE 5-58. Problem 18.

19. Because of the rotation of the Earth, a plumb bob may not hang exactly along the direction of the Earth's gravitational force on the plumb bob but may deviate slightly from that direction. (*a*) Show that the deflection θ in radians at a point at latitude L is given by

$$\theta = \left(\frac{2\pi^2 R}{gT^2}\right) \sin 2L,$$

where R is the radius of the Earth and T is the period of the Earth's rotation. (*b*) At what latitude is the deflection a maximum? How much is this deflection? (*c*) What is the deflection at the poles? At the equator?

20. A particle of mass m sits at rest at $x = 0$. At time $t = 0$ a force given by $F = F_0 e^{-t/T}$ is applied in the $+x$ direction; F_0 and T are constants. When $t = T$ the force is removed. At this instant when the force is removed, (*a*) what is the speed of the particle and (*b*) where is it?

COMPUTER PROBLEMS

1. An automobile is moving at a constant speed while pulling a block of wood of mass $m = 200$ kg with an elastic cord. The force exerted on the block of wood by the cord depends on the length of the cord, and is given by $F = 400(l - 10)$ where F is measured in newtons when l is measured in meters. This force, however, is 0 if $l < 10$ m. The coefficient of static friction between the block and ground is $\mu_s = 0.60$, while the coefficient of kinetic friction is $\mu_k = 0.50$. Set up a computer program to numerically evaluate the motion of the block under the following scenarios: (*a*) Assume the block is originally at rest and the automobile is originally 10 m distant and moving away from the block with a constant speed of 5 m/s. (*b*) Assume the block is originally at rest and the automobile is originally 10 m distant and moving away from the block with a constant speed of 20 m/s.

2. Starting from rest a person pushes a 95-kg crate across a rough floor with a force given by $F = 200e^{-0.15t}$ where F is in newtons and t is in seconds. The force decreases exponentially because the person tires. As long as the crate is moving a constant frictional force of 80 N opposes the motion. (*a*) How long after starting does the crate stop? (*b*) How far does it go? (*c*) How accurate are your results? (Try using the Euler method with an initial time interval of $\Delta t = 0.01$ s. Repeat the process, but use a time interval of $\Delta t = 0.001$ s. Compare the results to get an estimate of your accuracy.)

CHAPTER 6

MOMENTUM

Newton's laws are useful for solving a wide range of problems in dynamics. However, there is one class of problems in which, even though Newton's laws still apply as we have defined them, we may have insufficient knowledge of the forces to permit us to analyze the motion. These problems involve collisions *between one object and another.*

In this chapter we will learn how to analyze collisions between two objects. In doing so, we will find that we need a new dynamic variable, called linear momentum. *We will see that the law of conservation of linear momentum, one of the fundamental conservation laws of physics, can be used to study the collisions of objects from the scale of subatomic particles to the scale of galaxies.*

6-1 COLLISIONS

In a collision, two objects exert forces on each other for an identifiable time interval, so that we can separate the motion into three parts: before, during, and after the collision. Before and after the collision, we assume that the objects are far enough apart that they do not exert any forces on each other. During the collision, the objects exert forces on each other; these forces are equal in magnitude and opposite in direction, according to Newton's third law. We assume that these forces are much larger than any forces exerted on the two objects by other objects in their environment. The motion of the objects (or at least one of them) changes rather abruptly during the collision, so that we can make a relatively clear separation of the situation before the collision from the situation after the collision.

When a bat strikes a baseball, for example, the time between the beginning and the end of the collision can be determined fairly precisely. The bat is in contact with the ball for an interval that is quite short compared with the time during which we are watching the ball. During the collision the bat exerts a large force on the ball (Fig. 6-1). This force varies with time in a complex way that we can measure only with difficulty. Both the ball and the bat are deformed during the collision. Forces that act for a time that is short compared with the time of observation of the system are called *impulsive* forces.

When an alpha particle (^{4}He nucleus) collides with another nucleus (Fig. 6-2), the force exerted on each by the other may be the repulsive electrostatic force associated with the charges on the particles. The particles may not actually come into direct contact with each other, but we still may speak of this interaction as a collision because a relatively

FIGURE 6-1. A high-speed photograph of a bat striking a baseball. Note the deformation of the ball, indicating the large impulsive force exerted by the bat.

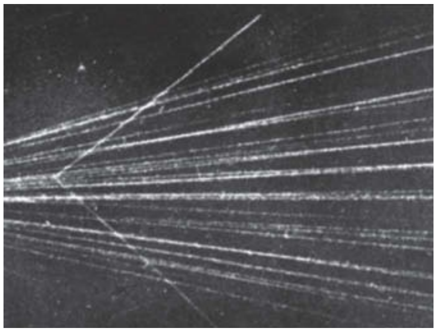
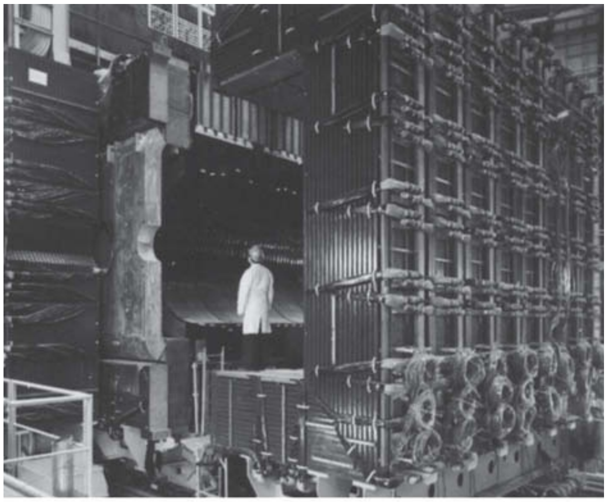
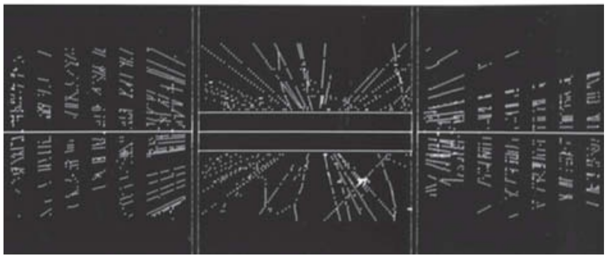


FIGURE 6-2. An alpha particle collides with a helium nucleus in a cloud chamber. Most of the incident particles (coming from the left) pass through without colliding.



(*a*)



(*b*)

FIGURE 6-4. (*a*) The massive detector UA1 used at the proton–proton collider at CERN, the particle physics research facility near Geneva, Switzerland. (*b*) A computer reconstruction of the paths of the particles produced in one proton–proton collision. Such reconstructions were used in 1983 to confirm the existence of the very short-lived particles called W and Z, which verified a theory that treats the electromagnetic force and the weak nuclear force as different aspects of a single more basic force.

strong force, acting for a time that is short compared with the time that the alpha particle is under observation, has a substantial effect on the motion of the alpha particle.

We can even speak about a collision between two galaxies (Fig. 6-3), if we are prepared to observe them over a time scale of the order of millions or billions of years. (However, a more feasible alternative is to shorten this long time span by computer modeling!)

Collisions between elementary particles provide the principal source of information about their internal structure. When two particles collide at high energy, often the products of the collision are very different from the original particles (Fig. 6-4). Sometimes these collisions produce hundreds of product particles. By studying the trajectories of the outgoing particles and applying fundamental laws, we can reconstruct the original event.

On a different scale, those who study traffic accidents also try to reconstruct collisions. From the paths and impact patterns of the colliding vehicles (Fig. 6-5), it is often possible to deduce such important details as the speeds and directions of motion of the two vehicles before the collision.

Another kind of collision is one that takes place between a space probe and a planet, called the "slingshot effect," in which the speed of the space probe can be increased in a "close encounter" with a (moving) planet. The probe does not actually touch the planet, but it does come strongly under its gravitational influence for a time that is very short compared to the duration of the space probe's journey. Thus we are justified in calling such encounters "collisions." For example, the Venus and Earth encounters of the Cassini mission (see Fig. 5-24) increased the spacecraft's speed by the equivalent of 75 tons of launch rocket fuel! Without this gravity assist, the Cassini spacecraft could not reach Saturn. (See Problem 14.)

FIGURE 6-3. Two galaxies colliding.

In principle it would be possible to analyze each of these collisions using Newton's second law. Given the initial motion of each object and the force that acts between them, we could use the methods of Chapter 5 to find the velocity and position of each colliding object as a function of the time. However, there are two reasons why this is not possible for the collisions shown in Figs. 6-1 to 6-5: (1) For some of the collisions, we don't know the exact form of the expression for the force between the objects. (2) The colliding objects are composed of many particles, and it is hopelessly complicated to keep track of the application of Newton's laws for the force between each particle of one colliding object and each particle of the other.

Here is the basic problem: We have two objects with different initial motions that are originally so far apart that neither exerts a measurable force on the other. Eventually they

Figure 6-5. A collision between two automobiles. Momentum conservation is used by accident reconstruction experts to calculate the velocities before the collision.

approach one another, so that each exerts a force on the other and alters its motion. This force occurs for a time that is relatively short compared with the entire motion of the objects. Finally, they separate again with new motions and no longer interact. We observe the motion before the collision and the motion after the collision, but during the brief collision we do not observe or measure what is happening.

If we know the initial motions, can we find the final motions even though we do not know the force that acts to change the motions? Surprisingly, the answer is "yes!" In the next section, we define a new dynamic variable that enables us to analyze such collisions.

6-2 LINEAR MOMENTUM

To analyze collisions, we need a new dynamic variable, the *linear momentum* of a body. (Later we will introduce a similar variable for rotational motion, called the *angular* momentum. For the time being, we will refer to linear momentum simply as "momentum.") The momentum $\vec{\mathbf{p}}$ of a body is defined as the product of its mass and its velocity:

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}}. \tag{6-1}$$

As the product of a vector and a scalar, momentum must also be a vector. Equation 6-1 indicates that the direction of $\vec{\mathbf{p}}$ is the same as the direction of $\vec{\mathbf{v}}$. Because $\vec{\mathbf{p}}$ depends on $\vec{\mathbf{v}}$, *the momentum (like the velocity) depends on the reference frame of the observer, and we must always specify this frame.*

Newton, in his famous *Principia,* expressed the second law of motion in terms of momentum (which he called "quantity of motion"). Expressed in modern terminology Newton's second law reads:

The rate of change of momentum of a body is equal to the resultant force acting on the body and is in the direction of that force.

In symbolic form this becomes

$$\sum \vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt}. \tag{6-2}$$

Here $\Sigma\,\vec{\mathbf{F}}$ represents the resultant force acting on the particle.

For a single particle of constant mass, this form of the second law is equivalent to the form $\Sigma\,\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ that we have used up to now. That is, if m is constant, then

$$\sum \vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} = \frac{d}{dt}(m\vec{\mathbf{v}}) = m\frac{d\vec{\mathbf{v}}}{dt} = m\vec{\mathbf{a}}.$$

The relations $\Sigma\,\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ and $\Sigma\,\vec{\mathbf{F}} = d\vec{\mathbf{p}}/dt$ for single particles are completely equivalent in classical mechanics.

The equivalence of $\Sigma\,\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ and $\Sigma\,\vec{\mathbf{F}} = d\vec{\mathbf{p}}/dt$ depends, as you can see from the above equation, on the mass being a constant so that it passes through the derivative: $d(m\vec{\mathbf{v}})/dt = m(d\vec{\mathbf{v}}/dt)$. We shall assume that this is the case for the problems we discuss in this chapter. Section 7-6 covers applications of Newton's laws to systems in which the mass changes, such as a rocket that exhausts burning gases.

6-3 IMPULSE AND MOMENTUM

In this section we consider the relationship between the force that acts on a body during a collision and the change in the momentum of that body. During a collision, the force varies with time. For example, Fig. 6-6 shows how the magnitude of the force might change with time during a collision. The force is exerted only during the collision, which begins at time t_i and ends at time t_f. The force is zero before and after the collision.

From Newton's second law in the form of Eq. 6-2 ($\Sigma\,\vec{\mathbf{F}} = d\vec{\mathbf{p}}/dt$), we can write the change in momentum $d\vec{\mathbf{p}}$

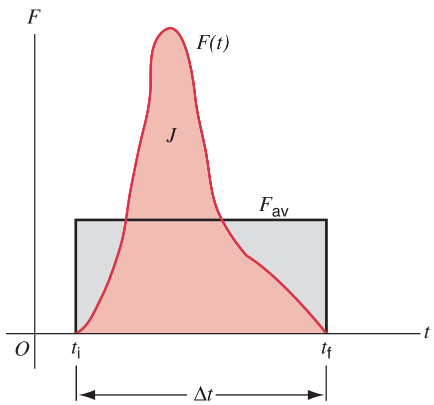


Figure 6-6. An impulsive force $F(t)$ varies in an arbitrary way with time during a collision that lasts from t_i to t_f. The area under the $F(t)$ curve is the impulse J, and the rectangle bounded by the average force F_av has an equal area.

of a particle in a time interval dt during which a net force $\sum \vec{\mathbf{F}}$ acts on it as

$$d\vec{\mathbf{p}} = \sum \vec{\mathbf{F}}\, dt.$$

To find the total change in momentum during the entire collision, we integrate over the time of collision, starting at time t_i (when the momentum is $\vec{\mathbf{p}}_i$) and ending at time t_f (when the momentum is $\vec{\mathbf{p}}_f$):

$$\int_{\vec{\mathbf{p}}_i}^{\vec{\mathbf{p}}_f} d\vec{\mathbf{p}} = \int_{t_i}^{t_f} \sum \vec{\mathbf{F}}\, dt. \tag{6-3}$$

The left side of Eq. 6-3 is the change in momentum, $\Delta\vec{\mathbf{p}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i$. The right side defines a new quantity called the *impulse.* For any arbitrary force $\vec{\mathbf{F}}$, the impulse $\vec{\mathbf{J}}$ is defined as

$$\vec{\mathbf{J}} = \int_{t_i}^{t_f} \vec{\mathbf{F}}\, dt. \tag{6-4}$$

The impulse depends on the strength of the force and on its duration. The impulse is a vector and, as Eq. 6-3 shows, the impulse has the same units and dimensions as momentum.

The right side of Eq. 6-3 is the impulse of the *net* force, $\vec{\mathbf{J}}_{net} = \int \Sigma\vec{\mathbf{F}}\, dt$. We can therefore write Eq. 6-3 as

$$\vec{\mathbf{J}}_{net} = \Delta\vec{\mathbf{p}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i. \tag{6-5}$$

Equation 6-5 is the mathematical statement of the *impulse-momentum theorem:*

> *The impulse of the net force acting on a particle during a given time interval is equal to the change in momentum of the particle during that interval.*

As a vector relationship, Eq. 6-5 contains within it the three component equations:

$$\begin{aligned} J_{net,x} &= \Delta p_x = p_{fx} - p_{ix}, \\ J_{net,y} &= \Delta p_y = p_{fy} - p_{iy}, \\ J_{net,z} &= \Delta p_z = p_{fz} - p_{iz}. \end{aligned} \tag{6-6}$$

Although in this chapter we use Eq. 6-5 only in situations involving impulsive forces (that is, those of short duration compared with the time of observation), no such limitation is built into that equation. Equation 6-5 is just as general as Newton's second law, from which it was derived. We could, for example, use Eq. 6-5 to find the momentum acquired by a body falling in the Earth's gravity.

We defined the impulse in terms of a single force, but the impulse-momentum theorem deals with the change in momentum due to the impulse of the *net* force—that is, the combined effect of all the forces that act on the particle. In the case of a collision involving two particles, there is often no distinction because each particle is acted upon by only one force, which is due to the other particle. In this case, the change in momentum of one particle is equal to the impulse of the force exerted by the other particle.

The impulsive force whose magnitude is represented in Fig. 6-6 is assumed to have a constant direction. The magnitude of the impulse of this force is represented by the area under the $F(t)$ curve. We can represent that same area by the rectangle in Fig. 6-6 of width Δt and height F_{av}, where F_{av} is the magnitude of the *average force* that acts during the interval Δt. Thus

$$J = F_{av}\,\Delta t. \tag{6-7}$$

In a collision such as that of the ball and bat of Fig. 6-1, it is difficult to measure $F(t)$ directly, but we can estimate Δt (perhaps a few milliseconds) and obtain a reasonable value for F_{av} based on the impulse computed according to Eq. 6-6 from the change in momentum of the ball (see Sample Problem 6-1).

We have defined a collision as an interaction that occurs in a time Δt that is negligible compared to the time during which we are observing the system. We can also characterize a collision as an event in which the external forces that may act on the system during the time of the collision are negligible compared to the impulsive collision forces. While a bat strikes a baseball, a golf club strikes a golf ball, or one billiard ball strikes another, external forces act on the system. Gravity or friction may exert forces on these bodies, for example; these external forces may not be the same on each colliding body nor are they necessarily canceled by other external forces. Even so, it is quite safe to neglect these external forces during the collision. As a result, the change in momentum of a particle arising from an external force during a collision is negligible compared to the change in momentum of that particle arising from the impulsive collisional force (Fig. 6-7).

For example, when a bat strikes a baseball, the collision lasts only a few milliseconds. Because the change in momentum of the ball is large and the time of collision is small, it follows from

$$\Delta\vec{\mathbf{p}} = \vec{\mathbf{F}}_{av}\,\Delta t \tag{6-8}$$

that the average impulsive force $\vec{\mathbf{F}}_{av}$ is relatively large. Compared to this force, the external force of gravity is neg-

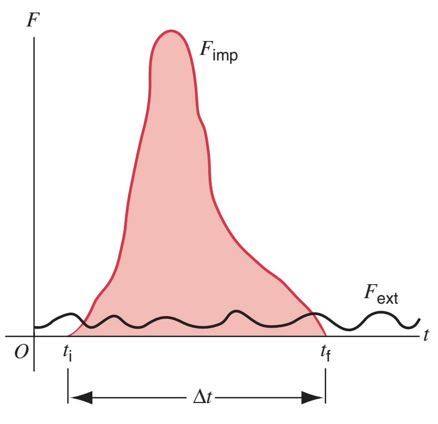


FIGURE 6-7. The impulsive force F_{imp} that acts during a collision is generally much stronger than any external force F_{ext} that may also act.

ligible. During the collision we can safely ignore this external force in determining the change in motion of the ball; the shorter the duration of the collision is, the more likely this is to be true.

SAMPLE PROBLEM 6-1. A baseball (which has an official weight of about 5 oz or a mass of 0.14 kg) is moving horizontally at a speed of 93 mi/h (about 42 m/s) when it is struck by the bat (see Fig. 6-1). It leaves the bat in a direction at an an angle $\phi = 35°$ above its incident path and with a speed of 50 m/s. (*a*) Find the impulse of the force exerted on the ball. (*b*) Assuming the collision lasts for 1.5 ms (= 0.0015 s), what is the average force? (*c*) Find the change in the momentum of the bat.

Solution (*a*) Figure 6-8*a* shows the initial momentum vector $\vec{\mathbf{p}}_i$ and the final momentum vector $\vec{\mathbf{p}}_f$ of the baseball. The components of the final momentum are given by

$$p_{fx} = mv_f \cos\phi = (0.14\text{ kg})(50\text{ m/s})(\cos 35°) = 5.7\text{ kg}\cdot\text{m/s},$$

$$p_{fy} = mv_f \sin\phi = (0.14\text{ kg})(50\text{ m/s})(\sin 35°) = 4.0\text{ kg}\cdot\text{m/s}.$$

In this coordinate system, the initial momentum has only an x component, whose (negative) value is

$$p_{ix} = mv_i = (0.14\text{ kg})(-42\text{ m/s}) = -5.9\text{ kg}\cdot\text{m/s}.$$

The impulse can now be found using Eq. 6-6:

$$J_x = p_{fx} - p_{ix} = 5.7\text{ kg}\cdot\text{m/s} - (-5.9\text{ kg}\cdot\text{m/s}) = 11.6\text{ kg}\cdot\text{m/s},$$

$$J_y = p_{fy} - p_{iy} = 4.0\text{ kg}\cdot\text{m/s} - 0 = 4.0\text{ kg}\cdot\text{m/s}.$$

In other terms, the impulse has magnitude

$$J = \sqrt{J_x^2 + J_y^2} = \sqrt{(11.6\text{ kg}\cdot\text{m/s})^2 + (4.0\text{ kg}\cdot\text{m/s})^2} = 12.3\text{ kg}\cdot\text{m/s}$$

and acts in a direction determined by

$$\theta = \tan^{-1}(J_y/J_x) = \tan^{-1}[(4.0\text{ kg}\cdot\text{m/s})/(11.6\text{ kg}\cdot\text{m/s})] = 19°$$

above the horizontal. Figure 6-8*b* shows the impulse vector $\vec{\mathbf{J}}$ and verifies graphically that, as the definition of Eq. 6-6 requires,

$$\vec{\mathbf{J}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i = \vec{\mathbf{p}}_f + (-\vec{\mathbf{p}}_i).$$

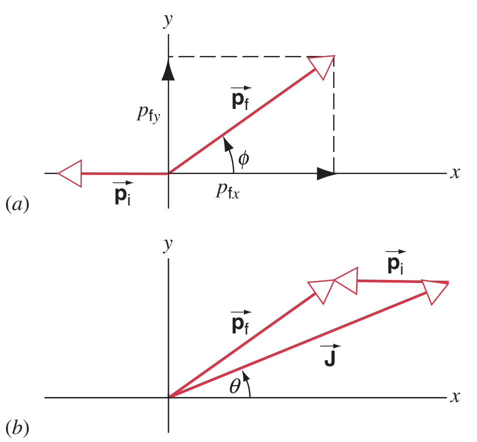


FIGURE 6-8. Sample Problem 6-1. (*a*) The initial and final momenta of the baseball. (*b*) The difference $\vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i$ is equal to the impulse $\vec{\mathbf{J}}$.

(*b*) Using Eq. 6-7, we obtain

$$F_{av} = J/\Delta t = (12.3\text{ kg}\cdot\text{m/s})/0.0015\text{ s} = 8200\text{ N},$$

which is nearly 1 ton. This force acts in the same direction as $\vec{\mathbf{J}}$—that is, 19° above the horizontal. Note that this is the *average* force; the *maximum* force is considerably greater, as Fig. 6-6 shows. Also, note that F_{av} (= 8200 N) $\gg mg$ (= 1.4 N). Thus we are quite safe in assuming that the impulsive force greatly exceeds the external force (gravity, in this case) and therefore is very nearly equal to the net force that acts during the collision.

(*c*) From Newton's third law, the force exerted on the bat by the ball is equal and opposite to the force exerted on the ball by the bat. Therefore, according to Eq. 6-8, the change in momentum of the bat is equal and opposite to that of the ball. Thus, for the bat,

$$\Delta p_x = -11.6\text{ kg}\cdot\text{m/s},$$

$$\Delta p_y = -4.0\text{ kg}\cdot\text{m/s}.$$

Is this a large change or a small one? Try to estimate the momentum of the bat in motion to answer this question.

SAMPLE PROBLEM 6-2. A cart of mass $m_1 = 0.24$ kg moves on a linear track without friction with an initial velocity of 0.17 m/s. It collides with another cart of mass $m_2 = 0.68$ kg that is initially at rest. The first cart carries a force probe that registers the magnitude of the force exerted by one cart on the other during the collision. The output of the force probe is shown in Fig. 6-9. Find the velocity of each cart after the collision.

Solution Our strategy in this problem is to find the impulse from the force graph. The impulse gives the change in momentum, which allows us to find the final momentum of each cart. The impulse $\int F\,dt$ is the area under the graph of $F(t)$ in Fig. 6-9, which can be determined as the area of a triangle:

$$J = \int F\,dt = \tfrac{1}{2}(0.014\text{ s} - 0.003\text{ s})(10\text{ N}) = 0.055\text{ N}\cdot\text{s} = 0.055\text{ kg}\cdot\text{m/s}.$$

Since the graph gives the magnitude of the force, this integral gives the magnitude of the impulse. We take the direction of motion of the first cart as the positive x direction. Then the x component force exerted on cart 1 by cart 2 is negative, and so the corresponding component of $\vec{\mathbf{J}}$ is negative. Since $J_x = \Delta p_x$, for the first

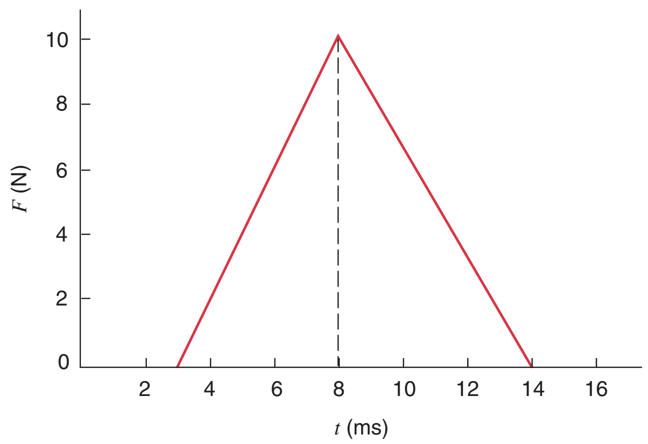


FIGURE 6-9. Sample Problem 6-2.

cart we have $\Delta p_{1x} = -0.055\ \text{kg}\cdot\text{m/s}$ and so its final momentum and velocity are

$$p_{1fx} = p_{1ix} + \Delta p_{1x} = (0.24\ \text{kg})(0.17\ \text{m/s}) - 0.055\ \text{kg}\cdot\text{m/s}$$
$$= -0.014\ \text{kg}\cdot\text{m/s}$$

$$v_{1fx} = \frac{p_{1fx}}{m_1} = \frac{-0.014\ \text{kg}\cdot\text{m/s}}{0.24\ \text{kg}} = -0.058\ \text{m/s} = -5.8\ \text{cm/s}.$$

Cart 1 moves in the negative x direction after the collision.

The force on cart 2 is, by Newton's third law, equal and opposite to the force on cart 1, so it is in the positive x direction. Because the forces are equal in magnitude, the impulses are equal in magnitude but opposite in direction. So $J_{2x} = \Delta p_{2x} = +0.055\ \text{kg}\cdot\text{m/s}$, and the final momentum and velocity of cart 2 are

$$p_{2fx} = p_{2ix} + \Delta p_{2x} = 0 + 0.055\ \text{kg}\cdot\text{m/s} = +0.055\ \text{kg}\cdot\text{m/s}$$

$$v_{2fx} = \frac{p_{2fx}}{m_2} = \frac{+0.055\ \text{kg}\cdot\text{m/s}}{0.68\ \text{kg}} = +0.081\ \text{m/s} = +8.1\ \text{cm/s}.$$

Cart 2 moves in the positive x direction after the collision.

6-4 CONSERVATION OF MOMENTUM

In this section we consider the analysis of collisions between two objects, each of which may be moving. In contrast to Sample Problem 6-2, the objects may be moving in any direction, so we must use vectors to describe the motion.

Figure 6-10*a* illustrates the general problem. A body with mass m_1 moves initially with velocity $\vec{\mathbf{v}}_{1i}$ and momentum $\vec{\mathbf{p}}_{1i} = m_1\vec{\mathbf{v}}_{1i}$. It collides with body 2, which is moving initially with velocity $\vec{\mathbf{v}}_{2i}$ and momentum $\vec{\mathbf{p}}_{2i} = m_2\vec{\mathbf{v}}_{2i}$. We focus our attention on the motions of the two bodies, which we define as our *system.* We assume that the system consisting of the two bodies is isolated from its environment, so that no forces act on either body during the collision except the impulsive force that each body exerts on the other. After the collision (Fig. 6-10*b*), m_1 moves with velocity $\vec{\mathbf{v}}_{1f}$ and momentum $\vec{\mathbf{p}}_{1f} = m_1\vec{\mathbf{v}}_{1f}$ and m_2 with velocity $\vec{\mathbf{v}}_{2f}$ and momentum $\vec{\mathbf{p}}_{2f} = m_2\vec{\mathbf{v}}_{2f}$.

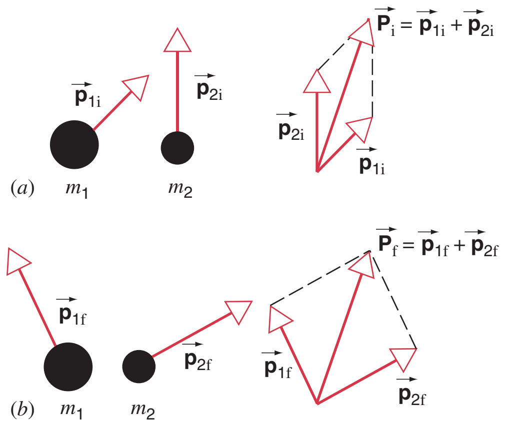


FIGURE 6-10. (*a*) Two objects and their momenta before they collide. (*b*) The objects and their momenta after they collide. Note that the total momentum vectors $\vec{\mathbf{P}}_i$ and $\vec{\mathbf{P}}_f$ are the same before and after the collision.

At any particular time, the total momentum of the system consisting of the two bodies is

$$\vec{\mathbf{P}} = \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2, \tag{6-9}$$

which we can evaluate before, during, or after the collision. Taking the time derivative of Eq. 6-9, we obtain

$$\frac{d\vec{\mathbf{P}}}{dt} = \frac{d\vec{\mathbf{p}}_1}{dt} + \frac{d\vec{\mathbf{p}}_2}{dt} = \sum \vec{\mathbf{F}}_1 + \sum \vec{\mathbf{F}}_2, \tag{6-10}$$

where we have used Eq. 6-2 to replace $d\vec{\mathbf{p}}/dt$ for each body with the net force acting on that body. Before the collision, no forces act on the bodies, so $\Sigma\,\vec{\mathbf{F}}_1 = 0$ and $\Sigma\,\vec{\mathbf{F}}_2 = 0$, and therefore $d\vec{\mathbf{P}}/dt = 0$. Similarly, after the collision $d\vec{\mathbf{P}}/dt = 0$ because again no forces act on the bodies. During the collision, the only force acting on body 1 is $\vec{\mathbf{F}}_{12}$, which is due to body 2. Similarly, $\vec{\mathbf{F}}_{21}$ is the only force acting on body 2 during the collision. $\vec{\mathbf{F}}_{12}$ and $\vec{\mathbf{F}}_{21}$ form an action–reaction pair, so $\vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21}$ and $\vec{\mathbf{F}}_{12} + \vec{\mathbf{F}}_{21} = 0$. Thus $d\vec{\mathbf{P}}/dt = 0$ during the collision, too. So we get the same result when we evaluate Eq. 6-10 before, during, and after the collision: at all times,

$$\frac{d\vec{\mathbf{P}}}{dt} = 0. \tag{6-11}$$

If the time derivative of a quantity is zero, then that quantity does not change with time and must be a constant:

$$\vec{\mathbf{P}} = \text{constant}. \tag{6-12}$$

That is, the total momentum of m_1 and m_2 before the collision must be the same in magnitude and direction as the total momentum of m_1 and m_2 after the collision. Even though $\vec{\mathbf{p}}_1$ and $\vec{\mathbf{p}}_2$ may both change as a result of the collision, their vector sum stays the same (as in Fig. 6-10).

Another way of expressing this result is

$$\vec{\mathbf{P}}_i = \vec{\mathbf{P}}_f, \tag{6-13}$$

where $\vec{\mathbf{P}}_i = \vec{\mathbf{p}}_{1i} + \vec{\mathbf{p}}_{2i}$ is the total initial momentum of the system before the collision and $\vec{\mathbf{P}}_f = \vec{\mathbf{p}}_{1f} + \vec{\mathbf{p}}_{2f}$ is the total final momentum after the collision.

Equations 6-11, 6-12, and 6-13 are equivalent mathematical statements of the *law of conservation of linear momentum* for an isolated system consisting of two bodies:

> *When the net external force acting on a system is zero, the total linear momentum of the system remains constant.*

This is a general result, valid for any type of interaction between the bodies. It is not even necessary that the bodies behave like particles for this law to be valid (as in the collision of Fig. 6-5). Although we obtained this result for a two-body system, the law of conservation of momentum is perfectly general and applies to any collection or system of bodies in which the only forces that act are those that the bodies in the system exert on each other.

Momentum is a vector quantity so for momentum to be conserved all three components must be conserved independently. For example, Eq. 6-12 gives

$$P_x = \text{constant}, \qquad P_y = \text{constant}, \qquad P_z = \text{constant}. \tag{6-14}$$

The total x component of the momentum stays the same before and after the collision, as do the y and z components.

Because we derived the law of conservation of momentum using Newton's laws, the law is valid in any inertial frame of reference. Observers in different inertial frames watching the same collision may disagree on the values they measure for the initial and final momenta, but each will agree that the initial and final momenta are equal. If momentum is conserved in one inertial frame, it is conserved in *every* inertial frame.

Conservation laws have an important role in our analysis and understanding of physical processes. They allow us to compare the behavior of a system "before" and "after," without having any detailed knowledge of the processes that happen "in between." For example, the law of conservation of linear momentum makes no assumptions about the type of force that the two bodies exert on one another; the total linear momentum before the collision will equal the total momentum after the collision, no matter what type of force acts on the colliding objects.

Later in this text we shall encounter other conservation laws, including those for energy, angular momentum, and electric charge. These laws are of great practical and theoretical importance. There is a deep theoretical connection between conserved quantities and symmetries of nature. For instance, the law of conservation of linear momentum is connected to the *spatial* symmetry of nature, which requires that an experiment done at one location should yield the identical result as the same experiment done at another location. Later we will discuss another conservation law, the conservation of energy, which is connected to *temporal* (time) symmetry: the result of an experiment done today should agree with the result of the same experiment done yesterday. Because of these connections, we believe that these two conservation laws are universally valid—if the nature of space and time is the same everywhere in the universe, then the same conservation laws should apply everywhere and at all times.

SAMPLE PROBLEM 6-3. Fred ($m_F = 75$ kg) and Ginger ($m_G = 55$ kg) are ice skating side by side at a common velocity of 3.2 m/s (Fig. 6-11*a*) when they push off from one another in a direction perpendicular to their original velocity. After they break contact, Ginger is skating in a direction at an angle of 32° from her original direction (Fig. 6-11*b*). In what direction is Fred now skating?

Solution We take Fred and Ginger together to be our system. Fred and Ginger exert forces on each other when they push apart,

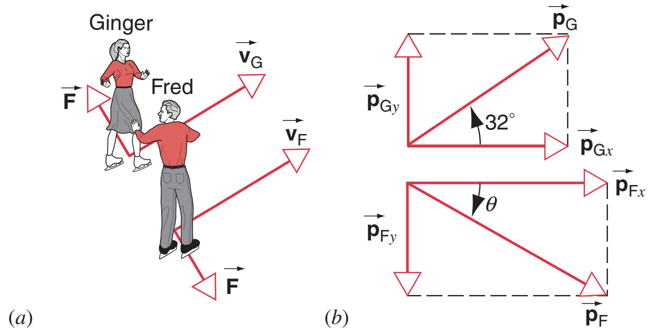


FIGURE 6-11. Sample Problem 6-3. (*a*) Two skaters push off from one another in a direction perpendicular to their original motion. (*b*) The momenta of the skaters after they push off.

but if we neglect the effect of any external forces (such as friction with the ice), their total momentum before they push apart must be the same as their total momentum after they push apart. The x components of their original momenta are (taking their original motion to be in the positive x direction):

$$p_{Gx} = m_G v_G = (55 \text{ kg})(3.2 \text{ m/s}) = 176 \text{ kg}\cdot\text{m/s}$$

$$p_{Fx} = m_F v_F = (75 \text{ kg})(3.2 \text{ m/s}) = 240 \text{ kg}\cdot\text{m/s}.$$

After they push off, Ginger's momentum acquires a y component so that her total momentum makes an angle of 32° with the positive x direction:

$$p_{Gy} = p_{Gx} \tan 32° = +110 \text{ kg}\cdot\text{m/s}.$$

Before they push apart, the y component of their total momentum is zero. For momentum to be conserved, the total y component must remain zero after they separate. Therefore, the y component of Fred's momentum must be -110 kg·m/s, and the direction of Fred's motion is determined from

$$\tan\theta = \frac{p_{Fy}}{p_{Fx}} = \frac{-110 \text{ kg}\cdot\text{m/s}}{240 \text{ kg}\cdot\text{m/s}} = -0.458$$

or

$$\theta = -25°.$$

Note that the x component of the momentum of either Fred or Ginger is unchanged by the force of their pushing away from each other; this force is exerted in the y direction, and the impulse-momentum theorem (Eq. 6-6) tells us that a force on either of them in the y direction cannot change the x component of their momentum.

SAMPLE PROBLEM 6-4. A man of mass 65 kg is running along a pier at a speed of 4.9 m/s (Fig. 6-12). He jumps from the pier into a rowboat of mass 88 kg that is drifting without friction in the same direction at a speed of 1.2 m/s. When the man is seated in the rowboat, what is its final velocity?

Solution As the man enters the boat, he and the boat exert forces on each other that cause them to acquire the same final speed (the man slows down and the boat speeds up). If there are no external forces acting on the system of man + boat, the total momentum of the man and the boat before he jumps must equal the total momentum after he is seated in the boat. We choose the positive x

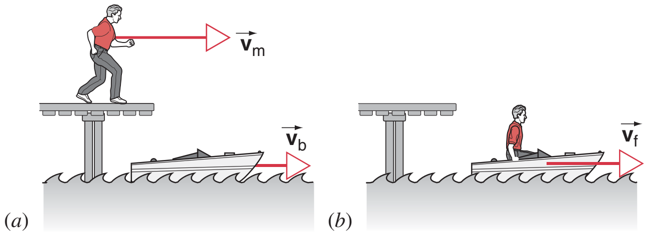


FIGURE 6-12. Sample Problem 6-4. (*a*) A man runs with velocity $\vec{\mathbf{v}}_m$ and jumps into a boat moving in the same direction with velocity $\vec{\mathbf{v}}_b$. (*b*) The man and boat are together moving with velocity $\vec{\mathbf{v}}_f$.

axis in the direction of the man's original velocity, and because all motion is in the x direction we need consider only the x components of all velocities and momenta. Before he jumps, the man has momentum $p_{mx} = m_m v_{mx}$ and the boat has momentum $p_{bx} = m_b v_{bx}$. The total initial momentum is

$$P_{ix} = p_{mx} + p_{bx} = m_m v_{mx} + m_b v_{bx}$$
$$= (65\ \text{kg})(4.9\ \text{m/s}) + (88\ \text{kg})(1.2\ \text{m/s}) = 424\ \text{kg}\cdot\text{m/s}.$$

After he jumps and is seated in the boat, they move together with the same velocity v_{fx}. Their combined final momentum is $P_{fx} = m_m v_{fx} + m_b v_{fx} = (m_m + m_b)v_{fx}$. With $P_{ix} = P_{fx}$, we obtain

$$v_{fx} = \frac{P_{ix}}{m_m + m_b} = \frac{424\ \text{kg}\cdot\text{m/s}}{65\ \text{kg} + 88\ \text{kg}} = 2.8\ \text{m/s}.$$

6-5 TWO-BODY COLLISIONS

In this section we examine different types of two-body collisions using momentum conservation to relate the motion of the bodies before and after the collision.

Figure 6-13*a* shows a general two-body collision. Before the collision, m_1 moves with initial velocity $\vec{\mathbf{v}}_{1i}$ and m_2 with initial velocity $\vec{\mathbf{v}}_{2i}$. After the collision, the final velocities are $\vec{\mathbf{v}}_{1f}$ and $\vec{\mathbf{v}}_{2f}$ respectively. According to conservation of momentum, the total momentum of m_1 and m_2 *before* the collision equals their total momentum *after* the collision:

$$m_1\vec{\mathbf{v}}_{1i} + m_2\vec{\mathbf{v}}_{2i} = m_1\vec{\mathbf{v}}_{1f} + m_2\vec{\mathbf{v}}_{2f}. \qquad (6\text{-}15)$$

Another way of writing Eq. 6-15 is

$$m_1(\vec{\mathbf{v}}_{1f} - \vec{\mathbf{v}}_{1i}) = -m_2(\vec{\mathbf{v}}_{2f} - \vec{\mathbf{v}}_{2i}) \qquad (6\text{-}16)$$

or

$$\Delta\vec{\mathbf{p}}_1 = -\Delta\vec{\mathbf{p}}_2. \qquad (6\text{-}17)$$

The changes in momentum of the two objects have equal magnitudes and opposite signs, a necessary consequence of the law of conservation of momentum. This result also follows directly from Newton's third law: according to the impulse-momentum theorem (Eq. 6-5), the change in momentum of either body equals the impulse of the net force that acts on that body. Equation 6-17 can thus be written as $\vec{\mathbf{J}}_1 = -\vec{\mathbf{J}}_2$ where $\vec{\mathbf{J}}_1$ means the impulse of the force on body 1 due to body 2 and $\vec{\mathbf{J}}_2$ means the impulse of the force on body 2 due to body 1. This equality follows directly from the definition of the impulse (Eq. 6-4) with $\vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21}$, as required by Newton's third law.

In some collisions, the bodies stick together (Fig. 6-13*b*) and move with a common final velocity. With $\vec{\mathbf{v}}_{1f} = \vec{\mathbf{v}}_{2f} = \vec{\mathbf{v}}_f$, Eq. 6-15 becomes

$$m_1\vec{\mathbf{v}}_{1i} + m_2\vec{\mathbf{v}}_{2i} = (m_1 + m_2)\vec{\mathbf{v}}_f. \qquad (6\text{-}18)$$

Equations 6-15 and 6-18 are vector equations, which implies that conservation of momentum must be valid for each of the components, as suggested by Eq. 6-14. Thus

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$

and similarly for the y and z components. If all motion takes place in a plane (the xy plane) and if we know the initial velocities of m_1 and m_2, then Eq. 6-15 gives two relationships among the four unknowns (the x and y components of $\vec{\mathbf{v}}_{1f}$ and $\vec{\mathbf{v}}_{2f}$). If we also know *one* of the final velocities, we can find the other; or if we know the *directions* of the two final velocities, we can find their magnitudes. Equation 6-18, on the other hand, has only two unknowns (the x and y components of $\vec{\mathbf{v}}_f$), so the two component equations contained in Eq. 6-18 are therefore sufficient to solve for these two unknowns.

In many applications, m_2 is initially at rest ($\vec{\mathbf{v}}_{2i} = 0$). This simplifies the calculation somewhat. Since momentum conservation is valid in any inertial frame, we can always find a frame of reference in which m_2 is at rest and apply momentum conservation in that frame, returning to the original frame of reference if we wish to evaluate the final velocities in that frame.

Often we have a "head-on" collision in which all of the motion occurs only in one direction, which we take to be

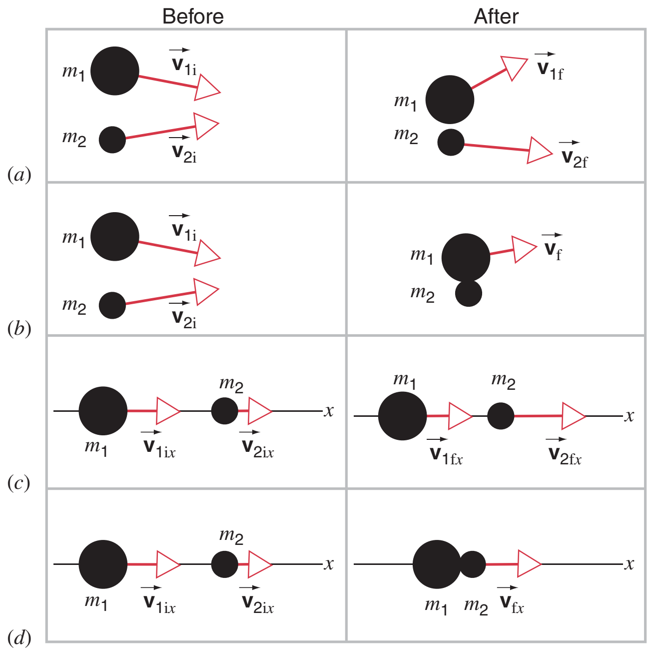


FIGURE 6-13. The initial and final velocities in various two-body collisions.

the x direction of our coordinate system (Fig. 6-13c). Conservation of momentum in this case can be written

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}. \qquad (6\text{-}19)$$

If three of the velocities are known, the fourth can be found from Eq. 6-19. If m_1 and m_2 stick together after the collision and move with a common final velocity v_{fx} (Fig. 6-13d), Eq. 6-19 becomes

$$m_1 v_{1ix} + m_2 v_{2ix} = (m_1 + m_2) v_{fx}. \qquad (6\text{-}20)$$

The x components of the velocities in Eqs. 6-19 and 6-20 can be positive or negative, depending on how we define the positive direction of the x axis.

Sample Problem 6-5. A glider of mass $m_1 = 1.25$ kg moves with a velocity of 3.62 m/s on a frictionless, level air track and collides with a second glider of mass $m_2 = 2.30$ kg that is initially at rest. After the collision, the first glider is found to be moving at 1.07 m/s in a direction opposite to that of its initial motion. What is the velocity of m_2 after the collision?

Solution Equation 6-19 gives the general momentum conservation result in one dimension. We choose the positive x direction to be that of the initial motion of m_1, so that $v_{1ix} = +3.62$ m/s and $v_{1fx} = -1.07$ m/s. With $v_{2ix} = 0$, we can solve Eq. 6-19 for the unknown v_{2fx} and obtain

$$v_{2fx} = \frac{m_1}{m_2}(v_{1ix} - v_{1fx})$$

$$= \frac{1.25 \text{ kg}}{2.30 \text{ kg}}[3.62 \text{ m/s} - (-1.07 \text{ m/s})] = 2.55 \text{ m/s}.$$

We can check this result by finding the change in momentum for each of the gliders:

$$\Delta p_{1x} = m_1(v_{1fx} - v_{1ix}) = (1.25 \text{ kg})(-1.07 \text{ m/s} - 3.62 \text{ m/s}) = -5.86 \text{ kg}\cdot\text{m/s},$$

$$\Delta p_{2x} = m_2(v_{2fx} - v_{2ix}) = (2.30 \text{ kg})(2.55 \text{ m/s} - 0) = +5.86 \text{ kg}\cdot\text{m/s}.$$

As expected, $\Delta p_{1x} = -\Delta p_{2x}$.

Sample Problem 6-6. Suppose the two gliders moving initially as in Sample Problem 6-5 stick together after the collision. What is the final velocity of the combination?

Solution In this case we can use Eq. 6-20 with $v_{2ix} = 0$:

$$v_{fx} = \frac{m_1 v_{1ix}}{m_1 + m_2} = \frac{(1.25 \text{ kg})(3.62 \text{ m/s})}{1.25 \text{ kg} + 2.30 \text{ kg}} = 1.27 \text{ m/s}.$$

Following the same method as in the previous Sample Problem, you should show that $\Delta p_{1x} = -2.93$ kg·m/s and $\Delta p_{2x} = +2.93$ kg·m/s, thereby satisfying conservation of momentum ($\Delta p_{1x} = -\Delta p_{2x}$).

Sample Problem 6-7. A spaceship of total mass m is coasting at a speed of 8.45 km/s (measured with respect to a particular inertial reference frame) in a region of space of negligible gravity. The ship consists of two craft of masses $m/4$ and $3m/4$ that are clamped together with a spring between them (Fig. 6-14). Upon a signal from the ship's commander, the bolts holding the two craft together are released, and the spring drives them apart such that the smaller craft moves forward (in the direction of the ship's original motion) at a speed of 11.63 km/s. What is the final speed of the larger craft?

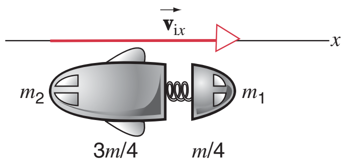


Figure 6-14. Sample Problem 6-7.

Solution This problem is the reverse of the previous one—instead of the two bodies colliding and sticking together, we have two bodies initially stuck together that come apart. The force that drives them apart is an internal force in the two-body system, so the law of conservation of momentum can be applied. We choose the positive x direction to be that of the original motion of the spaceship (and also that of the final velocity of the smaller craft). Let us turn Eq. 6-20 around, so that the velocity of the initial combined system is v_{ix} and the final velocities of the two pieces are v_{1fx} and v_{2fx}. Then momentum conservation gives

$$(m_1 + m_2)v_{ix} = m_1 v_{1fx} + m_2 v_{2fx}. \qquad (6\text{-}21)$$

We are given $v_{ix} = +8.45$ km/s and $v_{1fx} = +11.63$ km/s, and we wish to find v_{2fx}:

$$v_{2fx} = \frac{(m_1 + m_2)v_{ix} - m_1 v_{1fx}}{m_2}$$

$$= \frac{(m)(8.45 \text{ km/s}) - (m/4)(11.63 \text{ km/s})}{3m/4} = +7.39 \text{ km/s}.$$

Note that we do not need to know the actual mass of the ship for this calculation, only the relative masses of the two pieces. Is this true for all two-body collisions? Review Eqs. 6-19 and 6-20 to decide whether this is true in general.

It is instructive to analyze this problem from the perspective of another spaceship that is moving parallel to the first at the same velocity ($v_x = 8.45$ km/s). Relative to this ship, the smaller craft after its release moves with a velocity of $v'_{1fx} = v_{1fx} - v_x = 11.63$ km/s $- 8.45$ km/s $= 3.18$ km/s in the forward direction (the same direction as the ship's velocity). The larger craft moves with a velocity of $v'_{2fx} = v_{2fx} - v_x = 7.39$ km/s $- 8.45$ km/s $= -1.06$ km/s. In this frame of reference, the larger craft moves backward with a speed of 1.06 km/s. According to this observer, the initial momentum of the first spaceship (before the separation) is zero, because the relative velocity of the first ship is zero. After the separation, the total final momentum of the two craft is zero: $P'_{fx} = m_1 v'_{1fx} + m_2 v'_{2fx} = 0$, as you can show. According to this observer, the initial and final momenta are both zero, and thus momentum is conserved.

If momentum is conserved in one inertial reference frame, then it is conserved in every inertial reference frame. Often it is convenient to solve a problem in one reference frame and then transform the results to another. In the remainder of this section we discuss how the second reference frame used in this problem, in which the total momentum is zero, can often yield insight into the analysis of collisions.

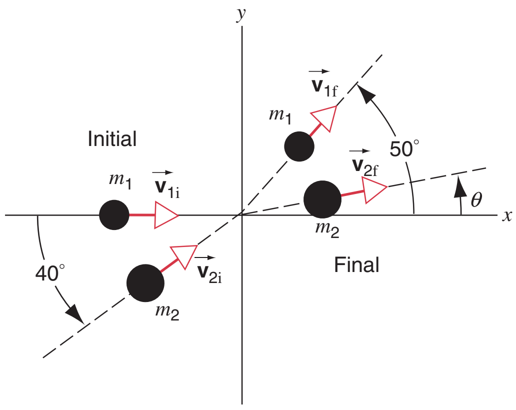


FIGURE 6-15. Sample Problem 6-8.

SAMPLE PROBLEM 6-8. A puck is sliding without friction on the ice at a speed of 2.48 m/s. It collides with a second puck of mass 1.5 times that of the first and moving initially with a velocity of 1.86 m/s in a direction 40° away from the direction of the first puck (Fig. 6-15). After the collision, the first puck moves at a velocity of 1.59 m/s in a direction at an angle of 50° from its initial direction (as shown in Fig. 6-15). Find the speed and direction of the second puck after the collision.

Solution In this problem we must use the law of conservation of momentum in its two-dimensional vector form. We define the x axis as the direction of the initial motion of the first puck. Let the second puck move with velocity $\vec{v}_{2f}$ at an angle θ with the x axis. Then the x component of the conservation of momentum equation (Eq. 6-15) gives $m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$, or (with $m_1 = m$ and $m_2 = 1.5m$)

$$m(2.48\ \text{m/s}) + 1.5m(1.86\ \text{m/s})\cos 40° = m(1.59\ \text{m/s})\cos 50° + 1.5m v_{2f}\cos\theta,$$

which reduces to

$$v_{2f}\cos\theta = 2.40\ \text{m/s},$$

and the y component is $m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$, so

$$m(0) + 1.5m(1.86\ \text{m/s})\sin 40° = m(1.59\ \text{m/s})\sin 50° + 1.5m v_{2f}\sin\theta,$$

which reduces to

$$v_{2f}\sin\theta = 0.38\ \text{m/s}.$$

Solving the two reduced equations for the two unknowns, we find

$$v_{2f} = 2.43\ \text{m/s}, \qquad \theta = 9.0°.$$

One-Dimensional Collisions in the Center-of-Mass Reference Frame

Earlier in this section we analyzed a one-dimensional collision between two bodies, viewed from an arbitrary inertial reference frame. Often we choose that frame to be fixed in the laboratory in which the collision is observed, so it is called the *laboratory reference frame* or lab frame. There is, however, another special reference frame in which the collision displays a particular symmetry and in which the analysis can often be carried out with relative ease. This frame, which is widely adopted by physicists for analyzing collisions of atoms and subatomic particles, is called the *center-of-mass reference frame* or cm frame. (The reason for this choice of name will be explained in the next chapter.) In Sample Problem 6-7, the second method of analysis was carried out in the cm frame, in which the spaceship appeared to be at rest before the separation.

The (one-dimensional) collision in the lab frame is shown in Fig. 6-16*a*. We now analyze that collision in the cm frame. All velocities are in the x direction, so for convenience we drop the x subscript from the velocities and momenta; however, it must be remembered that these are x components of vectors and so their signs must be chosen consistently relative to the direction we defined as the positive x direction. We use Eq. 4-32 in one dimension, and we let S represent the lab frame and S' represent the cm frame. The velocity $v_{S'S}$ (the velocity of the cm frame relative to the lab frame) is represented simply as v. According to an observer in the cm frame, the initial velocities of the two colliding objects are

$$m_1:\quad v'_{1i} = v_{1i} - v \qquad \text{and} \qquad m_2:\quad v'_{2i} = v_{2i} - v,$$

where the primes represent quantities measured in the cm (S') frame.

The total initial momentum of the two bodies in the cm frame is then

$$P'_i = m_1 v'_{1i} + m_2 v'_{2i} = m_1(v_{1i} - v) + m_2(v_{2i} - v). \tag{6-22}$$

We now *define* the cm frame to be *the frame in which the initial momentum of the two-body system is zero.* To find the particular value of v that will bring this about, we set $P'_i = 0$ in Eq. 6-22 and solve for v, obtaining

$$v = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2}. \tag{6-23}$$

If we travel at this velocity and observe the collision, the motion of the two bodies *before* the collision would appear as in Fig. 6-16*b*. Even though their masses may be different, the momenta of the two bodies are equal and opposite, so that their total is zero.

Because momentum is conserved, the total momentum *after* the collision ($P'_f = p'_{1f} + p'_{2f}$) must also be zero in the cm frame, so after the collision p'_{1f} and p'_{2f} must also be equal

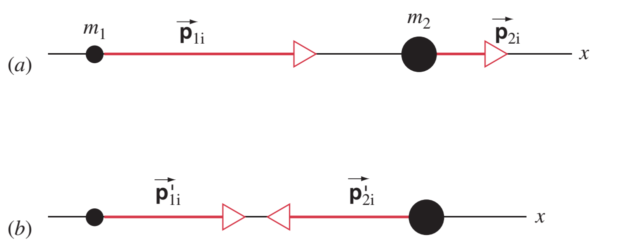


FIGURE 6-16. The momenta of two bodies before their collision in (*a*) the original frame of reference and (*b*) the center-of-mass frame of reference.

and opposite. The final momentum vectors may have any length, as long as they are equal to one another in magnitude.

Line 1 of Fig. 6-17, which is identical to Fig. 6-16*b*, shows the initial momenta of the two bodies in the cm frame. The outcome of the collision depends on the properties of the colliding bodies and on the nature of the forces they exert on each other; lines 2–5 of Fig. 6-17 show several different possibilities for the final momenta p'_{1f} and p'_{2f} in the cm frame. The total final momentum is zero in this frame, so that $p'_{1f} = -p'_{2f}$. Regardless of the type of collision, this symmetry is revealed in the cm frame.

In the case shown in line 2 of Fig. 6-17, the bodies simply bounce off one another, with their momenta unchanged in magnitude and reversed in direction. This type of collision is known as an *elastic* collision. Rigid solid objects, such as billiard balls or hockey pucks, usually experience collisions that can be regarded as elastic. In *inelastic* collisions (line 3), the bodies rebound with smaller momenta in the cm frame. This is the case for nonrigid objects, such as the baseball in Fig. 6-1. If the bodies stick together after the collision (line 4), the combination will be at rest in the cm frame; we call this a *completely inelastic* collision. The collision of two putty balls is an example. Finally (line 5), the bodies might rebound with momenta larger than their initial values. This might occur, for example, if a coiled spring or an explosive charge were released between the bodies at the instant of collision.

Elastic Collisions. We have defined an elastic collision as one in which, *in the cm frame,* the velocity of each body changes in direction but not in magnitude. Thus for m_1, $v'_{1f} = -v'_{1i}$ in the cm frame, and similarly for m_2. Now we use these results to derive expressions for the final velocities of the two elastically colliding bodies in the lab frame.

For m_1, the velocities in the two frames are related by $v'_{1i} = v_{1i} - v$ and $v'_{1f} = v_{1f} - v$, where v is the relative velocity between the frames (Eq. 6-23). Solving the latter of these two equations for the velocity in the lab frame, we obtain $v_{1f} = v'_{1f} + v$. Substituting the condition for elastic collisions ($v'_{1f} = -v'_{1i}$) then gives $v_{1f} = -v'_{1i} + v$. Finally, using the relationship between v'_{1i} and v_{1i}, we obtain

$$v_{1f} = -(v_{1i} - v) + v = -v_{1i} + 2v$$
$$= -v_{1i} + 2\frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2},$$

where the latter result comes from using Eq. 6-23. After some rearrangement, we get

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}. \quad (6\text{-}24)$$

To find v_{2f}, the final velocity of m_2, we could repeat the analysis that led to Eq. 6-24, interchanging the subscripts "1" and "2" everywhere they appear. In fact, we can simply make those changes to Eq. 6-24, which gives

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i}. \quad (6\text{-}25)$$

Equations 6-24 and 6-25 are general results for one-dimensional elastic collisions and allow us to calculate the final velocities in *any* inertial reference frame in terms of the initial velocities in that frame. Here are some special cases of interest:

1. *Equal masses.* When the colliding particles have equal masses ($m_1 = m_2$), Eqs. 6-24 and 6-25 become simply

$$v_{1f} = v_{2i} \quad \text{and} \quad v_{2f} = v_{1i}. \quad (6\text{-}26)$$

That is, the particles exchange velocities: the final velocity of one particle is equal to the initial velocity of the other.

2. *Target particle at rest.* Another case of interest is that in which particle m_2 is initially at rest. Then $v_{2i} = 0$ and

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \quad \text{and} \quad v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}. \quad (6\text{-}27)$$

Combining this special case with the previous one (that is, a collision between equal mass particles in which one is initially at rest), we see that the first particle is "stopped cold" and the second one "takes off" with the velocity the first one originally had. It is often possible to observe this effect in collisions of nonrotating billiard balls.

3. *Massive target.* If $m_2 >> m_1$, then Eqs. 6-24 and 6-25 reduce to

$$v_{1f} \approx -v_{1i} + 2v_{2i} \quad \text{and} \quad v_{2f} \approx v_{2i}. \quad (6\text{-}28)$$

When the massive particle is moving slowly or at rest, then

$$v_{1f} \approx -v_{1i} \quad \text{and} \quad v_{2f} \approx 0. \quad (6\text{-}29)$$

That is, when a light projectile collides with a very much more massive one at rest, the velocity of the light particle is approximately reversed, and the massive particle remains

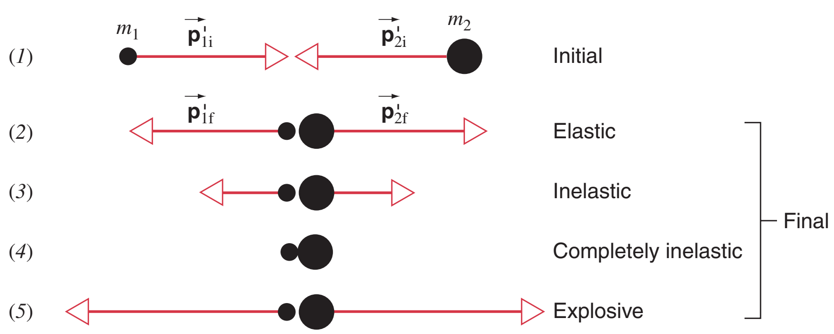


Figure 6-17. The momenta of two colliding objects in the center-of-mass frame for various types of collisions. Line 1 shows the initial momenta in this frame, and lines 2–5 show some possible final momenta.

approximately at rest. For example, a ball dropped from a height h rebounds from the Earth after the collision with reversed velocity and, if the collision were perfectly elastic and there were no air resistance, it would reach the same height h. Similarly, an electron rebounds from a (relatively massive) atom in a head-on collision with its motion reversed, while the target atom is essentially unaffected by the collision.

4. *Massive projectile.* When $m_1 >> m_2$, Eqs. 6-24 and 6-25 become

$$v_{1f} \approx v_{1i} \quad \text{and} \quad v_{2f} \approx 2v_{1i} - v_{2i}. \qquad (6\text{-}30)$$

If the light target particle is initially at rest (or moving much slower than m_1), then after the collision the target particle moves at twice the speed of m_1. The motion of m_1 is nearly unaffected by the collision with the much lighter target.

In alpha-particle scattering (Fig. 6-2), the incident alpha particle (whose mass is about 7000 times the electron mass) is essentially unaffected by collisions with the electrons of the target atoms (as indicated by the many straight-line paths in Fig. 6-2). The alpha particle is deflected only in the rare encounters with the massive nucleus of a target atom.

Equations 6-26 to 6-30 hold only for elastic collisions. For partially inelastic or explosive collisions, it is not possible to obtain a set of general equations like 6-24 or 6-25 for the final velocities unless we have more information about the momentum of each particle that is added or lost in the center-of-mass frame. In Chapter 11 we will see how considerations based on energy also allow us to analyze these different types of collisions. In partially inelastic or explosive collisions, the gain or loss of momentum (or energy) by one of the colliding bodies can be used to deduce properties related to the interaction between the bodies. This is a common technique used in nuclear physics, in which information about the properties of nuclei can be deduced by observing the momentum of the outgoing particles in nuclear collisions.

SAMPLE PROBLEM 6-9. An alpha particle (a nucleus of an atom of helium, $m = 4.0$ u) is accelerated to a velocity of 1.52×10^7 m/s and collides head-on with an oxygen nucleus ($m = 16.0$ u) at rest. After the collision, the oxygen nucleus moves with a velocity of 6.08×10^6 m/s along the original direction of motion of the alpha particle. (*a*) What is the velocity of the alpha particle after the collision? (*b*) Which type of collision listed in Fig. 6-17 best describes this process?

Solution (*a*) Conservation of momentum allows us to find the velocity of the alpha particle. Equation 6-19 expresses conservation of momentum for general one-dimensional collisions. We take the positive x direction to be that of the initial velocity of the alpha particle. Letting particle 1 be the alpha particle and particle 2 be the oxygen, we can write Eq. 6-19 with $v_{2ix} = 0$ as

$$v_{\alpha fx} = \frac{m_\alpha v_{\alpha ix} - m_O v_{Ofx}}{m_\alpha}$$

$$= \frac{(4.0\ \text{u})(1.52 \times 10^7\ \text{m/s}) - (16.0\ \text{u})(6.08 \times 10^6\ \text{m/s})}{4.0\ \text{u}}$$

$$= -9.12 \times 10^6\ \text{m/s}.$$

The alpha particle rebounds in the negative x direction.

Note that the mass units cancel in this equation, so we are free to use any convenient unit to express the masses of the particles. (*b*) The relative velocity between the lab and cm frames is given by Eq. 6-23:

$$v_x = \frac{m_\alpha v_{\alpha ix} + m_O v_{Oix}}{m_\alpha + m_O} = \frac{(4.0\ \text{u})(1.52 \times 10^7\ \text{m/s}) + 0}{4.0\ \text{u} + 16.0\ \text{u}}$$

$$= +0.304 \times 10^7\ \text{m/s}.$$

The initial momentum of the alpha particle in the cm frame is then $p'_{\alpha ix} = m_\alpha v'_{\alpha ix} = m_\alpha(v_{\alpha ix} - v_x) = (4.0\ \text{u})(1.52 \times 10^7\ \text{m/s} - 0.304 \times 10^7\ \text{m/s}) = +4.86 \times 10^7\ \text{u} \cdot \text{m/s}$. The final momentum of the alpha particle is $p'_{\alpha fx} = m_\alpha v'_{\alpha fx} = m_\alpha(v_{\alpha fx} - v_x) = (4.0\ \text{u})(-9.12 \times 10^6\ \text{m/s} - 0.304 \times 10^7\ \text{m/s}) = -4.86 \times 10^7\ \text{u} \cdot \text{m/s}$. Thus $p'_{\alpha ix} = -p'_{\alpha fx}$, and the alpha particle only reverses the direction of its momentum with its magnitude unchanged. You can show that the momentum of the oxygen nucleus is also simply reversed in the collision. The reversal of the momentum of both particles, with their magnitudes remaining the same, is the characteristic of an *elastic* collision.

MULTIPLE CHOICE

6-1 Collisions

6-2 Linear Momentum

1. Which of the following objects has the largest momentum?
   (A) A bullet fired from a rifle
   (B) A football quarterback running at top speed
   (C) A horse walking at about 2 miles/hour
   (D) An elephant standing still

2. A 2-kg ball moving straight down strikes the floor at 8 m/s. It rebounds upward at 6 m/s. What is the magnitude of the change in momentum of the ball?
   (A) 2 kg · m/s (B) 4 kg · m/s
   (C) 14 kg · m/s (D) 28 kg · m/s

3. An object is moving in a circle at constant speed v. The magnitude of the rate of change of momentum of the object
   (A) is zero. (B) is proportional to v.
   (C) is proportional to v^2. (D) is proportional to v^3.

4. If the net force acting on a body is constant, what can we conclude about its momentum?
   (A) The magnitude and/or the direction of $\vec{\mathbf{p}}$ may change.
   (B) The magnitude of $\vec{\mathbf{p}}$ remains fixed, but its direction may change.
   (C) The direction of $\vec{\mathbf{p}}$ remains fixed, but its magnitude may change.
   (D) $\vec{\mathbf{p}}$ remains fixed in both magnitude and direction.

6-3 Impulse and Momentum

5. An object is moving in a circle at constant speed v. From time t_i to time t_f the object moves halfway around the circle. The magnitude of the impulse due to the net force on the object during this time interval
   (A) is zero. (B) is proportional to v.
   (C) is proportional to v^2. (D) is proportional to v^3.

6. If $\vec{\mathbf{J}}$ is the impulse of a particular force, what is $d\vec{\mathbf{J}}/dt$?
   (A) The momentum (B) The change in momentum
   (C) The force (D) The change in the force

7. A variable force acts on an object from $t_i = 0$ to t_f. The impulse of the force is zero. One can conclude that
   (A) $\Delta\vec{\mathbf{r}} = 0$ and $\Delta\vec{\mathbf{p}} = 0$.
   (B) $\Delta\vec{\mathbf{r}} = 0$ but possibly $\Delta\vec{\mathbf{p}} \neq 0$.
   (C) possibly $\Delta\vec{\mathbf{r}} \neq 0$ but $\Delta\vec{\mathbf{p}} = 0$.
   (D) possibly $\Delta\vec{\mathbf{r}} \neq 0$ and possibly $\Delta\vec{\mathbf{p}} \neq 0$.

8. A small car traveling along a road at high speed loses control. The driver has a choice—collide with a solid concrete wall or with an oncoming fully loaded 10-ton truck, also moving at high speed. Which choice results in the more serious collision? Assume in both cases that the small car is at rest after the collision.
   (A) The collision with the truck
   (B) The collision with the concrete wall
   (C) The collisions would be equally serious, since the same impulse is imported to the car in either case.
   (D) More information is needed to assess the collisions.

9. Riot police often use rubber bullets instead of lead bullets. Assume that neither bullet penetrates the skin, both have the same mass, time of contact, and initial speed. The difference is that the lead bullets "stick" while the rubber bullets bounce off. Which "hurts" more?
   (A) The lead bullet
   (B) The rubber bullet
   (C) The bullets hurt the same amount.
   (D) It depends on where you get hit.

6-4 Conservation of Momentum

10. Can the law of momentum conservation ever be violated?
    (A) No
    (B) Yes, if there are more than two particles
    (C) Yes, when the forces between the particles are varying in time
    (D) Yes, if the two particles stick together after a collision

11. A basketball player jumps up to "shoot" a basket. Is momentum conserved?
    (A) Yes, but only if you choose the correct system
    (B) Yes, but only in the horizontal direction
    (C) No, because the velocity of the basketball player changes with time
    (D) It is a bad question, because momentum conservation is for objects moving at constant speed, and the basketball player is accelerating.

6-5 Two-Body Collisions

12. Consider a one-dimensional collision that involves a body of mass m_1 originally moving in the positive x direction with speed v_0 colliding with a second body of mass m_2 originally at rest. The collision could be completely inelastic, with the two bodies sticking together, completely elastic, or somewhere in between. After the collision, m_1 moves with velocity v_1 while m_2 moves with velocity v_2.
    (*a*) If $m_1 > m_2$, then
    (A) $-v_0 < v_1 < 0$ (B) $0 < v_1 < v_0$
    (C) $0 < v_1 < 2v_0$ (D) $v_0 < v_1 < 2v_0$
    (*b*) and
    (A) $-v_0 < v_2 < 0$. (B) $0 < v_2 < v_0$.
    (C) $v_0/2 < v_2 < 2v_0$. (D) $v_0 < v_2 < 2v_0$.
    (*c*) If $m_1 < m_2$ then
    (A) $-v_0 < v_1 < 0$ (B) $-v_0 < v_1 < v_0/2$
    (C) $0 < v_1 < v_0/2$ (D) $0 < v_1 < v_0$
    (*d*) and
    (A) $-v_0 < v_2 < 0$. (B) $-v_0 < v_2 < v_0/2$.
    (C) $0 < v_2 < v_0/2$. (D) $0 < v_2 < v_0$.

QUESTIONS

1. Justify the following statement. "The law of conservation of linear momentum, as applied to a single particle, is equivalent to Newton's first law of motion."

2. A particle with mass $m = 0$ (a neutrino, possibly) carries momentum. How can this be in view of Eq. 6-1, in which we see that the momentum is directly proportional to the mass?

3. Although the acceleration of a baseball after it has been hit does not depend on who hit it, something about the baseball's flight must depend on the batter. What is it?

4. Explain how an airbag in an automobile may help to protect a passenger from serious injury in case of a collision.

5. It is said that, during a 30-mi/h collision, a 10-lb child can exert a 300-lb force against a parent's grip. How can such a large force come about?

6. Can the impulse of a force be zero, even if the force is not zero? Explain why or why not.

7. Figure 6-18 shows a popular carnival device, in which the contestant tries to see how high a weighted marker can be raised by hitting a target with a sledge hammer. What physi-

FIGURE 6-18. Question 7.

cal quantity does the device measure? Is it the average force, the maximum force, the work done, the impulse, the energy transferred, the momentum transferred, or something else? Discuss your answer.

8. An hourglass is being weighed on a sensitive balance, first when sand is dropping in a steady stream from the upper to the lower part, and then again after the upper part is empty. Are the two weights the same or not? Explain your answer.

9. Give a plausible explanation for the breaking of wooden boards or bricks by a karate punch. (See "Karate Strikes," by Jearl D. Walker, *American Journal of Physics,* October 1975, p. 845.)

10. Explain how conservation of momentum applies to a handball bouncing off a wall.

11. A football player, momentarily at rest on the field, catches a football as he is tackled by a running player on the other team. This is certainly a collision (inelastic!) and momentum must be conserved. In the reference frame of the football field, there is momentum before the collision but there seems to be none after the collision. Is linear momentum really conserved? If so, explain how. If not, explain why.

12. You are driving along a highway at 50 mi/h, followed by another car moving at the same speed. You slow to 40 mi/h but the other driver does not and there is a collision. What are the initial velocities of the colliding cars as seen from the reference frame of (*a*) yourself, (*b*) the other driver, and (*c*) a state trooper, who is in a patrol car parked by the roadside? (*d*) A judge asks whether you bumped into the other driver or the other driver bumped into you. As a physicist, how would you answer?

13. C. R. Daish has written that, for professional golfers, the initial speed of the ball off the clubhead is about 140 mi/h. He also says: (*a*) "If the Empire State Building could be swung at the ball at the same speed as the clubhead, the initial ball velocity would only be increased by about 2%" and (*b*) "once the golfer has started his or her downswing, camera clicking, sneezing, and so on can have no effect on the motion of the ball." Can you give qualitative arguments to support these two statements?

14. Two identical cubical blocks, moving in the same direction with a common speed v, strike a third such block initially at rest on a horizontal frictionless surface. What is the motion of the blocks after the collision? Does it matter whether or not the two initially moving blocks were in contact? Does it matter whether these two blocks were glued together? Assume that the collisions are (*a*) completely inelastic or (*b*) elastic.

15. In a two-body collision in the center-of-mass reference frame the momenta of the particles are equal and opposite to one another both before and after the collision. Is the line of relative motion necessarily the same after collision as before? Under what conditions would the magnitudes of the velocities of the bodies increase? decrease? remain the same as a result of the collision?

EXERCISES

6-1 Collisions

6-2 Linear Momentum

1. How fast must an 816-kg Volkswagen travel to have the same momentum as (*a*) a 2650-kg Cadillac going 16.0 km/h? (*b*) a 9080-kg truck also going 16.0 km/hr?

2. A 2000-kg truck traveling north at 40.0 km/h turns east and accelerates to 50.0 km/h. What is the magnitude and direction of the change of the truck's momentum?

3. A 4.88-kg object with a speed of 31.4 m/s strikes a steel plate at an angle of 42.0° and rebounds at the same speed and angle (Fig. 6-19). What is the change (magnitude and direction) of the linear momentum of the object?

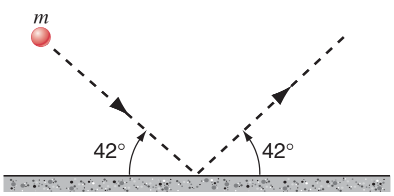


FIGURE 6-19. Exercise 3.

6-3 Impulse and Momentum

4. The bumper of a new car is being tested. The 2300-kg vehicle, moving at 15 m/s, is allowed to collide with a bridge abutment, being brought to rest in a time of 0.54 s. Find the average force that acted on the car during impact.

5. A ball of mass m and speed v strikes a wall perpendicularly and rebounds with undiminished speed. (*a*) If the time of collision is Δt, what is the average force exerted by the ball on the wall? (*b*) Evaluate this average force numerically for a rubber ball with mass 140 g moving at 7.8 m/s; the duration of the collision is 3.9 ms.

6. A golfer hits a golf ball, imparting to it an initial velocity of magnitude 52.2 m/s directed 30° above the horizontal. Assuming that the mass of the ball is 46.0 g and the club and ball are in contact for 1.20 ms, find (*a*) the impulse imparted to the ball, (*b*) the impulse imparted to the club, and (*c*) the average force exerted on the ball by the club.

7. A 150-g (weight = 5.30 oz) baseball pitched at a speed of 41.6 m/s (= 136 ft/s) is hit straight back to the pitcher at a speed of 61.5 m/s (= 202 ft/s). The bat is in contact with the ball for 4.70 ms. Find the average force exerted by the bat on the ball.

8. A force that averages 984 N is applied to a 420-g steel ball moving at 13.8 m/s by a collision lasting 27.0 ms. If the force is in a direction opposite to the initial velocity of the ball, find the final speed of the ball.

9. Figure 6-20 shows an approximate representation of force versus time during the collision of a 58-g tennis ball with a wall. The initial velocity of the ball is 32 m/s perpendicular to the wall; it rebounds with the same speed, also perpendicular to the wall. What is the value of F_{max}, the maximum contact force during the collision?

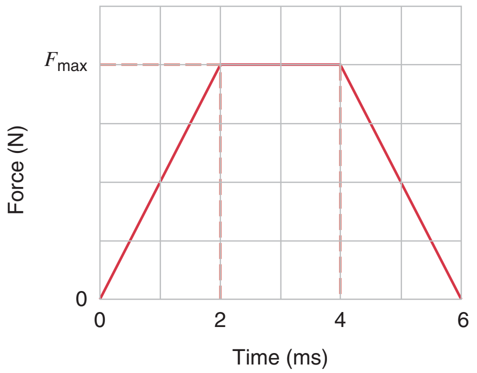


FIGURE 6-20. Exercise 9.

10. Two parts of a spacecraft are separated by detonating the explosive bolts that hold them together. The masses of the parts are 1200 kg and 1800 kg; the magnitude of the impulse delivered to each part is 300 N·s. What is the relative speed of separation of the two parts?

11. A croquet ball with a mass 0.50 kg is struck by a mallet, receiving the impulse shown in the graph (Fig. 6-21). What is the ball's velocity just after the force has become zero?

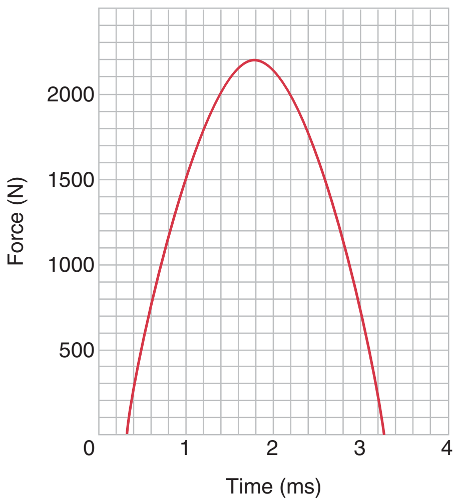


FIGURE 6-21. Exercise 11.

12. A karate expert breaks a pine board, 2.2 cm thick, with a hand chop. Strobe photography shows that the hand, whose mass may be taken as 540 g, strikes the top of the board with a speed of 9.5 m/s and comes to rest 2.8 cm below this level. (*a*) What is the time duration of the chop (assuming a constant force)? (*b*) What average force is applied?

13. A 2500-kg unmanned space probe is moving in a straight line at a constant speed of 300 m/s. A rocket engine on the space probe executes a burn in which a thrust of 3000 N acts for 65.0 s. What is the change in momentum (magnitude only) of the probe if the thrust is backward, forward, or sideways? Assume that the mass of the ejected fuel is negligible compared to the mass of the space probe.

14. A pellet gun fires ten 2.14-g pellets per second with a speed of 483 m/s. The pellets are stopped by a rigid wall. (*a*) Find the momentum of each pellet. (*b*) Calculate the average force exerted by the stream of pellets on the wall. (*c*) If each pellet is in contact with the wall for 1.25 ms, what is the average force exerted on the wall by each pellet while in contact? Why is this so different from (*b*)?

15. After launch from Earth orbit, a robot spacecraft of mass 5400 kg is coasting at constant speed halfway through its six-month flight to Mars when a NASA engineer discovers that, instead of heading for a 100-km-high orbit above the Martian surface, it is headed on a collision course directly toward the center of the planet. To correct the course, the engineer orders a short burst from the spacecraft's thrusters transverse to the direction of its motion. The thrust engines provide a constant force of 1200 N. For how long a time must the thrusters fire to achieve the correct course? Take needed data from Appendix C, and assume the distance between Earth and Mars to remain constant at its smallest possible value.

6-4 Conservation of Momentum

16. A 195-lb man standing on a surface of negligible friction kicks forward a 0.158-lb stone lying at his feet so that it acquires a speed of 12.7 ft/s. What velocity does the man acquire as a result?

17. A 75.2-kg man is riding on a 38.6-kg cart traveling at a speed of 2.33 m/s. He jumps off in such a way as to land on the ground with zero horizontal speed. Find the resulting change in the speed of the cart.

18. A railroad flatcar of weight W can roll without friction along a straight horizontal track. Initially, a man of weight w is standing on the car, which is moving to the right with speed v_0. What is the change in velocity of the car if the man runs to the left (Fig. 6-22) so that his speed relative to the car is v_{rel} just before he jumps off at the left end?

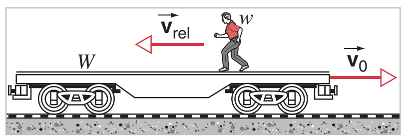


FIGURE 6-22. Exercise 18.

6-5 Two-Body Collisions

19. A space vehicle is traveling at 3860 km/h with respect to the Earth when the exhausted rocket motor is disengaged and sent backward with a speed of 125 km/h with respect to the command module. The mass of the motor is four times the mass of the module. What is the speed of the command module after the separation?

20. The blocks in Fig. 6-23 slide without friction. What is the velocity $\vec{\mathbf{v}}$ of the 1.6-kg block after the collision?

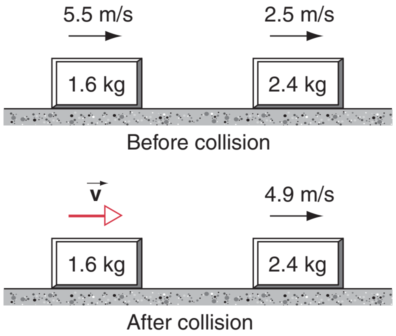


FIGURE 6-23. Exercises 20 and 21.

21. Refer to Fig. 6-23. Suppose the initial velocity of the 2.4-kg block is reversed; it is headed directly toward the 1.6-kg block. What would be the velocity $\vec{\mathbf{v}}$ of the 1.6-kg block after the collision?

22. Meteor Crater in Arizona (see Fig. 6-24) is thought to have been formed by the impact of a meteorite with the Earth some 20,000 years ago. The mass of the meteorite is estimated to be 5×10^{10} kg and its speed to have been 7.2 km/s. What speed would such a meteorite impart to the Earth in a head-on collision?

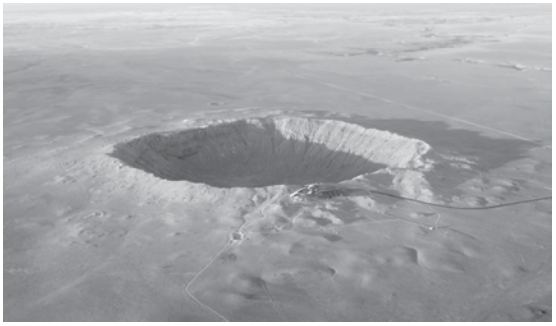


FIGURE 6-24. Exercise 22.

23. A 5.18-g bullet moving at 672 m/s strikes a 715-g wooden block at rest on a frictionless surface. The bullet emerges with its speed reduced to 428 m/s. Find the resulting speed of the block.

24. An alpha particle collides with an oxygen nucleus, initially at rest. The alpha particle is scattered at an angle of 64.0° above its initial direction of motion and oxygen nucleus recoils at an angle of 51.0° below this initial direction. The final speed of the oxygen nucleus is 1.20×10^5 m/s. What is the final speed of the alpha particle? (The mass of an alpha particle is 4.00 u and the mass of an oxygen nucleus is 16.0 u.)

25. Two objects, A and B, collide. A has mass 2.0 kg, and B has mass 3.0 kg. The velocities before the collision are $\vec{\mathbf{v}}_{iA} = (15\text{ m/s})\hat{\mathbf{i}} + (30\text{ m/s})\hat{\mathbf{j}}$ and $\vec{\mathbf{v}}_{iB} = (-10\text{ m/s})\hat{\mathbf{i}} + (5.0\text{ m/s})\hat{\mathbf{j}}$. After the collision, $\vec{\mathbf{v}}_{fA} = (-6.0\text{ m/s})\hat{\mathbf{i}} + (30\text{ m/s})\hat{\mathbf{j}}$. What is the final velocity of B?

26. A radioactive nucleus, initially at rest, decays by emitting an electron and a neutrino at right angles to one another. The momentum of the electron is 1.2×10^{-22} kg·m/s and that of the neutrino is 6.4×10^{-23} kg·m/s. Find the direction and magnitude of the momentum of the recoiling nucleus.

27. A barge with mass 1.50×10^5 kg is proceeding downriver at 6.20 m/s in heavy fog when it collides broadside with a barge heading directly across the river; see Fig. 6-25. The second barge has mass 2.78×10^5 kg and was moving at 4.30 m/s. Immediately after impact, the second barge finds its course deflected by 18.0° in the downriver direction and its speed increased to 5.10 m/s. The river current was practially zero at the time of the accident. What is the speed and direction of motion of the first barge immediately after the collision?

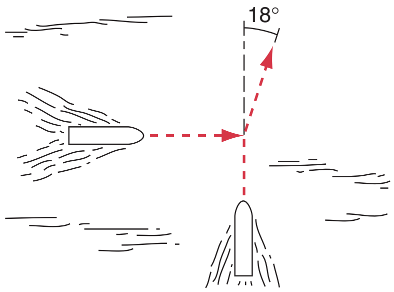


FIGURE 6-25. Exercise 27.

28. A hovering fly is approached by an enraged elephant charging at 2.1 m/s. Assuming that the collision is elastic, at what speed does the fly rebound? Note that the projectile (the elephant) is much more massive than the target (the fly).

29. Two titanium spheres approach each other head-on with the same speed and collide elastically. After the collision, one of the spheres, whose mass is 300 g, remains at rest. What is the mass of the other sphere?

30. A cart with mass 342 g moving on a frictionless linear air-track at an initial speed of 1.24 m/s strikes a second cart of unknown mass at rest. The collision between the carts is elastic. After the collision, the first cart continues in its original direction at 0.636 m/s. (*a*) What is the mass of the second cart? (*b*) What is its speed after impact?

31. An object of 2.0-kg mass makes an elastic collision with another object at rest and continues to move in the original direction but with one-fourth of its original speed. What is the mass of the struck object?

32. A railroad freight car weighing 31.8 tons and traveling at 5.20 ft/s overtakes one weighing 24.2 tons and traveling at 2.90 ft/s in the same direction. (*a*) Find the speeds of the cars after collision if the cars couple together. (*b*) If instead, as is very unlikely, the collision is elastic, find the speeds of the cars after collision.

33. After a totally inelastic collision, two objects of the same mass and initial speed are found to move away together at half their initial speed. Find the angle between the initial velocities of the objects.

34. A proton (atomic mass 1.01 u) with a speed of 518 m/s collides elastically with another proton at rest. The original proton is scattered 64.0° from its initial direction. (*a*) What is the direction of the velocity of the target proton after the collision? (*b*) What are the speeds of the two protons after the collision?

35. In the laboratory, a particle of mass 3.16 kg moving at 15.6 m/s to the left collides head-on with a particle of mass 2.84 kg moving at 12.2 m/s to the right. Find the velocity of the center of mass of the system of two particles after the collision.

PROBLEMS

1. A stream of water impinges on a stationary "dished" turbine blade, as shown in Fig. 6-26. The speed of the water is u, both before and after it strikes the curved surface of the blade, and the mass of water striking the blade per unit time is constant at the value μ. Find the force exerted by the water on the blade.

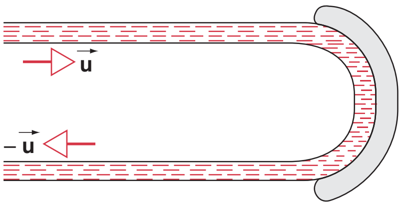


FIGURE 6-26. Problem 1.

2. A 1420-kg car moving at 5.28 m/s is initially traveling north. After completing a 90° right-hand turn in 4.60 s, the inattentive operator drives into a tree, which stops the car in 350 ms. What is the magnitude of the impulse delivered to the car (*a*) during the turn and (*b*) during the collision? What average force acts on the car (*c*) during the turn and (*d*) during the collision?

3. A 325-g ball with a speed v of 6.22 m/s strikes a wall at an angle θ of 33.0° and then rebounds with the same speed and angle (Fig. 6-27). It is in contact with the wall for 10.4 ms. (*a*) What impulse was experienced by the ball? (*b*) What was the average force exerted by the ball on the wall?

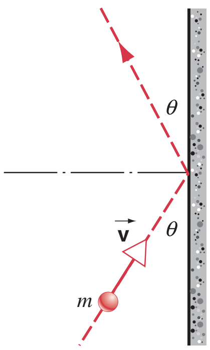


FIGURE 6-27. Problem 3.

4. It is well known that bullets and other missiles fired at Superman simply bounce off his chest as in Fig. 6-28. Suppose that a gangster sprays Superman's chest with 3.0-g bullets at the rate of 100 bullets/min, the speed of each bullet being 500 m/s. Suppose too that the bullets rebound straight back with no loss in speed. Find the average force exerted by the stream of bullets on Superman's chest.

5. During a violent thunderstorm, hail the size of marbles (diameter = 1.0 cm) falls at a speed of 25 m/s. There are estimated to be 120 hailstones per cubic meter of air. Ignore the bounce of the hail on impact. (*a*) What is the mass of each hailstone? (*b*) What force is exerted by hail on a 10 m × 20 m flat roof during the storm? Assume that, as for ice, 1.0 cm^3 of hail has a mass of 0.92 g.

FIGURE 6-28. Problem 4.

6. A very flexible uniform chain of mass M and length L is suspended from one end so that it hangs vertically, the lower end just touching the surface of a table. The upper end is suddenly released so that the chain falls onto the table and coils up in a small heap, each link coming to rest the instant it strikes the table; see Fig. 6-29. Find the force exerted by the table on the chain at any instant, in terms of the weight of chain already on the table at that moment.

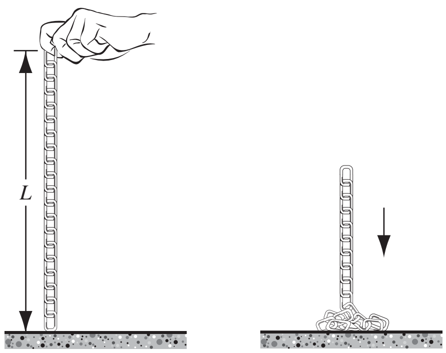


FIGURE 6-29. Problem 6.

7. A box is put on a scale that is adjusted to read zero when the box is empty. A stream of marbles is then poured into the box from a height h above its bottom at a rate of R (marbles per second). Each marble has a mass m. The collisions are completely inelastic; assume that the marbles stick to the box without bouncing when they hit. Find the scale reading of weight at time t after the marbles begin to fill the box. Determine a numerical answer when $R = 115\ s^{-1}$, $h = 9.62$ m, $m = 4.60$ g, and $t = 6.50$ s.

8. A 1930-kg railroad flatcar, which can move on the tracks with virtually no friction, is sitting motionless next to a station platform. A 108-kg football player is running along the platform parallel to the tracks at 9.74 m/s. He jumps onto the back of the flatcar. (*a*) What is the speed of the flatcar after he is aboard and at rest on the flatcar? (*b*) Now he starts to walk, at 0.520 m/s relative to the flatcar, to the front of the car. What is the speed of the flatcar as he walks?

9. A 2.9-ton weight falling through a distance of 6.5 ft drives a 0.50-ton pile 1.5 in. into the ground. (*a*) Assuming that the weight–pile collision is completely inelastic, find the average force of resistance exerted by the ground. (*b*) Assuming the force of resistance by the ground remains constant at the value found in (*a*), how far into the ground would the pile be driven if the collision were elastic? (*c*) Which is more effective in this case—elastic or inelastic collisions?

10. Two 22.7-kg ice sleds are placed a short distance apart, one directly behind the other, as shown in Fig. 6-30. A 3.63-kg cat, standing on one sled, jumps across to the other and immediately back to the first. Both jumps are made at a speed of 3.05 m/s relative to the sled the cat is standing on when the jump is made. Find the final speeds of the two sleds.

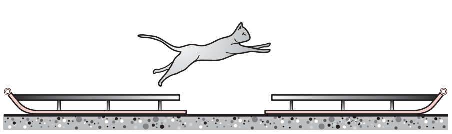


FIGURE 6-30. Problem 10.

11. Two vehicles *A* and *B* are traveling west and south, respectively, toward the same intersection where they collide and lock together. Before the collision, *A* (weight 2720 lb) is moving with a speed of 38.5 mi/h and *B* (weight 3640 lb) has a speed of 58.0 mi/h. Find the magnitude and direction of the velocity of the (interlocked) vehicles immediately after the collision.

12. Two balls *A* and *B*, having different but unknown masses, collide. *A* is initially at rest and *B* has a speed v. After collision, *B* has a speed $v/2$ and moves at right angles to its original motion. (*a*) Find the direction in which ball *A* moves after the collision. (*b*) Can you determine the speed of *A* from the information given? Explain.

13. In a game of pool, the cue ball strikes another ball initially at rest. After the collision, the cue ball moves at 3.50 m/s along a line making an angle of 65.0° with its original direction of motion. The second ball acquires a speed of 6.75 m/s. Using momentum conservation, find (*a*) the angle between the direction of motion of the second ball and the original direction of motion of the cue ball and (*b*) the original speed of the cue ball.

14. Spacecraft *Voyager 2* (mass m and speed v relative to the Sun) approaches the planet Jupiter (mass M and speed V relative to the Sun) as shown in Fig. 6-31. The spacecraft rounds the planet and departs in the opposite direction. What is its speed, relative to the Sun, after this "slingshot" encounter? Assume that $v = 12$ km/s and $V = 13$ km/s (the orbital speed of Jupiter), and that this is an elastic collision. The mass of Jupiter is very much greater than the mass of the spacecraft, $M \gg m$. (See "The Slingshot Effect: Explanation and Analogies," by Albert A. Bartlett and Charles W. Hord, *The Physics Teacher,* November 1985, p. 466.)

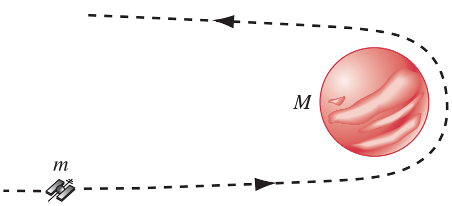


FIGURE 6-31. Problem 14.

15. The head of a golf club moving at 45.0 m/s strikes a golf ball (mass = 46.0 g) resting on a tee. The effective mass of the clubhead is 220 g. (*a*) With what speed does the ball leave the tee? (*b*) With what speed would it leave the tee if you doubled the mass of the clubhead? If you tripled it? What conclusions can you draw about the use of heavy clubs? Assume that the collisions are perfectly elastic and that the golfer can bring the heavier clubs up to the same speed at impact. See Question 13.

16. The two spheres on the right of Fig. 6-32 are slightly separated and initially at rest; the left sphere is incident with speed v_0. Assuming head-on elastic collisions, (*a*) if $M \leq m$, show that there are two collisions and find all final velocities; (*b*) if $M > m$, show that there are three collisions and find all final velocities.

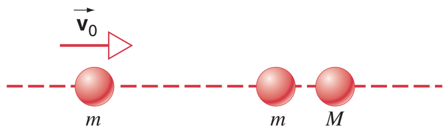


FIGURE 6-32. Problem 16.

17. A ball with an initial speed of 10.0 m/s collides elastically with two identical balls whose centers are on a line perpendicular to the initial velocity and that are initally in contact with each other (Fig. 6-33). The first ball is aimed directly at the contact point and all the balls are frictionless. Find the velocities of all three balls after the collision. (Hint: With friction absent, each impulse is directed along the line of centers of the balls, normal to the colliding surfaces.)

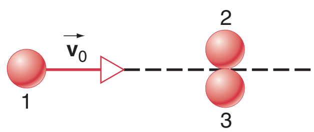


FIGURE 6-33. Problem 17.

18. Show that, in the case of an elastic collision of a particle of mass m_1 with a particle of mass m_2, initially at rest, (*a*) the maximum angle θ_m through which m_1 can be deflected by the collision is given by $\cos^2 \theta_m = 1 - (m_2/m_1)^2$, so that $0 \leq \theta_m \leq \pi/2$ when $m_1 > m_2$; (*b*) $\theta_1 + \theta_2 = \pi/2$, when $m_1 = m_2$; (*c*) θ_1 can take on all values between 0 and π when $m_1 < m_2$.

19. A 3.54-g bullet is fired horizontally at two blocks resting on a frictionless tabletop, as shown in Fig. 6-34*a*. The bullet passes through the first block, with mass 1.22 kg, and embeds itself in the second, with mass 1.78 kg. Speeds of 0.630 m/s and 1.48 m/s, respectively, are thereby imparted to the blocks, as shown in Fig. 6-34*b*. Neglecting the mass removed from the first block by the bullet, find (*a*) the speed of the bullet immediately after emerging from the first block and (*b*) the original speed of the bullet.

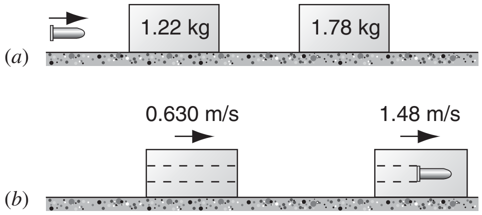


FIGURE 6-34. Problem 19.

20. A 2.0-kg block is released from rest at the top of a 22° frictionless inclined plane of height 0.65 m (Fig. 6-35). At the bottom of the plane it collides with and sticks to a block of mass 3.5 kg. The two blocks together slide a distance of 0.57 m across a horizontal plane before coming to rest. What is the coefficient of friction of the horizontal surface?

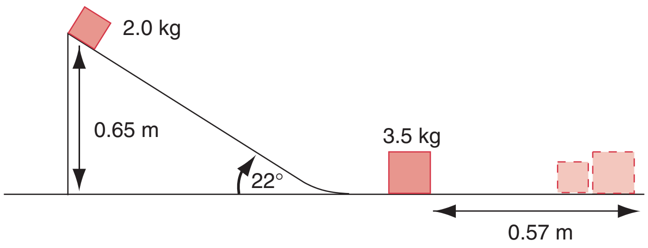


FIGURE 6-35. Problem 20.

21. Two cars *A* and *B* slide on an icy road as they attempt to stop at a traffic light. The mass of *A* is 1100 kg and the mass of *B* is 1400 kg. The coefficient of kinetic friction between the locked wheels of both cars and the road is 0.130. Car *A* succeeds in coming to rest at the light, but car *B* cannot stop and rear-ends car *A*. After the collision, *A* comes to rest 8.20 m ahead of the impact point and *B* 6.10 m ahead: see Fig. 6-36. Both drivers had their brakes locked throughout the incident. (*a*) From the distances each car moved after the collision, find the speed of each car immediately after impact. (*b*) Use conservation of momentum to find the speed at which car *B* struck car *A*. On what grounds can the use of momentum conservation be criticized here?

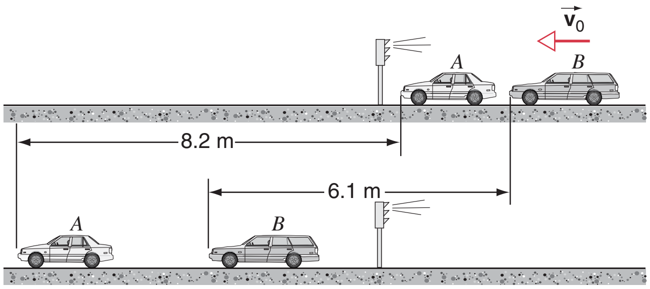


FIGURE 6-36. Problem 21.

COMPUTER PROBLEM

1. An interesting toy called the *Astro Blaster* (See Fig. 6-37) consists of four plastic balls on a stick. When the stick is dropped vertically the bottom ball bounces off the ground and then collides with the ball above. The second ball then collides with the third ball, which collides with the fourth ball. The speed of the top ball after the last collision is considerably larger than the speed at which the first ball hits the ground. Assuming all collisions are elastic, find the ratio of the masses of the four balls that will result in the largest final speed of the fourth ball, given that the lightest ball has 1/64 the mass of the heavest ball. (Note: This problem should be solved numerically, but it can also be solved analytically.)

FIGURE 6-37. Computer problem 1.

CHAPTER 7

SYSTEMS OF PARTICLES

So far we have treated objects as if they were point particles, having mass but no size. This is really not such a serious restriction, because all points of an object in simple translational motion move in identical fashion, and it makes no difference whether we treat the object as a particle or as an extended body. For many objects in motion, however, this restriction is not valid. When an object rotates as it moves, for instance, or when its parts vibrate relative to one another, it would not be valid to treat the entire object as a single particle. Even in these more complicated cases, there is one point of the object whose motion under the influence of external forces can be analyzed as that of a simple particle. This point is called the center of mass. *In this chapter we describe how to find the center of mass of an object, and we show that Newton's laws can be used to describe the motion of the center of mass of a complex system.*

7-1 THE MOTION OF A COMPLEX OBJECT

Figure 7-1 shows the motion of a baton being tossed between two jugglers. At first glance, the motion looks very complicated, and it may not be obvious how to apply Newton's laws to analyze the motion. The baton is certainly not behaving like a particle (all parts of it do *not* move in the same way), and it is not apparent that *any* part of the baton is traveling the parabolic path that we expect for particle-like projectiles.

Our formulation of Newton's laws was based on particle behavior. Sometimes we are able to treat complex objects like particles if all parts of the object move in the same way. Since this is not true for the motion of the baton in Fig. 7-1, we must find a new way to analyze its motion.

It is clear that the baton is undergoing two kinds of motion simultaneously: the translational motion associated with a projectile and the rotational motion of a rigid body (which will be treated starting with the next chapter). These combined motions may appear complicated; however, if we fix our attention on a special point associated with the object, the analysis becomes simple once again. We can consider the motion of the baton to be a combination of a parabolic trajectory of that point (as if there were no rotational motion) plus a rotation about that point (as if there were no translational motion). That special point is called the *center of mass.* Figure 7-2 shows a time-exposure photograph of the motion of a baton with an indicator light marking the

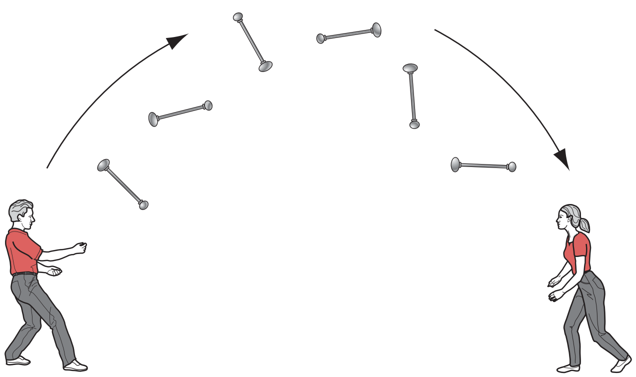


FIGURE 7-1. The complex motion of a baton as it is tossed between two jugglers.

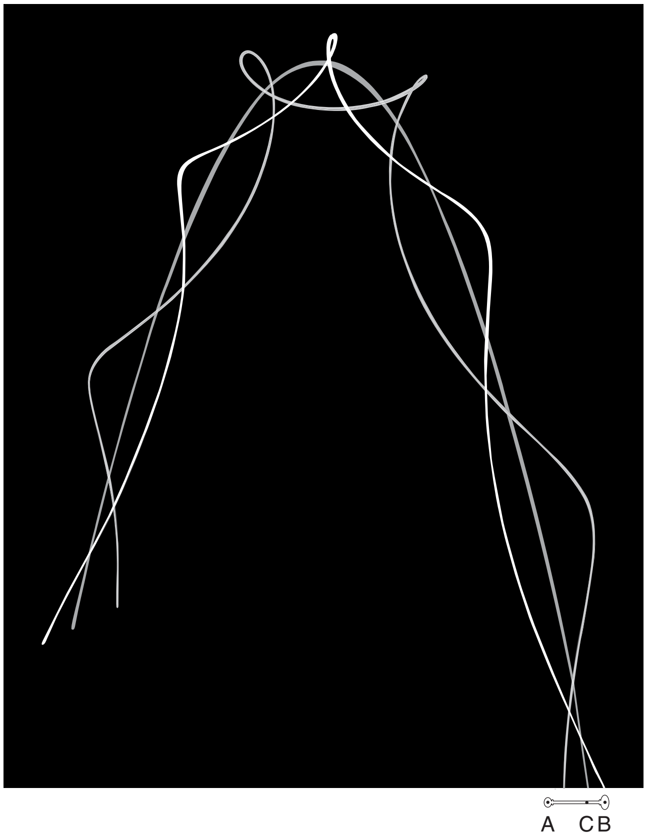


FIGURE 7-2. Time-exposure photograph of tossed baton with the motion of three points (*A*, *B*, *C*) indicated by lights. Points *A* and *B* show complex motions, but point *C* (the center of mass) follows a simple parabolic path. See "Center-of-Mass Baton" by Manfred Bucher *et al*., *The Physics Teacher,* February 1991, p. 74.

center of mass. This point moves in a simple parabolic trajectory, but no such simple description can be applied to the motion of other points of the baton.

Recall that in Section 6-5 we introduced the center of mass concept and found it useful in analyzing collisions between particles, but in that section we did not explain how to locate the center of mass of a system of particles. In this chapter we discuss how to find the center of mass of a solid object and how to use it to reduce a complex motion to a simpler one.

7-2 TWO-PARTICLE SYSTEMS

Let us try to simplify the problem discussed in the previous section. We consider the baton to consist of two particles, located at the ends *A* and *B*, connected by a thin rigid rod of fixed length and negligible mass. The mass of the particle at *B* is twice the mass of the particle at *A*.

We further simplify by sliding the baton on a frictionless horizontal surface instead of throwing it upward. This in effect eliminates gravity from the analysis.

In using Newton's laws to study the motion of a single object that we treated as a particle, we separated the problem into two parts: the particle and its environment. For a more complex object, such as our two-particle baton, it is usually more convenient to separate the problem into a *system* and its environment. The system can consist of any number of objects; we can define the system in any way that simplifies the problem, as long as we are consistent in the analysis and carefully account for all interactions between the system and its environment. These interactions are called *external forces.* Interactions between objects that are totally within the system are called *internal forces.* In the case of the baton, we define the system to consist of the two particles and the connecting rod; gravity and the normal force would then be classified as external forces, and the tension force exerted by the connecting rod on either of the particles would be an internal force.

We give the rod a push along the frictionless horizontal surface and examine its motion. Figure 7-3 shows a series of "snapshots" of the motions of the particles at *A* and *B* and the center of mass at *C*. Clearly each of the particles at *A* and *B* is accelerated, and so (according to Newton's second law) must be subject to a net force. However, point *C* shows no acceleration—its velocity is constant in both magnitude and direction. No other point associated with the baton moves in this simple way.

It is also interesting to view the motion of the baton from a frame of reference that is moving with the same velocity as point *C*. (As we shall see, this is the same as the center-of-mass reference frame we discussed in Section 6-5.) In this reference frame, point *C* would appear to be at rest. Figure 7-4 shows the resulting motion, with the positions of the baton drawn at the instants marked 1, 2, 3, and

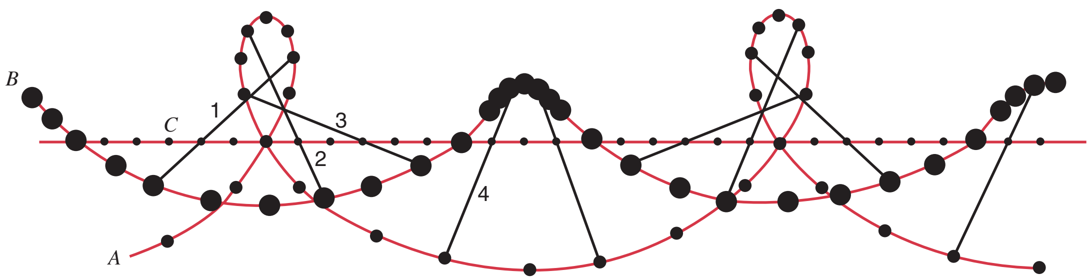


FIGURE 7-3. The motion of two particles attached to a connecting rod. The dots represent "snapshots" showing the locations of points *A*, *B*, and *C* at successive intervals of time. Point *C* on the rod follows a straight-line path and its successive positions are equally spaced, showing that it moves at constant velocity.

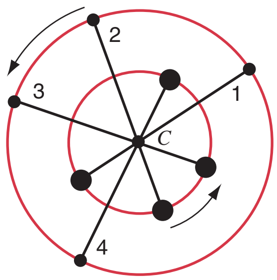


Figure 7-4. If we view the motion of Fig. 7-3 from a frame of reference moving along with point C, the rod appears to rotate about point C and the two particles move in circles of different radii.

4 in Fig. 7-3. The motion is a simple rotation with each particle having a constant rotational speed.

By focusing our attention on the center of mass C, we have been able to separate the complex motion of the system into two simple motions—the center of mass moves with constant velocity, and the system rotates with constant rotational speed about C. In the next chapter we shall consider the rotational motion; here we concentrate on the linear motion of the center of mass.

To find the location of the center of mass, we set up a coordinate system in the horizontal plane, as shown in Fig. 7-5. We let m_1 represent the mass of the particle at A and m_2 represent the particle at B. The vectors $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$ locate m_1 and m_2 at a particular instant of time, relative to the origin we have chosen for the coordinate system. The center of mass is then located at that time by the vector $\vec{\mathbf{r}}_{\rm cm}$:

$$\vec{\mathbf{r}}_{\rm cm} = \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2}{m_1 + m_2} \qquad (7\text{-}1)$$

or

$$x_{\rm cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \quad \text{and} \quad y_{\rm cm} = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}. \qquad (7\text{-}2)$$

The center of mass is a fixed point in any solid object whose location is determined by the way the mass of the object is distributed.

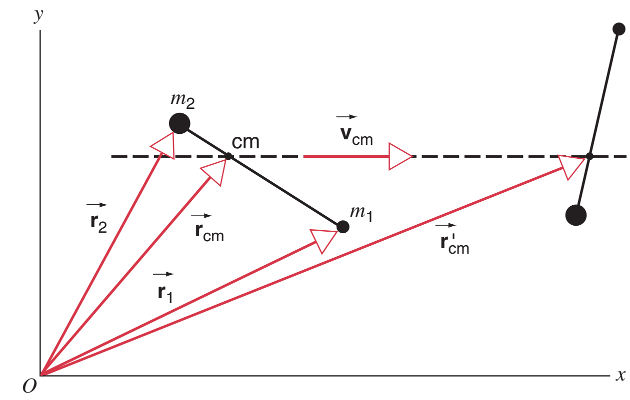


Figure 7-5. A coordinate system for locating the center of mass of our system at a particular time. At a later time, the center of mass is at $\vec{\mathbf{r}}\,'_{\rm cm}$.

At a later time (as shown in Fig. 7-5), the system has moved to a new location and the center of mass has similarly changed location. To see why the motion of the center of mass is special, let us find its velocity and acceleration:

$$\vec{\mathbf{v}}_{\rm cm} = \frac{d\vec{\mathbf{r}}_{\rm cm}}{dt} = \frac{m_1\dfrac{d\vec{\mathbf{r}}_1}{dt} + m_2\dfrac{d\vec{\mathbf{r}}_2}{dt}}{m_1 + m_2} \qquad (7\text{-}3)$$

or

$$\vec{\mathbf{v}}_{\rm cm} = \frac{m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2}{m_1 + m_2}, \qquad (7\text{-}4)$$

and

$$\vec{\mathbf{a}}_{\rm cm} = \frac{d\vec{\mathbf{v}}_{\rm cm}}{dt} = \frac{m_1\dfrac{d\vec{\mathbf{v}}_1}{dt} + m_2\dfrac{d\vec{\mathbf{v}}_2}{dt}}{m_1 + m_2} \qquad (7\text{-}5)$$

or

$$\vec{\mathbf{a}}_{\rm cm} = \frac{m_1\vec{\mathbf{a}}_1 + m_2\vec{\mathbf{a}}_2}{m_1 + m_2}. \qquad (7\text{-}6)$$

From Eq. 7-6 we can see why the motion of the center of mass of our system is so simple. Figure 7-6 shows free-body diagrams for the two particles and for the rod (assumed massless). For both particles, the vertical component of the acceleration is zero; thus the vertical component of the net force is zero, and the magnitudes of the normal force and weight are equal. The net force on m_1 is then $\vec{\mathbf{F}}_{1\rm r}$ (the force on m_1 due to the connecting rod), and Newton's second law gives $\vec{\mathbf{F}}_{1\rm r} = m_1\vec{\mathbf{a}}_1$. Similarly for m_2, $\vec{\mathbf{F}}_{2\rm r} = m_2\vec{\mathbf{a}}_2$. By Newton's third law, the force on m_1 due to the connecting rod is equal and opposite to the force on the connecting rod by m_1, or $\vec{\mathbf{F}}_{1\rm r} = -\vec{\mathbf{F}}_{\rm r1}$; similarly, $\vec{\mathbf{F}}_{2\rm r} = -\vec{\mathbf{F}}_{\rm r2}$. Combining these results, the numerator of Eq. 7-6 becomes $m_1\vec{\mathbf{a}}_1 + m_2\vec{\mathbf{a}}_2 = \vec{\mathbf{F}}_{1\rm r} + \vec{\mathbf{F}}_{2\rm r} = -\vec{\mathbf{F}}_{\rm r1} + (-\vec{\mathbf{F}}_{\rm r2}) = -(\vec{\mathbf{F}}_{\rm r1} + \vec{\mathbf{F}}_{\rm r2})$. Finally, because we assumed that the connecting rod is massless, the net force on it must be zero ($\sum \vec{\mathbf{F}}_{\rm rod} = m_{\rm rod}\vec{\mathbf{a}}_{\rm rod} = 0$ because $m_{\rm rod} = 0$); see Fig. 7-6*b*. With $\sum \vec{\mathbf{F}}_{\rm rod} = \vec{\mathbf{F}}_{\rm r1} + \vec{\mathbf{F}}_{\rm r2}$, the numerator of Eq. 7-6 becomes

$$m_1\vec{\mathbf{a}}_1 + m_2\vec{\mathbf{a}}_2 = -(\vec{\mathbf{F}}_{\rm r1} + \vec{\mathbf{F}}_{\rm r2}) = -\sum \vec{\mathbf{F}}_{\rm rod} = 0.$$

We therefore have $\vec{\mathbf{a}}_{\rm cm} = 0$, and so the center of mass moves with constant velocity.

In this discussion, we have so far assumed that no net external force acts on the system ($\vec{\mathbf{F}}_{1\rm r}$ and $\vec{\mathbf{F}}_{2\rm r}$ are *internal*

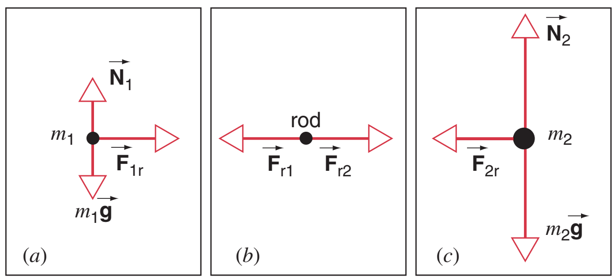


Figure 7-6. Free-body diagrams for (*a*) m_1, (*b*) the connecting rod, and (*c*) m_2.

forces, exerted by one part of the system on another part). Suppose now that there is an external force on each particle, perhaps a frictional force due to the surface. The net force on each particle is then the vector sum of the external force plus the internal force due to the rod:

$$\sum \vec{\mathbf{F}}_1 = \vec{\mathbf{F}}_{1,\text{ext}} + \vec{\mathbf{F}}_{1\text{r}} \quad \text{and} \quad \sum \vec{\mathbf{F}}_2 = \vec{\mathbf{F}}_{2,\text{ext}} + \vec{\mathbf{F}}_{2\text{r}}. \qquad (7\text{-}7)$$

When we analyze Eq. 7-6 we then obtain

$$m_1\vec{\mathbf{a}}_1 + m_2\vec{\mathbf{a}}_2 = \sum \vec{\mathbf{F}}_1 + \sum \vec{\mathbf{F}}_2$$
$$= \vec{\mathbf{F}}_{1,\text{ext}} + \vec{\mathbf{F}}_{1\text{r}} + \vec{\mathbf{F}}_{2,\text{ext}} + \vec{\mathbf{F}}_{2\text{r}}. \qquad (7\text{-}8)$$

Once again, $\vec{\mathbf{F}}_{1\text{r}} + \vec{\mathbf{F}}_{2\text{r}} = 0$, and defining the *net external force* to be $\sum \vec{\mathbf{F}}_{\text{ext}} = \vec{\mathbf{F}}_{1,\text{ext}} + \vec{\mathbf{F}}_{2,\text{ext}}$, Eq. 7-8 reduces to $m_1\vec{\mathbf{a}}_1 + m_2\vec{\mathbf{a}}_2 = \sum \vec{\mathbf{F}}_{\text{ext}}$. Using Eq. 7-6 we then obtain

$$\sum \vec{\mathbf{F}}_{\text{ext}} = (m_1 + m_2)\vec{\mathbf{a}}_{\text{cm}}. \qquad (7\text{-}9)$$

This looks very much like Newton's second law, but it is applied to something that does not exist: a particle of mass $m_1 + m_2$ located at the center of mass.

Summarizing our conclusions for the two-particle system, we have seen that we can simplify our analysis if we decompose the complex motions into a motion of the center of mass of the system and another motion about the center of mass. If no net external force acts on the system, then the center of mass moves with constant velocity. If there is a net external force, then the motion of the center of mass can be found by assuming that the net external force acts on a particle located at the center of mass and having a mass equal to the total mass of the system. In the next section, we develop more general expressions that lead to the same conclusions for even more complex systems.

SAMPLE PROBLEM 7-1. (*a*) Suppose the baton of Fig. 7-3 is at rest along the x axis with the more massive particle m_2 $(= 2m)$ at coordinate x and the less massive particle m_1 $(= m)$ at coordinate $x + L$ (where L is the length of the rod connecting the particles), as shown in Fig. 7-7*a*. Find the center of mass. (*b*) Suppose now that m_2 is at the origin and the rod makes an angle of 45° between the x and y axes (Fig. 7-7*b*). Find the location of the center of mass.

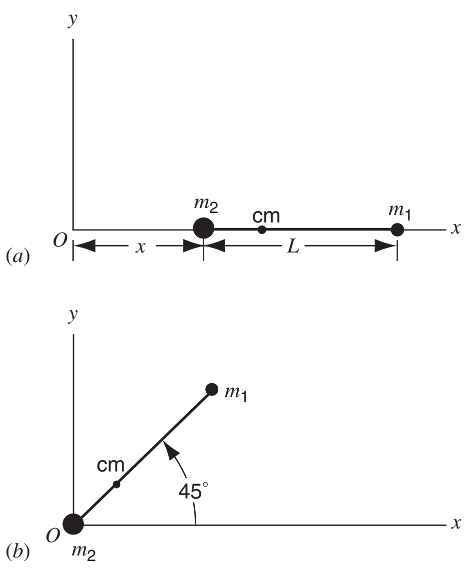


FIGURE 7-7. Sample Problem 7-1.

Solution (*a*) With $y_1 = 0$ and $y_2 = 0$, Eq. 7-2 gives $y_{\text{cm}} = 0$. The x coordinate of the center of mass is also found from Eq. 7-2:

$$x_{\text{cm}} = \frac{(m)(x + L) + (2m)(x)}{m + 2m} = x + \frac{L}{3}.$$

The center of mass is located along the rod a distance $L/3$ from the larger particle.

(*b*) In this case $x_1 = L/\sqrt{2}$, $y_1 = L/\sqrt{2}$, $x_2 = 0$, and $y_2 = 0$, and so we have

$$x_{\text{cm}} = \frac{(m)(L/\sqrt{2}) + (2m)(0)}{m + 2m} = \frac{L}{3\sqrt{2}},$$

$$y_{\text{cm}} = \frac{(m)(L/\sqrt{2}) + (2m)(0)}{m + 2m} = \frac{L}{3\sqrt{2}}.$$

Once again, the center of mass is along the connecting rod and $\frac{1}{3}$ of its length from the larger particle.

7-3 MANY-PARTICLE SYSTEMS

In this section we generalize the results of the previous section to systems in three dimensions that contain more than two particles.

We consider a system consisting of N particles of masses $m_1, m_2, \ldots, m_N$. The total mass is

$$M = m_1 + m_2 + \cdots + m_N = \sum m_n. \qquad (7\text{-}10)$$

Each particle in the system can be represented by its mass m_n (where $n = 1, 2, \ldots, N$), its location at the coordinate $\vec{\mathbf{r}}_n$ (whose components are x_n, y_n, and z_n), its velocity $\vec{\mathbf{v}}_n$ (whose components are v_{nx}, v_{ny}, and v_{nz}), and its acceleration $\vec{\mathbf{a}}_n$. The net force on particle m_n is $\sum \vec{\mathbf{F}}_n$, which in general differs from one particle to another. This force may arise partly from the other $N - 1$ particles and partly from an external agent.

The center of mass of the system can be defined by a logical extension of Eq. 7-1:

$$\vec{\mathbf{r}}_{\text{cm}} = \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2 + \cdots + m_N\vec{\mathbf{r}}_N}{m_1 + m_2 + \cdots + m_N}$$

or

$$\vec{\mathbf{r}}_{\text{cm}} = \frac{1}{M}\sum m_n\vec{\mathbf{r}}_n. \qquad (7\text{-}11)$$

In terms of components, the vector relationship of Eq. 7-11 can be written as

$$x_{\text{cm}} = \frac{1}{M}(m_1x_1 + m_2x_2 + \cdots + m_Nx_N)$$
$$= \frac{1}{M}\sum m_nx_n, \qquad (7\text{-}12a)$$

$$y_{\text{cm}} = \frac{1}{M}(m_1y_1 + m_2y_2 + \cdots + m_Ny_N)$$
$$= \frac{1}{M}\sum m_ny_n, \qquad (7\text{-}12b)$$

$$z_{\rm cm} = \frac{1}{M}(m_1 z_1 + m_2 z_2 + \cdots + m_N z_N)$$
$$= \frac{1}{M}\sum m_n z_n. \qquad (7\text{-}12c)$$

Taking the derivative of Eq. 7-11, we find the velocity of the center of mass:

$$\vec{\mathbf{v}}_{\rm cm} = \frac{d\vec{\mathbf{r}}_{\rm cm}}{dt}$$
$$= \frac{1}{M}\left(m_1\frac{d\vec{\mathbf{r}}_1}{dt} + m_2\frac{d\vec{\mathbf{r}}_2}{dt} + \cdots + m_N\frac{d\vec{\mathbf{r}}_N}{dt}\right)$$

or

$$\vec{\mathbf{v}}_{\rm cm} = \frac{1}{M}(m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 + \cdots + m_N\vec{\mathbf{v}}_N)$$
$$= \frac{1}{M}\sum m_n\vec{\mathbf{v}}_n. \qquad (7\text{-}13)$$

Differentiating once again, we find the acceleration of the center of mass:

$$\vec{\mathbf{a}}_{\rm cm} = \frac{d\vec{\mathbf{v}}_{\rm cm}}{dt} = \frac{1}{M}(m_1\vec{\mathbf{a}}_1 + m_2\vec{\mathbf{a}}_2 + \cdots + m_N\vec{\mathbf{a}}_N)$$
$$= \frac{1}{M}\sum m_n\vec{\mathbf{a}}_n. \qquad (7\text{-}14)$$

We can rewrite Eq. 7-14 as

$$M\vec{\mathbf{a}}_{\rm cm} = m_1\vec{\mathbf{a}}_1 + m_2\vec{\mathbf{a}}_2 + \cdots + m_N\vec{\mathbf{a}}_N$$

or

$$M\vec{\mathbf{a}}_{\rm cm} = \sum\vec{\mathbf{F}}_1 + \sum\vec{\mathbf{F}}_2 + \cdots + \sum\vec{\mathbf{F}}_N, \qquad (7\text{-}15)$$

where the last result follows from applying Newton's second law, $\Sigma\,\vec{\mathbf{F}}_n = m_n\vec{\mathbf{a}}_n$, to each individual particle. The total force acting on a system of particles is thus equal to the total mass of the system times the acceleration of the center of mass. Equation 7-15 is just Newton's second law for the system of N particles treated as a single particle of mass M located at the center of mass, moving with velocity $\vec{\mathbf{v}}_{\rm cm}$ and experiencing acceleration $\vec{\mathbf{a}}_{\rm cm}$.

It is helpful to simplify Eq. 7-15 even a bit more. We can separate the force acting on each particle in the system into internal forces, which arise from the interactions with other particles that are part of the system, and external forces, which originate with the environment of the system under consideration. Any given particle m_n may experience force exerted on it by particle m_k, which we write as $\vec{\mathbf{F}}_{nk}$. This particular force is one among the many that make up $\Sigma\,\vec{\mathbf{F}}_n$, the total force on m_n. Similarly, the total force on particle m_k includes a term $\vec{\mathbf{F}}_{kn}$ due to the interaction with particle m_n. By Newton's third law, $\vec{\mathbf{F}}_{nk} = -\vec{\mathbf{F}}_{kn}$, and thus these two particular forces cancel when we carry out the sum of all the forces in Eq. 7-15. In fact, all such internal forces are part of action–reaction pairs and cancel. (In Chapter 3 we cautioned that the action and reaction forces must apply to different particles and thus do not oppose one another on a given particle. We are not violating that caution here, because we are applying the action to one particle and the reaction to another. The distinction here is that we are adding to get the net force on the *two* particles, in which case the action and reaction components, which still apply to different particles, do indeed cancel.)

All that remains in Eq. 7-15 is the total of all the *external* forces, and Eq. 7-15 reduces to

$$\sum\vec{\mathbf{F}}_{\rm ext} = M\vec{\mathbf{a}}_{\rm cm}, \qquad (7\text{-}16)$$

which can be written in terms of its components as

$$\sum F_{{\rm ext},x} = Ma_{{\rm cm},x}, \qquad \sum F_{{\rm ext},y} = Ma_{{\rm cm},y},$$

and

$$\sum F_{{\rm ext},z} = Ma_{{\rm cm},z}. \qquad (7\text{-}17)$$

We can summarize this important result as follows:

The overall translational motion of a system of particles can be analyzed using Newton's laws as if all the mass were concentrated at the center of mass and the total external force were applied at that point.

A corollary follows immediately in the case $\Sigma\,\vec{\mathbf{F}}_{\rm ext} = 0$:

If the net external force on a system of particles is zero, then the center of mass of the system moves with constant velocity.

These are general results that apply equally well to collections of individual particles as they do to particles joined together by internal forces, as in a solid object. The object itself may be executing any sort of complicated motion, but the center of mass moves according to Eq. 7-16. Consider, for example, the motion of the baton of Fig. 7-1. As it travels, it also rotates. Its center of mass, however, follows a simple parabolic path. As far as the external force (gravity) is concerned, the system behaves as if it were a particle of mass M located at the center of mass. A complicated problem is therefore reduced to two relatively simple problems—the parabolic path of the center of mass and a rotation about the center of mass.

For another example, consider the Earth–Moon system moving under the Sun's gravity (the external force). Figure 7-8 shows that the center of mass of the system follows a

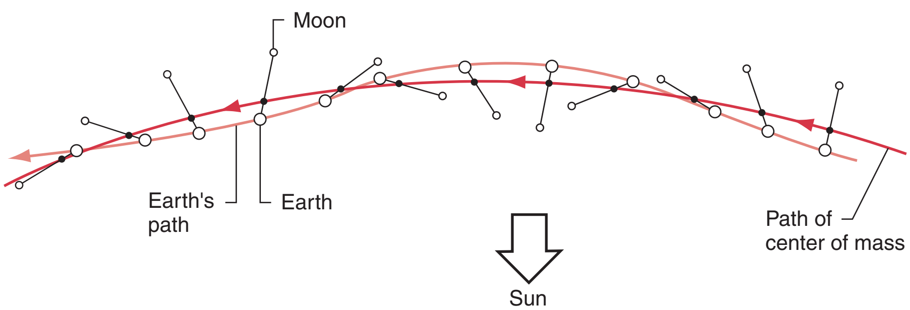


Figure 7-8. The center of mass of the Earth–Moon system follows a nearly circular orbit about the Sun, while the Earth and Moon rotate about their common center of mass, just like the baton of Fig. 7-3. This effect, which causes a slight "wobble" in the orbit of the Earth, is greatly exaggerated in the figure. The center of mass of the Earth–Moon system actually lies within the Earth, so the Earth always overlaps the orbital path of the center of mass.

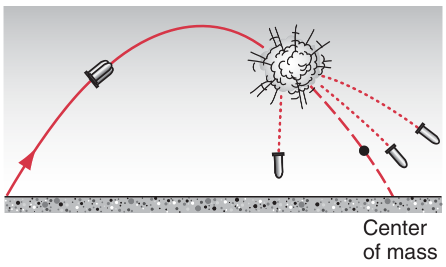


FIGURE 7-9. A projectile follows a parabolic path (solid line). An explosion breaks the projectile into three fragments, which travel so that their center of mass follows the original parabolic path.

stable orbit around the Sun; this is the path that would be followed by a particle of mass $m_{\text{Earth}} + m_{\text{Moon}}$. The Earth and Moon also rotate about their center of mass, resulting in a slight oscillation of the Earth about the path of the stable orbit. Using the data in Appendix C, you should be able to show that the center of mass of the Earth–Moon system lies within the Earth (see Exercise 1).

Figure 7-9 shows the motion of a projectile that breaks apart into three fragments. In effect, an explosion releases the three separate pieces, but since the explosion produces only internal forces it does not affect the motion of the center of mass. The center of mass continues to follow the parabolic path as if the explosion had not occurred, until one or more of the fragments experiences an external force, such as from atmospheric drag or impact with the ground.

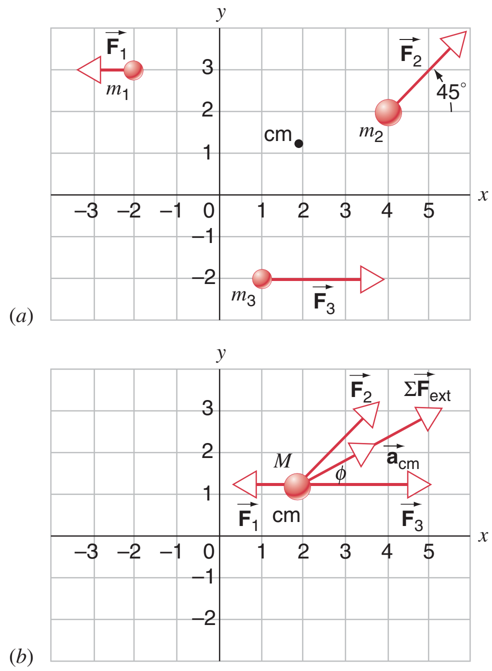


FIGURE 7-10. Sample Problem 7-2. (*a*) Three particles, placed at rest at the positions shown, are acted on by the forces shown. The center of mass of the system is marked. (*b*) The translational motion of the entire system can be represented by the motion of a particle with the total mass *M* located at the center of mass and acted on by the three external forces. The resultant force and the acceleration of the center of mass are shown. (The *x* and *y* axes are marked in centimeters.)

SAMPLE PROBLEM 7-2. Figure 7-10*a* shows a system of three initially resting particles of masses $m_1 = 4.1$ kg, $m_2 = 8.2$ kg, and $m_3 = 4.1$ kg. The particles are each acted on by different net external forces, which have magnitudes $F_1 = 6$ N, $F_2 = 12$ N, and $F_3 = 14$ N. The directions of the forces are shown in the figure. Where is the center of mass of this system, and what is the acceleration of the center of mass?

Solution The position of the center of mass is marked by a dot in the figure. As Fig. 7-10*b* suggests, we treat this point as if it held a real particle, assigning to it a mass *M* equal to the system mass of $m_1 + m_2 + m_3 = 16.4$ kg and assuming that all external forces are applied at that point. We find the center of mass from Eqs. 7-12*a* and 7-12*b*:

$$\begin{aligned} x_{cm} &= \frac{1}{M}(m_1x_1 + m_2x_2 + m_3x_3) \\ &= \frac{1}{16.4\text{ kg}}[(4.1\text{ kg})(-2\text{ cm}) + (8.2\text{ kg})(4\text{ cm}) \\ &\quad + (4.1\text{ kg})(1\text{ cm})] = 1.8\text{ cm}, \\ y_{cm} &= \frac{1}{M}(m_1y_1 + m_2y_2 + m_3y_3) \\ &= \frac{1}{16.4\text{ kg}}[(4.1\text{ kg})(3\text{ cm}) + (8.2\text{ kg})(2\text{ cm}) \\ &\quad + (4.1\text{ kg})(-2\text{ cm})] = 1.3\text{ cm}. \end{aligned}$$

The *x* component of the net external force acting on the center of mass is (see Fig. 7-10*b*)

$$\begin{aligned} \sum F_{\text{ext},x} &= F_{1x} + F_{2x} + F_{3x} \\ &= -6\text{ N} + (12\text{ N})(\cos 45°) + 14\text{ N} = 16.5\text{ N}, \end{aligned}$$

and the *y* component is

$$\begin{aligned} \sum F_{\text{ext},y} &= F_{1y} + F_{2y} + F_{3y} \\ &= 0 + (12\text{ N})(\sin 45°) + 0 = 8.5\text{ N}. \end{aligned}$$

The net external force thus has a magnitude of

$$\left|\sum \vec{F}_{\text{ext}}\right| = \sqrt{(F_{\text{ext},x})^2 + (F_{\text{ext},y})^2} = \sqrt{(16.5\text{ N})^2 + (8.5\text{ N})^2} = 18.6\text{ N}$$

and makes an angle with the *x* axis given by

$$\phi = \tan^{-1}\frac{\sum F_{\text{ext},y}}{\sum F_{\text{ext},x}} = \tan^{-1}\frac{8.5\text{ N}}{16.5\text{ N}} = 27°.$$

This is also the direction of the acceleration vector. From Eq. 7-16, the magnitude of the acceleration of the center of mass is given by

$$|\vec{a}_{cm}| = \frac{F_{\text{ext}}}{M} = \frac{18.6\text{ N}}{16.4\text{ kg}} = 1.1\text{ m/s}^2.$$

If the external forces are constant, then the acceleration of the center of mass is constant, even if the internal forces (and thus the accelerations of the individual particles) change with time.

SAMPLE PROBLEM 7-3. A projectile of mass 9.6 kg is launched from the ground with initial velocity of 12.4 m/s at an angle of 54° above the horizontal (Fig. 7-11). At some time after its launch, an explosion splits the projectile into two pieces. One piece, of mass 6.5 kg, is observed at 1.42 s after the launch at a height of 5.9 m and a horizontal distance of 13.6 m from the launch point. Find the location of the second fragment at that same time.

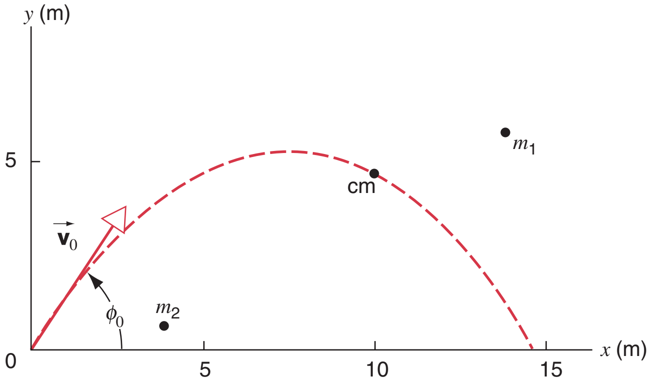


FIGURE 7-11. Sample Problem 7-3. The dashed line shows the parabolic trajectory of the center of mass of the two fragments. The locations of the center of mass and the two fragments are shown at $t = 1.42$ s.

Solution According to Eq. 7-16, the motion of the two fragments can be analyzed in terms of the motion of the combined system. Therefore, at $t = 1.42$ s after the launch, the center of mass of the two fragments must be at the same location where the original projectile would have been if it had not exploded. Let us first find that location. The location of the original projectile at $t = 1.42$ s can be found using Eqs. 4-10 with $v_{0x} = v_0 \cos \phi_0 = (12.4 \text{ m/s})\cos 54° = 7.3$ m/s and $v_{0y} = v_0 \sin \phi_0 = (12.4 \text{ m/s})\sin 54° = 10.0$ m/s. With the origin of the coordinate system at the original launch point, we have

$$x = v_{0x}t = (7.3 \text{ m/s})(1.42 \text{ s}) = 10.4 \text{ m},$$
$$y = v_{0y}t - \tfrac{1}{2}gt^2 = (10.0 \text{ m/s})(1.42 \text{ s}) - \tfrac{1}{2}(9.80 \text{ m/s}^2)(1.42 \text{ s})^2 = 4.3 \text{ m}$$

Because the motion of the center of mass of the system of the two fragments is the same as the motion of the original combined system would have been, the center of mass of the fragments at $t = 1.42$ s must be at $x_{cm} = 10.4$ m, $y_{cm} = 4.3$ m. We are given the location of one fragment, m_1, at that time: $x_1 = 13.6$ m, $y_1 = 5.9$ m. We can find the location of the other fragment, of mass $m_2 = M - m_1 = 9.6 \text{ kg} - 6.5 \text{ kg} = 3.1$ kg, by solving Eqs. 7-12*a* and 7-12*b* for x_2 and y_2:

$$x_2 = \frac{Mx_{cm} - m_1x_1}{m_2} = \frac{(9.6 \text{ kg})(10.4 \text{ m}) - (6.5 \text{ kg})(13.6 \text{ m})}{3.1 \text{ kg}} = 3.7 \text{ m},$$

$$y_2 = \frac{My_{cm} - m_1y_1}{m_2} = \frac{(9.6 \text{ kg})(4.3 \text{ m}) - (6.5 \text{ kg})(5.9 \text{ m})}{3.1 \text{ kg}} = 0.9 \text{ m}.$$

Figure 7-11 shows the location of the fragment m_2.

If we know the velocity of one fragment, we can use similar methods to find the velocity of the other (see Exercise 12).

In our analysis, we have assumed that gravity is the only external force that acts on the system, which allows us to represent the motion of the center of mass of the two fragments as the parabolic path of a projectile in the Earth's gravity. If one fragment struck the ground, there would be an additional force in the problem (the force of the ground on one fragment), and the center of mass would follow a different path. To use this method in that case, it would be necessary to know the force exerted by the ground.

7-4 CENTER OF MASS OF SOLID OBJECTS

It is far too tedious to find the center of mass of a solid object by using Eqs. 7-12 and summing over every atom in the system. Instead we divide the object into tiny elements of mass δm_n. As these elements become infinitesimally small, the sums of Eqs. 7-12 transform into integrals:

$$x_{cm} = \frac{1}{M} \lim_{\delta m \to 0} \sum x_n \, \delta m_n = \frac{1}{M} \int x \, dm, \qquad (7\text{-}18a)$$

$$y_{cm} = \frac{1}{M} \lim_{\delta m \to 0} \sum y_n \, \delta m_n = \frac{1}{M} \int y \, dm, \qquad (7\text{-}18b)$$

$$z_{cm} = \frac{1}{M} \lim_{\delta m \to 0} \sum z_n \, \delta m_n = \frac{1}{M} \int z \, dm. \qquad (7\text{-}18c)$$

In vector form (compare Eq. 7-11) these equations can be written

$$\vec{\mathbf{r}}_{cm} = \frac{1}{M} \int \vec{\mathbf{r}} \, dm. \qquad (7\text{-}19)$$

In many cases it is possible to use arguments based on geometry or symmetry to simplify the calculation of the center of mass of solid objects. If an object has *spherical symmetry*, the center of mass must lie at the geometrical center of the sphere. (It is not necessary that its density be uniform; a baseball, for example, has spherical symmetry even though it is composed of layers of different materials. Its center of mass is at its geometric center. When we refer to spherical symmetry, we mean that the density may vary with r but it must have the same variation in every direction.) If a solid has *cylindrical symmetry* (that is, if its mass is distributed symmetrically about an axis), then the center of mass must lie on the axis. If its mass is distributed symmetrically about a plane, then the center of mass must be in the plane.

Often we encounter solid, irregular objects that can be divided into several parts. We can find the center of mass of each part, and then by treating each part as a particle located at its own center of mass we can find the center of mass of the combination.

As an example, consider the triangular plate shown in Fig. 7-12. We divide the plate into a large number of thin strips parallel to the base of the triangle, as in Fig. 7-12*a*. The

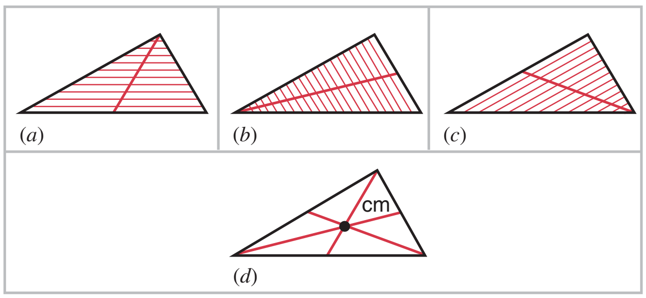


FIGURE 7-12. In (*a*), (*b*), and (*c*), the triangle is divided into thin strips, parallel to each of the three sides. The center of mass must lie along the symmetrical dividing lines shown. (*d*) The dot, the only point common to all three lines, is the position of the center of mass.

center of mass of each strip must lie at its geometrical center, and therefore the center of mass of the plate must lie somewhere along the line connecting the centers of the strips. (Replace each strip with a point mass located at the center of mass of the strip. The row of point masses forms in effect a one-dimensional object whose center of mass must surely lie along its length). Repeating this procedure for strips drawn parallel to the other two sides (Fig. 7-12*b* and 7-12*c*), we obtain two additional lines, each of which must also include the center of mass of the plate. Superimposing all three lines, as in Fig. 7-12*d*, we find they have only one point in common, which must therefore be the center of mass.

SAMPLE PROBLEM 7-4. Figure 7-13*a* shows a circular metal plate of radius $2R$ from which a disk of radius R has been removed. Let us call it object X. Its center of mass is shown as a dot on the x axis. Find the location of this point.

Solution Object X has symmetry about the x axis; that is, the portion above the x axis is the mirror image of the portion below the x axis. Because of this symmetry, the center of mass must lie along the x axis. Furthermore, because there is more of object X to the right of the y axis than to the left, the center of mass must lie to the right of the y axis. It is thus very reasonable that point X represents the center of mass of object X.

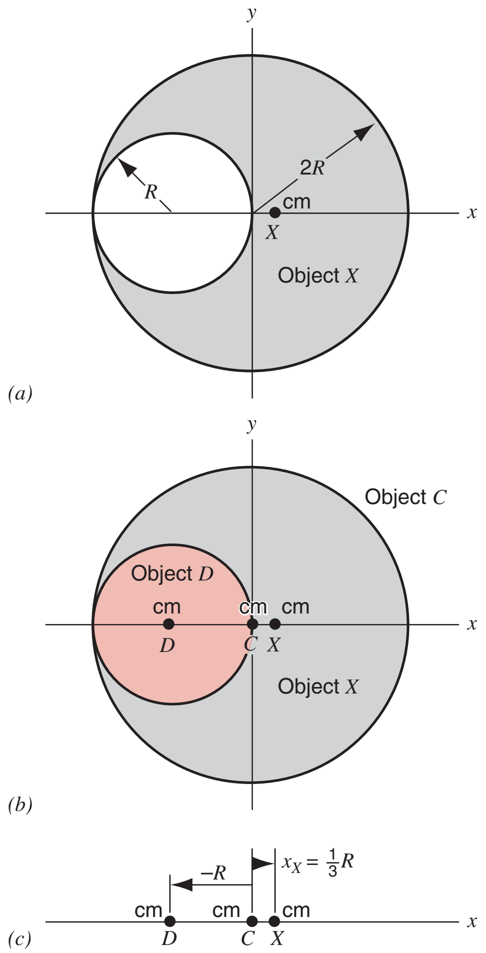


FIGURE 7-13. Sample Problem 7-4. (*a*) Object X is a metal disk of radius $2R$ with a hole of radius R cut in it. (*b*) Object D is a metal disk that fills the hole in object X; its center of mass is at $x_D = -R$. Object C is the composite disk made up of objects X and D; its center of mass is at the origin. (*c*) The centers of mass of the three objects.

Figure 7-13*b* shows object X, its hole filled with a disk of the same material of radius R, which we call object D. Let us label as object C the large uniform composite disk so formed. From symmetry, the center of mass of object C is at the origin of the coordinate system, as shown.

In finding the center of mass of a composite object, we can assume that the masses of its components are concentrated at their individual centers of mass. Thus object C can be treated as equivalent to two particles, representing objects X and D. Figure 7-13*c* shows the positions of the centers of mass of these three objects.

The position of the center of mass of object C is given from Eq. 7-12*a* as

$$x_C = \frac{m_D x_D + m_X x_X}{m_D + m_X},$$

in which x_D and x_X are the positions of the centers of mass of objects D and X, respectively. Noting that $x_C = 0$ and solving for x_X, we obtain

$$x_X = -\left(\frac{m_D}{m_X}\right) x_D.$$

The ratio m_D/m_X must be the same as the ratio of the areas of objects D and X (assuming the plate is of uniform density and thickness). That is,

$$\frac{m_D}{m_X} = \frac{\text{area of } D}{\text{area of } X} = \frac{\text{area of } D}{\text{area of } C - \text{area of } D}$$

$$= \frac{\pi R^2}{\pi(2R)^2 - \pi R^2} = \frac{1}{3}.$$

With $x_D = -R$, we obtain

$$x_X = \tfrac{1}{3}R.$$

SAMPLE PROBLEM 7-5. A thin strip of material is bent into the shape of a semicircle of radius R (Fig. 7-14). Find its center of mass.

Solution This case has symmetry about the y axis (that is, for every particle to the left of the y axis there is a particle in a similar loca-

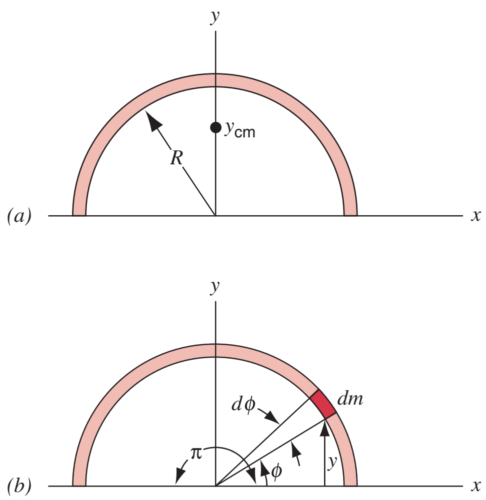


FIGURE 7-14. Sample Problem 7-5. (*a*) A thin strip of metal bent into the shape of a semicircle. (*b*) An element of the strip of mass dm located at the angular coordinate ϕ.

tion to the right of the y axis). The center of mass must therefore lie on the y axis; that is, $x_{\rm cm} = 0$. However, there is no symmetry about the x axis, so we must use Eq. 7-18*b* to find $y_{\rm cm}$. Using an angular coordinate simplifies the integration to be performed. Consider the small element of mass dm shown in Fig. 7-14*b*. It subtends an angle $d\phi$, and since the total mass M of the strip subtends an angle π (a full circle would subtend an angle 2π), the mass dm must be the same fraction of M as $d\phi$ is of π. That is, $dm/M = d\phi/\pi$, or $dm = (M/\pi)d\phi$. The element dm is located at the coordinate $y = R \sin \phi$. In this case we can write Eq. 7-18*b* as

$$y_{\rm cm} = \frac{1}{M}\int y\,dm = \frac{1}{M}\int_0^{\pi}(R\sin\phi)\frac{M}{\pi}\,d\phi$$
$$= \frac{R}{\pi}\int_0^{\pi}\sin\phi\,d\phi = \frac{2R}{\pi} = 0.637R.$$

The center of mass is roughly two-thirds of a radius along the y axis. Note that, as this case illustrates, the center of mass does not need to be within the volume or the material of an object.

Sample Problem 7-6. A ball of mass m and radius R is placed inside a spherical shell of the same mass m and inner radius $2R$. The combination is at rest on a table top as shown in Fig. 7-15*a*. The ball is released, rolls back and forth inside, and finally comes to rest at the bottom, as in Fig. 7-15*c*. What will be the displacement d of the shell during this process?

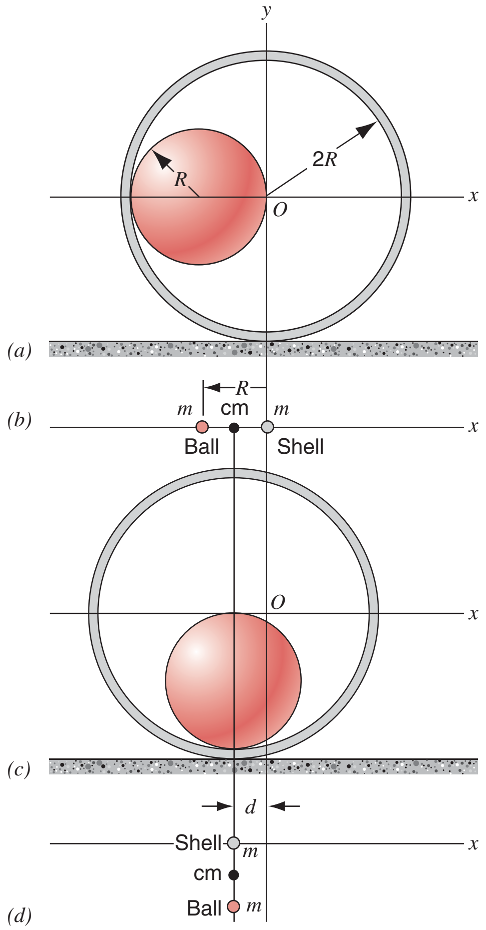


Figure 7-15. Sample Problem 7-6. (*a*) A ball of radius R is released from this initial position and is free to roll inside a spherical shell of radius $2R$. (*b*) The centers of mass of the ball, the shell, and their combination. (*c*) The final state after the ball has come to rest. The shell has moved so that the horizontal coordinate of the center of mass of the system remains in place. (*d*) The centers of mass of the ball, the shell, and their combination.

Solution The only external forces acting on the ball–shell system are the downward force of gravity and the normal force exerted vertically upward by the table. Neither force has a horizontal component so that $\sum F_{{\rm ext},x} = 0$. From Eq. 7-16 the acceleration component $a_{{\rm cm},x}$ of the center of mass must also be zero. Thus the horizontal position of the center of mass of the system must remain fixed, and the shell must move in such a way as to make sure that this happens.

We can represent both ball and shell by single particles of mass m, located at their respective centers. Figure 7-15*b* shows the system before the ball is released and Fig. 7-15*d* after the ball has come to rest at the bottom of the shell. We choose our origin to coincide with the initial position of the center of the shell. Figure 7-15*b* shows that, with respect to this origin, the center of mass of the ball–shell system is located a distance $\frac{1}{2}R$ to the left, halfway between the two particles. Figure 7-15*d* shows that the displacement of the shell is given by

$$d = \tfrac{1}{2}R.$$

The shell must move to the left through this distance as the ball comes to rest.

The ball is brought to rest by the frictional force that acts between it and the shell. Why does this frictional force not affect the final location of the center of mass?

7-5 CONSERVATION OF MOMENTUM IN A SYSTEM OF PARTICLES

Suppose we have a system containing N particles. The particles have masses m_n ($n = 1, 2, \ldots, N$) and move with velocities $\vec{\mathbf{v}}_n$ and momenta $\vec{\mathbf{p}}_n = m_n\vec{\mathbf{v}}_n$. The total momentum $\vec{\mathbf{P}}$ of the system is then

$$\vec{\mathbf{P}} = \sum_{n=1}^{N}\vec{\mathbf{p}}_n = \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 + \cdots + \vec{\mathbf{p}}_N$$
$$= \sum_{n=1}^{N} m_n\vec{\mathbf{v}}_n = m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 + \cdots + m_N\vec{\mathbf{v}}_N, \quad (7\text{-}20)$$

which, according to Eq. 7-13, can be written as

$$\vec{\mathbf{P}} = M\vec{\mathbf{v}}_{\rm cm}. \quad (7\text{-}21)$$

Here $M = m_1 + m_2 + \cdots + m_N$ is the total mass of the system. Equation 7-21 gives us a different but equivalent definition of the total momentum of a system of particles:

The total momentum of a system of particles is equal to the product of the total mass of a system and the velocity of its center of mass.

Assuming a constant mass M, the derivative of the momentum is

$$\frac{d\vec{\mathbf{P}}}{dt} = M\frac{d\vec{\mathbf{v}}_{\rm cm}}{dt} = M\vec{\mathbf{a}}_{\rm cm}, \quad (7\text{-}22)$$

using Eq. 7-14. Comparison of Eq. 7-22 with Eq. 7-16, $\sum \vec{\mathbf{F}}_{\text{ext}} = M\vec{\mathbf{a}}_{\text{cm}}$, allows us to write Newton's second law for a system of particles as

$$\sum \vec{\mathbf{F}}_{\text{ext}} = \frac{d\vec{\mathbf{P}}}{dt}. \tag{7-23}$$

Equation 7-23 states that, in a system of particles, the net external force equals the rate of change of the linear momentum of the system. This equation is the generalization of the single-particle expression $\sum \vec{\mathbf{F}} = d\vec{\mathbf{p}}/dt$, Eq. 6-2, to a system of many particles. Equation 7-23 reduces to Eq. 6-2 for the special case of a single particle, since only external forces can act on a one-particle system.

The law of conservation of linear momentum, which we derived in Section 6-4 for a system of two particles, also applies to a system of many particles, as we can see immediately from Eq. 7-23: If the net external force acting on a system is zero, then $d\vec{\mathbf{P}}/dt = 0$ and so the total linear momentum $\vec{\mathbf{P}}$ of the system remains constant.

If we view the system from a frame of reference that is moving with the center of mass, then in this frame the velocity $\vec{\mathbf{v}}_n'$ of a particle in the system is

$$\vec{\mathbf{v}}_n' = \vec{\mathbf{v}}_n - \vec{\mathbf{v}}_{\text{cm}}. \tag{7-24}$$

In this center-of-mass reference frame, the total momentum is, using Eq. 7-13,

$$\begin{aligned}\vec{\mathbf{P}}' &= \sum_{n=1}^{N} m_n\vec{\mathbf{v}}_n' = \sum_{n=1}^{N} m_n\vec{\mathbf{v}}_n - \sum_{n=1}^{N} m_n\vec{\mathbf{v}}_{\text{cm}} \\ &= M\vec{\mathbf{v}}_{\text{cm}} - \vec{\mathbf{v}}_{\text{cm}}\sum_{n=1}^{N} m_n \\ &= M\vec{\mathbf{v}}_{\text{cm}} - \vec{\mathbf{v}}_{\text{cm}}M = 0, \end{aligned} \tag{7-25}$$

so the center-of-mass reference frame is also the frame in which the total momentum is zero. This justifies our choice in Section 6-5 of the center-of-mass velocity (compare Eq. 6-23 with Eq. 7-4) for the reference frame from which to view the two-particle collision—only in this reference frame is the total momentum of the colliding particles zero both before and after the collision.

So far we are considering only systems in which the total mass M remains constant. Special care must be taken when applying Eq. 7-23 to systems in which the mass can change. The use of Eq. 7-23 in analyzing variable-mass systems is discussed in Section 7-6.

SAMPLE PROBLEM 7-7. A stream of bullets whose mass m is each 3.8 g is fired horizontally with a speed v of 1100 m/s into a large wooden block of mass M (= 12 kg) that is initially at rest on a horizontal table; see Fig. 7-16. If the block is free to slide without friction across the table, what speed will it acquire after it has absorbed eight bullets?

Solution For the moment we consider only the horizontal direction, which we define as the x axis, with positive to the right in Fig. 7-16. The x component of Eq. 7-23 is $\sum F_{\text{ext},x} = dP_x/dt$. The net external force on the block has a nonzero horizontal component (due to the bullets), and the net external force on a bullet has a horizontal component (due to the block). However, if we choose our system to include both block and bullets, the forces between them are internal forces. No net horizontal external force acts on this system, and so the x component of the momentum must remain constant. We have identified the boundary of this system in Fig. 7-16. The initial (horizontal) momentum, measured while the bullets are still in flight and the block is at rest, is

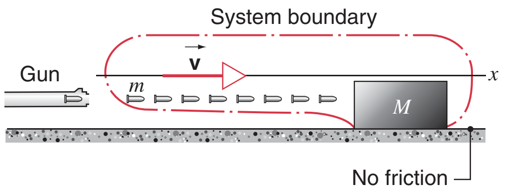


FIGURE 7-16. Sample Problem 7-7. A gun fires a stream of bullets toward a block of wood. We analyze the system that we define to be the block plus the bullets in flight.

$$P_{ix} = N(mv),$$

in which mv is the momentum of an individual bullet and $N = 8$. The final horizontal momentum, measured when all the bullets are in the block and the block is sliding over the table with horizontal velocity V, is

$$P_{fx} = (M + Nm)V.$$

Conservation of momentum requires that

$$P_{ix} = P_{fx}$$

or

$$N(mv) = (M + Nm)V.$$

Solving for V yields

$$V = \frac{Nm}{M + Nm}v = \frac{(8)(3.8 \times 10^{-3}\text{ kg})}{12\text{ kg} + (8)(3.8 \times 10^{-3}\text{ kg})}(1100\text{ m/s})$$
$$= 2.8\text{ m/s}.$$

In the vertical direction, the external forces are the weight of the bullets, the weight of the block, and the normal force on the block. While the bullets are in flight, they acquire a small vertical momentum component as a result of the action of gravity. When the bullets strike the block, the block must exert on each bullet a force with both horizontal and vertical components. Along with the vertical force on the bullet, which is necessary to change its vertical momentum to zero, there must (according to Newton's third law) be a corresponding increase in the normal force exerted on the block by the horizontal surface. This increase is not only from the weight of the imbedded bullet; it has an additional contribution arising from the rate of change of the vertical momentum of the bullet. When all the bullets have come to rest relative to the block, the normal force will equal the combined weights of block and imbedded bullets.

For simplicity in solving this problem, we have assumed that the bullets are fired so rapidly that all eight are in flight before the first bullet strikes the block. Can you solve this problem without making this assumption?

Suppose the system boundary is enlarged so that it includes the gun, which is fixed to the Earth. Does the horizontal momentum of this system change before and after the firing? Is there a horizontal external force?

SAMPLE PROBLEM 7-8. As Fig. 7-17 shows, a cannon whose mass M is 1300 kg fires a 72-kg ball in a horizontal direction with a speed v of 55 m/s relative to the cannon. The cannon is mounted so that it can recoil freely. (*a*) What is the velocity V of the recoiling cannon with respect to the Earth? (*b*) What is the initial velocity v_E of the ball with respect to the Earth?

Solution (*a*) The coordinate system is set up with the positive x axis to the right in Fig. 7-17. Our inertial reference frame is fixed with respect to the Earth.

We choose the cannon plus the ball as our system. By doing so, the forces associated with the firing of the cannon are internal to the system, and we do not have to deal with them. The external forces acting on the system have no horizontal components. Thus the horizontal component of the total linear momentum of the system must remain unchanged as the cannon is fired.

In terms of vectors, $\vec{\mathbf{v}}_E = \vec{\mathbf{v}} + \vec{\mathbf{V}}$; that is, the velocity of the ball with respect to the Earth equals the velocity of the ball with respect to the cannon plus the velocity of the cannon with respect to the Earth. In the horizontal direction, we have $v_{Ex} = v_x + V_x$ where, as shown in the figure, we expect that the x component of $\vec{\mathbf{V}}$ is negative.

In the reference frame of the Earth, the horizontal component of the initial momentum P_{ix} is zero. After the cannon has fired, the final momentum of the system with respect to the Earth is that of the cannon ball plus the recoiling cannon:

$$P_{fx} = MV_x + mv_{Ex} = MV_x + m(v_x + V_x).$$

With $\Sigma F_{ext,x} = 0$, we must have $P_{ix} = P_{fx}$ and so

$$MV_x + m(v_x + V_x) = 0.$$

Solving for V_x gives

$$V_x = -\frac{mv_x}{M + m} = -\frac{(72 \text{ kg})(55 \text{ m/s})}{1300 \text{ kg} + 72 \text{ kg}} = -2.9 \text{ m/s}.$$

The negative sign tells us that the cannon recoils to the left in Fig. 7-17, as we expect it should.

(*b*) With respect to the Earth, the horizontal component of the velocity of the ball is

$$\begin{aligned} v_{Ex} &= v_x + V_x \\ &= 55 \text{ m/s} + (-2.9 \text{ m/s}) = 52 \text{ m/s}. \end{aligned}$$

Because of the recoil, the ball is moving a little slower with respect to the Earth than it otherwise would. Note the importance in this problem of choosing the system (cannon + ball) wisely and being absolutely clear about the reference frame (Earth or recoiling cannon) to which the various measurements are referred.

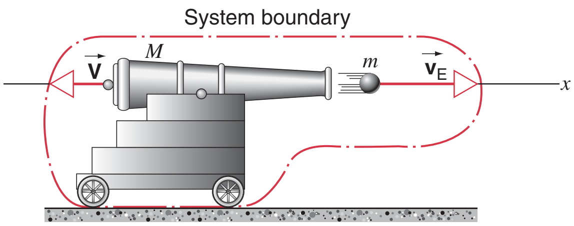


FIGURE 7-17. Sample Problem 7-8. A cannon of mass M fires a ball of mass m. The velocities of the ball and the recoiling cannon are shown in a reference frame fixed with respect to the Earth. Velocities are taken as positive to the right.

7-6 SYSTEMS OF VARIABLE MASS* (Optional)

Imagine that the cart holding the cannon of Fig. 7-17 also holds a large stack of cannonballs. As the cannon is repeatedly fired, the cart (which we assume to move without friction) recoils to the left, and with each recoil its speed increases. With the system boundary drawn as in Fig. 7-17, we know that the total horizontal momentum must be zero and that there is no net horizontal force on the system. If, however, we consider a system including only the cannon plus cart, then the previous statement is no longer true. The momentum of the cannon increases each time it is fired, and it is appropriate for us to use the familiar language of Newtonian physics to account for the change in momentum through the action of a suitable force. In this case, the force that accelerates the cannon is a reaction force: the cannon, by virtue of its exploding charge, pushes on the cannonballs to eject them, and the reaction force (the cannonballs pushing back on the cannon) moves the cannon to the left.

As the cannon is repeatedly fired, the total mass on the cart decreases by the quantity of cannonballs that have been ejected. The methods of Sample Problem 7-8 cannot easily be used to solve this problem, because the mass M of the recoiling object is different every time the cannon fires.

We refer to the system S consisting of cannon plus cart (including the unfired cannon balls) in this example as a "variable-mass" system. Of course, the larger system S' consisting of cannon plus all fired cannonballs is a constant-mass system *and* a constant-momentum system (in the absence of external force). The smaller system S, however, does not have constant mass. Moreover, the ejected cannonballs carry momentum, and there is a net outflow of momentum from S that is responsible for its acceleration.

The above example gives a reasonably good mental image of how a rocket works. Fuel is burned and ejected at high speed; the combustion products correspond to the cannonballs. The rocket (less the consumed fuel) experiences an acceleration that depends on the rate at which fuel is consumed and the speed with which it is ejected.

The goal in analyzing systems similar to the rocket is *not* to consider the kinematics of the entire system S'. Instead, we focus our attention on one particular subsystem S, and we ask how S moves as the mass within the entire system S' is redistributed so that the mass within subsystem S changes. The total mass within S' remains constant, but the particular subsystem S we consider can change its state of motion as it gains or loses mass (and momentum).

Figure 7-18 shows a schematic view of a generalized system. At time t, the subsystem S has a mass M and moves with velocity $\vec{\mathbf{v}}$ in the particular inertial frame of reference from which we are observing. At a time $t + \Delta t$, the mass of

* See "Force, Momentum Change, and Motion," by Martin S. Tiersten, *American Journal of Physics,* January 1969, p. 82, for an excellent general reference on systems of fixed and variable mass.

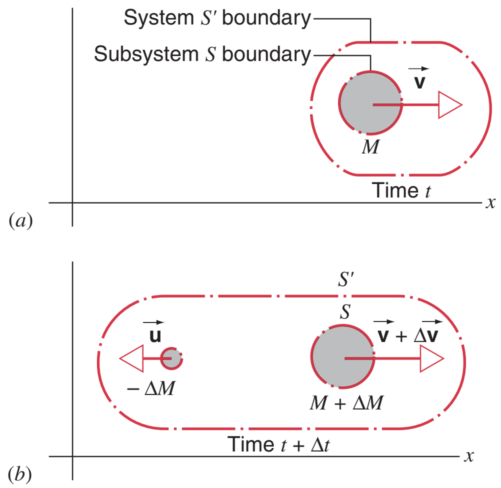


FIGURE 7-18. (*a*) A system S' at time t consists of a mass M moving with velocity $\vec{v}$. (*b*) At a time Δt later, the original mass M has ejected some mass $-\Delta M$. The remaining mass $M + \Delta M$, which we call the subsystem S, now moves with velocity $\vec{v} + \Delta\vec{v}$.

S has changed by an amount ΔM (a negative quantity, in the case of ejected mass) to $M + \Delta M$, while the mass of the remainder of the full system S' has changed by a corresponding amount $-\Delta M$. The system S now moves with a velocity $\vec{v} + \Delta\vec{v}$, and the ejected matter moves with velocity $\vec{u}$, both measured from our frame of reference.

To make the situation as general as possible, we also allow for an external force that may act on the entire system. In the case of a rocket, this is not the force that propels the rocket (which is an internal force for the system S'), but instead is the force due to some external agent, perhaps gravity or atmospheric drag. The total momentum of the entire system S' is $\vec{P}$, and Newton's second law can be written

$$\sum \vec{F}_{\text{ext}} = \frac{d\vec{P}}{dt}. \tag{7-26}$$

In the time interval Δt, the change in momentum $\Delta\vec{P}$ is

$$\Delta\vec{P} = \vec{P}_f - \vec{P}_i, \tag{7-27}$$

where $\vec{P}_f$, the final momentum of the system S' at time $t + \Delta t$, and $\vec{P}_i$, the initial momentum of S' at time t, are given by

$$\vec{P}_i = M\vec{v}, \tag{7-28a}$$

$$\vec{P}_f = (M + \Delta M)(\vec{v} + \Delta\vec{v}) + (-\Delta M)\vec{u}. \tag{7-28b}$$

The change in momentum of S' is thus

$$\begin{aligned}\Delta\vec{P} &= \vec{P}_f - \vec{P}_i\\ &= (M + \Delta M)(\vec{v} + \Delta\vec{v}) + (-\Delta M)\vec{u} - M\vec{v}.\end{aligned} \tag{7-29}$$

Rewriting the derivative in Eq. 7-26 as a limit and substituting this expression for $\Delta\vec{P}$, we obtain

$$\begin{aligned}\sum \vec{F}_{\text{ext}} &= \lim_{\Delta t\to 0} \frac{\Delta\vec{P}}{\Delta t}\\ &= \lim_{\Delta t\to 0} \frac{(M + \Delta M)(\vec{v} + \Delta\vec{v}) + (-\Delta M)\vec{u} - M\vec{v}}{\Delta t}\\ &= \lim_{\Delta t\to 0}\left[M\frac{\Delta\vec{v}}{\Delta t} + (\vec{v} - \vec{u})\frac{\Delta M}{\Delta t} + \Delta\vec{v}\frac{\Delta M}{\Delta t}\right]\\ &= M\frac{d\vec{v}}{dt} + (\vec{v} - \vec{u})\frac{dM}{dt}.\end{aligned} \tag{7-30}$$

Note that, in taking the limit, the last term in the square brackets vanishes, because $\Delta\vec{v} \to 0$ as $\Delta t \to 0$. In Eq. 7-30, M is the mass of the subsystem S at time t, and $d\vec{v}/dt$ is its acceleration as it gains or loses mass at velocity $\vec{u}$ (in our frame of reference) and at a rate $|dM/dt|$. If $dM/dt > 0$, the mass of the subsystem increases; if $dM/dt < 0$, its mass decreases.

We can write Eq. 7-30 in a more instructive form as

$$M\frac{d\vec{v}}{dt} = \sum \vec{F}_{\text{ext}} + \vec{v}_{\text{rel}}\frac{dM}{dt}, \tag{7-31}$$

where $\vec{v}_{\text{rel}} = \vec{u} - \vec{v}$ is the velocity of the gained or lost matter relative to the subsystem S. For example, if subsystem S is a rocket, $\vec{v}_{\text{rel}}$ is the velocity of the ejected gases *relative to the rocket.* This is a reasonable quantity to introduce, because the speed of the ejected gases is a fundamental design characteristic of a rocket engine and should not be expressed in a form that depends on any frame of reference other than the rocket itself.

Equation 7-31 shows that the acceleration $d\vec{v}/dt$ of the subsystem S (the rocket, for example) is in part determined by the net external force and in part by the momentum transferred by the mass that is gained or lost. Note that $\vec{v}_{\text{rel}} = \vec{u} - \vec{v}$ points to the left in Fig. 7-18; since dM/dt is negative for a rocket, the second term in Eq. 7-31 is represented by a vector that points to the right and so is responsible for accelerating the subsystem in that direction. This term is called the *thrust* of the rocket and can be interpreted as the force exerted on the rocket by the ejected gas. The thrust of a rocket can be increased by increasing either the speed of the ejected gas or the rate at which it is ejected.

If mass is ejected at a constant rate and at a constant speed relative to S, the thrust is constant but the acceleration is not constant, because M is decreasing. If $dM/dt = 0$, so that the mass of the subsystem does not change, Eq. 7-31 reduces to our familiar form of Newton's second law, $\Sigma\, \vec{F}_{\text{ext}} = M\vec{a}$.

The analogy between a rocket and a recoiling gun is apparent from Fig. 7-19. In each case momentum is conserved for the entire system, consisting of the ejected mass (bullets or fuel) plus the object that ejects the mass. When we focus our attention on the gun or the rocket within the larger system, we see that its mass changes and that there is a force that drives it, a recoil in the case of the gun and a thrust in the case of the rocket. If we view the system from a reference frame at the center of mass, then as time passes there is more ejected mass, and it has traveled further to the

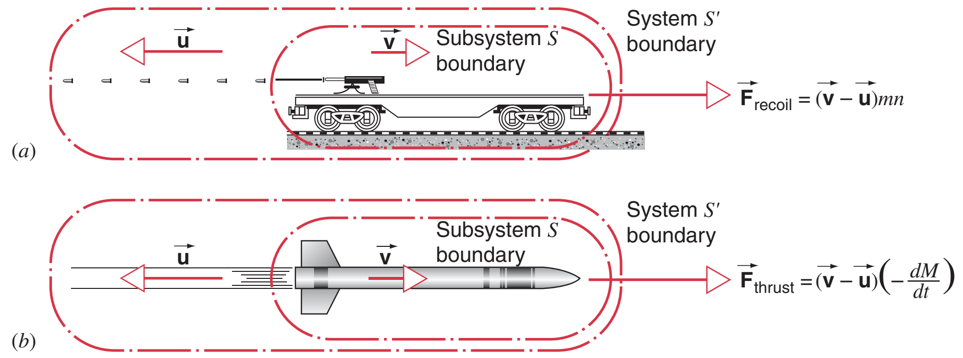


FIGURE 7-19. (*a*) A machine gun fires a stream of bullets at a rate of *n* per unit time. The total momentum of the system S' remains constant, but the subsystem S experiences a recoil force that changes its momentum. Its change in momentum in a time dt is exactly equal to the opposite momentum $mn\vec{u}\,dt$ carried by the bullets. (*b*) A rocket ejects a stream of combustion products. The total momentum of the system S' remains constant, but the subsystem S experiences a thrust that changes its momentum. Its change in momentum in a time dt is exactly equal to the opposite momentum $\vec{u}\,dM$ carried by the ejected gas.

left in Fig. 7-19, meaning that the object must travel to the right to keep the center of mass fixed.

SAMPLE PROBLEM 7-9. A spaceship with a total mass of 13,600 kg is moving relative to a certain inertial reference frame with a speed of 960 m/s in a region of space of negligible gravity. It fires its rocket engines to give an acceleration parallel to the initial velocity. The rockets eject gas at a constant rate of 146 kg/s with a constant speed (relative to the spaceship) of 1520 m/s, and they are fired until 9100 kg of fuel has been burned and ejected. (*a*) What is the thrust produced by the rockets? (*b*) What is the velocity of the spaceship after the rockets have fired?

Solution (*a*) The thrust is given by the last term of Eq. 7-31. Its magnitude is

$$F = \left| v_{\text{rel}} \frac{dM}{dt} \right| = (1520 \text{ m/s})(146 \text{ kg/s}) = 2.22 \times 10^5 \text{ N}.$$

(*b*) Choosing the positive *x* direction to be that of the spaceship's initial velocity, we can write Eq. 7-31 (with $\Sigma \vec{F}_{\text{ext}} = 0$) as

$$M \frac{dv_x}{dt} = v_{\text{rel},x} \frac{dM}{dt}.$$

Because the gas is ejected relative to the rocket in a direction opposite to its velocity (which we have chosen as our positive *x* direction), $v_{\text{rel},x}$ is negative. Since dM/dt is also negative, the right side of this equation is positive, and the spaceship's velocity increases. Rewriting the equation as $dv_x = v_{\text{rel},x}\,(dM/M)$, we can integrate on the left from the initial velocity of 960 m/s to the final velocity we seek to determine. On the right we integrate from the initial mass (13,600 kg) to the final mass (13,600 kg $-$ 9100 kg $=$ 4500 kg)

$$\int_{v_{ix}}^{v_{fx}} dv_x = v_{\text{rel},x} \int_{M_i}^{M_f} \frac{dM}{M},$$

which we can evaluate as

$$v_{fx} - v_{ix} = v_{\text{rel},x} \ln \frac{M_f}{M_i}. \qquad (7\text{-}32)$$

Solving for v_{fx} we find

$$v_{fx} = 960 \text{ m/s} + (-1520 \text{ m/s}) \ln \frac{4500 \text{ kg}}{13{,}600 \text{ kg}} = 2640 \text{ m/s}.$$

SAMPLE PROBLEM 7-10. Sand drops from a stationary hopper at a rate of 0.134 kg/s onto a conveyor belt moving with a speed of 0.96 m/s, as shown in Fig. 7-20. What net force must be applied to the conveyor belt to keep it moving at constant speed?

Solution We choose the direction of motion of the belt as our positive *x* direction, and we fix our coordinate system in the laboratory that locates our inertial reference frame (and in which the hopper is at rest). The system S' includes the belt and all of the sand in the hopper. The subsystem S represents the belt and only the sand that has dropped onto it. The mass of S is increasing ($dM/dt > 0$) as more sand drops onto the belt.

We can apply Eq. 7-31 to this situation, with $dv_x/dt = 0$ (because the belt moves with constant velocity) and also with $u_x = 0$

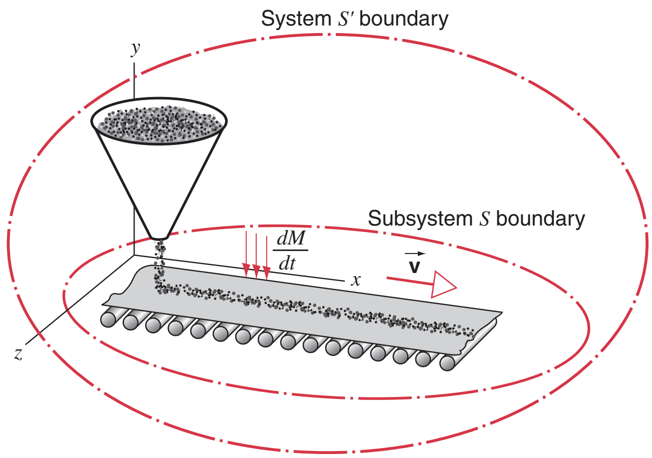


FIGURE 7-20. Sample Problem 7-10. Sand drops from a hopper at a rate dM/dt onto a conveyor belt moving with constant velocity $\vec{v}$ in the reference frame of the laboratory. The hopper is at rest in the reference frame of the laboratory.

(because the sand dropping onto the belt has no component of velocity in the x direction. Thus $v_{\text{rel},x} = -v_x$; that is, an observer traveling with the belt would see the sand leaving the hopper (and the hopper itself) moving in the negative x direction. Solving for the net external force, we find

$$\sum F_{\text{ext},x} = -v_{\text{rel},x}\frac{dM}{dt} = v_x\frac{dM}{dt} = (0.96\text{ m/s})(0.134\text{ kg/s})$$
$$= 0.129\text{ N}.$$

The force has a positive x component; that is, it must be applied in the direction of motion of the belt to increase the x component of the velocity of each grain of sand that drops onto the belt from 0 to 0.96 m/s.

MULTIPLE CHOICE

7-1 The Motion of a Complex Object

7-2 Two-Particle Systems

1. Two frictionless pucks are connected by a rubber band. One of the pucks is projected across an air table, the rubber band tightens, and the second puck follows—in an apparently random way—the first puck. The center of mass of this two-particle system is located
- (A) at a fixed distance from one of the pucks.
- (B) usually, but not always, between the two pucks.
- (C) at a distance from one of the pucks that is a fixed ratio to the distance between the two pucks.
- (D) sometimes closer to one puck, and sometimes closer to the other.

2. Two objects are moving on a surface. The center of mass exists only if
- (A) the two objects are physically connected.
- (B) the surface is level.
- (C) the surface is frictionless.
- (D) There is always a center of mass.

3. Two objects are sitting on a level, frictionless surface. The objects are not connected or touching. A force F is applied to *one* of the objects, which then moves with acceleration a. Which of the following statements is most correct?
- (A) The center of mass concept cannot be applied because the external force does not act on both objects.
- (B) The center of mass moves with acceleration that could be greater than a.
- (C) The center of mass moves with acceleration that must be equal to a.
- (D) The center of mass moves with acceleration that must be less than a.

4. Two objects of unequal mass are connected by a light string that passes over a pulley. One of the objects is given an initial upward velocity. The center of mass of the two objects will
- (A) accelerate up or down, depending on the relative masses of the two objects.
- (B) accelerate downward only after it has reached some highest point.
- (C) accelerate downward at some value less than g.
- (D) accelerate downward with a value of g.

5. Two objects of unequal mass are connected by a compressed spring. The combined object is thrown vertically into the air. At the highest point of the trajectory the spring releases, resulting in one of the objects being projected even higher into the air; the spring remains attached to the other object. Shortly after the spring releases, the center of mass of the objects is
- (A) moving upward and accelerating upward.
- (B) moving upward but accelerating downward.
- (C) moving downward but accelerating upward.
- (D) moving downward and accelerating downward.
- (E) There is not enough information given to answer the question.

7-3 Many-Particle Systems

6. Three objects are on a table. You can find the center of mass by
- (A) combining the three objects together according to Eq. 7-12.
- (B) combining the two lightest objects first according to Eq. 7-12, calling it a new "particle," and then combining the third object to this new particle.
- (C) combining the two heaviest objects first, calling it a new "particle," and then adding the third object to this new particle.
- (D) any of the above methods; they are equivalent.

7. Seven identical geese are flying south together at constant speed. A hunter shoots one of them, which immediately dies and falls to the ground. The other six continue flying south at the original speed. After the one goose has hit the ground, the center of mass of all seven geese
- (A) continues south at the original speed, but is now located some distance behind the flying geese.
- (B) continues south, but at $\frac{6}{7}$ the original speed.
- (C) continues south, but at $\frac{1}{7}$ the original speed.
- (D) stops with the dead goose.

8. Measure the height that a person can jump by how far above the ground her head rises. Can a person jump higher with her hands fixed above her head or with her hands fixed at her sides?
- (A) With her hands fixed above head.
- (B) With hands fixed at sides.
- (C) The result is the same in both cases.
- (D) The answer depends on the relative size of the person's hands compared to her overall mass.

7-4 Center of Mass of Solid Objects

9. A solid body has a center of mass located inside the body. A hole is drilled somewhere in the body, but not near the center of mass. After the hole has been drilled, the center of mass of the remaining body moves from the original position

(A) away from the hole. (B) toward the hole.
(C) not at all. (D) in a random direction.

10. Consider Sample Problem 7-6. The inner ball is released and rolls around inside the spherical shell for some time before coming to rest. During this time the center of mass
(A) moves both horizontally and vertically.
(B) moves both horizontally and vertically, but returns to the original horizontal position when the system comes to rest.
(C) moves only vertically.
(D) moves vertically only downward, because the center of mass can never move upward.

7-5 Conservation of Momentum in a System of Particles

11. A system of N particles is free from any external forces.
(*a*) Which of the following is true for the magnitude of the total momentum of the system?
(A) It must be zero.
(B) It could be non-zero, but it must be constant.
(C) It could be non-zero, and it might not be constant.
(D) The answer depends on the nature of the internal forces in the system.
(*b*) Which of the following must be true for the sum of the *magnitudes* of the momenta of the individual particles in the system?
(A) It must be zero.
(B) It could be non-zero, but it must be constant.
(C) It could be non-zero, and it might not be constant.
(D) It could be zero, even if the magnitude of the total momentum is not zero.

12. An isolated rail car of mass M is moving along a straight, frictionless track at an initial speed v_0. The car is passing under a bridge when a crate filled with N bowling balls, each of mass m, is dropped from the bridge into the bed of the rail car. The crate splits open and the bowling balls bounce around inside the rail car, but none of them fall out.
(*a*) Is the momentum of the rail car + bowling balls system conserved in this collision?
(A) Yes, the momentum is completely conserved.
(B) Only the momentum component in the vertical direction is conserved.
(C) Only the momentum component parallel to the track is conserved.
(D) No components are conserved.
(*b*) What is the average speed of the rail car + bowling balls system some time after the collision?
(A) $(M + Nm)v_0/M$ (B) $Mv_0/(Nm + M)$
(C) Nmv_0/M
(D) The speed cannot be determined because there is not enough information.

13. An isolated rail car originally moving with speed v_0 on a straight, frictionless, level track contains a large amount of sand. A release valve on the bottom of the car malfunctions, and sand begins to pour out *straight down* relative to the rail car.
(*a*) Is momentum conserved in this process?
(A) The momentum of the rail car alone is conserved.
(B) The momentum of the rail car + sand remaining within the car is conserved.
(C) The momentum of the rail car + *all* of the sand, both inside and outside the rail car, is conserved.
(D) None of the three previous systems have momentum conservation.
(*b*) What happens to the speed of the rail car as the sand pours out?
(A) The car begins to roll faster.
(B) The car maintains the same speed.
(C) The car begins to slow down.
(D) The problem cannot be solved since momentum is not conserved.

7-6 Systems of Variable Mass

QUESTIONS

1. A canoeist in a still pond can reach shore by jerking sharply on the rope attached to the bow of the canoe. How do you explain this? (It really can be done.)

2. How might a person sitting at rest on a frictionless horizontal surface get altogether off it?

3. A box is sitting on a frictionless, level surface. A small spring cannon, which is capable of firing a lump of clay, is lowered into the box on a string attached to the ceiling. Before the cannon is fired, the center of mass of the cannon, clay, and box is fixed at a point A. The cannon fires; the clay shoots from the cannon and sticks to the wall of the box. Does the center of mass of the cannon, clay, and box system move? Explain.

4. Does the center of mass of a solid object necessarily lie within the object? If not, give examples.

5. Figure 7-21 shows (*a*) an isosceles triangular prism and (*b*) a right circular cone whose diameter is the same length as the base of the triangle. The center of mass of the triangle is one-third of the way up from the base but that of the cone is only one-fourth of the way up. Can you explain the difference?

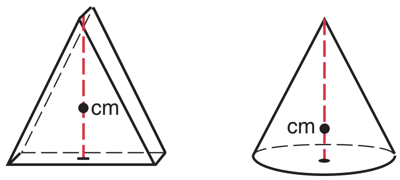


FIGURE 7-21. Question 5.

6. How is the center of mass concept related to the concept of geographic center of the country? To the population center of the country? What can you conclude from the fact that the geographic center differs from the population center?

7. Where is the center of mass of the Earth's atmosphere?

8. The center of mass of a full can of soda is approximately at the center. (*a*) Part of the soda is consumed and the can is replaced on the table. What happens to the center of mass of the can + soda system? (*b*) Eventually all of the soda is consumed and the can is replaced on the table. What happens to the center of mass of the can? See Problem 5.

9. An amateur sculptor decides to portray a bird (Fig. 7-22). Luckily, the final model is actually able to stand upright. The model is formed of a single thick sheet of metal of uniform thickness. Of the points shown, which is most likely to be the center of mass?

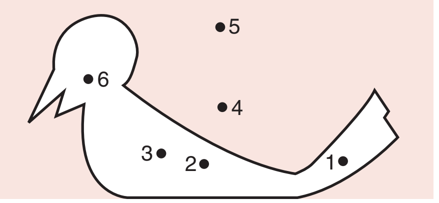


FIGURE 7-22. Question 9.

10. Someone claims that when a skillful high jumper clears the bar, the jumper's center of mass actually goes under the bar. Is this possible?

11. A ballet dancer doing a grand jete (great leap; see Fig. 7-23) seems to float horizontally in the central portion of her leap. Show how the dancer can maneuver her legs in flight so that, although her center of mass does indeed follow the expected parabolic trajectory, the top of her head moves more or less horizontally. (See "The Physics of Dance," by Kenneth Laws, *Physics Today,* February 1985, p. 24.)

FIGURE 7-23. Question 11.

12. Can a sailboat be propelled by air blown at the sails from a fan attached to the boat? Explain your answer.

13. If only an external force can change the state of motion of the center of mass of a body, how does it happen that the internal force of the brakes can bring a car to rest?

14. A man stands still on a large sheet of slick ice; in his hand he holds a lighted firecracker. He throws the firecracker at an angle (that is, not vertically) into the air. Describe briefly, but as exactly as you can, the motion of the center of mass of the firecracker and the motion of the center of mass of the system consisting of man and firecracker. It will be most convenient to describe each motion during each of the following periods: (*a*) after he throws the firecracker, but before it explodes; (*b*) between the explosion and the first piece of firecracker hitting the ice; (*c*) between the first fragment hitting the ice and the last fragment landing; and (*d*) during the time when all fragments have landed but none has reached the edge of the ice.

15. You throw an ice cube with velocity $\vec{\mathbf{v}}$ into a hot, gravity-free, evacuated space. The cube gradually melts to liquid water and then boils to water vapor. (*a*) Is it a system of particles all the time? (*b*) If so, is it the same system of particles? (*c*) Does the motion of the center of mass undergo any abrupt changes? (*d*) Does the total linear momentum change?

16. An evacuated box is at rest on a frictionless table. You punch a small hole in one face so that air can enter. (See Fig. 7-24.) How will the box move? What argument did you use to arrive at your answer?

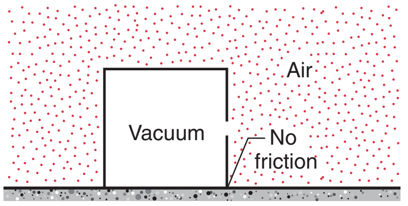


FIGURE 7-24. Question 16.

17. A railroad flatcar is initially at rest. It holds N people each of weight w. If each person in succession runs parallel to the tracks with a relative velocity $v_{\rm rel}$ and jumps off the end, do they impart to the car a greater velocity than if they all run and jump at the same time?

18. Can you think of variable-mass systems other than the examples given in the text?

19. It is not correct to use the equation $\sum \vec{\mathbf{F}}_{\rm ext} = d(M\vec{\mathbf{v}})/dt$ for a system of variable mass. To show this (*a*) put the equation in the form $(\sum \vec{\mathbf{F}}_{\rm ext} - M\,d\vec{\mathbf{v}}/dt)/(dM/dt) = \vec{\mathbf{v}}$ and (*b*) show that one side of this equation has the same value in all inertial frames, where as the other side does not. Hence the equation cannot be generally valid. (*c*) Show that Eq. 7-31 leads to no such contradiction.

20. In 1920 a prominent newspaper editorialized as follows about the pioneering rocket experiments of Robert H. Goddard, dismissing the notion that a rocket could operate in a vacuum: "That Professor Goddard, with his 'chair' in Clark College and the countenancing of the Smithsonian Institution, does not know the relation of action to reaction, and of the need to have something better than a vacuum against which to react—to say that would be absurd. Of course, he seems only to lack the knowledge ladled out daily in high schools." What is wrong with this argument?

21. The final velocity of the final stage of a multi-stage rocket is much greater than the final velocity of a single-stage rocket of the same total mass and fuel supply. Explain this fact.

22. Can a rocket reach a speed greater than the speed of the exhaust gases that propel it? Explain why or why not.

23. Are there any possible methods of propulsion in outer space other than rockets? If so, what are they and why are they not used?

24. Equation 7-32 suggests that the speed of a rocket can increase without limit if enough fuel is burned. Is this reasonable? What is the limit of applicability of Eq. 7-32? Where in our derivation of Eq. 7-32 did we introduce this limit? (See "The Equation of Motion for Relativistic Particles and Systems with Variable Rest Mass," by Kalman B. Pomeranz, *American Journal of Physics,* December 1964, p. 955.)

EXERCISES

7-1 The Motion of a Complex Object

7-2 Two-Particle Systems

1. How far is the center of mass of the Earth–Moon system from the center of the Earth? (From Appendix C, obtain the masses of the Earth and Moon and the distance between the centers of the Earth and Moon. It is interesting to compare the answer to the Earth's radius.)

2. Show that the ratio of the distances x_1 and x_2 of two particles from their center of mass is the inverse ratio of their masses; that is, $x_1/x_2 = m_2/m_1$.

3. A Plymouth with a mass of 2210 kg is moving along a straight stretch of road at 105 km/h. It is followed by a Ford with mass 2080 kg moving at 43.5 km/h. How fast is the center of mass of the two cars moving?

4. Two skaters, one with mass 65 kg and the other with mass 42 kg, stand on an ice rink holding a pole with a length of 9.7 m and a mass that is negligible. Starting from the ends of the pole, the skaters pull themselves along the pole until they meet. How far will the 42-kg skater move?

5. Two particles *P* and *Q* are initially at rest 1.64 m apart. *P* has a mass of 1.43 kg and *Q* a mass of 4.29 kg. *P* and *Q* attract each other with a constant force of 1.79×10^{-2} N. No external forces act on the system. (*a*) Describe the motion of the center of mass. (*b*) At what distance from *P*'s original position do the particles collide?

6. A shell is fired from a gun with a muzzle velocity of 466 m/s, at an angle of 57.4° with the horizontal. At the top of the trajectory, the shell explodes into two fragments of equal mass. One fragment, whose speed immediately after the explosion is zero, falls vertically. How far from the gun does the other fragment land, assuming level terrain?

7. A dog weighing 10.8 lb is standing on a flatboat so that he is 21.4 ft from the shore. He walks 8.50 ft on the boat toward shore and then halts. The boat weighs 46.4 lb, and one can assume there is no friction between it and the water. How far is he from the shore at the end of this time? (Hint: The center of mass of boat + dog does not move. Why?) The shoreline is also to the left in Fig. 7-25.

FIGURE 7-25. Exercise 7.

8. Richard, mass 78.4 kg, and Judy, who is less massive, are enjoying Lake George at dusk in a 31.6-kg canoe. When the canoe is at rest in the placid water, they change seats, which are 2.93 m apart and symmetrically located with respect to the canoe's center. Richard notices that the canoe moved 41.2 cm relative to a submerged log and calculates Judy's mass. What is it?

9. An 84.4-kg man is standing at the rear of a 425-kg iceboat that is moving at 4.16 m/s across ice that may be considered to be frictionless. He decides to walk to the front of the 18.2-m-long boat and does so at a speed of 2.08 m/s with respect to the boat. How far does the boat move across the ice while he is walking?

7-3 Many-Particle Systems

10. Where is the center of mass of the three particles shown in Fig. 7-26?

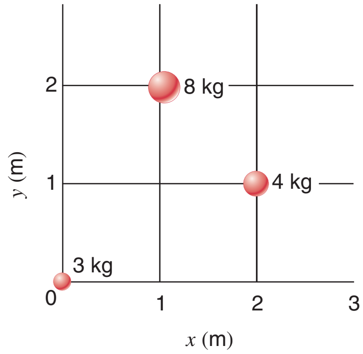


FIGURE 7-26. Exercise 10.

11. In the ammonia (NH_3) molecule, the three hydrogen (H) atoms form an equilateral triangle, the distance between centers of the atoms being 16.28×10^{-11} m, so that the center of the triangle is 9.40×10^{-11} m from each hydrogen atom. The nitrogen (N) atom is at the apex of a pyramid, the three hydrogens constituting the base (see Fig. 7-27). The nitrogen/hydrogen distance is 10.14×10^{-11} m and the nitro-

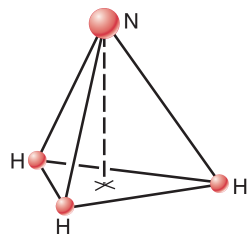


FIGURE 7-27. Exercise 11.

gen/hydrogen atomic mass ratio is 13.9. Locate the center of mass relative to the nitrogen atom.

12. Consider Sample Problem 7-3. The 6.5-kg fragment is observed at $t = 1.42$ s to be moving with a velocity whose horizontal component is 11.4 m/s in the same direction as that of the launch of the original projectile and whose vertical component is 4.6 m/s downward. Find the velocity of the 3.1-kg fragment at that time.

7-4 Center of Mass of Solid Objects

13. Three thin rods each of length L are arranged in an inverted U, as shown in Fig. 7-28. The two rods on the arms of the U each have mass M; the third rod has mass $3M$. Where is the center of mass of the assembly?

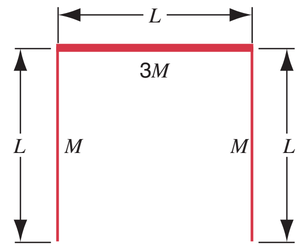


FIGURE 7-28. Exercise 13.

14. Fig. 7-29 shows a composite slab with dimensions 22.0 cm × 13.0 cm × 2.80 cm. Half of the slab is made of aluminum (density = 2.70 g/cm³) and half of iron (density = 7.85 g/cm³), as shown. Where is the center of mass of the slab?

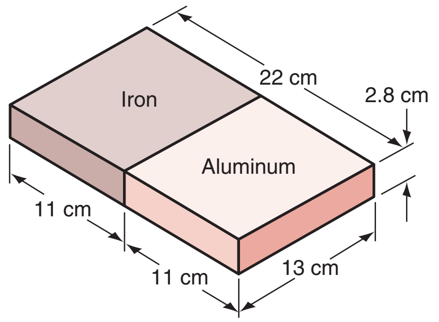


FIGURE 7-29. Exercise 14.

15. A box, open at the top, in the form of a cube of edge length 40 cm, is constructed from thin metal plate. Find the coordinates of the center of mass of the box with respect to the coordinate system shown in Fig. 7-30.

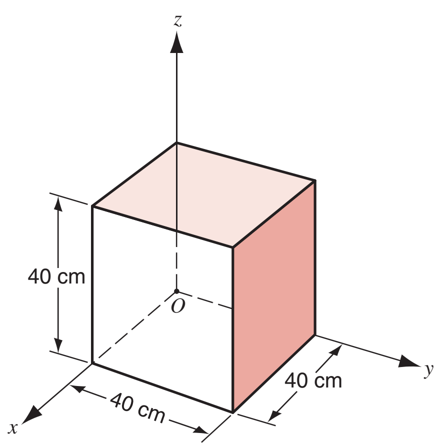


FIGURE 7-30. Exercise 15.

7-5 Conservation of Momentum in a System of Particles

16. A vessel at rest explodes, breaking into three pieces. Two pieces, one with twice the mass of the other, fly off perpendicular to one another with the same speed of 31.4 m/s. The third piece has three times the mass of the lightest piece. Find the magnitude and direction of its velocity immediately after the explosion. (Specify the direction by giving the angle from the line of travel of the least massive piece.)

17. Each minute, a special game warden's machine gun fires 220, 12.6-g rubber bullets with a muzzle velocity of 975 m/s. How many bullets must be fired at an 84.7-kg animal charging toward the warden at 3.87 m/s in order to stop the animal in its tracks? (Assume that the bullets travel horizontally and drop to the ground after striking the target.)

18. A railway flat car is rushing along a level frictionless track at a speed of 45 m/s. Mounted on the car and aimed forward is a cannon that fires 65-kg cannon balls with a muzzle speed of 625 m/s. The total mass of the car, the cannon, and the large supply of cannon balls on the car is 3500 kg. How many cannon balls must be fired to bring the car as close to rest as possible?

19. Twelve 100.0-kg containers of rocket parts in empty space are loosely tethered by ropes tied together at a common point. The center of mass of the twelve containers is originally at rest. A 50-kg lump of "space-goo" moving at 80 m/s collides with one of the containers and sticks to it. (*a*) Assuming that none of the tethers break, find the speed of the center of mass of the twelve containers after the collision with the space goo. (*b*) Assuming instead that the tether of the struck container does break, find the speed of the center of mass of the twelve containers after the collision.

7-6 Systems of Variable Mass

20. A rocket at rest in space, where there is virtually no gravity, has a mass of 2.55×10^5 kg, of which 1.81×10^5 kg is fuel. The engine consumes fuel at the rate of 480 kg/s, and the exhaust speed is 3.27 km/s. The engine is fired for 250 s. (*a*) Find the thrust of the rocket engine. (*b*) What is the mass of the rocket after the engine burn? (*c*) What is the final speed attained?

21. Consider a rocket at rest in empty space. What must be its mass ratio (ratio of initial to final mass) in order that, after firing its engine, the rocket's speed is (*a*) equal to the exhaust speed and (*b*) equal to twice the exhaust speed?

22. During a lunar mission, it is necessary to make a midcourse correction of 22.6 m/s in the speed of the spacecraft, which is moving at 388 m/s. The exhaust speed of the rocket engine is 1230 m/s. What fraction of the initial mass of the spacecraft must be discarded as exhaust?

23. A rocket of total mass 1.11×10^5 kg, of which 8.70×10^4 kg is fuel, is to be launched vertically. The fuel will be burned at the constant rate of 820 kg/s. Relative to the rocket, what is the minimum exhaust speed that allows liftoff at launch?

24. A 5.4-kg toboggan carrying 35 kg of sand slides from rest down an icy slope 93 m long, inclined 26° below the horizontal. The sand leaks from the back of the toboggan at the rate of 2.3 kg/s. How long does it take the toboggan to reach the bottom of the slope?

25. A freight car, open at the top, weighing 9.75 metric tons, is coasting along a level track with negligible friction at

1.36 m/s when it begins to rain hard. The raindrops fall vertically with respect to the ground. What is the speed of the car when it has collected 0.50 metric tons of rain? What assumptions, if any, must you make to get your answer?

PROBLEMS

1. A man of mass m clings to a rope ladder suspended below a balloon of mass M; see Fig. 7-31. The balloon is stationary with respect to the ground. (*a*) If the man begins to climb the ladder at a speed v (with respect to the ladder), in what direction and with what speed (with respect to the Earth) will the balloon move? (*b*) What is the state of motion after the man stops climbing?

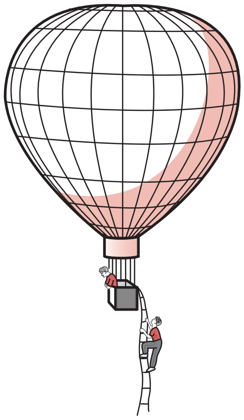


FIGURE 7-31. Problem 1.

2. Two bodies, each made up of weights from a set, are connected by a light cord that passes over a light, frictionless pulley with a diameter of 56.0 mm. The two bodies are at the same level. Each originally has a mass of 850 g. (*a*) Locate their center of mass. (*b*) Thirty-four grams are transferred from one body to the other, but the bodies are prevented from moving. Locate the center of mass. (*c*) The two bodies are now released. Describe the motion of the center of mass and determine its acceleration.

3. A uniform flexible chain of length L, with weight per unit length λ, passes over a small, frictionless peg; see Fig. 7-32. It is released from a rest position with a length of chain x hanging from one side and a length $L - x$ from the other side. Find the acceleration a as a function of x.

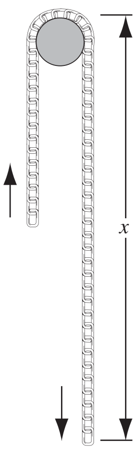


FIGURE 7-32. Problem 3.

4. A cannon and a supply of cannonballs are inside a sealed railroad car of length L, as in Fig. 7-33. The cannon fires to the right; the car recoils to the left. The cannonballs remain in the car after hitting the far wall. (*a*) After all the cannonballs have been fired, what is the greatest distance the car can have moved from its original position? (*b*) What is the speed of the car after all the cannonballs have been fired?

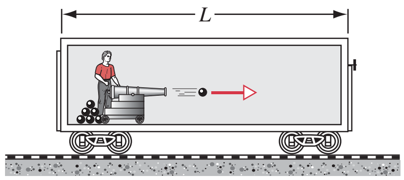


FIGURE 7-33. Problem 4.

5. A cylindrical storage tank is initially filled with aviation gasoline. The tank is then drained through a valve on the bottom. See Fig. 7-34. (*a*) As the gasoline is withdrawn, describe qualitatively the motion of the center of mass of the tank and its remaining contents. (*b*) What is the depth x to which the tank is filled when the center of mass of the tank and its remaining contents reaches its lowest point? Express your answer in terms of H, the height of the tank; M, its mass; and m, the mass of gasoline it can hold.

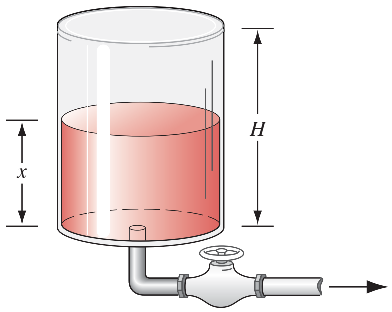


FIGURE 7-34. Problem 5.

6. Find the center of mass of a homogeneous semicircular plate. Let R be the radius of the circle.

7. A 1400-kg cannon, which fires a 70.0-kg shell with a muzzle speed of 556 m/s, is set at an elevation angle of 39.0° above the horizontal. The cannon is mounted on frictionless rails, so that it recoils freely. (*a*) What is the speed of the shell with respect to the Earth? (*b*) At what angle with the ground is the shell projected? (Hint: The horizontal component of the momentum of the system remains unchanged as the gun is fired.)

8. A rocket sled with a mass of 2870 kg moves at 252 m/s on a set of rails. At a certain point, a scoop on the sled dips into a trough of water located between the tracks and scoops water into an empty tank on the sled. Determine the speed of the sled after 917 kg of water have been scooped up.

9. To keep a conveyor belt moving when it transports luggage requires a greater driving force than for an empty belt. What additional driving force is needed if the belt moves at a constant speed of 1.5 m/s and the rate at which luggage is placed on one end of the belt and removed at the other end is 20 kg/s? Assume that the luggage is dropped vertically onto the belt; persons removing luggage grab hold of it and bring it to rest relative to themselves before lifting it off the belt.

10. A 5860-kg rocket is set for vertical firing. The exhaust speed is 1.17 km/s. How much gas must be ejected each second to supply the thrust needed (*a*) to overcome the weight of the rocket and (*b*) to give the rocket an initial upward acceleration of 18.3 m/s^2? Note that, in contrast to the situation described in Sample Problem 7-9, gravity is present here as an external force.

11. Two long barges are floating in the same direction in still water, one with a speed of 9.65 km/h and the other with a speed of 21.2 km/h. While they are passing each other, coal is shoveled from the slower to the faster one at a rate of 925 kg/min; see Fig. 7-35. How much additional force must be provided by the driving engines of each barge if neither is to change speed? Assume that the shoveling is always perfectly sideways and that the frictional forces between the barges and the water do not depend on the weight of the barges.

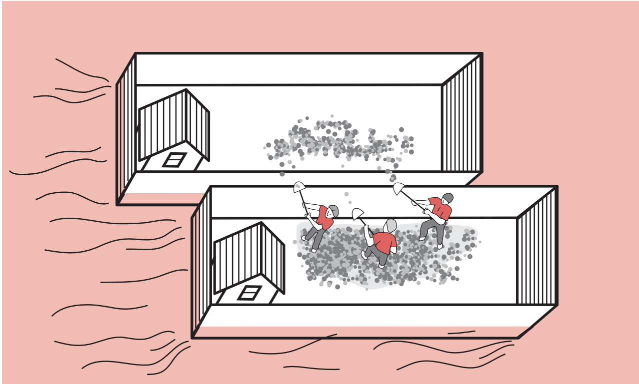


FIGURE 7-35. Problem 11.

12. A flexible inextensible string of length L is threaded into a smooth tube, into which it snugly fits. The tube contains a right-angled bend and is positioned in the vertical plane so that one arm is vertical and the other horizontal. Initially, at $t = 0$, a length y_0 of the string is hanging down in the vertical arm. The string is released and slides through the tube, so that at any subsequent time t later, it is moving with a speed dy/dt, where $y(t)$ is the length of the string that is then hanging vertically. (*a*) Show that in terms of the variable-mass problem $v_{rel} = 0$, so that the equation of motion has the form $m\,dv/dt = F_{ext}$. (*b*) Show that the specific equation of motion is $d^2y/dt^2 = gy/L$. (*c*) Show that

$$y = (y_0/2)(e^{\sqrt{g/L}\,t} + e^{-\sqrt{g/L}\,t})$$

is a solution to the equation of motion [by substitution into (*b*)] and discuss the solution.

COMPUTER PROBLEM

1. A medium-sized rocket has a mass of 4000 kg when empty and can hold 27,000 kg of fuel and oxidizer. The engine burns fuel at a rate of 230 kg/s and the speed of the exhaust gas is a constant 2,500 m/s. Assume the rocket is launched vertically and that there is minimal air friction. Numerically solve for the trajectory of the rocket. Include effects caused by the variation of free-fall acceleration with altitude,

$$g = (9.8\ \text{m/s}^2)\left(\frac{R_E}{R_E + y}\right)^2,$$

where y is the altitude of the rocket above the ground and R_E is the radius of the Earth. (*a*) At what altitude does burn-out occur? (*b*) What is the speed of the rocket at this point? (*c*) What will be the highest point of the trajectory?

CHAPTER 8

ROTATIONAL KINEMATICS

Up to this point, we have studied only the transla-tional *motion of objects. We considered both solid rigid bodies (such as a tossed baton) and systems in which parts of the body are in relative motion (such as a cannon and its ejected cannonball).*

The most general motion of a rigid body includes rotational *as well as translational motions. In this chapter we begin to consider this general motion. We start by describing the rotation with appropriate variables and relating them to one another; this is* rotational kinematics *and is the subject of this chapter. Relating rotational motion to the interaction of an object with its environment (rotational dynamics) is discussed in the next two chapters.*

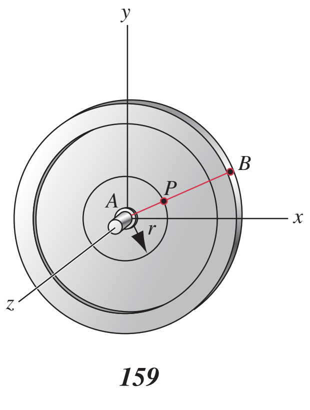
8-1 ROTATIONAL MOTION

Figure 8-1 shows a fixed exercise bicycle. The axle of the spinning front wheel is fixed in space; let it define the z axis of our coordinate system. An arbitrary point P on the wheel is a perpendicular distance r from point A on the z axis. The line AB is drawn through P from A. The motion of point P traces out the arc of a circle as the wheel turns. It does not necessarily do so at constant speed, because the rider might be changing the rate at which she is pedaling.

The motion of the wheel is an example of *pure rotation of a rigid body;* which we define as follows:

> *A rigid body moves in pure rotation if every point of the body (such as P in Fig. 8-1) moves in a circular path. The centers of these circles must lie on a common*

(*a*)



(*b*)

FIGURE 8-1. (*a*) The wheel of a fixed exercise bicycle is an example of the pure rotation of a rigid body. (*b*) The coordinates used to describe the rotation of the wheel. The axis of rotation, which is perpendicular to the xy plane, is the z axis. An arbitrary point P at a distance r from the axis A moves in a circle of radius r.

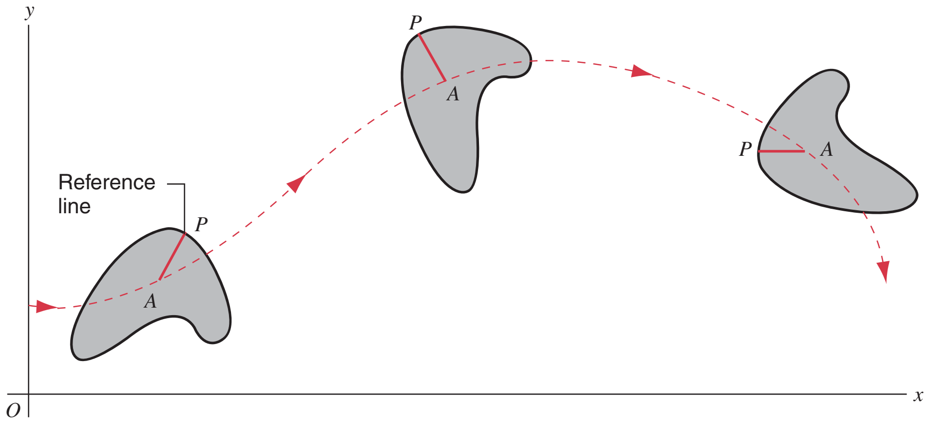


FIGURE 8-2. An arbitrary rigid body in both rotational and translational motion. In this special two-dimensional case, the translational motion is confined to the xy plane. The dashed line shows the path in the xy plane corresponding to the translational motion of the axis of rotation, which is parallel to the z axis through point A. The rotational motion is indicated by the line AP.

straight line called the axis of rotation (the z axis of Fig. 8-1).

We can also characterize the motion of the wheel by the reference line AB in Fig. 8-1. As the wheel rotates, the line AB moves through a certain angle in the xy plane. Another way to define a pure rotation is the following:

A rigid body moves in pure rotation if a reference line perpendicular to the axis (such as AB in Fig. 8-1) moves through the same angle in a given time interval as any other reference line perpendicular to the axis in the body.

In the case of an ordinary bicycle wheel, the line AB might represent one of the (assumed radial) spokes of the wheel. The above definition thus means that, for a wheel in pure rotation, if one spoke turns through a certain angle $\Delta\phi$ in a time interval Δt, then any other spoke must also turn through $\Delta\phi$ during that same interval.

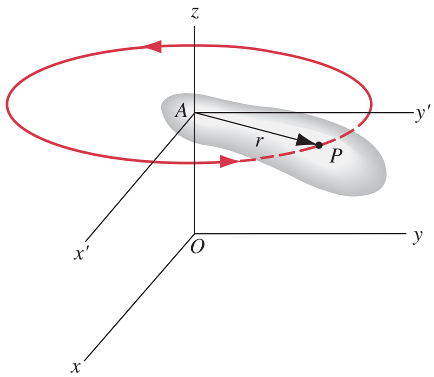
The general motion of a rigid object will include both rotational and translational components, as, for example, in the case of a wheel on a *moving* bicycle. A point P on such a wheel moves in a circle according to an observer in the same reference frame as the wheel (the bicycle rider, for instance), but another observer fixed to the ground would describe the motion differently. In even more complex cases, such as a wobbling football in flight, we may have a combination of translational motion, rotational motion about an axis, *and* a variation in the direction of the axis. In general, the three-dimensional description of a rigid body requires six coordinates: three to locate the center of mass, two angles (such as latitude and longitude) to orient the axis of rotation, and one angle to describe rotations about the axis. Figure 8-2 shows a two-dimensional arbitrary rigid body undergoing both rotational and translational motion. In this case only three coordinates are needed: two for the center of mass and one for the angular coordinate of a reference line in the body.

In this chapter only pure rotational motion is considered. (In the next chapter the more complicated case of combined rotation and translation is discussed.) We consider only rigid objects, in which there is no relative motion of the parts as the object rotates; a liquid in a spinning container, for instance, is excluded.

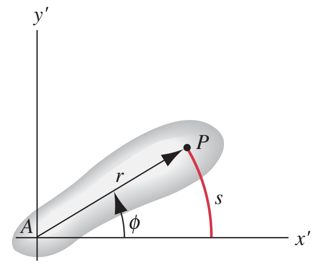
8-2 THE ROTATIONAL VARIABLES

Figure 8-3a shows a body of arbitrary shape rotating about the z axis. We can tell exactly where the entire rotating body is in our reference frame if we know the location of any single point P of the body in this frame. Thus, for the kinematics of this problem, we need consider only the (two-dimensional) motion of a point in a circle of radius r equal to the perpendicular distance from P to the point A on the z axis. Figure 8-3b shows a slice through the body parallel to the xy plane that includes the point P.

The angle ϕ in Fig. 8-3b is the angular position of the reference line AP with respect to the x' axis. *We arbitrarily*



(*a*)



(*b*)

FIGURE 8-3. (*a*) An arbitrary rigid body rotating about the z axis. (*b*) A cross-sectional slice through the body. The x' and y' axes are parallel to the x and y axes, respectively, but pass through point A. The reference line AP, which connects a point P of the body to the axis, is instantaneously located at an angle ϕ with respect to the x' axis. The point P moves through an arc length s as the line AP rotates through the angle ϕ.

choose the positive sense of the rotation to be counterclockwise, so that (in Fig. 8-3*b*) ϕ increases for counterclockwise rotation and decreases for clockwise rotation, according to an observer who is farther along the positive z axis than the rotating object.

As the body rotates, the point P moves through an arc of length s, as shown in Fig. 8-3*b*. The arc length and the radius (the distance from P to the axis of rotation) determine the angle through which the reference line rotates:

$$\phi = s/r. \tag{8-1}$$

Angles defined in this way are given in units of *radians* (rad). The angle ϕ, being the ratio of two lengths, is a pure number and has no dimensions. It does, however, have units (radians, in this case). When the arc length is numerically equal to r, then $\phi = r/r = 1$ radian. We can often treat radians as "unity" in equations, introducing the unit when necessary. Certainly not all equations that are the ratio of two lengths (like Eq. 8-1) will be measures of angles and require radian measure!

Because the circumference of a circle of radius r is $2\pi r$, it follows from Eq. 8-1 that a particle that moves in an arc length of one circumference must trace out an angle of 2π rad. Thus

$$1 \text{ revolution} = 2\pi \text{ radians} = 360°,$$

or

$$1 \text{ radian} = 57.3° = 0.159 \text{ revolution}.$$

Often we can express angles and associated rotational quantities in units based on either radians, degrees, or revolutions. However, when an equation mixes angular and linear quantities, such as Eq. 8-1, the angular variables must *always* be expressed in radians.

Let the body of Fig. 8-3*b* rotate counterclockwise. At time t_1 the angular position of the line AP is ϕ_1, and at a later time t_2 its angular position is ϕ_2. This is shown in Fig. 8-4, which gives the positions of P and of the reference line at these times; the outline of the body itself has been omitted for simplicity.

The *angular displacement* of P is $\phi_2 - \phi_1 = \Delta\phi$ during the time interval $t_2 - t_1 = \Delta t$. We define the *average angular velocity* ω_{av} of particle P in this time interval as

$$\omega_{av} = \frac{\phi_2 - \phi_1}{t_2 - t_1} = \frac{\Delta\phi}{\Delta t}. \tag{8-2}$$

The *instantaneous angular velocity* ω is the limit approached by this ratio as Δt approaches zero:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\phi}{\Delta t}$$

or

$$\omega = \frac{d\phi}{dt}. \tag{8-3}$$

Angular velocity can be positive or negative, according to whether ϕ is increasing or decreasing. (Later we will show that ω is a vector quantity, which can have positive or negative components relative to a particular axis, just like the translational velocity $\vec{\mathbf{v}}$.) As in the case of translational motion, when we use the term "angular velocity," we mean the *instantaneous* angular velocity. When we refer to *angular speed,* we mean the magnitude of the angular velocity.

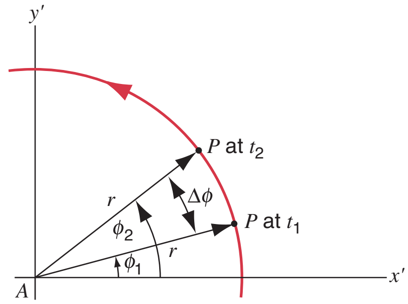


Figure 8-4. The reference line AP of Fig. 8-3*b* is at the angular coordinate ϕ_1 at time t_1 and at the angular coordinate ϕ_2 at time t_2. In the time interval $\Delta t = t_2 - t_1$, the net angular displacement is $\Delta\phi = \phi_2 - \phi_1$.

Angular velocity has the dimensions of inverse time (T^{-1}); its units may be radians per second (rad/s) or revolutions per second (rev/s).

For a rigid body in pure rotation, the angular velocity is the same for every point of the body. All lines like AP in Fig. 8-3, which are fixed in the body and run perpendicular to the axis of rotation to any points of the body, rotate at the same angular velocity.

If the angular velocity of P is not constant, then the point has an angular acceleration. Let ω_1 and ω_2 be the instantaneous angular velocities at the times t_1 and t_2, respectively; then the *average angular acceleration* α_{av} of the point P is defined as

$$\alpha_{av} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}. \tag{8-4}$$

The *instantaneous angular acceleration* is the limit of this ratio as Δt approaches zero:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\phi}{\Delta t}$$

or

$$\alpha = \frac{d\omega}{dt}. \tag{8-5}$$

Angular acceleration can be positive or negative, depending on whether the angular velocity is increasing or decreasing. When we refer to "angular acceleration," we mean the *instantaneous* angular acceleration. Its dimensions are inverse time squared (T^{-2}), and its units might be rad/s^2 or rev/s^2. Because ω is the same for each point of a rigid body, it follows from Eq. 8-5 that α must also be the same for each point and so α is, like ω, characteristic of the body as a whole.

Instead of the rotation of a rigid body, we could have considered the motion of a single particle in a circular path. That is, P in Fig. 8-4 can represent a particle of mass m, constrained to move in a circle of radius r (perhaps held by a rigid massless rod of length r pivoted on the z axis). All the results derived in this section are valid whether we regard P as a mathematical point or as a physical particle; we could, for example, refer to the angular velocity or angular acceleration of the *particle* P as it rotates about the z axis. Later, we shall find it useful to regard the rotating rigid body of Fig. 8-3 as a collection of particles, each of which is rotating about the z axis with the same angular velocity and angular acceleration.

SAMPLE PROBLEM 8-1. A fan blade is initially rotating an angular speed of 48.6 rpm (revolutions per minute). It slows down and eventually comes to rest in a time of 32 seconds after turning through a total of 8.8 revolutions. Find (*a*) the average angular velocity and (*b*) the average angular acceleration of the fan blade.

Solution (*a*) As the fan comes to rest, the net displacement $\Delta\phi$ is 8.8 revolutions in a time $\Delta t = 32$ s. The average angular velocity is found from Eq. 8-2:

$$\omega_{av} = \frac{\Delta\phi}{\Delta t} = \frac{8.8 \text{ rev}}{32 \text{ s}} = 0.28 \text{ rev/s}.$$

(*b*) The initial angular velocity is $\omega_i = 48.6$ rev/min $= 0.81$ rev/s. The final angular velocity ω_f is 0. The average angular acceleration is given by Eq. 8-4:

$$\alpha_{av} = \frac{\Delta\omega}{\Delta t} = \frac{0 - 0.81 \text{ rev/s}}{32 \text{ s}} = -0.025 \text{ rev/s}^2.$$

In this problem it is quite acceptable to express the angular quantities in revolutions. However, as we shall see, in equations that mix angular and linear variables (such as Eq. 8-1), the angular variables must *always* be expressed in radians.

The positive angular velocity and negative angular acceleration suggest, in analogy with translational kinematics, that the fan is slowing down. In the case of translational kinematics, it was necessary to define a direction for the coordinate system to give meaning to positive and negative quantities. In the next section we will show that rotational variables behave like vectors and that likewise defining a coordinate system allows us to associate positive and negative values with directions of rotation in the coordinate system.

SAMPLE PROBLEM 8-2. A wheel with a fixed axle (such as on the exercise bicycle of Fig. 8-1) is rotating so that the instantaneous angular velocity of a reference line painted along a radius is given as a function of the time by $\omega = At + Bt^2$, where $A = 6.2$ rad/s^2 and $B = 8.7$ rad/s^3. (*a*) If the reference line is initially at $\phi = 0$ when $t = 0$, find its angular position when $t = 2.0$ s. (*b*) What is the instantaneous angular acceleration of the reference line at $t = 0.50$ s?

Solution (*a*) To obtain ϕ from ω, we must carry out an integral. Writing Eq. 8-3 as $d\phi = \omega\, dt$, we can integrate to find $\phi = \int d\phi = \int \omega\, dt$, or

$$\phi = \int (At + Bt^2)\, dt = \tfrac{1}{2}At^2 + \tfrac{1}{3}Bt^3 + C,$$

where C is a constant of integration that must be determined from the initial condition. In order to have $\phi = 0$ when $t = 0$, we must have $C = 0$. Evaluating the resulting expression at $t = 2.0$ s, we obtain

$$\phi = \tfrac{1}{2}(6.2 \text{ rad/s}^2)(2.0 \text{ s})^2 + \tfrac{1}{3}(8.7 \text{ rad/s}^3)(2.0 \text{ s})^3 = 35.6 \text{ rad}.$$

The wheel rotates through 35.6 radians or 5.7 revolutions in 2.0 s.
(*b*) To obtain the angular acceleration from the angular velocity, we must find the derivative as specified by Eq. 8-5:

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(At + Bt^2) = A + 2Bt.$$

Evaluating this expression at $t = 0.50$ s gives $\alpha = 6.2$ rad/s^2 + 2(8.7 rad/s^3)(0.50 s) = 14.9 rad/s^2.

8-3 ROTATIONAL QUANTITIES AS VECTORS

When we deal with displacement, velocity, and acceleration in translational motion, whether in one dimension or more than one, our first step is always to set up a coordinate system and to specify the positive direction for each of the axes. Only in this way can we define what it means for a displacement, velocity, or acceleration component to be positive or negative. This step is necessary because these quantities are represented by vectors. Other quantities, such as mass or temperature, carry no directional information; they are scalars and their values are independent of any choice of coordinate system.

So now we must ask whether the variables of angular kinematics (angular displacement, angular velocity, angular acceleration) also behave like vectors. If so, then we must specify a coordinate system and define the variables with respect to that system. To be represented as a vector, a physical quantity must not only have magnitude and direction; it must also obey the laws of vector addition. Only through experiment can we learn whether the angular variables obey these laws.

Let us begin with the angular displacement $\Delta\phi$, which specifies the angle through which a body rotates. One particular law that vectors must obey is the commutative addition law: for two arbitrary vectors, we must have $\vec{\mathbf{A}} + \vec{\mathbf{B}} = \vec{\mathbf{B}} + \vec{\mathbf{A}}$; that is, the order of the vectors does not affect their sum. Let us examine this law for angular displacements. We shall apply two successive rotations $\Delta\phi_1$ and $\Delta\phi_2$ to an object, such as the book illustrated in Fig. 8-5, which initially lies in the yz plane. As shown in Fig. 8-5*a*, we first rotate by $\Delta\phi_1$, a 90° turn about the x axis, followed by $\Delta\phi_2$, a 90° turn about the z axis. In Fig. 8-5*b* we show the situation if the order of the two rotations is reversed: first $\Delta\phi_2$ (90° about the z axis) and then $\Delta\phi_1$ (90° about the x axis). As you can see, the final positions of the book are very different. Thus we conclude in this case that $\Delta\phi_1 + \Delta\phi_2 \neq \Delta\phi_2 + \Delta\phi_1$, and so *finite angular displacements cannot be represented as vector quantities.*

The situation changes as the angular displacements are made smaller. Figures 8-5*c*, *d* show the effect of successive

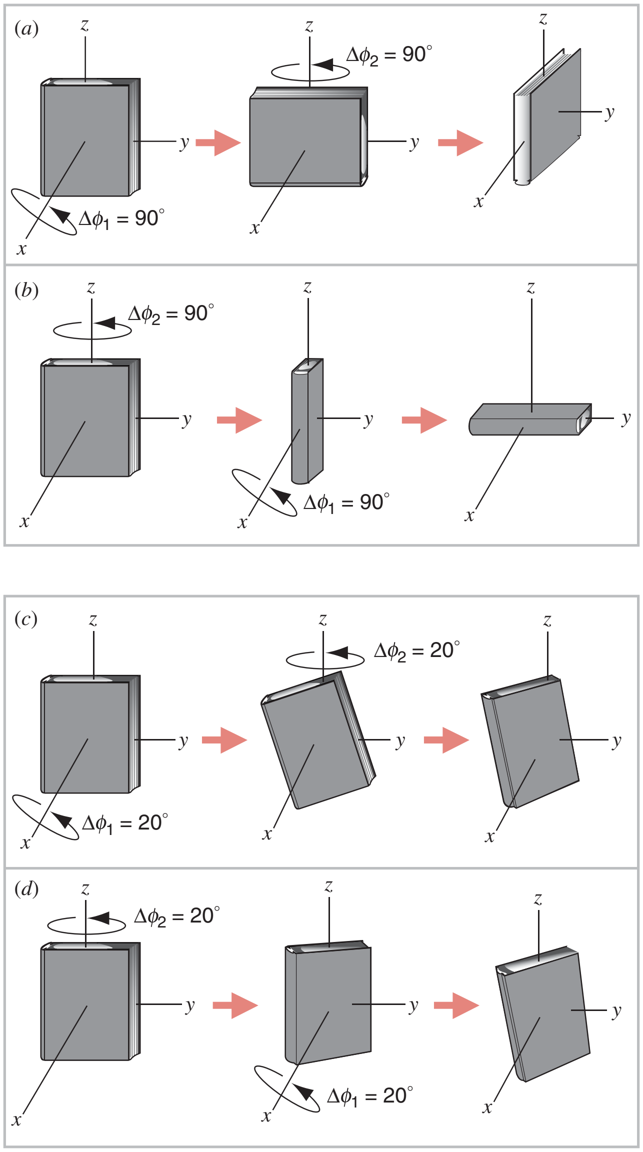


Figure 8-5. (*a*) The book is given two successive rotations: $\Delta\phi_1 = 90°$ about the x axis and $\Delta\phi_2 = 90°$ about the z axis. (*b*) If the order of the rotations is reversed, the final position of the book is different. (*c*) Now the book is rotated as in (*a*) but by two smaller angles: $\Delta\phi_1 = 20°$ about the x axis and $\Delta\phi_2 = 20°$ about the z axis. (*d*) If the order of the rotations in (*c*) is reversed, the final position more closely resembles that of (*c*).

20° rotations, and now the two final positions of the book are more nearly the same. The smaller we make the rotation angle, the more similar the final positions become. If the angular displacements are made infinitesimal, the positions are identical and the order of the rotations no longer affects the final outcome; that is, $d\phi_1 + d\phi_2 = d\phi_2 + d\phi_1$. Hence *infinitesimal angular rotations can be represented as vectors.*

Quantities defined in terms of infinitesimal angular displacements may also be vectors. For example, the angular velocity is $\vec{\omega} = d\vec{\phi}/dt$. Since $d\vec{\phi}$ is a vector and dt is a scalar, the quotient $\vec{\omega}$ is a vector. *Angular velocity can therefore be represented as a vector.* In Fig. 8-6*a*, for example, we represent the angular velocity $\vec{\omega}$ of the rotating rigid body by an arrow drawn along the axis of rotation; in Fig. 8-6*b* we represent the rotation of a particle P about a fixed axis in just the same way. The length of the arrow is made proportional to the magnitude of the angular velocity. The sense of the rotation determines the direction in which the arrow points along the axis. By convention, if the fingers of the *right hand* curl around the axis in the direction of rotation of the body, the extended thumb points along the direction of the angular velocity vector (Fig 8-6*c*). For the wheel of Fig. 8-1, therefore, the angular velocity vector

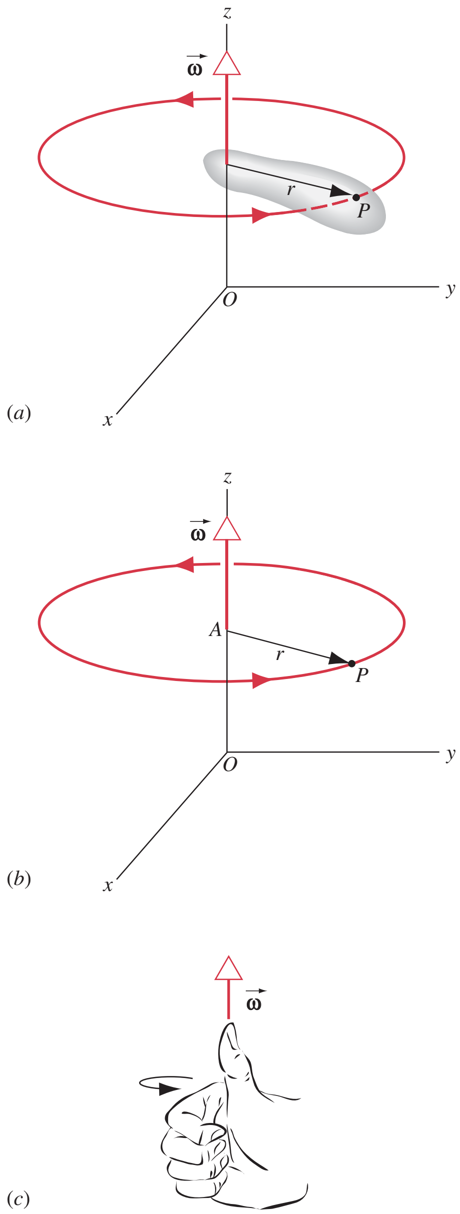


Figure 8-6. The angular velocity vector of (*a*) a rotating rigid body and (*b*) a rotating particle, both taken about a fixed axis. (*c*) The right-hand rule determines the direction of the angular velocity vector.

points perpendicularly into the page (in the negative z direction) if the rider is pedaling forward. In Fig. 8-3b, $\vec{\boldsymbol{\omega}}$ is perpendicular to the page pointing up out of the page, corresponding to the counterclockwise rotation. Note that the object does not move in the direction of the angular velocity vector. The vector represents the angular velocity of the rotational motion taking place in a plane perpendicular to it.

Angular acceleration is also a vector quantity. This follows from the definition $\vec{\boldsymbol{\alpha}} = d\vec{\boldsymbol{\omega}}/dt$, in which $d\vec{\boldsymbol{\omega}}$ is a vector and dt a scalar. Later we shall encounter other rotational quantities that are vectors, such as torque and angular momentum. The use of the right-hand rule to define the direction of the vectors $d\vec{\boldsymbol{\phi}}$, $\vec{\boldsymbol{\omega}}$, and $\vec{\boldsymbol{\alpha}}$ leads to a consistent vector formalism for all rotational quantities.

8-4 ROTATION WITH CONSTANT ANGULAR ACCELERATION

For translational motion of a particle or a rigid body along a fixed direction, such as the x axis, we have seen (in Chapter 2) that the simplest type of motion is that in which the acceleration a_x is zero. The next simplest type corresponds to a_x = a constant (other than zero); for this motion we derived Eqs. 2-26 and 2-28, which describe the velocity and the position as functions of the time.

For the rotational motion of a particle or a rigid body around a fixed axis (which we take to be the z axis), the simplest type of motion is that in which the angular acceleration α_z is zero (such as uniform circular motion). The next simplest type of motion, in which α_z = a constant (other than zero), corresponds exactly to translational motion with a_x = a constant (other than zero). As before, we can derive equations that give the angular velocity ω and angular displacement ϕ as functions of the time t. These angular equations can be derived, using methods we used to derive the translational equations, or they can simply be written down, by substituting angular quantities for the corresponding quantities in the translational equations.

We first derive the expression for ω_z as a function of t. We begin by rewriting Eq. 8-5 as

$$d\omega_z = \alpha_z dt.$$

We now integrate on the left from ω_{0z} (the angular velocity at time $t = 0$) to ω_z (the angular velocity at time t), and on the right from time 0 to time t:

$$\int_{\omega_{0z}}^{\omega_z} d\omega_z = \int_0^t \alpha_z dt = \alpha_z \int_0^t dt,$$

where the last step can be taken *only* when the angular acceleration α_z is constant. Carrying out the integration, we obtain

$$\omega_z - \omega_{0z} = \alpha_z t$$

or

$$\omega_z = \omega_{0z} + \alpha_z t. \qquad (8\text{-}6)$$

This is the rotational analogue of Eq. 2-26, $v_x = v_{0x} + a_x t$. Note that we could obtain the rotational expression by substituting ω_z for v_x and α_z for a_x in the translational expression.

Setting $\omega_z = d\phi/dt$ in Eq. 8-6 and integrating again, we obtain an expression for the angular displacement in the case of constant angular acceleration:

$$\int_{\phi_0}^{\phi} d\phi = \int_0^t (\omega_{0z} + \alpha_z t)\, dt,$$

or

$$\phi = \phi_0 + \omega_{0z} t + \tfrac{1}{2}\alpha_z t^2, \qquad (8\text{-}7)$$

which is similar to the corresponding result for translational motion with constant acceleration, Eq. 2-28, $x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$.

The positive sense of the angular quantities ω_z and α_z is determined by the direction in which ϕ is increasing. From Eq. 8-3 we see that ω_z is positive if ϕ is increasing with time (that is, the object is rotating in a counterclockwise direction). Similarly, from Eq. 8-5, we see that α_z is positive if ω_z is increasing with time, even if ω_z is negative and becoming less negative. These are similar to the corresponding sign conventions for the linear quantities. You can see from Fig. 8-3b that this association of positive ω_z with increasing ϕ is consistent with the use of the right-hand rule: if the fingers of the right hand curl in the direction of increasing ϕ, then the thumb is pointing out of the page—that is, in the positive z direction—indicating that ω_z is positive.

The rotation of a particle (or a rigid body) *about a fixed axis* has a formal correspondence to the translational motion of a particle (or a rigid body) *along a fixed direction.* The kinematical variables are ϕ, ω_z, and α_z in the first case and x, v_x, and a_x in the second. These quantities correspond in pairs: ϕ to x, ω_z to v_x, and α_z to a_x. Note that the angular quantities differ dimensionally from the corresponding linear quantities by a length factor. Note, too, that all six quantities may be treated as components of one-dimensional vectors in this special case. For example, a particle at any instant can be moving in one direction or the other along its straight-line path, corresponding to a positive or a negative value for v_x; similarly, a particle at any instant can be rotating in one direction or another about its fixed axis, corresponding to a positive or a negative value for ω_z.

When, in translational motion, we remove the restriction that the motion be along a straight line and consider the general case of motion in three dimensions along a curved path, the components x, v_x, and a_x must be replaced by the vectors $\vec{\mathbf{r}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{a}}$. In Section 8-5, we see to what extent the rotational kinematic variables reveal themselves as vectors when we remove the restriction of a fixed axis of rotation.

SAMPLE PROBLEM 8-3. Starting from rest at time $t = 0$, a grindstone has a constant angular acceleration of 3.2 rad/s^2. At $t = 0$ the reference line AB in Fig. 8-7 is horizontal. Find (a) the

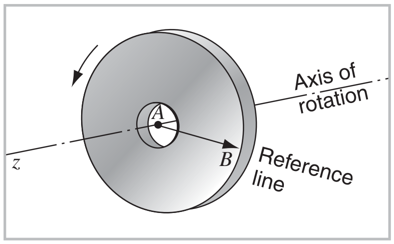


FIGURE 8-7. Sample Problem 8-3. The reference line AB is horizontal at $t = 0$ and rotates with the grindstone in the xy plane.

angular displacement of the line AB (and hence of the grindstone) and (*b*) the angular speed of the grindstone 2.7 s later.

Solution (*a*) We chose a coordinate system so that $\vec{\omega}$ is along the positive z direction (so the grindstone and the line AB rotate in the xy plane).

At $t = 0$, we have $\phi_0 = 0$, $\omega_{0z} = 0$, and $\alpha_z = 3.2 \text{ rad/s}^2$. Therefore, after 2.7 s, Eq. 8-7 gives

$$\begin{aligned}\phi &= \phi_0 + \omega_{0z}t + \tfrac{1}{2}\alpha_z t^2\\ &= 0 + (0)(2.7 \text{ s}) + \tfrac{1}{2}(3.2 \text{ rad/s}^2)(2.7 \text{ s})^2\\ &= 11.7 \text{ rad} = 1.9 \text{ rev}.\end{aligned}$$

(*b*) From Eq. 8-6,

$$\begin{aligned}\omega_z &= \omega_{0z} + \alpha_z t = 0 + (3.2 \text{ rad/s}^2)(2.7 \text{ s})\\ &= 8.6 \text{ rad/s} = 1.4 \text{ rev/s}.\end{aligned}$$

SAMPLE PROBLEM 8-4. Suppose the power driving the grindstone of Sample Problem 8-3 is turned off when it is spinning with an angular speed of 8.6 rad/s. A small frictional force on the shaft causes a constant angular deceleration, and the grindstone eventually comes to rest in a time of 192 s. Find (*a*) the angular acceleration and (*b*) the total angle turned through during the slowing down.

Solution (*a*) Given $\omega_{0z} = 8.6$ rad/s, $\omega_z = 0$, and $t = 192$ s, we find α_z from Eq. 8-6:

$$\alpha_z = \frac{\omega_z - \omega_{0z}}{t} = \frac{0 - 8.6 \text{ rad/s}}{192 \text{ s}} = -0.045 \text{ rad/s}^2.$$

Here the negative value of α_z shows that ω_z (which is positive) is decreasing in magnitude.

(*b*) From Eq. 8-7 we have

$$\begin{aligned}\phi &= \phi_0 + \omega_{0z}t + \tfrac{1}{2}\alpha_z t^2\\ &= 0 + (8.6 \text{ rad/s})(192 \text{ s}) + \tfrac{1}{2}(-0.045 \text{ rad/s}^2)(192 \text{ s})^2\\ &= 822 \text{ rad} = 131 \text{ rev}.\end{aligned}$$

8-5 RELATIONSHIPS BETWEEN LINEAR AND ANGULAR VARIABLES

In Section 4-5 we discussed the linear velocity and acceleration of a particle moving in a circle. When a rigid body rotates about a fixed axis, every particle in the body moves in a circle. Hence we can describe the motion of such a particle either in linear variables or in angular variables. The relationship between the linear and angular variables enables us to pass back and forth from one description to another and is very useful.

Consider a particle at P in the rigid body, a perpendicular distance r from the axis through A, as in Fig. 8-3*a*. This particle moves in a circle of radius r. The angular position ϕ of the reference line AP is measured with respect to the x or x' axis, as in Fig. 8-3*b*. The particle moves through a distance s along the arc when the body rotates through an angle ϕ, such that

$$s = \phi r, \tag{8-8}$$

where ϕ is in radians.

Differentiating both sides of this equation with respect to the time, and noting that r is constant, we obtain

$$\frac{ds}{dt} = \frac{d\phi}{dt}r.$$

However, ds/dt is the (tangential) linear speed v_T of the particle at P and $d\phi/dt$ is the angular speed ω of the rotating body, so that

$$v_T = \omega r. \tag{8-9}$$

This is a relation between the *magnitudes* of the tangential linear velocity and the angular velocity; the linear speed of a particle in circular motion is the product of the angular speed and the distance r of the particle from the axis of rotation.

Differentiating Eq. 8-9 with respect to the time, we have

$$\frac{dv_T}{dt} = \frac{d\omega}{dt}r.$$

However, dv_T/dt is the magnitude of the *tangential* component a_T of the acceleration of the particle (see Section 8-6), and $d\omega/dt$ is the magnitude of the angular acceleration of the rotating body, so that

$$a_T = \alpha r. \tag{8-10}$$

Hence the magnitude of the tangential component of the linear acceleration of a particle in circular motion is the product of the magnitude of the angular acceleration and the distance r of the particle from the axis of rotation.

We have seen in Section 4-5 that the *radial* (or centripetal) component a_R of the acceleration is v_T^2/r for a particle moving in a circle. This can be expressed in terms of angular speed by use of Eq. 8-9. We have

$$a_R = \frac{v_T^2}{r} = \omega^2 r. \tag{8-11}$$

The resultant acceleration $\vec{\mathbf{a}}$ of point P is shown in Fig. 8-8.

Equations 8-8 through 8-11 enable us to describe the motion of one point on a rigid body rotating about a fixed axis *either* in angular variables *or* in linear variables. We might ask why we need the angular variables when we are already familiar with the equivalent linear variables. The answer is that the angular description offers a distinct

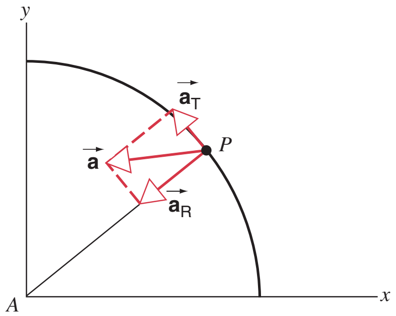


FIGURE 8-8. The radial and tangential components of the acceleration of a particle at point P of the rigid body rotating about the z axis.

advantage over the linear description when various points on the same rotating body must be considered. On a rotating body, points that are at different distances from the axis do not have the same *linear* displacement, speed, or acceleration, but *all* points on a rigid body rotating about a fixed axis do have the same *angular* displacement, speed, or acceleration at any instant. By the use of angular variables we can describe the motion of the entire body in a simple way.

Figure 8-9 shows an interesting example of the relation between linear and angular variables. When a tall chimney is toppled by an explosive charge at its base, it will often break as it falls, the rupture starting on the downward side of the falling chimney.

Before rupture, the chimney is a rigid body, rotating about an axis near its base with a certain angular acceleration α. According to Eq. 8-10, the top of the chimney has a tangential acceleration a_T given by αL, where L is the length of the chimney. The vertical component of a_T can easily exceed g, the acceleration of free fall. That is, the top of the chimney is falling downward with a vertical acceleration greater than that of a freely falling brick.

This can happen only as long as the chimney remains a single rigid body. Put simply, the bottom part of the chimney, acting through the mortar that holds the bricks together, must "pull down" on the top part of the chimney to cause it to fall so fast. This shearing force is often more than the mortar can tolerate, and the chimney breaks, The chimney has now become two rigid bodies, its top part being in free fall and reaching the ground later than it would if the chimney had not broken.

FIGURE 8-9. A falling chimney often is not strong enough to provide the tangential acceleration at large radius that is necessary if the entire object is to rotate like a rigid body with constant angular acceleration. See "More on the Falling Chimney," by A. A. Bartlett, *The Physics Teacher,* September 1976, p. 351, for an account of this phenomenon.

SAMPLE PROBLEM 8-5. If the radius of the grindstone of Sample Problem 8-3 is 0.24 m, calculate (*a*) the linear or tangential speed of a point on the rim, (*b*) the tangential acceleration of a point on the rim, and (*c*) the radial acceleration of a point on the rim, at the end of 2.7 s. (*d*) Repeat for a point halfway in from the rim—that is, at $r = 0.12$ m.

Solution We have $\alpha = 3.2\ \text{rad/s}^2$, $\omega = 8.6$ rad/s after 2.7 s, and $r = 0.24$ m. Then,

(*a*) $v_T = \omega r = (8.6\ \text{rad/s})(0.24\ \text{m}) = 2.1\ \text{m/s}$,

(*b*) $a_T = \alpha r = (3.2\ \text{rad/s}^2)(0.24\ \text{m}) = 0.77\ \text{m/s}^2$,

(*c*) $a_R = \omega^2 r = (8.6\ \text{rad/s})^2(0.24\ \text{m}) = 18\ \text{m/s}^2$.

(*d*) The *angular* variables are the same for this point at $r = 0.12$ m as for a point on the rim. That is, once again $\alpha = 3.2\ \text{rad/s}^2$ and $\omega = 8.6$ rad/s. Using Eqs. 8-9 to 8-11, with $r = 0.12$ m, we obtain for this point

$$v_T = 1.0\ \text{m/s}, \qquad a_T = 0.38\ \text{m/s}^2, \qquad a_R = 8.9\ \text{m/s}^2.$$

These are each half of their respective values for the point on the rim. The linear variables scale in proportion to the radius from the axis of rotation.

Note once again that, in equations that involve *only* angular variables, you may express the angular quantities in any angular unit (degrees, radians, revolutions), as long as you do so consistently. However, in equations in which angular and linear quantities are mixed, such as Eqs. 8-8 to 8-11, you *must* express the angular quantities in radians, as we have done in this sample problem. We must do so because Eqs. 8-9 to 8-11 were based on Eq. 8-8, which in effect defines radian measure.

SAMPLE PROBLEM 8.6. A pulsar is a rapidly rotating neutron star, the result of the gravitational collapse of an ordinary star that has used up its fuel supply. Pulsars emit light or other electromagnetic radiation in a narrow beam, which can sweep by the Earth once in each revolution. A certain pulsar has a rotational period of $T = 0.033$ s and a radius of $r = 15$ km. What is the tangential speed of a point on its equator?

Solution The angular speed is

$$\omega = \frac{2\pi\ \text{radians}}{T} = \frac{2\pi\ \text{rad}}{0.033\ \text{s}} = 190\ \text{rad/s}$$

and the tangential speed is

$$v_T = \omega r = (190\ \text{rad/s})(15\ \text{km}) = 2900\ \text{km/s}.$$

It is interesting to note that this is about 1% of the speed of light and also nearly 4 orders of magnitude larger than the tangential speed of a point at the Earth's equator.

8-6 VECTOR RELATIONSHIPS BETWEEN LINEAR AND ANGULAR VARIABLES (Optional)

In the previous section we developed relationships between the angular velocity ω, tangential velocity v_T, angular acceleration α, tangential acceleration a_T, and radial (or centripetal) acceleration a_R. All of these quantities are represented by vectors, and we now examine their relationships in vector form.

We have often found it helpful to express vectors in terms of their components by using unit vectors. In rectangular (Cartesian) coordinates, the unit vectors are $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$, which respectively identify the x, y, z directions (see Appendix H). In analyzing rotational motion, it is more useful to use new unit vectors that identify the radial and tangential directions. We assume that the rotation is described by Fig. 8-3, and we focus our attention on an arbitrary particle of the rotating body. This particle, at point P, moves in a circular path parallel to the xy plane; that is, its tangential velocity has only x and y components. (Its angular velocity, as we have seen, points in the z direction.)

Figure 8-10a shows the rotating particle and the radial and tangential unit vectors, which we call $\hat{\mathbf{u}}_r$ and $\hat{\mathbf{u}}_\phi$. The radial unit vector $\hat{\mathbf{u}}_r$ points in the direction of increasing r—that is, radially outward from the center of the circle. The tangential unit vector $\hat{\mathbf{u}}_\phi$ points in the direction of increasing ϕ—tangent to the circle and in the counterclockwise direction. Like $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, the unit vectors $\hat{\mathbf{u}}_r$ and $\hat{\mathbf{u}}_\phi$ are dimensionless, have unit length, and are perpendicular to one another. Unlike $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, the directions of $\hat{\mathbf{u}}_r$ and $\hat{\mathbf{u}}_\phi$ change as the particle moves around the circle. When we take derivatives of expressions involving $\hat{\mathbf{u}}_r$ and $\hat{\mathbf{u}}_\phi$, we must take this change of direction into account. The unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, on the other hand, can be treated as constants for differentiation.

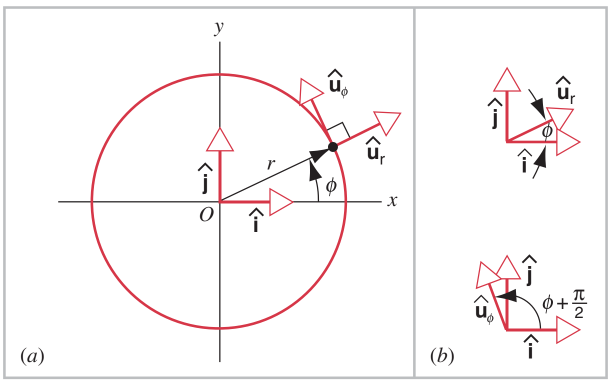


FIGURE 8-10. (a) A particle moving counterclockwise in a circle or radius r. (b) The unit vectors $\hat{\mathbf{u}}_r$ and $\hat{\mathbf{u}}_\phi$ and their relation to $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.

Using Fig. 8-10b we can express $\hat{\mathbf{u}}_r$ and $\hat{\mathbf{u}}_\phi$ in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$:

$$\hat{\mathbf{u}}_r = (\cos\phi)\hat{\mathbf{i}} + (\sin\phi)\hat{\mathbf{j}}, \tag{8-12a}$$

$$\hat{\mathbf{u}}_\phi = (-\sin\phi)\hat{\mathbf{i}} + (\cos\phi)\hat{\mathbf{j}}. \tag{8-12b}$$

The velocity of the particle has only a tangential component (no radial component), and can thus be written in vector form as its magnitude times the unit vector in the tangential direction:

$$\vec{\mathbf{v}} = v_T\,\hat{\mathbf{u}}_\phi. \tag{8-13}$$

Because the rotating object may in general have an angular acceleration, the tangential velocity may change in magnitude as well as direction.

The acceleration $\vec{\mathbf{a}}$ of the particle can be found in the usual way as $d\vec{\mathbf{v}}/dt$:

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d(v_T\hat{\mathbf{u}}_\phi)}{dt} = \frac{dv_T}{dt}\hat{\mathbf{u}}_\phi + v_T\frac{d\hat{\mathbf{u}}_\phi}{dt}. \tag{8-14}$$

In the first term, the derivative dv_T/dt is just the tangential acceleration a_T. To evaluate the second term, we must find an expression for the derivative of the unit vector $\hat{\mathbf{u}}_\phi$. Using Eq. 8-12b we have

$$\frac{d\hat{\mathbf{u}}_\phi}{dt} = -\frac{d(\sin\phi)}{dt}\hat{\mathbf{i}} + \frac{d(\cos\phi)}{dt}\hat{\mathbf{j}}. \tag{8-15}$$

Now $d(\sin\phi)/dt = (\cos\phi)d\phi/dt = \omega\cos\phi$, where $\omega = d\phi/dt$ (Eq. 8-3). Similarly, $d(\cos\phi)/dt = -\omega\sin\phi$. Making these substitutions in Eq. 8-15 and removing common factors, we obtain

$$\frac{d\hat{\mathbf{u}}_\phi}{dt} = -\omega[(\cos\phi)\hat{\mathbf{i}} + (\sin\phi)\hat{\mathbf{j}}] = -\omega\hat{\mathbf{u}}_r, \tag{8-16}$$

where we have used Eq. 8-12a in the last step. We can now write Eq. 8-14 as

$$\vec{\mathbf{a}} = a_T\hat{\mathbf{u}}_\phi - v_T\omega\hat{\mathbf{u}}_r. \tag{8-17}$$

The first term in Eq. 8-17 is the tangential acceleration $\vec{\mathbf{a}}_T = a_T\hat{\mathbf{u}}_\phi$, a vector with magnitude a_T pointing in the tangential direction (the direction of increasing ϕ). We can write the second term in a more instructive form using Eq. 8-11: $-v_T\omega\hat{\mathbf{u}}_r = -v_T(v_T/r)\,\hat{\mathbf{u}}_r = -(v_T^2/r)\,\hat{\mathbf{u}}_r$. The quantity v_T^2/r is, according to Eq. 8-11, the radial (or centripetal) acceleration a_R. The radial acceleration can be represented in vector form as $\vec{\mathbf{a}}_R = -a_R\hat{\mathbf{u}}_r$, the minus sign indicating that this vector points in the direction of decreasing r—that is, toward the center of the circle. In terms of $\vec{\mathbf{a}}_T$ and $\vec{\mathbf{a}}_R$, Eq. 8-17 is

$$\vec{\mathbf{a}} = \vec{\mathbf{a}}_T + \vec{\mathbf{a}}_R. \tag{8-18}$$

These three acceleration vectors are shown in Fig. 8-8.

The Vectors $\vec{\omega}$ and $\vec{a}$

The spatial relationship between the angular vectors $\vec{\omega}$ and $\vec{\alpha}$ and the linear vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{a}}$ can be written in a compact form using the *vector cross product,* which is defined and discussed in Appendix H. The vector cross product of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is another vector $\vec{\mathbf{C}}$, which we write as $\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$. This vector $\vec{\mathbf{C}}$ has two properties that are important for our discussion: (1) The magnitude of $\vec{\mathbf{C}}$ is $AB \sin\theta$, where A is the magnitude of $\vec{\mathbf{A}}$, B is the magnitude of $\vec{\mathbf{B}}$, and θ is the angle between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. (2) The vector $\vec{\mathbf{C}}$ is perpendicular to the plane formed by $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ in a direction determined by the right-hand rule (see Appendix H).

Figure 8-11*a* shows the rotating particle and the vectors $\vec{\omega}$ and $\vec{\mathbf{v}}$ representing its angular and linear velocities. The vector $\vec{\mathbf{R}}$ locates the particle with respect to the origin of an *xyz* coordinate system. As shown in the figure, the particle moves in a circle of radius $r = R \sin\theta$.

Let's consider the vector cross product $\vec{\omega} \times \vec{\mathbf{R}}$. According to the definition of the vector cross product, the magnitude of this cross product is $\omega R \sin\theta = \omega r$, which equals the magnitude of the tangential velocity v_T, according to Eq. 8-9. Figure 8-11*a* shows that the direction of this cross product is the same as the direction of $\vec{\mathbf{v}}$: if we rotate the fingers of the right hand from $\vec{\omega}$ to $\vec{\mathbf{R}}$ through the angle θ, the thumb points in the direction of $\vec{\mathbf{v}}$. We have thus shown that the magnitude and direction of the cross product $\vec{\omega} \times \vec{\mathbf{R}}$ are identical to the magnitude and direction of $\vec{\mathbf{v}}$, and we can therefore write

$$\vec{\mathbf{v}} = \vec{\omega} \times \vec{\mathbf{R}}. \qquad (8\text{-}19)$$

This is the vector form of Eq. 8-9.

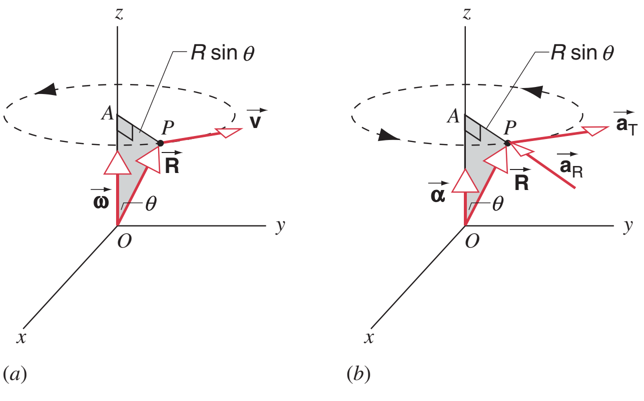


FIGURE 8-11. (*a*) A particle at P in the rotating rigid body of Fig. 8-3*a* is located at $\mathbf{R}$ with respect to the origin O. The particle has angular velocity $\vec{\omega}$ (directed along the z axis) and tangential velocity $\vec{\mathbf{v}}$. (*b*) The particle at P has angular acceleration $\vec{\alpha}$ along the z axis. The particle also has tangential acceleration $\vec{\mathbf{a}}_T$ and radial acceleration $\vec{\mathbf{a}}_R$.

We now evaluate the acceleration by taking the derivative of Eq. 8-19. In doing so we must be careful to preserve the order of the vectors $\vec{\omega}$ and $\vec{\mathbf{R}}$, because the order of the vectors in the cross product is important ($\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}}$). Using the usual method for taking the derivative of a product, we have

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{\mathbf{R}}) = \frac{d\vec{\omega}}{dt} \times \vec{\mathbf{R}} + \vec{\omega} \times \frac{d\vec{\mathbf{R}}}{dt}. \qquad (8\text{-}20)$$

Note that in both terms on the right side of Eq. 8-20, $\vec{\omega}$ comes before $\vec{\mathbf{R}}$, so we have correctly preserved the order of $\vec{\omega}$ and $\vec{\mathbf{R}}$.

Consider the first term on the right of Eq. 8-20. Just like the analogous linear equation $a = dv/dt$, the equation $\alpha = d\omega/dt$ (Eq. 8-5) holds for any component of $\vec{\alpha}$ and $\vec{\omega}$, and thus also for the vectors themselves: $\vec{\alpha} = d\vec{\omega}/dt$. In the last term of Eq. 8-20, $d\vec{\mathbf{R}}/dt$ is equal to the velocity $\vec{\mathbf{v}}$ of the particle. Making these substitutions in Eq. 8-20, we have

$$\vec{\mathbf{a}} = \vec{\alpha} \times \vec{\mathbf{R}} + \vec{\omega} \times \vec{\mathbf{v}}. \qquad (8\text{-}21)$$

According to the rule for finding the magnitude of a cross product, the magnitude of the first term $\vec{\alpha} \times \vec{\mathbf{R}}$ is $\alpha R \sin\theta = \alpha r$, which is just the tangential acceleration a_T according to Eq. 8-10. To find the direction of this vector product, we note that the expression $\vec{\alpha} = d\vec{\omega}/dt$ shows that $\vec{\alpha}$ must have the same direction as $d\vec{\omega}$. With a fixed axis of rotation, which we have assumed, $\vec{\omega}$ always points in the same direction (along the axis of rotation), so any change in $\vec{\omega}$ must also point along the axis. Thus $\vec{\alpha}$ has the same direction as $\vec{\omega}$—namely, along the z axis as shown in Fig. 8-11*b*. The right-hand rule for cross products shows that $\vec{\alpha} \times \vec{\mathbf{R}}$ is in the direction of the tangent to the circle at the location of P. Because $\vec{\alpha} \times \vec{\mathbf{R}}$ has the same magnitude and direction as the tangential acceleration $\vec{\mathbf{a}}_T$, we must have $\vec{\alpha} \times \vec{\mathbf{R}} = \vec{\mathbf{a}}_T$.

The magnitude of the vector cross product in the second term of Eq. 8-21 ($\vec{\omega} \times \vec{\mathbf{v}}$) is ωv_T, because the angle between these two vectors is 90° as shown in Fig. 8-11*a*. Using Eq. 8-9 we can write this as $\omega v_T = \omega^2 r$, which is the radial acceleration (Eq. 8-11). The right-hand rule for vector products (Fig. 8-11*a*) shows that $\vec{\omega} \times \vec{\mathbf{v}}$ points radially inward at P. The product $\vec{\omega} \times \vec{\mathbf{v}}$ has the magnitude and direction of the radial acceleration and so $\vec{\omega} \times \vec{\mathbf{v}} = \vec{\mathbf{a}}_R$. Making these two substitutions in Eq. 8-21 ($\vec{\alpha} \times \vec{\mathbf{R}} = \vec{\mathbf{a}}_T$ and $\vec{\omega} \times \vec{\mathbf{v}} = \vec{\mathbf{a}}_R$) gives us again Eq. 8-18.

Equations 8-19 and 8-21 then give us the vector relationships between the angular and linear variables. The beauty of these compact expressions is that, like all vector equations, they contain information about the magnitudes *and the directions* of the relationships.

MULTIPLE CHOICE

8-1 Rotational Motion

1. You have a small globe, which is mounted so that it can spin on the polar axis *and* can be spun about a horizontal axis (so that the south pole can be on top). Give the globe a quick spin about the polar axis, and then, before it stops, give it a spin about the horizontal axis. Are there any points on the globe that are at rest?
(A) There are two points, fixed on the globe, that are at rest.
(B) There are two points that are instantaneously at rest, but these two points move around the globe in an apparently random fashion.
(C) At some times two points are instantaneously at rest, but at other times there are no points at rest.
(D) There are no points at rest until the globe stops spinning.

2. A bicycle wheel is rolling on a level surface. At any given instant in time the wheel
(A) is undergoing pure rotational motion.
(B) is undergoing pure translational motion.
(C) is undergoing both translational and rotational motion.
(D) is undergoing motion that can be described by the answers (A) or (C).

3. Consider rigid-body physics in a higher or lower dimension than three. How many coordinates are required to specify the location and orientation of a rigid body
(*a*) if the space is two-dimensional?
(A) 2 (B) 3 (C) 4 (D) 5
(*b*) if the space is one-dimensional?
(A) 0 (B) 1 (C) 2 (D) 3
(*c*) if the space is four-dimensional?
(A) 7 (B) 8 (C) 9 (D) 10

8-2 The Rotational Variables

8-3 Rotational Quantities as Vectors

4. Which way does $\vec{\omega}$ point for the Earth?
(A) Parallel to the NS axis and pointing north.
(B) Parallel to the NS axis and pointing south.
(C) Parallel to the NS axis and pointing east.
(D) Parallel to the NS axis and pointing west.

8-4 Rotation with Constant Angular Acceleration

5. Two different disks of radii $r_1 > r_2$ are free to spin separately about an axis through the center and perpendicular to the plane of each disk. Both disks start from rest, and both undergo the same angular acceleration for the same length of time. Which disk will have the larger final angular velocity?
(A) Disk 1
(B) Disk 2
(C) The disks will have the same angular velocity.
(D) The answer depends on the mass of the disks.

8-5 Relationships between Linear and Angular Variables

6. A disk is uniformly accelerated from rest with angular acceleration α. The magnitude of the *linear* acceleration of a point on the rim of the disk
(*a*) grows with the time t as
(A) t (B) t^2 (C) t^3 (D) t^4
for $\alpha t^2 \ll 1$ and
(*b*) grows as
(A) t (B) t^2 (C) t^3 (D) t^4
for $\alpha t^2 \gg 1$.

8-6 Vector Relationships between Linear and Angular Variables

7. A small bug of mass m is stationary on a horizontal turntable rotating with angular velocity $\vec{\omega}$. The bug's position is *fixed* relative to the rotating turntable, and is given by the vector $\vec{r}$, which is measured from the axis of the turntable to the bug. Consider the vector described by $m\vec{\omega} \times (\vec{\omega} \times \vec{r})$.
(*a*) Which direction does this vector point?
(A) Toward the axis of rotation
(B) Away from the axis of rotation
(C) Tangent to the circular path traced out by the bug
(D) In a vertical direction
(*b*) The dimensions of this vector are the same as that for
(A) angular acceleration. (B) force.
(C) linear momentum. (D) velocity squared.
(*c*) This vector is proportional to
(A) mvr. (B) mr^2. (C) mv^2/r. (D) mr/v^2.
where v is the speed of the bug as measured from a nonrotating frame.

8. The Coriolis force (see Section 5-6) is a pseudoforce that occurs in rotating coordinate systems (such as the Earth). The force is given by $-2m\vec{\omega} \times \vec{v}$ where $\vec{\omega}$ is the rotational velocity of the Earth and $\vec{v}$ is the velocity of a particle as measured from the Earth's (noninertial) frame of reference.
(*a*) A projectile is launched from the equator due north. The direction of the Coriolis force on this projectile is
(A) east. (B) west. (C) up.
(D) down. (E) The force is zero.
(*b*) A projectile is launched from the equator due east. The direction of the Coriolis force on this projectile is
(A) north. (B) south. (C) up.
(D) down. (E) The force is zero.
(*c*) A projectile is launched from the equator vertically upward. The direction of the Coriolis force on this projectile is
(A) north. (B) south. (C) east.
(D) west. (E) The force is zero.
(See Exercise 34.)

QUESTIONS

1. In Section 8-1 we stated that, in general, six variables are required to locate a rigid body with respect to a particular reference frame. How many variables are required to locate the body of Fig. 8-2 with respect to the *xy* frame shown in that figure? If this number is not six, account for the difference.

2. The rotation of the Sun can be monitored by tracking sunspots, magnetic storms on the Sun that appear dark against the otherwise bright solar disk. Figure 8.12*a* shows the initial positions of five spots and Fig. 8-12*b* the positions of these same spots one solar rotation later. What can we conclude about the physical nature of the Sun from these observations?

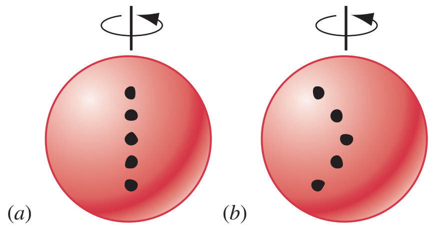


FIGURE 8-12. Question 2.

3. In what sense is the radian a "natural" measure of angle and the degree an "arbitrary" measure of that same quantity? Hence what advantages are there in using radians rather than degrees?

4. Could the angular quantities ϕ, ω_z, and α_z be expressed in terms of degrees instead of radians in Eqs. 8-6 and 8-7?

5. A rigid body is free to rotate about a fixed axis. Can the body have nonzero angular acceleration even if the angular velocity of the body is (perhaps instantaneously) zero? What is the linear equivalent of this question? Give physical examples to illustrate both the angular and linear situations.

6. A golfer swings a golf club, making a long drive from the tee. Do all points on the club have the same angular velocity at any instant while the club is in motion?

7. Does the vector representing the angular velocity of a wheel rotating about a fixed axis necessarily have to lie along that axis? Could it be pictured as merely parallel to the axis, but located anywhere? Recall that we are free to slide a displacement vector along its own direction or translate it sideways without changing its value.

8. Experiment rotating a book after the fashion of Fig. 8-5, but this time use angular displacements of 180° rather than 90°. What do you conclude about the final positions of the book? Does this change your mind about whether (finite) angular displacements can be treated as vectors?

9. A small cube is contained in a larger cube as shown in Fig. 8-13. Each corner of the small cube is attached to the corresponding corner of the large cube with an elastic string; this is a simple example of a *spinor.* Show that the inner cube can be rotated through 360° and the strings cannot be untangled, but if the inner cube is rotated through 720° then the strings *can* be untangled.

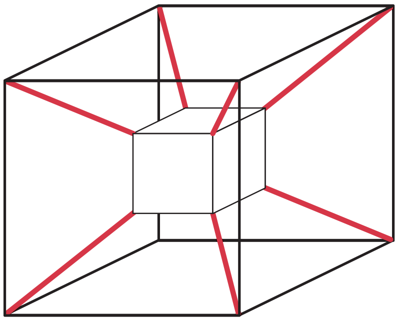


FIGURE 8-13. Question 9.

10. Is the relationship $\Delta\phi_1 + \Delta\phi_2 = \Delta\phi_2 + \Delta\phi_1$ valid if $\Delta\phi_1$ and $\Delta\phi_2$ refer to different axes of rotation? Is it valid if they refer to different rotations about the same axis?

11. The planet Venus (see Fig. 8-14) moves in a circular orbit around the Sun, completing one revolution every 225 days. Venus also rotates about a polar axis, completing one rotation every 243 days. The sense (direction) of the rotational motion is opposite, but parallel, to that of the orbital motion. (*a*) Describe a vector that represents the rotation of Venus about its axis. (*b*) Describe the vector that represents the angular velocity of Venus about the Sun. (*c*) Describe the resultant angular velocity, obtained by adding the orbital and rotational angular velocities.

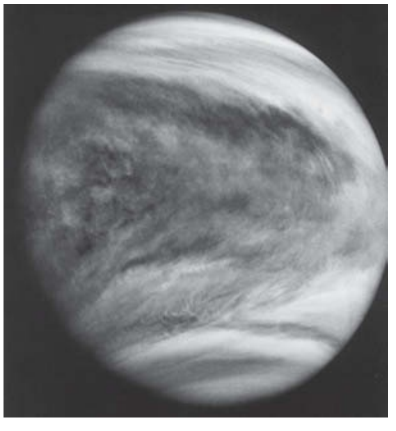


FIGURE 8-14. Question 11.

12. A disk is free to spin with a variable angular velocity. For a point on the rim of the disk (*a*) can $a_T = 0$ if $a_R \neq 0$? (*b*) Can $a_R = 0$ if $a_T \neq 0$? (*c*) Can $a_T = 0$ and $a_R = 0$?

13. Why is it suitable to express the angular acceleration in rev/s^2 in Eq. 8-7 ($\phi = \phi_0 + \omega_{0z}t + \frac{1}{2}\alpha_z t^2$) but not in Eq. 8-10 ($a_T = \alpha r$)?

14. When we say that a point on the equator of the Earth has an angular speed of 2π rad/day, what reference frame do we have in mind?
15. Taking the rotation and the revolution of the Earth into account, does a tree move faster during the day or during the night? With respect to what reference frame is your answer given? (The Earth's rotation and revolution are in the same direction.)
16. A wheel is rotating about its axle. Consider a point on the rim. When the wheel rotates with constant angular velocity, does the point have a radial acceleration? A tangential acceleration? When the wheel rotates with constant angular acceleration, does the point have a radial acceleration? A tangential acceleration? Do the magnitudes of these accelerations change with time?
17. Suppose that you were asked to determine the distance traveled by a needle in playing a vinyl phonograph record. What information do you need? Discuss from the point of view of reference frames (*a*) fixed in the room, (*b*) fixed on the rotating record, and (*c*) fixed on the arm of the record player.
18. What is the relation between the angular velocities of a pair of coupled gears of different radii?

EXERCISES

8-1 Rotational Motion

1. A rigid body exists in an *n*-dimensional space. How many coordinates are needed to specify the position and orientation of this body in this space?

8-2 The Rotational Variables

2. Show that 1 rev/min = 0.105 rad/s.
3. The angle turned through by the flywheel of a generator during a time interval t is given by

$$\phi = at + bt^3 - ct^4,$$

where a, b, and c are constants. What is the expression for its (*a*) angular velocity and (*b*) angular acceleration?
4. Our Sun is 2.3×10^4 ly (light-years) from the center of our Milky Way galaxy and is moving in a circle around this center at a speed of 250 km/s. (*a*) How long does it take the Sun to make one revolution about the galactic center? (*b*) How many revolutions has the Sun completed since it was formed about 4.5×10^9 years ago?
5. A wheel rotates with an angular acceleration α_z given by

$$\alpha_z = 4at^3 - 3bt^2,$$

where t is the time and a and b are constants. If the wheel has an initial angular velocity ω_0, write the equations for (*a*) the angular velocity and (*b*) the angle turned through as functions of time.
6. What is the angular speed of (*a*) the second hand, (*b*) the minute hand, and (*c*) the hour hand of a watch?
7. A good baseball pitcher can throw a baseball toward home plate at 85 mi/h with a spin of 1800 rev/min. How many revolutions does the baseball make on its way to home plate? For simplicity, assume that the 60-ft trajectory is a straight line.
8. A diver makes 2.5 complete revolutions on the way from a 10-m platform to the water below. Assuming zero initial vertical velocity, calculate the average angular velocity for this drive.
9. A wheel has eight spokes and a radius of 30 cm. It is mounted on a fixed axle and is spinning at 2.5 rev/s. You want to shoot a 24-cm arrow parallel to this axle and through the wheel without hitting any of the spokes. Assume that the arrow and the spokes are very thin; see Fig. 8-15. (*a*) What minimum speed must the arrow have? (*b*) Does it matter where between the axle and the rim of the wheel you aim? If so, where is the best location?

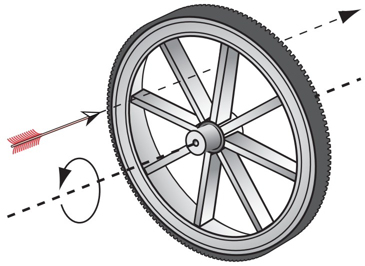


FIGURE 8-15. Exercise 9.

8-3 Rotational Quantities as Vectors

10. A planet P revolves around the Sun in a circular orbit, with the Sun at the center, which is coplanar with and concentric to the circular orbit of Earth E around the Sun. P and E revolve in the same direction. The times required for the revolution of P and E around the Sun are T_P and T_E. Let T_S be the time required for P to make one revolution around the Sun relative to E: show that $1/T_S = 1/T_E - 1/T_P$. Assume $T_P > T_E$.
11. Repeat the previous problem to find an expression for T_S when $T_P < T_E$.

8-4 Rotation with Constant Angular Acceleration

12. A turntable rotating at 78 rev/min slows down and stops in 32 s after the motor is turned off. (*a*) Find its (constant) angular acceleration in rev/min^2. (*b*) How many revolutions does it make in this time?
13. The angular speed of an automobile engine is increased uniformly from 1170 rev/min to 2880 rev/min in 12.6 s. (*a*) Find the angular acceleration in rev/min^2. (*b*) How many revolutions does the engine make during this time?
14. As part of a maintenance inspection, the compressor of a jet engine is made to spin according to the graph shown in

Fig. 8-16. How many revolutions does the compressor make during the test?

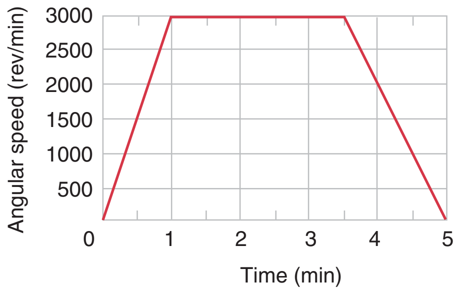


FIGURE 8-16. Exercise 14.

15. The flywheel of an engine is rotating at 25.2 rad/s. When the engine is turned off, the flywheel decelerates at a constant rate and comes to rest after 19.7 s. Calculate (*a*) the angular acceleration (in rad/s^2) of the flywheel, (*b*) the angle (in rad) through which the flywheel rotates in coming to rest, and (*c*) the number of revolutions made by the flywheel in coming to rest.

16. While waiting to board a helicopter, you notice that the rotor's motion changed from 315 rev/min to 225 rev/min in 1.00 min. (*a*) Find the average angular acceleration during the interval. (*b*) Assuming that this acceleration remains constant, calculate how long it will take for the rotor to stop. (*c*) How many revolutions will the rotor make after your second observation?

17. A certain wheel turns through 90 rev in 15 s, its angular speed at the end of the period being 10 rev/s. (*a*) What was the angular speed of the wheel at the beginning of the 15-s interval, assuming constant angular acceleration? (*b*) How much time had elapsed between the time the wheel was at rest and the beginning of the 15-s interval?

18. A pulley wheel 8.14 cm in diameter has a 5.63-m-long cord wrapped around its periphery. Starting from rest, the wheel is given an angular acceleration of 1.47 rad/s^2. (*a*) Through what angle must the wheel turn for the cord to unwind? (*b*) How long does it take?

19. A flywheel completes 42.3 rev as it slows from an angular speed of 1.44 rad/s to a complete stop. (*a*) Assuming constant acceleration, what is the time required for it to come to rest? (*b*) What is the angular acceleration? (*c*) How much time is required for it to complete the first one-half of the 42.3 rev?

20. Starting from rest at $t = 0$, a wheel undergoes a constant angular acceleration. When $t = 2.33$ s, the angular velocity of the wheel is 4.96 rad/s. The acceleration continues until $t = 23.0$ s, when it abruptly ceases. Through what angle does the wheel rotate in the interval $t = 0$ to $t = 46.0$ s?

8-5 Relationships between Linear and Angular Variables

21. What is the angular speed of a car rounding a circular turn of radius 110 m at 52.4 km/h?

22. A point on the rim of a 0.75-m-diameter grinding wheel changes speed uniformly from 12 m/s to 25 m/s in 6.2 s. What is the angular acceleration of the grinding wheel during this interval?

23. What are (*a*) the angular speed, (*b*) the radial acceleration, and (*c*) the tangential acceleration of a spaceship negotiating a circular turn of radius 3220 km at a constant speed of 28,700 km/h?

24. A threaded rod with 12.0 turns/cm and diameter 1.18 cm is mounted horizontally. A bar with a threaded hole to match the rod is screwed onto the rod; see Fig. 8-17. The bar spins at 237 rev/min. How long will it take for the bar to move 1.50 cm along the rod?

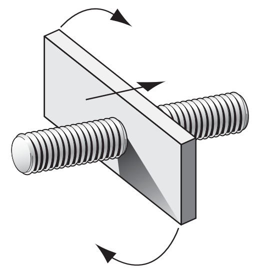


FIGURE 8-17. Exercise 24.

25. (*a*) What is the angular speed about the polar axis of a point on the Earth's surface at a latitude of 40° N? (*b*) What is the linear speed? (*c*) What are the values for a point at the equator?

26. A gyroscope flywheel of radius 2.83 cm is accelerated from rest at 14.2 rad/s^2 until its angular speed is 2760 rev/min. (*a*) What is the tangential acceleration of a point on the rim of the flywheel? (*b*) What is the radial acceleration of this point when the flywheel is spinning at full speed? (*c*) Through what distance does a point on the rim move during the acceleration?

27. If an airplane propeller of radius 5.0 ft (= 1.5 m) rotates at 2000 rev/min and the airplane is propelled at a ground speed of 300 mi/h (= 480 km/h), what is the speed of a point on the tip of the propeller, as seen by (*a*) the pilot and (*b*) an observer on the ground? Assume that the plane's velocity is parallel to the propeller's axis of rotation.

28. The blades of a windmill start from rest and rotate with an angular acceleration of 0.236 rad/s^2. How much time elapses before a point on a blade experiences the same value for the magnitudes of the centripetal acceleration and tangential acceleration?

29. A rigid body, starting at rest, rotates about a fixed axis with constant angular acceleration α. Consider a particle a distance r from the axis. Express (*a*) the radial acceleration and (*b*) the tangential acceleration of this particle in terms of α, r, and the time t. (*c*) If the resultant acceleration of the particle at some instant makes an angle of 57.0° with the tangential acceleration, through what total angle has the body rotated from $t = 0$ to that instant?

30. An automobile traveling at 97 km/h has wheels of diameter 76 cm. (*a*) Find the angular speed of the wheels about the axle. (*b*) The car is brought to a stop uniformly in 30 turns of the wheels. Calculate the angular acceleration. (*c*) How far does the car advance during this braking period?

31. A speedometer on the front wheel of a bicycle gives a reading that is directly proportional to the angular speed of the wheel. Suppose that such a speedometer is calibrated for a wheel of diameter 72 cm but is mistakenly used on a wheel of diameter

62 cm. Would the linear speed reading be wrong? If so, in what sense and by what fraction of the true speed?

8-6 Vector Relationships between Linear and Angular Variables

32. An object moves in the *xy* plane such that $x = R\cos\omega t$ and $y = R\sin\omega t$. Here x and y are the coordinates of the object, t is the time, and R and ω are constants. (*a*) Eliminate t between these equations to find the equation of the curve in which the object moves. What is this curve? What is the meaning of the constant ω? (*b*) Differentiate the equations for x and y with respect to the time to find the x and y components of the velocity of the body, v_x and v_y. Combine v_x and v_y to find the magnitude and direction of v. Describe the motion of the object. (*c*) Differentiate v_x and v_y with respect to the time to obtain the magnitude and direction of the resultant acceleration.

33. A rigid object rotating about the z axis is slowing down at 2.66 rad/s^2. Consider a particle located at $\vec{\mathbf{r}} = (1.83\text{ m})\hat{\mathbf{j}} + (1.26\text{ m})\hat{\mathbf{k}}$. At the instant that $\vec{\boldsymbol{\omega}} = (14.3\text{ rad/s})\hat{\mathbf{k}}$, find (*a*) the velocity of the particle and (*b*) its acceleration. (*c*) What is the radius of the circular path of the particle?

34. A 12-kg projectile is launched vertically upward with an initial speed of 35 m/s from a football field in Minneapolis, MN. (*a*) Calculate the magnitude and direction of the Coriolis force (see Multiple Choice question 8 and Section 5-6) on the projectile shortly after the projectile is launched. (*b*) What is the approximate direction of the Coriolis force on the projectile while the projectile is heading back toward the Earth? (*c*) Will the projectile return to the original launch point? If not, in which direction will it land relative to the launch point?

PROBLEMS

1. The angular position of a point on the rim of a rotating wheel is described by $\phi = (4.0\text{ rad/s})t - (3.0\text{ rad/s}^2)t^2 + (1.0\text{ rad/s}^3)t^3$. (*a*) What is the angular velocity at $t = 2.0$ s and at $t = 4.0$ s? (*b*) What is the average angular acceleration for the time interval that begins at $t = 2.0$ s and ends at $t = 4.0$ s? (*c*) What is the instantaneous angular acceleration at the beginning and end of this time interval?

2. A wheel with 16 spokes rotating in the clockwise direction is photographed on film. The film is passed through a projector at the rate of 24 frames/s, which is the proper rate for the projector. On the screen, however, the wheel appears to rotate counterclockwise at 4.0 rev/min. Find the smallest possible angular speed at which the wheel was rotating.

3. A solar day is the time interval between two successive appearances of the Sun overhead at a given longitude—that is, the time for one complete rotation of Earth relative to the Sun. A sidereal day is the time for one complete rotation of Earth relative to the fixed stars—that is, the time interval between two successive overhead observations of a fixed direction in the heavens called the vernal equinox. (*a*) Show that there is exactly one less (mean) solar day in a year than there are (mean) sidereal days in a year. (*b*) If the (mean) solar day is exactly 24 hours, how long is a (mean) sidereal day?

4. A pulsar is a rapidly rotating neutron star from which we receive radio pulses with precise synchronization, there being one pulse for each rotation of the star. The period T of rotation is found by measuring the time between pulses. At present, the pulsar in the central region of the Crab nebula (see Fig. 8-18) has a period of rotation of $T = 0.033$ s, and this is observed to be increasing at the rate of 1.26×10^{-5} s/y. (*a*) Show that the angular speed ω of the star is related to the period of rotation by $\omega = 2\pi/T$. (*b*) What is the value of the angular acceleration in rad/s^2? (*c*) If its angular acceleration is constant, when will the pulsar stop rotating? (*d*) The pulsar originated in a supernova explosion in the year A.D. 1054. What was the period of rotation of the pulsar when it was born? (Assume constant angular acceleration.)

5. Two students perform a simple experiment. The first student observes the orientation of a stationary disk with a single

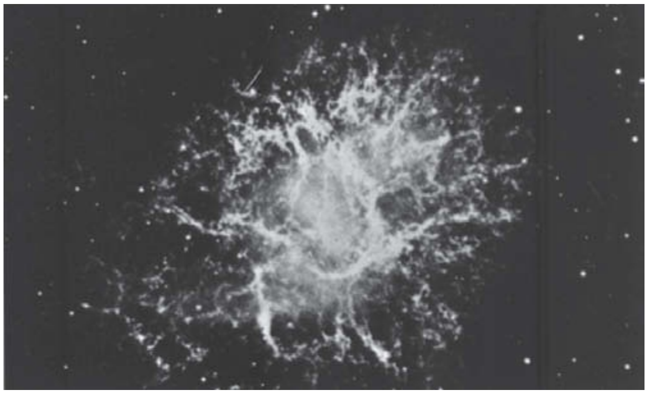


FIGURE 8-18. Problem 4.

mark on the edge of the rim. He then looks away. The second student then gives the disk a constant rotational acceleration of 3.0 rad/s^2 for a time of 4.0 seconds; she then brings the disk to a halt with constant angular acceleration in a time of 0.10 s. The first student is now allowed to look at the disk again. (*a*) From the point of view of the first student, through what angle did the disk move? (*b*) What was the average angular velocity of the disk?

6. An astronaut is being tested in a centrifuge. The centrifuge has a radius of 10.4 m and in starting, rotates according to $\phi = (0.326\text{ rad/s}^2)t^2$. When $t = 5.60$ s, what are the astronaut's (*a*) angular speed, (*b*) tangential speed, (*c*) tangential acceleration, and (*d*) radial acceleration?

7. Earth's orbit about the Sun is almost a circle. (*a*) What is the angular speed of Earth (regarded as a particle) about the Sun? (*b*) What is its linear speed in its orbit? (*c*) What is the acceleration of Earth with respect to the Sun?

8. The flywheel of a steam engine runs with a constant angular speed of 156 rev/min. When steam is shut off, the friction of the bearings and of the air brings the wheel to rest in 2.20 h. (*a*) What is the constant angular acceleration of the wheel, in rev/min^2? (*b*) How many rotations will the wheel make before coming to rest? (*c*) What is the tangential linear acceleration

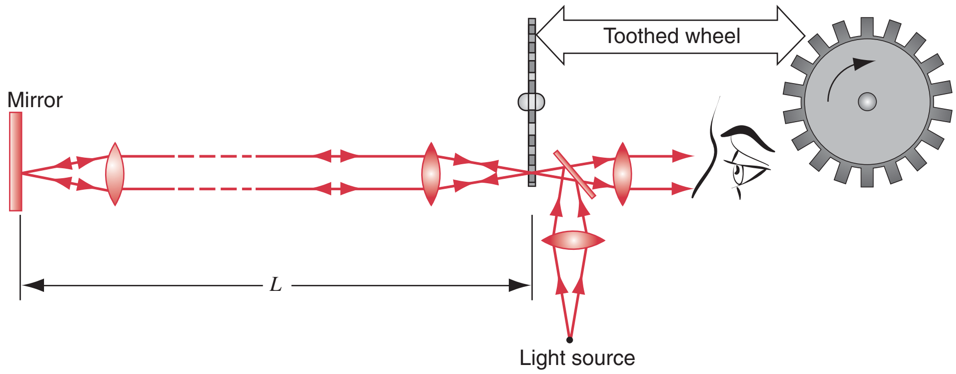


FIGURE 8-19. Problem 9.

of a particle 52.4 cm from the axis of rotation when the flywheel is turning at 72.5 rev/min? (*d*) What is the magnitude of the total linear acceleration of the particle in part (*c*)?

9. An early method of measuring the speed of light makes use of a rotating toothed wheel. A beam of light passes through a slot at the outside edge of the wheel, as in Fig. 8-19, travels to a distant mirror, and returns to the wheel just in time to pass through the next slot in the wheel. One such toothed wheel has a radius of 5.0 cm and 500 teeth at its edge. Measurements taken when the mirror was a distance $L = 500$ m from the wheel indicated a speed of light of 3.0×10^5 km/s. (*a*) What was the (constant) angular speed of the wheel? (*b*) What was the linear speed of a point on its edge?

10. Wheel *A* of radius $r_A = 10.0$ cm is coupled by a belt *B* to wheel *C* of radius $r_C = 25.0$ cm, as shown in Fig. 8-20. Wheel *A* increases its angular speed from rest at a uniform rate of 1.60 rad/s^2. Determine the time for wheel *C* to reach a rotational speed of 100 rev/min, assuming the belt does not slip. (Hint: If the belt does not slip, the linear speeds at the rims of the two wheels must be equal.)

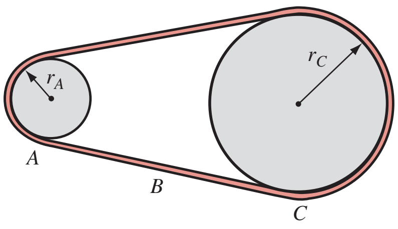


FIGURE 8-20. Problem 10.

11. The disc of a compact disc/digital audio system has an inner and outer radius for its recorded material (the Tchaikovsky and Mendelssohn violin concertos) of 2.50 cm and 5.80 cm, respectively. At playback, the disc is scanned at a constant linear speed of 130 cm/s, starting from its inner edge and moving outward. (*a*) If the initial angular speed of the disc is 50.0 rad/s, what is its final angular speed? (*b*) The spiral scan lines are 1.60 μm apart; what is the total length of the scan? (*c*) What is the playing time?

12. A car moves due east on a straight and level road at a constant velocity $\vec{\mathbf{v}}$. An observer stands a distance *b* north of the road. Find the angular velocity $\vec{\boldsymbol{\omega}}$ and angular acceleration $\vec{\boldsymbol{\alpha}}$ of the car as measured by the observer as a function of time. Assume that the car is closest to the observer at time $t = 0$.

13. A rocket sled moves on a straight horizontal track with a velocity $\vec{\mathbf{v}}(t)$. An observer standing a distance *b* from the track measures the angular velocity $\vec{\boldsymbol{\omega}}$ to be constant. (*a*) Find $\vec{\mathbf{v}}(t)$, assuming that the rocket sled is closest to the observer when $t = 0$. (*b*) At approximately what time t_c does the motion of the rocket sled become physically impossible?

COMPUTER PROBLEMS

1. The effective force on a projectile moving very close to the Earth is

$$\vec{\mathbf{F}} = m\vec{\mathbf{g}} - m\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}) - 2m\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}},$$

where $\vec{\mathbf{g}}$ is the acceleration of free fall, $\vec{\boldsymbol{\omega}}$ is the angular velocity of the Earth, and $\vec{\mathbf{v}}$ is the velocity of the projectile as measured from the Earth's (noninertial) frame of reference. (*a*) Write a computer program that will find the actual trajectory of a 1.0-kg projectile launched vertically upward with an initial velocity of 100 m/s from a point on the Earth's equator. How far from the launch point does the projectile land? (*b*) Write a computer program that will find the actual trajectory of a 1.0-kg projectile launched with an initial velocity of 100 m/s at a 45° angle above the horizontal due north from a point on the Earth's equator. How large is the error caused by the rotating Earth on the target location?

2. A flywheel slows down under the influence of a nonconstant angular acceleration. The angular position of a reference line on the flywheel is described by

$$\phi(t) = (A + Bt + Ct^3)e^{-\beta t}$$

from $t = 0$, when the flywheel began slowing down, to $t = T$, when it came to a rest. Here $A = +2.40$ rad, $B = +5.12$ rad/s, $C = -0.124$ rad/s^3, and $\beta = +0.100$ s^{-1}. (*a*) Find an expression for the angular velocity, and find the time *T* at which the velocity becomes zero. (*b*) Find the angle through which the wheel rotates in being brought to rest.

CHAPTER 9

ROTATIONAL DYNAMICS

In Chapter 8 we considered rotational kinematics and pointed out that it contained no basic new features, the rotational parameters ϕ, ω, and α being related to corresponding translational parameters x, v, and a for the particles that make up the rotating system. In this chapter, following the pattern of our study of translational motion, we consider the causes of rotation, a subject called rotational dynamics. *Rotating systems are made up of particles, and we have already learned how to apply the laws of classical mechanics to the motion of particles. For this reason rotational dynamics, like kinematics, should contain no features that are fundamentally new. As in Chapter 8, however, it is very useful to recast the concepts of translational motion into a new form, especially chosen for its convenience in describing rotating systems.*

9-1 TORQUE

We began our study of dynamics in Chapter 3 by defining a force in terms of the acceleration it produced when acting upon a body of standard mass (Section 3-3). We were then able to obtain the mass of any other body in relation to the standard mass by measuring the acceleration produced when the same force acts on each body (Section 3-4). We incorporated our observations about force, mass, and acceleration into Newton's second law, according to which the net force acting on a body is equal to its mass times its acceleration.

Our procedure for rotational dynamics is similar. We will begin by considering the angular acceleration produced when a force acts on a particular rigid body that is free to rotate about a fixed axis. In analogy with translational motion, we will find that the angular acceleration is proportional to the magnitude of the applied force. However, a new feature emerges that was not present in translational motion: the angular acceleration *also* depends on *where* the force is applied to the body. A given force applied at different locations on a body (or even at the same location but in different directions) will in general produce different angular accelerations.

The quantity in rotational dynamics that takes into account both the magnitude of the force and the direction and location at which it is applied is called the *torque.* The word torque comes from a Latin root meaning "twist," and you can think of a torque as a twist in the same sense that we think of a force as a push or a pull. Like force (and like angular acceleration), torque is a vector quantity. *In this chapter we will consider only cases in which the rotational axis is fixed in direction.* As a result it will be necessary to consider only one component of the torque vector. This restriction is similar to our discussion of translational dynamics in one dimension in Chapter 3.

In addition, we find that the angular acceleration of a body in response to a particular torque depends not only on the mass of the body but also on how that mass is distributed relative to the axis of rotation. For a given torque, we get a different angular acceleration when the mass is close to the axis of rotation than we get when it is farther away. The rotational quantity that describes the mass of a body and its distribution relative to the axis of rotation is called the *rotational inertia.** Unlike the mass, the rotational inertia of a body is not an intrinsic property of the body, but in-

* Also known as the *moment of inertia.*

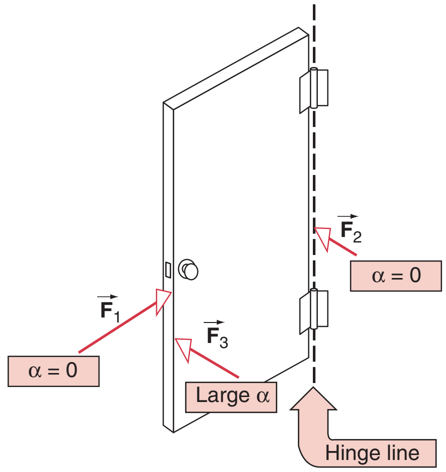


FIGURE 9-1. Applying a given force $\vec{F}$ to a door produces an angular acceleration α that varies with the point at which $\vec{F}$ is applied and with its direction relative to the hinge line. Force $\vec{F}_1$ is applied along a line that would pass through the hinge line, and it produces no angular acceleration (the door does not move). Force $\vec{F}_2$ is applied at the hinge line; it likewise produces no angular acceleration. Force $\vec{F}_3$ is applied at a point far from the hinge line and in a direction perpendicular to the line connecting the point of application of $\vec{F}_3$ with the hinge line; this force produces the largest possible angular acceleration.

stead depends on the choice of the axis of rotation. Just as the mass can be regarded as the property of a body that represents its resistance to linear acceleration, the rotational inertia represents the resistance of a body to angular acceleration. We discuss the rotational inertia of solid bodies in the next section; in this section we consider the torque on a body due to an applied force.

One of our most common experiences with rotational motion is opening a hinged door. We observe that a given force can produce various amounts of angular acceleration, depending on where the force is applied to the door and how it is directed (Fig. 9-1). A force (such as $\vec{F}_1$) applied at the edge and directed along the door cannot produce an angular acceleration, nor can a force (such as $\vec{F}_2$) applied at the hinge line. A force (such as $\vec{F}_3$) applied at right angles at the outer edge of the door produces the largest angular acceleration.

Figure 9-2 shows an arbitrary rigid body that is free to rotate about the z axis. A force $\vec{F}$ is applied to the body at point P, which is located a perpendicular distance r from the axis of rotation. Figure 9-3*a* shows a cross-sectional slice through the body in the xy plane; the vector $\vec{r}$ in this plane locates the point P relative to the axis. We assume that the force $\vec{F}$ also lies in this plane and thus has only x and y components; any possible z component of the force has no effect on rotations about the z axis, just as a vertical force applied to the door of Fig. 9-1 gives no resulting angular acceleration about the hinge line.

As Fig. 9-3*b* shows, the vectors $\vec{r}$ and $\vec{F}$ make an angle θ with each other. The force $\vec{F}$ can be resolved into its radial and tangential components. The radial component

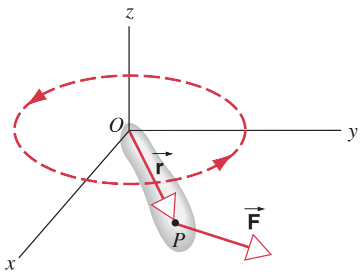


FIGURE 9-2. A rigid body is free to rotate about the z axis. A force $\vec{F}$ is applied at point P of the body.

$F_R = F\cos\theta$ has no effect on the rotation of the body about the z axis, just as the force $\vec{F}_1$ in Fig. 9-1 is not successful in rotating the door. Only the tangential component $F_T = F\sin\theta$ produces a rotation about the z axis (like the force $\vec{F}_3$ in Fig. 9-1).

In addition to the magnitude of the tangential component of $\vec{F}$, the angular acceleration of the body will depend on how far from the axis the force is applied. The greater is the distance from the axis, the greater is the angular acceleration produced by a given force.

The angular acceleration thus depends on both the tangential component of the force and the distance of the point of application of the force from the axis of rotation. The rotational quantity that includes both of these factors is called the *torque* τ. Its magnitude is defined as

$$\tau = rF\sin\theta. \tag{9-1}$$

Torque has dimensions of force times distance. The unit of torque may be the newton-meter (N·m) or pound-foot (lb·ft), among other possibilities.

According to Eq. 9-1, the torque is zero if: (1) $r = 0$—that is, the force is applied at or through the axis of rotation; (2) $F = 0$—that is, there is no applied force; or (3) $\theta = 0°$ or $180°$—that is, the force is applied in the radial direction either outward or inward. In each of these three cases, the force produces no angular acceleration about the z axis.

Had we chosen to place the axis of rotation at a different location in the body, the force applied at P could result in a different torque (because r or θ might be different). The torque produced by a given force thus depends on the choice of the axis of rotation, or equivalently of the origin of the coordinate system. To make this choice clear, we will always refer to the particular point about which we have calculated the torque. Thus Eq. 9-1 defines the torque about point O. Had we selected a different point O' midway between O and P, the resulting torque about O' would be half as great as the torque about point O (because the distance r would be half as great).

Figure 9-3*c* shows another way of interpreting the torque about O. The component of the force perpendicular to $\vec{r}$ is labeled as $F_\perp$; this is the same as the tangential component F_T in Fig. 9-3*b* and has magnitude $F\sin\theta$. The component of $\vec{r}$ perpendicular to $\vec{F}$ is labeled as $r_\perp$ and has

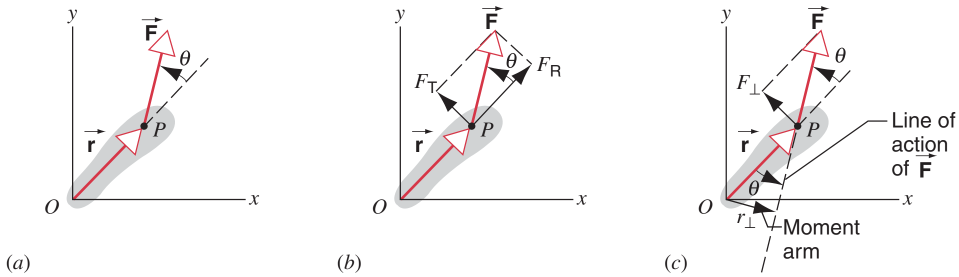


Figure 9-3. (*a*) A cross-sectional slice in the *xy* plane of the body shown in Fig. 9-2. The force $\vec{\mathbf{F}}$ is in the *xy* plane. (*b*) The force $\vec{\mathbf{F}}$ is resolved into its radial (F_R) and tangential (F_T) components. (*c*) The component of $\vec{\mathbf{F}}$ perpendicular to $\vec{\mathbf{r}}$ is $F_\perp$ (also identified as the tangential component F_T), and the component of $\vec{\mathbf{r}}$ perpendicular to $\vec{\mathbf{F}}$ (or to its line of action) is $r_\perp$.

magnitude $r \sin \theta$. We can then rewrite Eq. 9-1 in either of two ways:

$$\tau = r(F \sin \theta) = rF_\perp, \qquad (9\text{-}2a)$$

$$\tau = (r \sin \theta)F = r_\perp F. \qquad (9\text{-}2b)$$

In Eq. 9-2*a*, the magnitude of the torque depends on the component of the force perpendicular to $\vec{\mathbf{r}}$; if this component is zero then the torque is zero. In Eq. 9-2*b*, the torque depends on the *moment arm* $r_\perp$ which, as Fig. 9-3*c* shows, is the perpendicular distance from the origin to the line along which $\vec{\mathbf{F}}$ acts, called the *line of action* of $\vec{\mathbf{F}}$. If the moment arm of a force is zero, then the torque about *O* is zero; for example, the radial component F_R has a moment arm of zero and so gives no torque about *O*.

Torque as a Vector

Equation 9-1 gives the magnitude of the torque, but torque also has a direction, which we take to be the direction of the axis around which the force produces a rotation. In Fig. 9-2, that axis is the *z* axis. If a quantity has both magnitude and direction, we suspect that we may be able to represent it as a vector, provided that it satisfies the transformation and combination rules that we associate with vectors. The torque does indeed satisfy those rules, and thus it is convenient to represent torque as a vector.

To find the torque, we must combine the vector $\vec{\mathbf{r}}$ and the vector $\vec{\mathbf{F}}$ into a new vector $\vec{\boldsymbol{\tau}}$. One way of combining two arbitrary vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ in to a third vector $\vec{\mathbf{C}}$ is by means of the *vector product* (or *cross product*), written as $\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$ (and read as "A cross B"). The cross product of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is defined to be a vector $\vec{\mathbf{C}}$ whose magnitude is $C = AB \sin \theta$, where A is the magnitude of $\vec{\mathbf{A}}$, B is the magnitude of $\vec{\mathbf{B}}$, and θ is the smaller angle between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. This definition ($C = AB \sin \theta$) is in the same form as Eq. 9-1 for the torque ($\tau = rF \sin \theta$), which leads us to suspect that the torque may be written as the cross product of the vectors $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$.

What about the direction of the vector determined by the cross product? The direction of the vector $\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$ is defined to be perpendicular to the plane formed by $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ as determined by the right-hand rule: align the fingers of the right hand with the first vector ($\vec{\mathbf{A}}$), and rotate the fingers from $\vec{\mathbf{A}}$ to $\vec{\mathbf{B}}$ through the smaller of the two angles between those vectors. The extended thumb then points in the direction of $\vec{\mathbf{C}}$. Note that, according to this definition, the cross product $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ is not the same as the cross product $\vec{\mathbf{B}} \times \vec{\mathbf{A}}$; in fact, $\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}}$. The two cross products have the same magnitude but opposite directions. Further details about the cross product may be found in Appendix H.

In terms of the cross product, the torque is expressed as

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}, \qquad (9\text{-}3)$$

According to the definition of the cross product, the magnitude of the vector $\vec{\boldsymbol{\tau}}$ given by Eq. 9-3 is $rF \sin \theta$, in agreement with the definition of the magnitude of the torque in Eq. 9-1. To illustrate how the right-hand rule determines the direction of the torque vector, in Fig. 9-4 we have redrawn the vectors $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$ from Fig. 9-2; for simplicity the rigid body itself is not shown. According to the right-hand rule as

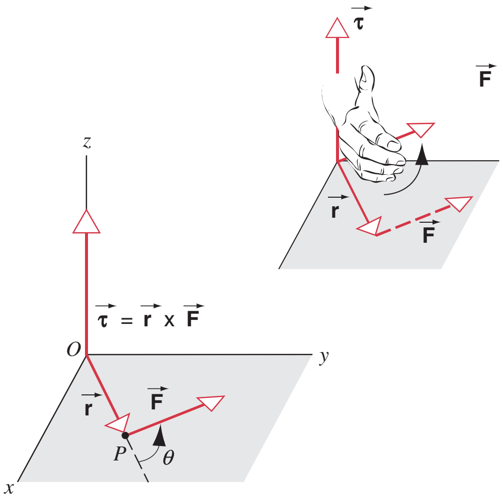


Figure 9-4. A force $\vec{\mathbf{F}}$ acts at point *P* in a rigid body (not shown). This force exerts a torque $\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$ on the body with respect to the origin *O*. The torque vector points in the direction of increasing *z*; it could be drawn anywhere we choose, as long as it is parallel to the *z* axis. The inset shows how the right-hand rule is used to find the direction of the torque. For convenience we can slide the force vector laterally, without changing its direction, until the tail of $\vec{\mathbf{F}}$ joins the tail of $\vec{\mathbf{r}}$.

illustrated in the inset to the figure, the fingers are aligned with $\vec{\mathbf{r}}$ and rotated through the angle θ to $\vec{\mathbf{F}}$. The thumb then points in the direction of the torque vector, which is parallel to the z axis. In terms of the components of $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $\vec{\mathbf{F}} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}$, we can write the torque (see Appendix H) as

$$\vec{\tau} = (yF_z - zF_y)\hat{\mathbf{i}} + (zF_x - xF_z)\hat{\mathbf{j}} + (xF_y - yF_x)\hat{\mathbf{k}}. \quad (9\text{-}4)$$

The torque as defined by the cross product in Eq. 9-3 is perpendicular to the plane determined by $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$. In the case of Fig. 9-4, that plane is the xy plane. The torque must then be perpendicular to the xy plane, or parallel to the z axis. It is not necessary to draw the torque vector *along* the z axis (which we have done in Fig. 9-4); we could locate the vector *anywhere* in the coordinate system of Fig. 9-4 without changing the validity of Eq. 9-3, as long as $\vec{\tau}$ remains parallel to the z axis.

With the rigid body and the applied force positioned as in Fig. 9-2, the torque has only a positive z component. Equation 9-1 defines τ_z in the geometry of Fig. 9-2, but that equation gives only the magnitude of τ_z and not its sign. Under the action of the applied force, the angular velocity of the rigid body will increase in the direction shown in Fig. 9-2, which corresponds to an angular acceleration in the z direction and having a positive z component (using the definitions given in Section 8-3 for the direction of the angular velocity and angular acceleration vectors). Thus a positive τ_z produces a positive α_z. This is very similar to the vector relationship in the linear form of Newton's second law, according to which a force component in a given direction produces an acceleration in that direction.

We can assign an algebraic sign to the vector component of a torque along any axis by regarding that torque component as positive if it tends to produce counterclockwise rotations when viewed from along that axis, and negative if it tends to produce clockwise rotations. From another point of view, to find the sign of the component of a torque vector along any axis, say the z axis, align the thumb of the right hand along the positive direction of that axis; then τ_z is positive for a force which, acting alone, would tend to produce a rotation in the direction of the fingers of the right hand; negative torques are those that tend to produce rotations in the opposite direction. Equation 9-4 gives the signs of the components directly.

SAMPLE PROBLEM 9-1. A pendulum consists of a body of mass $m = 0.17$ kg on the end of a rigid rod of length $L = 1.25$ m and negligible mass (Fig. 9-5). (*a*) What is the magnitude of the torque due to gravity about the pivot point O at the instant the pendulum is displaced as shown through an angle of $\theta = 10°$ from the vertical? (*b*) What is the direction of the torque about O at that instant? Does its direction depend on whether the pendulum is displaced to the left or right of vertical?

Solution (*a*) We can use Eq. 9-1 directly to find the magnitude of the torque, with $r = L$ and $F = mg$:

$$\tau = Lmg \sin\theta = (1.25\ \text{m})(0.17\ \text{kg})(9.8\ \text{m/s}^2)(\sin 10°)$$
$$= 0.36\ \text{N}\cdot\text{m}.$$

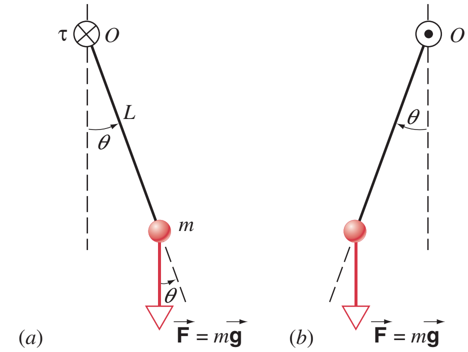


FIGURE 9-5. Sample Problem 9-1. A pendulum, consisting of a body of mass m on the end of a massless rigid rod of length L. (*a*) Gravity exerts a torque into the page at O, indicated there by the symbol ⊗ (suggesting the tail of an arrow). (*b*) When the pendulum is displaced to the left of the vertical, the torque at O is out of the page, indicated by the symbol ⊙ (suggesting the tip of an arrow).

(*b*) With the displacement as shown in Fig. 9-5*a*, the torque about the pivot is into the plane of the paper. You should be able to convince yourself that, if the pendulum is displaced on the opposite side of the vertical, the torque has the opposite direction. As we discuss later in this chapter, the effect of a torque is to produce an angular acceleration parallel to the torque. In the first instance, the angular acceleration into the paper tends to move the pendulum toward its equilibrium position. When the pendulum is displaced on the opposite side of the vertical (Fig. 9-5*b*), the torque out of the paper tends once again to restore the pendulum to its equilibrium position. Check these conclusions using the right-hand rule to relate the sense of the rotation to the direction of the angular acceleration vector (assumed parallel to the torque).

9-2 ROTATIONAL INERTIA AND NEWTON'S SECOND LAW

Hold a long stick in your hand, as in Fig. 9-6. By turning your wrist, you can rotate the stick about various axes. You will find that it takes considerably less effort to rotate the stick about an axis along its length (as in Fig. 9-6*a*) than it does to rotate it about an axis perpendicular to its length (as

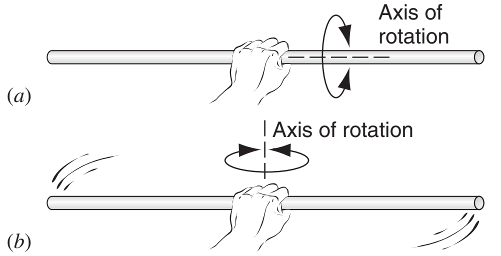


FIGURE 9-6. To rotate a long stick about the axis along its length, as in (*a*), takes less effort than it does to rotate it about an axis perpendicular to its length, as in (*b*). In (*a*), the particles of the stick lie closer to the axis of rotation than they do in (*b*), and so the stick has a smaller rotational inertia in (*a*).

in Fig. 9-6*b*). The difference occurs because the *rotational inertia* is different in the two cases. Unlike the mass of an object (the translational inertia), which has only a single value, the rotational inertia of an object can vary if we choose different axes of rotation. As we shall see, it depends on how the mass is distributed relative to the axis of rotation. In Fig. 9-6*a*, the mass lies relatively close to the axis of rotation; in Fig. 9-6*b*, the mass lies on the average much further from the axis. This difference results in a larger rotational inertia in Fig. 9-6*b*, which we feel as a greater resistance to rotation. In this section we consider the rotational inertia of a particle or a collection of particles; the next section deals with the rotational inertia of solid bodies such as the stick of Fig. 9-6.

Rotational Inertia of a Single Particle

Figure 9-7 shows a single particle of mass m. The particle is free to rotate about the z axis, to which it is attached by a thin rod of length r and of negligible mass. A force $\vec{\mathbf{F}}$ is applied to the particle in an arbitrary direction at an angle θ with the rod. As we have discussed in Section 9-1, a force component parallel to the axis of rotation (the z axis) has no effect on the rotation about that axis, so we need to consider only a force that lies in the xy plane.

The tangential component of $\vec{\mathbf{F}}$ is the only force on the particle that acts in the tangential direction, so the net tangential force is $\sum F_T = F \sin\theta$. Newton's second law applied to the tangential motion of the particle gives $\sum F_T = ma_T$. Substituting $F \sin\theta$ for the net tangential force, and also substituting $a_T = \alpha_z r$ (Eq. 8-10), we obtain

$$F \sin\theta = m\alpha_z r.$$

If we multiply both sides by the radius r, the left side of this equation becomes $rF \sin\theta$, which is the z component of the torque about point O as defined in Eq. 9-1. We therefore obtain

$$\tau_z = mr^2\alpha_z. \tag{9-5}$$

This equation establishes the proportionality between the z component of the torque and the z component of the angular acceleration for rotations about a fixed axis (the z axis). It is similar to Newton's second law for translational motion in one dimension (which can be written $F_z = ma_z$), and the quantity mr^2 in Eq. 9-5 is analogous to the mass in the translational equation. We define this quantity to be the *rotational inertia I* of the particle:

$$I = mr^2. \tag{9-6}$$

The rotational inertia depends on the mass of the particle and on the perpendicular distance between the particle and the axis of rotation. As the particle's distance from the axis increases, the rotational inertia increases, even though the mass does not change.

Rotational inertia has dimensions of mass times length squared (ML^2), and its units might be $kg \cdot m^2$, for instance. The rotational inertia may vary with the location or the direction of the axis of rotation, but it is *not* a vector (its directional properties are more complicated than those of ordinary vectors). However, as defined in Eq. 9-6, for rotations about a single axis the rotational inertia can be treated as a scalar, in analogy with mass.

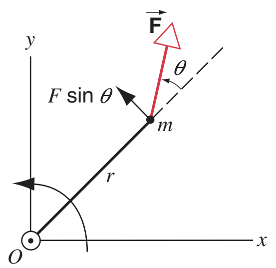


FIGURE 9-7. A force $\vec{\mathbf{F}}$ is applied to a particle attached to a rigid rod of negligible mass that rotates in the xy plane. The torque due to $\vec{\mathbf{F}}$ is in the positive z direction (out of the page) as indicated by the ⊙ symbol at the origin.

Newton's Second Law for Rotation

With this definition of rotational inertia, we can now examine in more detail the relationship between torque and angular acceleration. We shall do this in a more complicated system that may consist of many particles. Just as our quest for the relationship between force and linear acceleration (as discussed in Chapter 3) led us to the translational form of Newton's second law, our discussion here will lead us to a rotational form of Newton's second law.

Let us begin by considering the rotational inertia of a more complicated system consisting of many particles. We still apply only a single force to one of the particles, as before. For example, Fig. 9-8*a* shows a rigid body consisting of two particles of masses m_1 and m_2, each of which is free to rotate in the xy plane about the z axis. The particles are connected to the axis by thin rods of negligible mass of lengths r_1 and r_2, respectively, and they are also connected to one another by a similar rod. An external force $\vec{\mathbf{P}}$ in the xy plane is applied to particle 1. Each particle also experiences a tension acting along the rod that connects it to the origin ($\vec{\mathbf{T}}_1$ and $\vec{\mathbf{T}}_2$), as well as a tension acting along the rod connecting the two particles ($\vec{\mathbf{T}}_{1r}$ and $\vec{\mathbf{T}}_{2r}$), as shown in Fig. 9-8*b*. Because $\vec{\mathbf{T}}_{1r}$ (the force exerted on particle 1 by the rod) and $\vec{\mathbf{T}}_{r1}$ (the force on the rod by particle 1) form an action–reaction pair, and similarly for $\vec{\mathbf{T}}_{2r}$ and $\vec{\mathbf{T}}_{r2}$, and also because the net force on the rod $\vec{\mathbf{T}}_{r1} + \vec{\mathbf{T}}_{r2}$ must be zero (due to its negligible mass), we must have $\vec{\mathbf{T}}_{1r} = -\vec{\mathbf{T}}_{2r}$.

The net force acting on particle 1 is $\sum \vec{\mathbf{F}}_1 = \vec{\mathbf{P}} + \vec{\mathbf{T}}_1 + \vec{\mathbf{T}}_{1r}$, and on particle 2, $\sum \vec{\mathbf{F}}_2 = \vec{\mathbf{T}}_2 + \vec{\mathbf{T}}_{2r}$. We consider the radial and tangential components of the forces and accelerations. The components of the net forces are shown in Fig. 9-8*c*. Because the particles are connected to the origin by rigid rods, there is no radial motion. Furthermore, the radial components of the net forces $\sum \vec{\mathbf{F}}_1$ and $\sum \vec{\mathbf{F}}_2$ give no torque about the origin O, because their moment arms are zero. Only the tangential components of the

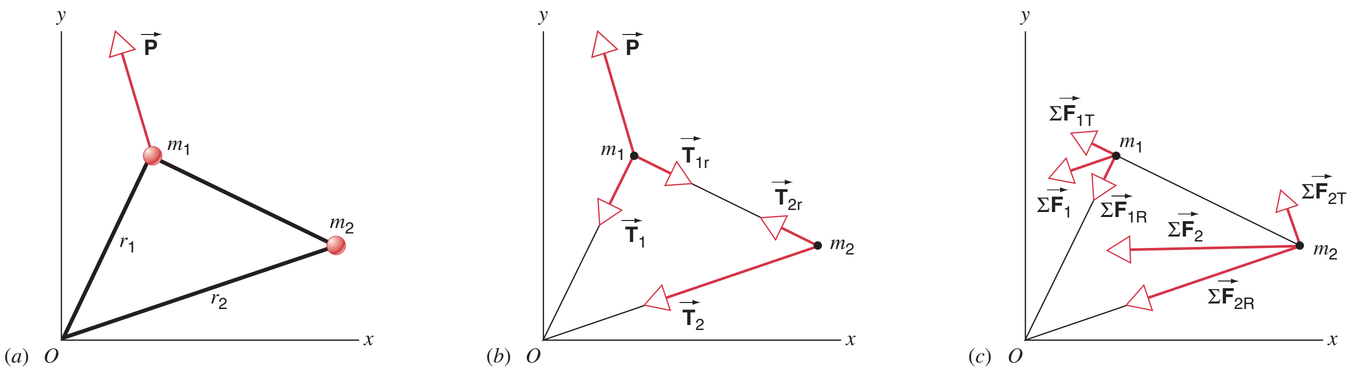


FIGURE 9-8. (*a*) A force $\vec{\mathbf{P}}$ is applied to a rigid body consisting of two particles connected to the axis of rotation (the z axis) and to each other by rigid rods of negligible mass. The entire assembly rotates in the xy plane. (*b*) The forces acting on each particle. (*c*) The net force on each particle and its radial and tangential components.

net forces contribute to the net torque about O. The net torque about O for the two-particle system is the sum of the net torques for each of the particles:

$$\begin{aligned}\sum \tau_z &= \sum \tau_{1z} + \sum \tau_{2z}\\ &= (\sum F_{1\mathrm{T}})r_1 + (\sum F_{2\mathrm{T}})r_2. \quad (9\text{-}7)\end{aligned}$$

For each particle, the net tangential force and the tangential acceleration are related by Newton's second law: $\sum F_{1\mathrm{T}} = m_1 a_{1\mathrm{T}}$ and $\sum F_{2\mathrm{T}} = m_2 a_{2\mathrm{T}}$. Making these substitutions into Eq. 9-7, we obtain

$$\begin{aligned}\sum \tau_z &= (\sum F_{1\mathrm{T}})r_1 + (\sum F_{2\mathrm{T}})r_2\\ &= (m_1 a_{1\mathrm{T}})r_1 + (m_2 a_{2\mathrm{T}})r_2\\ &= (m_1 \alpha_z r_1)r_1 + (m_2 \alpha_z r_2)r_2\\ &= (m_1 r_1^2 + m_2 r_2^2)\alpha_z, \quad (9\text{-}8)\end{aligned}$$

where the third line follows from using Eq. 8-10 for the tangential accelerations ($a_{1\mathrm{T}} = \alpha_z r_1$ and $a_{2\mathrm{T}} = \alpha_z r_2$). The angular accelerations α_z are the same for both particles, because the two-particle object rotates as a rigid body.

The quantity $m_1 r_1^2 + m_2 r_2^2$ in Eq. 9-8 is, by analogy with Eq. 9-6, the *total* rotational inertia of this two-particle system:

$$I = m_1 r_1^2 + m_2 r_2^2. \quad (9\text{-}9)$$

For rotations of two particles about a common axis, we can simply add their rotational inertias. The obvious extension to a rigid object consisting of N particles rotating about the same axis is

$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots + m_N r_N^2 = \sum m_n r_n^2. \quad (9\text{-}10)$$

We can make one further simplification in Eq. 9-8. Let us return to Fig. 9-8*b* to examine the contributions to the net torque about O. The tensions $\vec{\mathbf{T}}_1$ and $\vec{\mathbf{T}}_2$ have no torque about O, because their lines of action go through O. Moreover, the tensions $\vec{\mathbf{T}}_{1\mathrm{r}}$ and $\vec{\mathbf{T}}_{2\mathrm{r}}$ do not contribute to the net torque on the two-particle system, because they are equal and opposite *and* they have the same line of action. Thus the net torque about O is due only to the external force $\vec{\mathbf{P}}$, and we can replace $\sum \tau_z$ in Eq. 9-8 with $\sum \tau_{\mathrm{ext},z}$, the z component of the torque about O due only to the *external* force. Making this substitution, and using Eq. 9-9, we can write Eq. 9-8 as

$$\sum \tau_{\mathrm{ext},z} = I\alpha_z. \quad (9\text{-}11)$$

This is the *rotational form of Newton's second law.* It relates the *net external torque* about a particular fixed axis (the z axis in this case) to the angular acceleration about that axis. The rotational inertia I must be calculated about that same axis.

Equation 9-11 is very similar to the form of Newton's second law for translational motion in one dimension, $\sum F_z = ma_z$. There is, however, a very significant difference: this translational equation is one component of the vector equation $\sum \vec{\mathbf{F}} = m\vec{\mathbf{a}}$, but we cannot in general write the rotational equation in this vector form because the rotational inertia I may be different for rotations about the x, y, and z axes. This suggests that the rotational inertia is a more complicated quantity than the simple scalar form we are using. However, in using Eq. 9-11 for rotations about a single axis, we can regard I as a scalar.

In this calculation, we have taken the external force to be applied to one of the particles. If we instead applied the force elsewhere in the system of Fig. 9-8*a* (even to one of the connecting rods), we would find a different value of $\sum \tau_{\mathrm{ext},z}$, but Eq. 9-11 would remain valid. If many external forces act on a rigid body, we add up the torques due to all the external forces, taking each torque with respect to the same z axis.

SAMPLE PROBLEM 9-2. Three particles of masses m_1 (2.3 kg), m_2 (3.2 kg), and m_3 (1.5 kg) are connected by thin rods of negligible mass so that they lie at the vertices of a 3-4-5 right triangle in the xy plane (Fig. 9-9). (*a*) Find the rotational inertia about each of the three axes perpendicular to the xy plane and passing through one of the particles. (*b*) A force $\vec{\mathbf{F}}$ of magnitude 4.5 N is applied to m_2 in the xy plane and makes an angle of 30° with the horizontal. Find the angular acceleration if the system rotates about an axis perpendicular to the xy plane and passing through m_3.

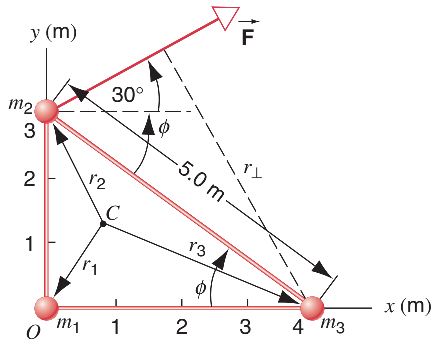


Figure 9-9. Sample Problem 9-2. Point C marks the center of mass of the system consisting of the three particles.

Solution (a) Consider first the axis through m_1. For point masses, m_1 lies on the axis, so $r_1 = 0$ and m_1 does not contribute to the rotational inertia. The distances from this axis to m_2 and m_3 are $r_2 = 3.0$ m and $r_3 = 4.0$ m. The rotational inertia about the axis through m_1 is then (using Eq. 9-10)

$$I_1 = \sum m_n r_n^2 = (2.3\text{ kg})(0\text{ m})^2 + (3.2\text{ kg})(3.0\text{ m})^2 + (1.5\text{ kg})(4.0\text{ m})^2 = 53\text{ kg}\cdot\text{m}^2.$$

Similarly for the axis through m_2, we have

$$I_2 = \sum m_n r_n^2 = (2.3\text{ kg})(3.0\text{ m})^2 + (3.2\text{ kg})(0\text{ m})^2 + (1.5\text{ kg})(5.0\text{ m})^2 = 58\text{ kg}\cdot\text{m}^2.$$

For the axis through m_3,

$$I_3 = \sum m_n r_n^2 = (2.3\text{ kg})(4.0\text{ m})^2 + (3.2\text{ kg})(5.0\text{ m})^2 + (1.5\text{ kg})(0\text{ m})^2 = 117\text{ kg}\cdot\text{m}^2.$$

If a given torque is applied to the system, for rotations about which axis will the torque produce the greatest angular acceleration? The least angular acceleration?

(b) Since the body rotates about an axis parallel to the z axis, only the z component of the torque is needed. We can use Eq. 9-2b ($\tau_z = r_\perp F$) for the magnitude of the torque, so we must find the value of the moment arm $r_\perp$ indicated in Fig. 9-9. From the triangle with the three particles at the vertices, we have $\phi = \sin^{-1} 3/5 = 37°$. The angle between $\vec{\mathbf{F}}$ and the line connecting m_3 with m_2 is $30° + 37° = 67°$, and thus $r_\perp = 5.0 \sin 67° = 4.6$ m. The magnitude of the torque about m_3 is then

$$\tau_z = r_\perp F = (4.6\text{ m})(4.5\text{ N}) = 20.7\text{ N}\cdot\text{m}.$$

Since Eqs. 9-1 and 9-2 give only the magnitude of the torque, we must decide independently whether its z component is positive or negative. Under the action of the force $\vec{\mathbf{F}}$, the system shown in Fig. 9-9 will tend to rotate in a clockwise direction. Using the right-hand rule with the fingers in the direction of the clockwise rotation, the thumb points into the paper—that is, in the negative z direction. We therefore conclude that $\tau_z = -20.7$ N·m.

Since this is the only external torque acting on the system, Eq. 9-11 gives the angular acceleration, using the rotational inertia about the axis through m_3 found in part (a):

$$\alpha_z = \frac{\sum \tau_{\text{ext},z}}{I_3} = \frac{-20.7\text{ N}\cdot\text{m}}{117\text{ kg}\cdot\text{m}^2} = -0.18\text{ rad/s}^2.$$

Once again, the negative sign indicates a clockwise angular acceleration using the right-hand rule.

Sample Problem 9-3. For the three-particle system of Fig. 9-9, find the rotational inertia about an axis perpendicular to the xy plane and passing through the center of mass of the system.

Solution First we must locate the center of mass:

$$x_{\text{cm}} = \frac{\sum m_n x_n}{\sum m_n} = \frac{(2.3\text{ kg})(0\text{ m}) + (3.2\text{ kg})(0\text{ m}) + (1.5\text{ kg})(4.0\text{ m})}{2.3\text{ kg} + 3.2\text{ kg} + 1.5\text{ kg}} = 0.86\text{ m},$$

$$y_{\text{cm}} = \frac{\sum m_n y_n}{\sum m_n} = \frac{(2.3\text{ kg})(0\text{ m}) + (3.2\text{ kg})(3.0\text{ m}) + (1.5\text{ kg})(0\text{ m})}{2.3\text{ kg} + 3.2\text{ kg} + 1.5\text{ kg}} = 1.37\text{ m}.$$

The squared distances from the center of mass to each of the particles are

$$r_1^2 = x_{\text{cm}}^2 + y_{\text{cm}}^2 = (0.86\text{ m})^2 + (1.37\text{ m})^2 = 2.62\text{ m}^2$$

$$r_2^2 = x_{\text{cm}}^2 + (y_2 - y_{\text{cm}})^2 = (0.86\text{ m})^2 + (3.0\text{ m} - 1.37\text{ m})^2 = 3.40\text{ m}^2,$$

$$r_3^2 = (x_3 - x_{\text{cm}})^2 + y_{\text{cm}}^2 = (4.0\text{ m} - 0.86\text{ m})^2 + (1.37\text{ m})^2 = 11.74\text{ m}^2.$$

The rotational inertia then follows directly from Eq. 9-10:

$$I_{\text{cm}} = \sum m_n r_n^2 = (2.3\text{ kg})(2.62\text{ m}^2) + (3.2\text{ kg})(3.40\text{ m}^2) + (1.5\text{ kg})(11.74\text{ m}^2) = 35\text{ kg}\cdot\text{m}^2.$$

Note that the rotational inertia about the center of mass is the smallest of those we have calculated for this system (compare the values in Sample Problem 9-2). This is a general result, which we shall prove next. It is easier to rotate a body about an axis through the center of mass than about any other parallel axis.

The Parallel-Axis Theorem

The result of the previous sample problem leads us to an important general result, the *parallel-axis theorem:*

> *The rotational inertia of any body about an arbitrary axis equals the rotational inertia about a parallel axis through the center of mass plus the total mass times the squared distance between the two axes.*

Mathematically, the parallel-axis theorem has the following form:

$$I = I_{\text{cm}} + Mh^2, \qquad (9\text{-}12)$$

where I is the rotational inertia about the arbitrary axis, I_{cm} is the rotational inertia about the parallel axis through the center of mass, M is the total mass of the object, and h is the perpendicular distance between the axes. Note that the two axes must be parallel.

Before we prove the parallel-axis theorem, let us show how it could have been used to obtain the results of Sample Problem 9-2. We start with the rotational inertia about the center of mass, which was found in Sample Problem 9-3. $I_{cm} = 35\ \text{kg}\cdot\text{m}^2$. The distance h between the axis through the center of mass and the axis through m_1 is just r_1, whose square was computed in Sample Problem 9-3. Thus

$$\begin{aligned} I_1 &= I_{cm} + Mh^2 \\ &= 35\ \text{kg}\cdot\text{m}^2 + (2.3\ \text{kg} + 3.2\ \text{kg} + 1.5\ \text{kg})(2.62\ \text{m}^2) \\ &= 53\ \text{kg}\cdot\text{m}^2, \end{aligned}$$

in agreement with the result of part (*a*) of Sample Problem 9-2. You should check that I_2 and I_3 are similarly verified.

The parallel-axis theorem has an important corollary: since the term Mh^2 is always positive, I_{cm} is always the smallest rotational inertia of any group of parallel axes. (It may not be the *absolute* smallest rotational inertia of the object; an axis pointing in a different direction may yield a smaller value.) Thus for rotations in a given plane, choosing an axis through the center of mass gives the greatest angular acceleration for a given torque.

Proof of Parallel-Axis Theorem. Figure 9-10 shows a thin slab in the xy plane, which can be regarded as a collection of particles. We wish to calculate the rotational inertia of this object about the z axis, which passes through the origin O in Fig. 9-10 at right angles to the plane of that figure. We represent each particle in the slab by its mass m_n, its coordinates x_n and y_n with respect to the origin O, and its coordinates x'_n and y'_n with respect to the center of mass C. The rotational inertia about an axis through O is

$$I = \sum m_n r_n^2 = \sum m_n(x_n^2 + y_n^2).$$

Relative to O, the center of mass has coordinates x_{cm} and y_{cm}, and from the geometry of Fig. 9-10 you can see that

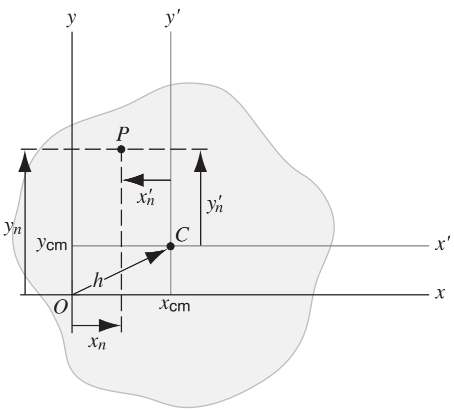


FIGURE 9-10. A thin slab in the xy plane is to be rotated about the z axis, which is perpendicular to the page at the origin O. Point C labels the center of mass of the slab. A particle P is located at coordinates x_n, y_n relative to the origin O and at coordinates x'_n, y'_n relative to the center of mass C.

the relationships between the coordinates x_n, y_n and x'_n, y'_n are $x_n = x'_n + x_{cm}$ and $y_n = y'_n + y_{cm}$. Substituting these transformations, we have

$$\begin{aligned} I &= \sum m_n[(x'_n + x_{cm})^2 + (y'_n + y_{cm})^2] \\ &= \sum m_n(x'^2_n + 2x'_n x_{cm} + x^2_{cm} + y'^2_n + 2y'_n y_{cm} + y^2_{cm}). \end{aligned}$$

Regrouping the terms, we can write this as

$$\begin{aligned} I = \sum m_n(x'^2_n + y'^2_n) + 2x_{cm}\sum m_n x'_n + 2y_{cm}\sum m_n y'_n \\ + (x^2_{cm} + y^2_{cm})\sum m_n. \end{aligned}$$

The first summation above is just $I_{cm} = \Sigma\, m_n r'^2_n$. The next two terms look like the formulas used to calculate the coordinates of a center of mass (Eq. 7-12), but (as Fig. 9-10 shows) they are calculated *in* the center-of-mass system. For instance, $\Sigma\, m_n x'_n = Mx'_{cm} = 0$ because $x'_{cm} = 0$, and similarly $\Sigma\, m_n y'_n = My'_{cm} = 0$; in the center-of-mass coordinate system, the center of mass is by definition at the origin, and so these terms vanish. In the last term, we let h represent the distance between the origin O and the center of mass C, so that $h^2 = x^2_{cm} + y^2_{cm}$; also, $\Sigma\, m_n = M$, the total mass. Thus

$$I = I_{cm} + Mh^2,$$

which proves the parallel-axis theorem.

SAMPLE PROBLEM 9-4. The object shown in Fig. 9-11 consists of two particles, of masses m_1 and m_2, connected by a light rigid rod of length L. (*a*) Neglecting the mass of the rod, find the rotational inertia I of this system for rotations of this object about an axis perpendicular to the rod and a distance x from m_1. (*b*) Show that I is a minimum when $x = x_{cm}$.

Solution (*a*) From Eq. 9-9, we obtain

$$I = m_1 x^2 + m_2(L - x)^2.$$

(*b*) We find the minimum value of I by setting dI/dx equal to 0:

$$\frac{dI}{dx} = 2m_1 x + 2m_2(L - x)(-1) = 0.$$

Solving, we find the value of x at which this minimum occurs:

$$x = \frac{m_2 L}{m_1 + m_2}.$$

This is identical to the expression for the center of mass of the object, and thus the rotational inertia does reach its minimum value at $x = x_{cm}$. This is consistent with the parallel-axis theorem, which requires that I_{cm} be the smallest rotational inertia among parallel axes.

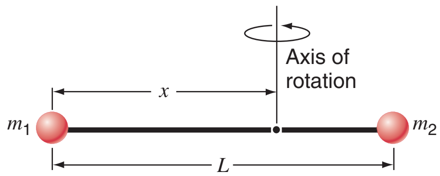


FIGURE 9-11. Sample Problem 9-4. The object is to be rotated about an axis perpendicular to the connecting rod and a distance x from m_1.

Points at which the first derivative of a function equals zero may not all be minima of the function; some may be maxima. Can you show, using the *second* derivative, that we have indeed found a minimum of I?

9-3 ROTATIONAL INERTIA OF SOLID BODIES

If we regard a body as made up of a number of discrete particles, we can calculate its rotational inertia about any axis from Eq. 9-10, in which the sum is taken over all the particles. If, however, we regard it as a continuous distribution of matter, we can imagine it divided into a large number of small mass elements δm_n. Each δm_n is located at a particular perpendicular distance r_n from the axis of rotation. By considering each δm_n as approximately a point mass, we can calculate the rotational inertia according to Eq. 9-10:

$$I = \sum r_n^2\, \delta m_n. \tag{9-13}$$

We shall soon take this to the limit of infinitesimally small δm_n so that the sum becomes an integral. For now, let us illustrate the transition to integral calculus by using Eq. 9-13 to approximate the rotational inertia of a uniform solid rod rotated about an axis perpendicular to the rod at its midpoint. Figure 9-12*a* illustrates the situation. The rod has length L and mass M. Let us imagine that the rod is divided into 10 pieces, each of length $L/10$ and mass $M/10$. The pieces are numbered from $n = 1$ to $n = 10$, so that the nth piece is a distance r_n from the axis; for this calculation, we take r_n to be measured from the axis to the center of the piece. Thus the pieces on each end have $r_1 = r_{10} = 0.45L$, the pieces next to the ends have $r_2 = r_9 = 0.35L$, and the pieces nearest the axis have $r_5 = r_6 = 0.05L$. We now carry out the sum over the 10 pieces according to Eq. 9-13:

$$\begin{aligned} I &= r_1^2\, \delta m_1 + r_2^2\, \delta m_2 + \cdots + r_{10}^2\, \delta m_{10} \\ &= (0.1M)(0.45L)^2 + (0.1M)(0.35L)^2 + (0.1M)(0.25L)^2 \\ &\quad + (0.1M)(0.15L)^2 + (0.1M)(0.05L)^2 + \cdots, \end{aligned}$$

where in the second equation the five terms listed correspond to half of the rod and $\cdots$ means we have five identical terms from the other half. Evaluating the total of the numerical factors, we obtain the result

$$I = 0.0825ML^2 = \frac{1}{12.12}ML^2 \qquad \text{(10 pieces)}.$$

Our reason for writing the result in this form will soon be apparent.

Suppose now we divide the rod into 20 pieces, each of length $L/20$ and mass $M/20$ (Fig. 9-12*b*). Repeating the above calculation, we obtain the result

$$I = 0.0831ML^2 = \frac{1}{12.03}ML^2 \qquad \text{(20 pieces)}.$$

As we increase the number of pieces, does the result approach a limiting value that we can regard as the rotational inertia? In Exercise 21, you are asked to derive the result for any arbitrary number N of pieces:

$$I = \frac{1}{12}ML^2\left(\frac{N^2 - 1}{N^2}\right) \qquad (N \text{ pieces}). \tag{9-14}$$

Clearly this approaches a limit of $ML^2/12$ as $N \to \infty$, and we can assign this as the value of the rotational inertia of the rod. Note that the numerical coefficients for $N = 10$ ($\frac{1}{12.12}$) and $N = 20$ ($\frac{1}{12.03}$) show the approach to the $N \to \infty$ limit ($\frac{1}{12}$).

The above algebraic method works easily in a few cases, and it is helpful in forming an image in our minds of how integral calculus divides a solid object into infinitesimal pieces and sums over the pieces. For calculations involving most solids, the algebraic method is cumbersome, and it is far easier to use calculus techniques directly. Let us take the limit of Eq. 9-13 as the number of pieces becomes very large or, equivalently, as their masses δm become very small:

$$I = \lim_{\delta m_n \to 0} \sum r_n^2\, \delta m_n,$$

and in the usual way the sum becomes an integral in the limit:

$$I = \int r^2\, dm. \tag{9-15}$$

The integration is carried out over the entire volume of the object, but often certain geometrical simplifications can reduce the integral to more manageable terms.

As an example, let us return to the rod rotated about an axis through its center. Figure 9-13 shows the problem drawn for the integral approach. We choose an *arbitrary* element of mass dm located a distance x from the axis. (We use x as the variable of integration.) The mass of this ele-

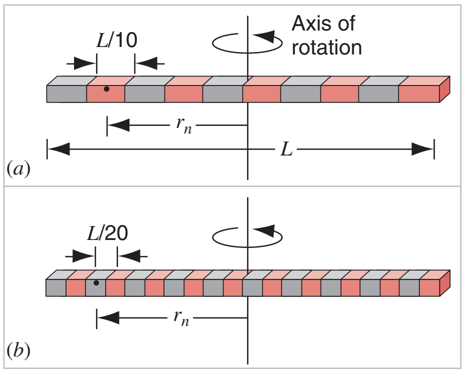


FIGURE 9-12. (*a*) The rotational inertia of a solid rod of length L, rotated about an axis through its center and perpendicular to its length, can be approximately computed by dividing the rod into 10 equal pieces, each of length $L/10$. Each piece is treated as a point mass a distance r_n from the axis. (*b*) A more accurate approximation to the rotational inertia of the rod is obtained by dividing it into 20 pieces.

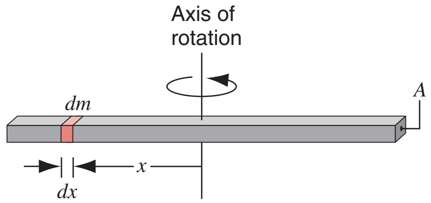


FIGURE 9-13. The rotational inertia of a solid rod is computed by integrating along its length. An element of mass dm is located at a perpendicular distance x from the axis of rotation.

ment is equal to its density (mass per unit volume) ρ times the volume element dV. The volume element is equal to the area times its thickness dx:

$$dV = A\,dx$$
$$dm = \rho\,dV = \rho A\,dx.$$

We assume the rod has uniform cross-sectional area A and uniform density ρ, the latter being equal to the total mass M divided by the total volume AL: $\rho = M/V = M/AL$. Evaluating Eq. 9-15, we obtain

$$I = \int r^2\,dm = \int x^2 \frac{M}{AL} A\,dx = \frac{M}{L}\int x^2\,dx.$$

With $x = 0$ at the midpoint of the rod, the limits of integration are from $x = -L/2$ to $x = +L/2$. The rotational inertia is then

$$I = \frac{M}{L}\int_{-L/2}^{+L/2} x^2\,dx = \frac{M}{L}\frac{x^3}{3}\Bigg|_{-L/2}^{+L/2}$$
$$I = \tfrac{1}{12}ML^2. \tag{9-16}$$

This result is identical with the one deduced from the algebraic method, Eq. 9-14, in the limit $N \rightarrow \infty$.

If we wish to rotate the rod about an axis through one end perpendicular to its length, we can use the parallel-axis theorem (Eq. 9-12). We have already found I_{cm}, and the distance h between the parallel axes is just half the length, so

$$I = \tfrac{1}{12}ML^2 + M(L/2)^2 = \tfrac{1}{3}ML^2.$$

Often we can calculate the rotational inertia of a solid body by decomposing it into elements of known rotational inertia. For example, suppose we have a uniform solid rectangular plate of length a and width b, as shown in Fig. 9-14. We wish to calculate the rotational inertia about an axis perpendicular to the plate and through its center.

The plate can be divided into a series of strips, each of which is to be regarded as a rod. Consider the strip of mass dm, length a, and width dx shown in Fig. 9-14. The mass dm of the strip is related to the total mass M as the surface area of the strip ($a\,dx$) is related to the total surface area ab:

$$\frac{dm}{M} = \frac{a\,dx}{ab} = \frac{dx}{b}$$
$$dm = \frac{M}{b}\,dx.$$

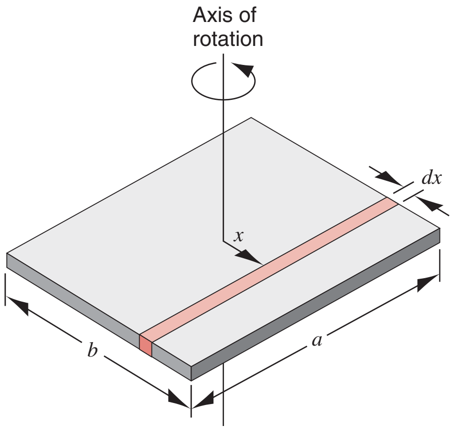


FIGURE 9-14. A solid rectangular plate of sides a and b is rotated about an axis through its center and perpendicular to its surface. To compute the rotational inertia, we consider the plate to be divided into strips. The shaded strip can be considered a rod, whose rotational inertia about the central axis can be found using the parallel-axis theorem.

The rotational inertia dI of the strip about the axis is, by the parallel-axis theorem, related to the rotational inertia of the strip (regarded as a rod) about its center of mass, given by Eq. 9-16 as $dI_{cm} = \frac{1}{12}dm\,a^2$:

$$dI = dI_{cm} + dm\,h^2$$
$$= \tfrac{1}{12}dm\,a^2 + dm\,x^2.$$

Substituting for dm yields

$$dI = \frac{Ma^2}{12b}\,dx + \frac{M}{b}x^2\,dx,$$

and I follows from the integral

$$I = \int dI = \frac{Ma^2}{12b}\int dx + \frac{M}{b}\int x^2\,dx.$$

The limits of integration on x are from $-b/2$ to $+b/2$. Carrying out the integrations, we obtain

$$I = \tfrac{1}{12}M(a^2 + b^2). \tag{9-17}$$

Note that this result is independent of the thickness of the plate: we would get the same result for a stack of plates of total mass M or, equivalently, for a solid rectangular block of the same surface dimensions. Note also that our result depends on the diagonal length of the plate rather than on a and b separately. Can you explain this?

Working in this way, we can evaluate the rotational inertia of almost any regular solid object. Figure 9-15 shows some common objects and their rotational inertias. Although it is relatively straightforward to use two- and three-dimensional integrals to calculate these rotational inertias, it is often possible, as we did in the above calculation, to decompose a complex solid into simpler solids of known rotational inertia. Problem 16 at the end of this chapter describes such a calculation for a solid sphere.

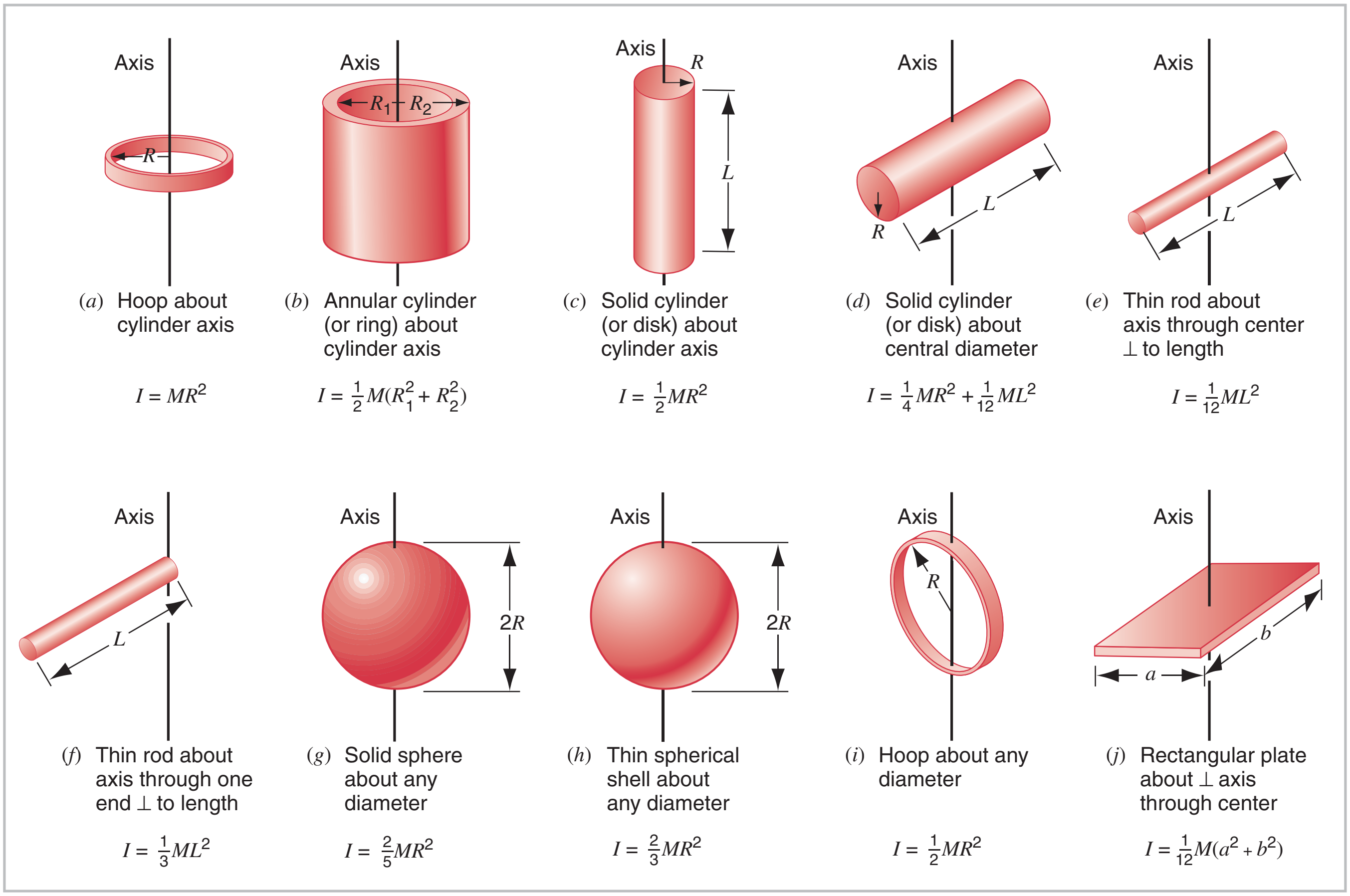


FIGURE 9-15. The rotational inertia of various solids about selected axes.

SAMPLE PROBLEM 9-5. Two identical solid spheres of mass M and radius R are joined together, and the combination is rotated about an axis tangent to one sphere and perpendicular to the line connecting them (Fig. 9-16). What is the rotational inertia of the combination?

Solution Like masses, rotational inertias of solid objects add like scalars, so the total for the two spheres is $I = I_1 + I_2$. For the first sphere (the one closer to the axis of rotation) we have, from the parallel axis theorem,

$$I_1 = I_{cm} + Mh^2 = \tfrac{2}{5}MR^2 + MR^2 = 1.4MR^2$$

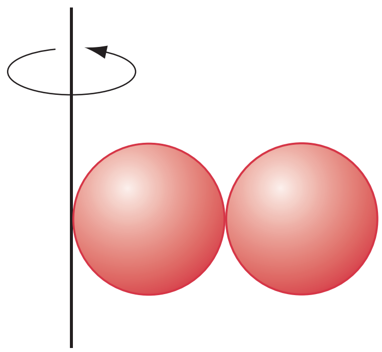


FIGURE 9-16. Sample Problem 9-5. Two spheres in contact are rotated about an axis.

and for the second sphere,

$$I_2 = I_{cm} + Mh^2 = \tfrac{2}{5}MR^2 + M(3R)^2 = 9.4MR^2.$$

The total is

$$I = I_1 + I_2 = 10.8MR^2.$$

9-4 TORQUE DUE TO GRAVITY

In Fig. 9-2, a force was applied to a single point on the body, and it was then possible to use Eq. 9-3 to find the torque due to that force. Suppose instead that you are holding one end of a long beam, the other end of which can pivot about a horizontal axis through the end (Fig. 9-17). If your hand were not supporting the end of the beam, it

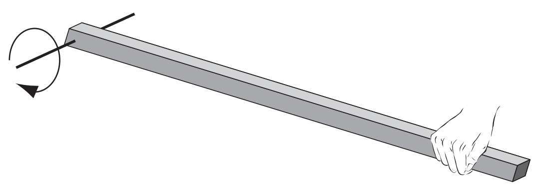


FIGURE 9-17. A hand holds one end of a beam that can rotate about a horizontal axis through the other end.

would rotate about the axis due to the downward force of gravity. If we regard the beam as a collection of point particles, then gravity acting downward gives a torque along the axis due to the weight of each particle. The net torque on the entire beam would be the sum of these individual torques, but that would be a hopelessly complicated problem to solve.

Fortunately, the problem usually can be simplified. We can replace the effect of gravity acting on all the particles of a body with a single force that has two characteristics: (1) it is equal to the weight of the object, and (2) it acts at a single point called the *center of gravity.* (As we show later, for most cases of interest—and for all cases considered in this book—the center of gravity of a body coincides with its center of mass.) Let us prove that the single force acting on an object has the two characteristics listed above.

Imagine the body of mass M (Fig. 9-18) to be divided into a large number of particles. The gravitational force exerted by the Earth on the nth particle of mass m_n is $m_n\vec{\mathbf{g}}$. This force is directed down toward the center of the Earth. The net force on the entire object due to gravity is the sum over all the individual particles, or

$$\sum \vec{\mathbf{F}} = \sum m_n\vec{\mathbf{g}}. \qquad (9\text{-}18)$$

Because we have assumed that $\vec{\mathbf{g}}$ has the same value for every particle of the body, we can factor $\vec{\mathbf{g}}$ out of the summation of Eq. 9-18, which gives

$$\sum \vec{\mathbf{F}} = \vec{\mathbf{g}} \sum m_n = M\vec{\mathbf{g}} \qquad (9\text{-}19)$$

This proves the first of the assertions we made above, that we can replace the resultant force of gravity acting on the entire body by the single force $M\vec{\mathbf{g}}$.

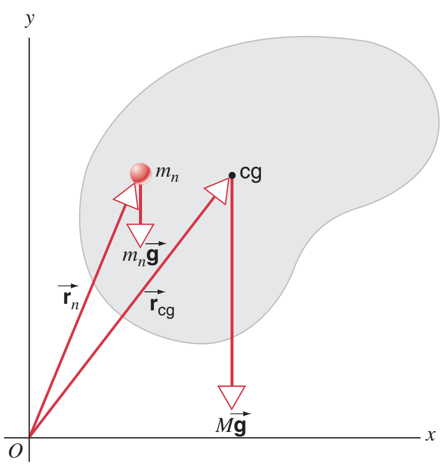


FIGURE 9-18. Each particle in a body, such as the one with mass m_n, experiences a gravitational force such as $m_n\vec{\mathbf{g}}$. The entire weight of a body, though distributed throughout its volume as the sum of the gravitational forces on all such particles, may be replaced by a single force of magnitude Mg acting at the center of gravity. If the gravitational field is uniform (that is, the same for all particles), the center of gravity coincides with the center of mass, and so $\vec{\mathbf{r}}_{cg}$ is the same as $\vec{\mathbf{r}}_{cm}$.

Let us now calculate the torque about an axis perpendicular to the page through the arbitrary point O, as shown in Fig. 9-18. The vector $\vec{\mathbf{r}}_n$ locates the particle of mass m_n relative to this origin. The net torque about this point due to gravity acting on all the particles is

$$\sum \vec{\boldsymbol{\tau}} = \sum (\vec{\mathbf{r}}_n \times m_n\vec{\mathbf{g}}) = \sum (m_n\vec{\mathbf{r}}_n \times \vec{\mathbf{g}}), \qquad (9\text{-}20)$$

where the last step is taken by moving the scalar m_n within the sum. Once again we use the constancy of $\vec{\mathbf{g}}$ to remove it from the summation, being careful not to change the order of the vectors $\vec{\mathbf{r}}_n$ and $\vec{\mathbf{g}}$ so that the sign of the cross product does not change. According to Eq. 7-11, the remaining summation, $\sum m_n\vec{\mathbf{r}}_n$, is just $M\vec{\mathbf{r}}_{cm}$, where $\vec{\mathbf{r}}_{cm}$ is the vector that locates the center of mass of the body relative to the origin O. Taking these two steps, we can write Eq. 9-20 as

$$\begin{aligned} \sum \vec{\boldsymbol{\tau}} &= \left(\sum m_n\vec{\mathbf{r}}_n\right) \times \vec{\mathbf{g}} = M\vec{\mathbf{r}}_{cm} \times \vec{\mathbf{g}} \\ &= \vec{\mathbf{r}}_{cm} \times M\vec{\mathbf{g}}. \end{aligned} \qquad (9\text{-}21)$$

The resultant torque on the body thus equals the torque that would be produced by the single force $M\vec{\mathbf{g}}$ acting at the center of mass of the body, and thus the center of gravity (cg) coincides with the center of mass, which proves the second assertion we made above. A useful corollary of Eq. 9-21 is that *the torque due to gravity about the center of mass of a body is zero.*

Center of Mass and Center of Gravity

In this section, we have used "center of mass" and "center of gravity" interchangeably. The center of mass is defined for any body and can be calculated, according to methods described in Chapter 7, from the distribution of mass within the body. The center of gravity, on the other hand, is defined only for bodies in a gravitational field. To calculate the center of gravity, we must know not only the mass distribution of the body, but also the variation of $\vec{\mathbf{g}}$ over the body. If $\vec{\mathbf{g}}$ is not constant over the body, then the center of gravity and the center of mass may not coincide, in which case $\vec{\mathbf{g}}$ cannot be removed from the sums in Eqs. 9-18 and 9-20.

Consider the "dumbbell" arrangement of Fig. 9-19, consisting of two spheres of equal mass connected by a rod of negligible mass. The axis of the rod is inclined at some nonzero angle from the horizontal. The center of mass lies

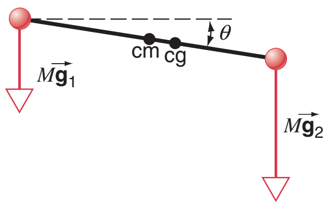


FIGURE 9-19. Two spheres of equal mass connected by a light rod. The center of mass lies halfway between the spheres. If the gravitational acceleration $\vec{\mathbf{g}}$ is greater at the location of the lower sphere, then the center of gravity is closer to that sphere.

at the geometrical center of the dumbbell. If its axis were horizontal, the center of gravity would coincide with the center of mass. When the axis is not horizontal, this is no longer true. Because g varies slightly with distance from the Earth, the lower sphere experiences a greater gravitational force than the higher sphere. As a result, the center of gravity is located slightly below the center of mass.

If the angle with the horizontal changes, or if we move the dumbbell to a place where g has a different value, the location of the center of gravity will change (while the center of mass remains fixed). Thus the location of the center of gravity depends on the orientation of the object as well as on the local gravitational attraction. For a dumbbell of length 1 m inclined at an angle of 45° near the Earth's surface, the distance between the center of mass and the center of gravity is about 55 nm, far smaller than the precision at which we normally work and therefore entirely negligible. We can safely assume that the center of gravity coincides with the center of mass.

If we suspend a body from an arbitrary point, it will come to rest in a position where the net force is zero and the net torque about any axis is zero. Because the net vertical force is zero, the downward weight must equal the upward force exerted at the point of support. The net torque must also be zero, so the two forces must be acting along the same vertical line.

The same conditions must be true if you try to balance a vertical meter stick on your hand. If the stick starts to tip, even slightly, the downward weight and the upward force of your hand will not be acting along the same line, and there will be a net torque on the stick that causes it to rotate and fall to the floor. You must therefore constantly move your hand to keep the upward force directly under the center of gravity of the stick.

We can use this property to find the center of gravity of an extended object. Consider a body of arbitrary shape suspended from a point S (Fig. 9-20). The point of support must be on a vertical line with the center of gravity. If we draw a vertical line through S, then we know that the center of gravity must lie somewhere on the line. We can repeat the procedure with a new choice of point S, as in Fig. 9-20*b*, and we find a second line that must contain the center of gravity. The center of gravity must therefore lie at the intersection of the two lines.

If we suspend the object from the center of gravity, as in Fig. 9-20*c*, and release it, the body will remain at rest no matter what its orientation. We can turn it any way we wish, and it remains at rest. This illustrates the corollary of Eq. 9-21: the torque due to gravity is zero about the center of gravity, because $\vec{\mathbf{r}}_{cm}$ is zero at that point.

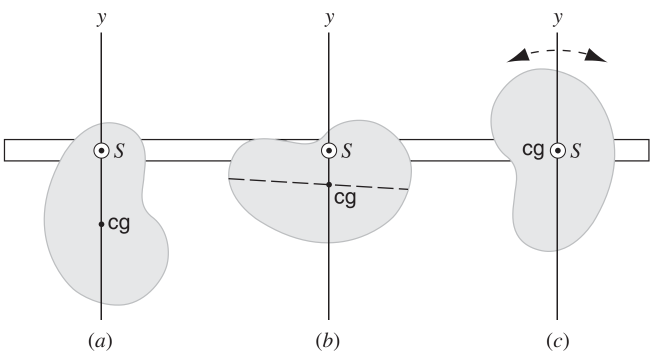


FIGURE 9-20. A body suspended from an arbitrary point S, as in (*a*) and (*b*), will be in stable equilibrium only if its center of gravity (cg) hangs vertically below its suspension point S. The dashed line in (*b*) represents the vertical line in (*a*), showing that the center of gravity can be located by suspending the body successively from two different points. (*c*) If a body is suspended at its center of gravity, it is in equilibrium no matter what its orientation.

9-5 EQUILIBRIUM APPLICATIONS OF NEWTON'S LAWS FOR ROTATION

It is possible for the net external force acting on a body to be zero, while the net external torque is nonzero. For example, consider two forces of equal magnitude that act on a body in opposite directions but not along the same line. This body will have an angular acceleration but no linear or translational acceleration. It is also possible for the net external torque on a body to be zero, while the net external force is not (a body falling in gravity); in this case there is a translational acceleration but no angular acceleration. For a body to be in equilibrium *both the net external force and the net external torque must be zero.* In this case the body will have *neither* an angular acceleration nor a translational acceleration. According to this definition, the body could have a linear or an angular velocity, as long as that velocity is constant. However, we will most often consider the special case in which the body is at rest.

We therefore have two conditions of equilibrium:

$$\sum \vec{\mathbf{F}}_{ext} = 0 \tag{9-22}$$

and

$$\sum \vec{\boldsymbol{\tau}}_{ext} = 0. \tag{9-23}$$

Each of these vector equations can be replaced with its equivalent three component (scalar) equations:

$$\sum F_x = 0, \qquad \sum F_y = 0, \qquad \sum F_z = 0 \tag{9-24}$$

and

$$\sum \tau_x = 0, \qquad \sum \tau_y = 0, \qquad \sum \tau_z = 0, \tag{9-25}$$

where for convenience we have dropped the subscript "ext" from these equations. At equilibrium, the sum of the external force components and the sum of the external torque components along each of the coordinate axes must be zero. This must be true for any choice of the directions of the coordinate axes.

The equilibrium condition for the torques is true for any choice of the axis about which the torques are calculated. To prove this statement, we consider a rigid body on which many forces act. Relative to the origin O, force $\vec{F}_1$ is applied at the point located at $\vec{r}_1$, force $\vec{F}_2$ at $\vec{r}_2$, and so on. The net torque about an axis through O is therefore

$$\begin{aligned}\vec{\tau}_O &= \vec{\tau}_1 + \vec{\tau}_2 + \cdots + \vec{\tau}_N \\ &= \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \cdots + \vec{r}_N \times \vec{F}_N. \end{aligned} \quad (9\text{-}26)$$

Suppose a point P is located at displacement $\vec{r}_P$ with respect to O (Fig. 9-21). The point of application of $\vec{F}_1$, with respect to P, is $(\vec{r}_1 - \vec{r}_P)$. The torque about P is

$$\begin{aligned}\vec{\tau}_P &= (\vec{r}_1 - \vec{r}_P) \times \vec{F}_1 + (\vec{r}_2 - \vec{r}_P) \times \vec{F}_2 \\ &\quad + \cdots + (\vec{r}_N - \vec{r}_P) \times \vec{F}_N \\ &= [\vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \cdots + \vec{r}_N \times \vec{F}_N] \\ &\quad - [\vec{r}_P \times \vec{F}_1 + \vec{r}_P \times \vec{F}_2 + \cdots + \vec{r}_P \times \vec{F}_N].\end{aligned}$$

The first group of terms in the brackets gives $\vec{\tau}_O$ according to Eq. 9-26. We can rewrite the second group by removing the constant factor of $\vec{r}_P$:

$$\begin{aligned}\vec{\tau}_P &= \vec{\tau}_O - [\vec{r}_P \times (\vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N)] \\ &= \vec{\tau}_O - [\vec{r}_P \times (\textstyle\sum \vec{F}_{\rm ext})] \\ &= \vec{\tau}_O,\end{aligned}$$

where we make the last step because $\Sigma \vec{F}_{\rm ext} = 0$ for a body in translational equilibrium. Thus the torque about any two points has the same value when the body is in translational equilibrium.

Often we deal with problems in which all the forces lie in the same plane. In this case the six conditions of Eqs. 9-24 and 9-25 are reduced to three. We resolve the forces into two components:

$$\sum F_x = 0, \qquad \sum F_y = 0, \qquad (9\text{-}27)$$

and, if we calculate torques about a point that also lies in the xy plane, all torques must be in the direction perpendicular to the xy plane. In this case we have

$$\sum \tau_z = 0, \qquad (9\text{-}28)$$

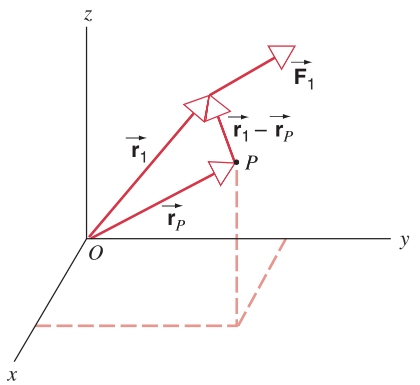


FIGURE 9-21. The force $\vec{F}_1$ is one of N external forces that act on a rigid body (not shown). The vector $\vec{r}_1$ locates the point of application of $\vec{F}_1$ relative to O and is used in calculating the torque of $\vec{F}_1$ about O. The vector $\vec{r}_1 - \vec{r}_P$ is used in calculating the torque of $\vec{F}_1$ about P.

We limit ourselves mostly to planar problems to simplify the calculations; this condition does not impose any fundamental restriction on the application of the general principles of equilibrium.

Equilibrium Analysis Procedures

Usually in equilibrium problems, we are interested in determining the values of one or more unknown forces by applying the conditions for equilibrium (zero net external force and zero net external torque). Here are the procedures you should follow:

1. *Draw a boundary around the system,* so that you can clearly separate the system you are considering from its environment.

2. *Draw a free-body diagram showing all external forces that act on the system and their points of application.* External forces are those that act through the system boundary that you drew in step 1; these often include gravity, friction, and forces exerted by wires or beams that cross the boundary. Internal forces (those that objects within the system exert on each other) should not appear in the diagram. Sometimes the direction of a force may not be obvious in advance. If you imagine making a cut through the beam or wire where it crosses the boundary, the ends of this cut will pull apart if the force acts outward from the boundary. If you are in doubt, choose the direction arbitrarily, and if you have guessed wrong your solution will result in negative values for the components of that force.

3. *Set up a coordinate system and choose the direction of the axes.* This coordinate system will be used to resolve the forces into their components.

4. *Set up a coordinate system and axes for resolving the torques into their components.* In equilibrium, the net external torque must be zero about *any* axis. Often you can choose to calculate torques about a point through which several forces act, thereby eliminating those forces from the torque equation. In adding torque components, we follow the sign convention that *the torque along any axis is positive if acting alone it would produce a counterclockwise rotation about that axis.* The right-hand rule for torques can also be used to establish this convention.

Once we have carried out these steps in setting up the problem, we can carry out the solution using Eqs. 9-22 and 9-23 or 9-27 and 9-28, as the following problems illustrate.

SAMPLE PROBLEM 9-6. A uniform beam of length L whose mass m is 1.8 kg rests with its ends on two digital scales, as in Fig. 9-22a. A block whose mass M is 2.7 kg rests on the beam, its center one-fourth of the way from the beam's left end. What do the scales read?

Solution We choose as our system the beam and the block, taken together. Figure 9-22b is a free-body diagram for this system, showing all the external forces that act on the system. The weight of the beam, $m\vec{g}$, acts downward at its center of mass, which is at its geometric center, since the beam is uniform. Similarly, $M\vec{g}$, the weight of the block, acts downward at its center of mass. The scales push

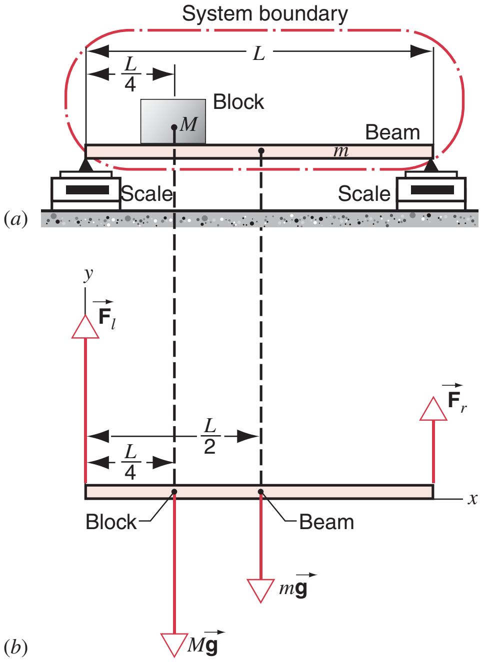


Figure 9-22. Sample Problem 9-6. (*a*) A beam of mass *m* supports a block of mass *M*. The digital scales display the vertical forces exerted on the two ends of the beam. (*b*) A free-body diagram showing the forces that act on the system, consisting of beam + block.

upward at the ends of the beam with forces $\vec{F}_l$ and $\vec{F}_r$. The magnitudes of these latter two forces are the scale readings that we seek.

Our system is in static equilibrium, so we can apply the conditions of Eqs. 9-27 and 9-28. The forces have no *x* components, so the equation $\Sigma F_x = 0$ provides no information. The *y* component of the net external force is $\Sigma F_y = F_l + F_r - Mg - mg$. With the equilibrium condition $\Sigma F_y = 0$, we have

$$F_l + F_r - Mg - mg = 0. \tag{9-29}$$

Further information about the unknown forces F_l and F_r comes from the torque equation (Eq. 9-28). We choose to take torques about an axis through the left end of the beam. The force F_l has a moment arm of zero. Using the right-hand rule, we conclude that F_r gives a positive torque, and Mg and mg give negative torques. The net torque is found by multiplying each force by its moment arm (in this case, its distance from the chosen axis): $\Sigma \tau_z = (F_l)(0) + (F_r)(L) - (mg)(L/2) - (Mg)(L/4)$. With $\Sigma \tau_z = 0$, we have

$$F_r L - \frac{mgL}{2} - \frac{MgL}{4} = 0 \tag{9-30}$$

or

$$F_r = \left(\frac{g}{4}\right)(M + 2m)$$

$$= \tfrac{1}{4}(9.8 \text{ m/s}^2)[2.7 \text{ kg} + 2(1.8 \text{ kg})] = 15 \text{ N}.$$

Note how our choice of axis eliminates the force F_l from the torque equation and allows us to solve directly for the other force. If we had chosen to take torques about any arbitrary point, we would have obtained an equation involving F_l and F_r which could be solved simultaneously with Eq. 9-29. Our choice of axis helps to simplify the algebra somewhat, but of course it in no way changes the ultimate solution.

If we substitute the value of F_r into Eq. 9-29 and solve for F_l, we find

$$F_l = (M + m)g - F_r$$
$$= (2.7 \text{ kg} + 1.8 \text{ kg})(9.8 \text{ m/s}^2) - 15 \text{ N} = 29 \text{ N}.$$

Note that the length of the beam and the height of the center of mass of the block do not enter the solution to this problem. Is this physically reasonable?

Try solving this problem by using only the balance of torques equation, once for an axis at the left end of the beam and again for an axis at the right end of the beam. This method, just like the method we used in solving this problem, gives two equations that can be solved for the unknowns F_l and F_r.

Sample Problem 9-7. A ladder whose length *L* is 12 m and whose mass *m* is 45 kg rests against a wall. Its upper end is a distance *h* of 9.3 m above the ground, as in Fig. 9-23*a*. The center of mass of the ladder is one-third of the way up the ladder. A firefighter whose mass *M* is 72 kg climbs halfway up the ladder. Assume that the wall, but not the ground, is frictionless. What forces are exerted on the ladder by the wall and by the ground?

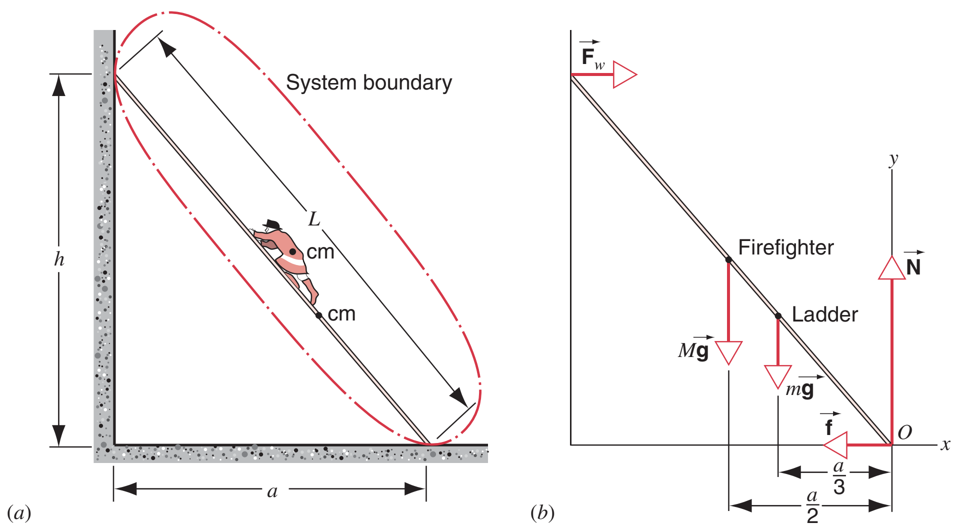


Figure 9-23. Sample Problem 9-7. (*a*) A firefighter climbs halfway up a ladder that is leaning against a frictionless wall. (*b*) A free-body diagram, showing (to scale) all the forces that act.

Solution Figure 9-23*b* shows a free-body diagram. The wall exerts a horizontal force $\vec{F}_w$ on the ladder; it can exert no vertical force because the wall–ladder contact is assumed to be frictionless. The ground exerts a force on the ladder with a horizontal component f due to friction and a vertical component N, the normal force. We choose coordinate axes as shown, with the origin O at the point where the ladder meets the ground. The distance a from the wall to the foot of the ladder is readily found from

$$a = \sqrt{L^2 - h^2} = \sqrt{(12\text{ m})^2 - (9.3\text{ m})^2} = 7.6\text{ m}.$$

The x and y components of the net force on the ladder are $\Sigma F_x = F_w - f$ and $\Sigma F_y = N - Mg - mg$. Equations 9-27 ($\Sigma F_x = 0$ and $\Sigma F_y = 0$) then give

$$F_w - f = 0 \qquad \text{and} \qquad N - Mg - mg = 0. \qquad (9\text{-}31)$$

From the second of these equations,

$$N = (M + m)g = (72\text{ kg} + 45\text{ kg})(9.8\text{ m/s}^2) = 1150\text{ N}.$$

Taking torques about an axis through point O and parallel to the z direction, we note that F_w gives a negative torque, Mg and mg give positive torques, and N and f give zero torques about O because their moment arms are zero. Multiplying each force by its moment arm, we find $\Sigma \tau_z = -(F_w)(h) + (Mg)(a/2) + (mg)(a/3) + (N)(0) + (f)(0)$. Using Eq. 9-28 ($\Sigma \tau_z = 0$), we find

$$-F_w h + \frac{Mga}{2} + \frac{mga}{3} = 0. \qquad (9\text{-}32)$$

This wise choice of location for the axis eliminated two variables, f and N, from the balance of torques equation. We find, solving Eq. 9-32 for F_w,

$$F_w = \frac{ga(M/2 + m/3)}{h} = \frac{(9.8\text{ m/s}^2)(7.6\text{ m})[(72\text{ kg})/2 + (45\text{ kg})/3]}{9.3\text{ m}} = 410\text{ N}.$$

From Eq. 9-31 we have at once

$$f = F_w = 410\text{ N}.$$

SAMPLE PROBLEM 9-8. A uniform beam of length $L = 3.3$ m and mass $m = 8.5$ kg is hinged at a wall as in Fig. 9-24*a*. A wire connected to the wall a distance $d = 2.1$ m above the hinge is connected to the other end of the beam, the length of the wire being such that the beam makes an angle of $\theta = 30°$ with the horizontal. A body of mass $M = 56$ kg is suspended from the upper end of the beam. Find the tension in the wire and the force exerted by the hinge on the beam.

Solution Figure 9-24*b* shows all the external forces that act on the beam, which we have chosen as our system. Because two of the forces are directed vertically downward, we choose our axes to be horizontal and vertical. The tension in the wire and the force exerted by the hinge on the beam are represented by their horizontal and vertical components.

The components of the net force on the beam are $\Sigma F_x = F_h - T_h$ and $\Sigma F_y = F_v + T_v - Mg - mg$, and the equilibrium condition for the force (Eq. 9-27) gives

$$F_h - T_h = 0 \qquad \text{and} \qquad F_v + T_v - Mg - mg = 0. \qquad (9\text{-}33)$$

To apply the equilibrium condition on the torque (Eq. 9-28), we choose our axis at the upper end of the beam (why?), and we find the net torque by multiplying each force by its moment arm:

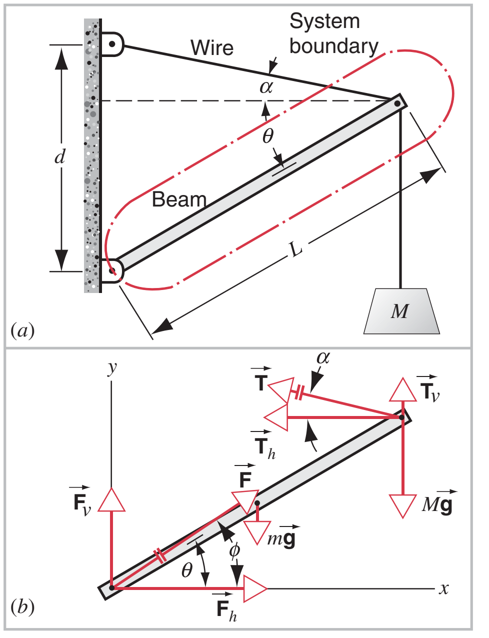


FIGURE 9-24. Sample Problem 9-8. (*a*) A beam is supported by a hinge on a wall at its lower end and by a wire to the wall at its upper end. An object of mass M hangs from the upper end of the beam. (*b*) A free-body diagram, showing the forces acting on the beam. A force $\vec{F}$ is exerted by the hinge and a force $\vec{T}$ is supplied by the tension in the wire.

$\Sigma \tau_z = -(F_v)(L\cos\theta) + (F_h)(L\sin\theta) + (mg)(\tfrac{1}{2}L\cos\theta) + (T_v)(0) + (T_h)(0) + (Mg)(0)$. Setting this equal to 0 and doing some manipulation, we find

$$F_v - F_h \tan\theta - mg/2 = 0. \qquad (9\text{-}34)$$

So far we have three equations in the four unknowns (F_v, F_h, T_v, T_h). A fourth relationship comes from the requirement that T_v and T_h must add to give a resultant tension $\vec{T}$ directed along the wire. The wire cannot support a force component perpendicular to its long dimension. (This is not true for the rigid beam.) The fourth equation is

$$T_v = T_h \tan\alpha, \qquad (9\text{-}35)$$

where $\tan\alpha = (d - L\sin\theta)/(L\cos\theta)$.

Combining the four equations we find, after doing the necessary algebra,

$$F_v = 506\text{ N}, \quad F_h = 804\text{ N}, \quad T_v = 126\text{ N}, \quad T_h = 804\text{ N}.$$

The tension in the wire will then be

$$T = \sqrt{T_h^2 + T_v^2} = 814\text{ N},$$

and the force exerted by the hinge on the beam is

$$F = \sqrt{F_h^2 + F_v^2} = 950\text{ N}.$$

Note that both T and F are considerably larger than the combined weights of the beam and the suspended body (632 N).

The vector $\vec{F}$ makes an angle with the horizontal of

$$\phi = \tan^{-1}\frac{F_v}{F_h} = 32.2°.$$

Thus the resultant force vector acting on the beam at the hinge does not point along the beam direction.

In the preceding examples we have been careful to limit the number of unknown forces to the number of independent equations relating the forces. When all the forces act in a plane, we can have only three independent equations of equilibrium, one for rotational equilibrium about any axis normal to the plane and two others for translational equilibrium in the plane. However, we often have more than three unknown forces. For example, in Sample Problem 9-7, if we drop the assumption of a frictionless wall, we have four unknown quantities—namely, the horizontal and vertical components of the force acting on the ladder at the wall and the horizontal and vertical components of the force acting on the ladder at the ground. Because we have only three equations, these forces cannot be determined. We must therefore be able to find another independent relation between the unknown forces if we hope to solve the problem uniquely. (In Sample Problem 9-8, this last equation came from a physical property of one of the elements of the system.) Taking torques about a second axis does not give a fourth independent equation; you can show that such an equation is a linear combination of the first torque equation and the two force equations, and so it contains no new information.

Another simple example of an undetermined structure occurs when we wish to determine the forces exerted by the ground on each of the four tires of an automobile when it is at rest on a horizontal surface. If we assume that these forces are normal to the ground, we have four unknown quantities. We have only three independent equations giving the equilibrium conditions—one for translational equilibrium in the single direction of all the forces and two for rotational equilibrium about the two axes perpendicular to each other in a horizontal plane. Again the solution of the problem is mathematically indeterminate.

Of course, since there is actually a unique solution to this physical problem, we must find a physical basis for the additional independent relation between the forces that enables us to solve the problem. The difficulty is removed when we realize that structures are never perfectly rigid, as we have assumed throughout. All structures are actually somewhat deformed. For example, the automobile tires and the ground are deformed, as are the ladder and wall. The laws of elasticity and the elastic properties of the structure provide the necessary additional relation between the four forces. A complete analysis therefore requires not only the laws of rigid body mechanics but also the laws of elasticity.

9-6 NONEQUILIBRIUM APPLICATIONS OF NEWTON'S LAWS FOR ROTATION

In this section we remove the restriction from the previous section in which the angular acceleration was zero because the net torque was zero. Here we consider cases in which a nonzero net torque acts on a body and imparts an angular acceleration to it.

In the case of linear motion in one dimension, we solve similar problems using Newton's second law, $\Sigma F_x = ma_x$, where one component of the net force produces a component of the acceleration along the same coordinate axis. To maintain the analogy with Newton's laws for linear motion, we continue the restriction that the body rotate about a single fixed axis. We use the rotational form of Newton's second law (Eq. 9-11), $\Sigma \tau_z = I\alpha_z$, where (as in the previous section) we have for convenience dropped the "ext" subscript with the understanding that we are considering *only* external torques in our analysis.

In this section we will analyze problems involving angular accelerations produced by a torque applied to an object with a fixed axis of rotation. In the next section we will broaden the discussion somewhat to include cases in which the object rotates and also moves linearly (but keeps the axis of rotation in a fixed direction). In Chapter 10 we consider rotations in which the axis is not fixed in direction.

SAMPLE PROBLEM 9-9. A playground merry-go-round is pushed by a parent who exerts a force $\vec{\mathbf{F}}$ of magnitude 115 N at a point P on the rim a distance of $r = 1.50$ m from the axis of rotation (Fig. 9-25). The force is exerted in a direction at an angle 32° below the horizontal, and the horizontal component of the force is in a direction 15° inward from the tangent at P. (*a*) Find the magnitude of the component of the torque that accelerates the merry-go-round. (*b*) Assuming that the merry-go-round can be represented as a steel disk 1.5 m in radius and 0.40 cm thick and that the child riding on it can be represented as a 25-kg "particle" 1.0 m from the axis of rotation, find the resulting angular acceleration of the system including the merry-go-round and child.

Solution (*a*) Only the horizontal component of $\vec{\mathbf{F}}$ produces a vertical torque. Let us find $F_\perp$, the component of $\vec{\mathbf{F}}$ along the horizontal line perpendicular to $\vec{\mathbf{r}}$. The horizontal component of $\vec{\mathbf{F}}$ is

$$F_\text{h} = F\cos 32° = 97.5 \text{ N}.$$

The component of F_h perpendicular to $\vec{\mathbf{r}}$ is

$$F_\perp = F_\text{h}\cos 15° = 94.2 \text{ N}.$$

The (vertical) torque along the axis of rotation is thus

$$\tau = rF_\perp = (1.50 \text{ m})(94.2 \text{ N}) = 141 \text{ N}\cdot\text{m}.$$

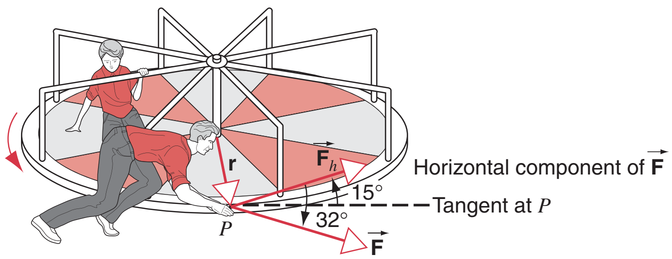


FIGURE 9-25. Sample Problem 9-9. A parent pushes a playground merry-go-round. The parent is leaning down, so the force has a downward component. Furthermore, because the parent is outside the rim, the force is directed slightly inward. The horizontal component of the force, F_h, is in the plane of the rotating platform and makes an angle of 15° with the tangent at P, the point at which the force is applied.

The component of F_h parallel to $r\,(= F_h \sin 15°)$ produces no torque at all about the axis of rotation, and the vertical component of $F\,(= F \sin 32°)$ produces a torque perpendicular to the axis that would tend to tip the rotating platform out of the horizontal plane (because the parent is pushing *down* on the platform) if that torque were not opposed by an equal and opposite torque from the bearings.

(*b*) The merry-go-round is a circular disk of radius $R = 1.5$ m and thickness $d = 0.40$ cm. Its volume is $\pi R^2 d = 2.83 \times 10^4\ \text{cm}^3$. The density of steel is 7.9 g/cm^3, so the mass of the merry-go-round is $(2.8 \times 10^4\ \text{cm}^3)(7.9\ \text{g/cm}^3) = 2.23 \times 10^5\ \text{g} = 223$ kg. From Figure 9-15*c* we obtain the rotational inertia of a disk rotated about an axis perpendicular to its center:

$$I_m = \tfrac{1}{2}MR^2 = \tfrac{1}{2}(223\ \text{kg})(1.5\ \text{m})^2 = 251\ \text{kg}\cdot\text{m}^2.$$

The rotational inertia of the child, whom we treat as a particle of mass $m = 25$ kg at a distance of $r = 1.0$ m from the axis of rotation, is

$$I_c = mr^2 = (25\ \text{kg})(1.0\ \text{m})^2 = 25\ \text{kg}\cdot\text{m}^2.$$

The total rotational inertia is $I_t = I_m + I_c = 251\ \text{kg}\cdot\text{m}^2 + 25\ \text{kg}\cdot\text{m}^2 = 276\ \text{kg}\cdot\text{m}^2$. The angular acceleration can now be found from Eq. 9-11:

$$\alpha_z = \frac{\tau_z}{I_t} = \frac{141\ \text{N}\cdot\text{m}}{276\ \text{kg}\cdot\text{m}^2} = 0.51\ \text{rad/s}^2.$$

Based on the direction of the force shown in Fig. 9-25, the right-hand rule indicates that both τ_z and α_z point vertically upward from the plane of the merry-go-round.

SAMPLE PROBLEM 9-10. Figure 9-26*a* shows a pulley, which can be considered as a uniform disk of mass $M = 2.5$ kg and radius $R = 20$ cm, mounted on a fixed (frictionless) horizontal axle. A block of mass $m = 1.2$ kg hangs from a light cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the tension in the cord, and the angular acceleration of the disk.

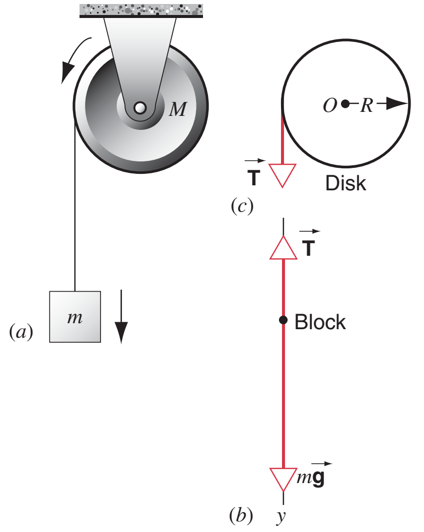


FIGURE 9-26. Sample Problem 9-10. (*a*) A falling block causes the disk to rotate. (*b*) A free-body diagram for the block. (*c*) A partial free-body diagram for the disk. The directions taken as positive are shown by the arrows in (*a*). The positive *z* axis is out of the page.

Solution Figure 9-26*b* shows a free-body diagram for the block. Note that, in drawing the free-body diagram for the analysis of rotations, it is necessary to show the forces *and* their points of application, so that we may determine the line of action for each force in calculating the corresponding torque. We choose the *y* axis to be positive downward, so that the net force is $\Sigma F_y = mg - T$, which is a positive quantity if the block accelerates downward. Using the *y* component of Newton's second law ($\Sigma F_y = ma_y$), we have

$$mg - T = ma_y.$$

Figure 9-26*c* shows a partial free-body diagram for the disk. Choosing the positive *z* axis to be out of the plane of the figure, the *z* component of the net torque about *O* is $\Sigma \tau_z = TR$ (neither the weight of the disk nor the upward force exerted at its point of support contribute to the torque about *O*, because both their lines of action pass through *O*). Applying the rotational form of Newton's second law (Eq. 9-11) gives $TR = I\alpha_z$, where α_z is positive for the counterclockwise rotation. With $I = \frac{1}{2}MR^2$ and $\alpha_z = a_T/R$, we obtain $TR = (\frac{1}{2}MR^2)(a_T/R)$ or

$$T = \tfrac{1}{2}Ma_T.$$

Because the cord does not slip or stretch, the acceleration a_y of the block must equal the tangential acceleration a_T of a point on the rim of the disk. With $a_y = a_T = a$, we can combine the equations for the block and the disk to obtain

$$a = g\frac{2m}{M + 2m} = (9.8\ \text{m/s}^2)\frac{(2)(1.2\ \text{kg})}{2.5\ \text{kg} + (2)(1.2\ \text{kg})} = 4.8\ \text{m/s}^2,$$

and

$$T = mg\frac{M}{M + 2m} = (1.2\ \text{kg})(9.8\ \text{m/s}^2)\frac{2.5\ \text{kg}}{2.5\ \text{kg} + (2)(1.2\ \text{kg})} = 6.0\ \text{N}.$$

As expected, the acceleration of the falling block is less than *g*, and the tension in the cord (= 6.0 N) is less than the weight of the hanging block ($= mg = 11.8$ N). We see also that the acceleration of the block and the tension depend on the mass of the disk but not on its radius. As a check, we note that the formulas derived above predict $a = g$ and $T = 0$ for the case of a massless disk ($M = 0$). This is what we expect; the block simply falls as a free body, trailing the cord behind it.

The angular acceleration of the disk follows from

$$\alpha_z = \frac{a}{R} = \frac{4.8\ \text{m/s}^2}{0.20\ \text{m}} = 24\ \text{rad/s}^2 = 3.8\ \text{rev/s}^2$$

and it is positive, corresponding to a rotation in the direction of the arrow in Fig. 9-26*a*.

For rotations about a fixed axis, the angular velocity and acceleration have only one component, and therefore only that same component of the torque enters into Newton's laws. However, we may apply a force to a rigid body in any direction, and there will in general be two or three components to the torque, only one of which actually produces rotations. What happens to the other components?

Consider the bicycle wheel shown in Fig. 9-27. The axle of the wheel is fixed in direction by the two bearings, so

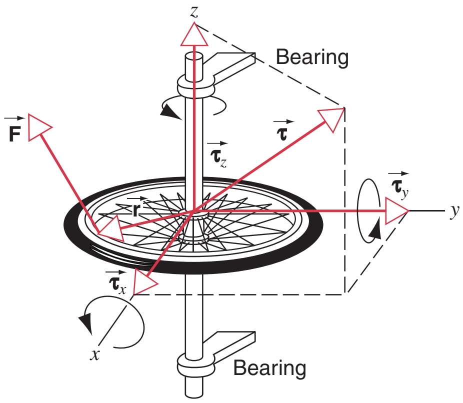


Figure 9-27. A rigid body, in this case a wheel, is free to rotate about the z axis. An arbitrary force $\vec{\mathbf{F}}$, shown acting at a point on the rim, can produce torque components along the three coordinate axes. Only the z component is successful in rotating the wheel. The x and y components of the torque would tend to tip the axis of rotation away from the z axis. This tendency must be opposed by equal and opposite torques (not shown) exerted by the bearings, which hold the axis in a fixed direction.

that the rotation axis corresponds to the z axis. A force $\vec{\mathbf{F}}$ is applied to the wheel in an arbitrary direction, and in general the associated torque may have x, y, and z components, as shown in Fig. 9-27. Each component of the torque tends to produce rotation about its corresponding axis. However, we have assumed that the body is fixed in such a way that rotation about only the z axis is possible. The x and y components of the torque produce no motion. In this case, the bearings serve to constrain the system to rotate about only the z axis, and they must therefore provide torques that cancel the x and y components of the torque from the applied force. This indicates what is meant by a body constrained to move about a fixed axis: only torque components parallel to that axis are effective in rotating the body, and torque components perpendicular to the axis are assumed to be balanced by other parts of the system. The bearings *must* provide torques with x and y components to keep the direction of the axis of rotation fixed; the bearings *may* also provide a torque in the z direction, such as in the case of nonideal bearings that exert frictional forces on the axle of the wheel. Since the center of mass of the wheel does not move, the forces exerted by the bearings must add to the external forces to give a net force of zero.

9-7 COMBINED ROTATIONAL AND TRANSLATIONAL MOTION

Figure 9-28 shows a time-exposure photograph of a rolling wheel. This is one example of a possibly complex motion in which an object simultaneously undergoes both rotational and translational displacements.

In general, the translational and rotational motions are completely independent. For example, consider a puck sliding across a horizontal surface (perhaps a sheet of ice). You can start the puck in translational motion only (no rotation), or you can spin it in one place so that it has only rotational and no translational motion. Alternatively, you can simultaneously push the puck (with any linear velocity) and rotate it (with any angular velocity), so it moves across the ice with both translational and rotational motion. The center of mass moves in a straight line (even in the presence of an external force such as friction), but the motion of any other point of the puck may be a complicated combination of the rotational and translational motions, like the point on the rim of the wheel in Fig. 9-28.

As represented by the sliding puck or the rolling wheel, we restrict our discussion of this combined motion to cases satisfying two conditions: (1) the axis of rotation passes through the center of mass (which serves as the reference point for calculating torque and angular momentum), and (2) the axis always has the same direction in space (that is, the axis at one instant is parallel to the axis at any other instant). If these two conditions are valid, we may apply Eq. 9-11 ($\Sigma\, \tau_z = I\alpha_z$, using only *external* torques) to the rotational motion. Independent of the rotational motion, we may apply Eq. 7-16 ($\Sigma\, \vec{\mathbf{F}} = M\vec{\mathbf{a}}_{cm}$, using only *external* forces) to the translational motion.

There is one special case of this type of motion that we often observe; this case is illustrated by the rolling wheel of Fig. 9-28. Note that where the illuminated point on the rim

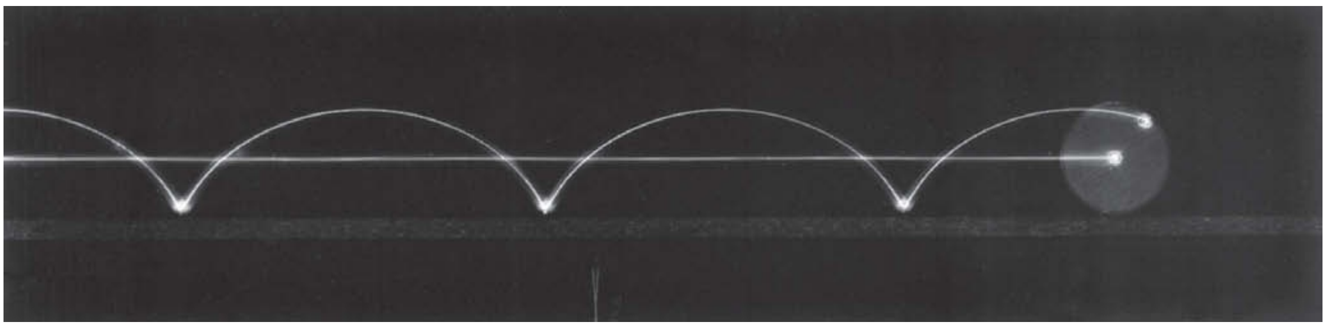


Figure 9-28. A time-exposure photo of a rolling wheel. Small lights have been attached to the wheel, one at its center and another at its edge. The latter traces out a curve called a *cycloid.*

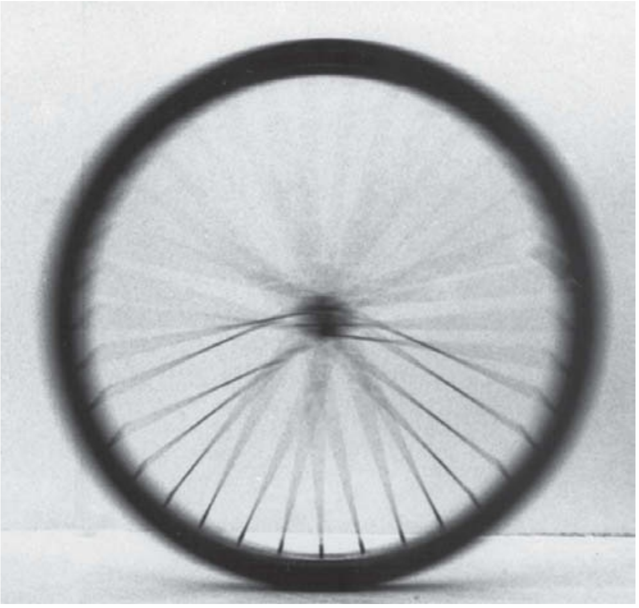


FIGURE 9-29. A photo of a rolling bicycle wheel. Note that the spokes near the top of the wheel are more blurred than those near the bottom. This is because the top has a greater linear velocity.

contacts the surface, the light seems especially bright, corresponding to a long exposure of the film. At these instants, that point is moving very slowly relative to the surface, or may perhaps be instantaneously at rest. This special case, in which an object rolls across a surface in such a way that there is no relative motion between the object and the surface at the instantaneous point of contact, is called *rolling without slipping.*

Figure 9-29 shows another example of rolling without slipping. Note that the spokes of the bicycle wheel near the bottom are in sharper focus than the spokes at the top, which appear blurred. The top of the wheel is clearly moving faster than the bottom! In rolling without slipping, the frictional force between the wheel and the surface is responsible for preventing the relative motion at the point of contact. Even though the wheel is moving, it is the force of *static* friction that applies.

Not all cases of rolling on a frictional surface result in rolling without slipping. For example, imagine a car trying to start on an icy street. At first, perhaps the wheels spin in place, so we have pure rotation with no translation. If sand is placed on the ice, the wheels still spin rapidly, but the car begins to inch forward. There is still some slipping between the tires and the ice, but we now have some translational motion. Eventually the tires stop slipping on the ice, so there is no relative motion between them; this is the condition of rolling without slipping.

Figure 9-30 shows one way to view rolling without slipping as a combination of rotational and translational motions. In pure translational motion (Fig. 9-30*a*), the center of mass C (along with every point on the wheel) moves with velocity v_{cm} to the right. In pure rotational motion (Fig. 9-30*b*) at angular speed ω, every point on the rim has tangential speed ωR. When the two motions are combined, the resulting velocity of point B (at the bottom of the wheel) is $v_{cm} - \omega R$. For rolling without slipping, the point where the wheel contacts the surface must be at rest; thus $v_{cm} - \omega R = 0$, or

$$v_{cm} = \omega R. \tag{9-36}$$

Superimposing the resulting translational and rotational motions, we obtain Fig. 9-30*c*. Note that the linear speed at the top of the wheel (point T) is exactly twice that at the center.

Equation 9-36 applies *only* in the case of rolling without slipping; in the general case of combined rotational and translational motion, v_{cm} does not equal ωR.

There is yet another instructive way to analyze rolling without slipping: we consider the point of contact B to be an instantaneous axis of rotation, as illustrated in Fig. 9-31. At each instant there is a new point of contact B and therefore a new axis of rotation, but instantaneously the motion consists of a pure rotation about B. The angular velocity of this rotation about B is the same as the angular velocity ω of the rotation about the center of mass. Since the distance from B to T is twice the distance from B to C, once again we conclude that the linear speed at T is twice that at C.

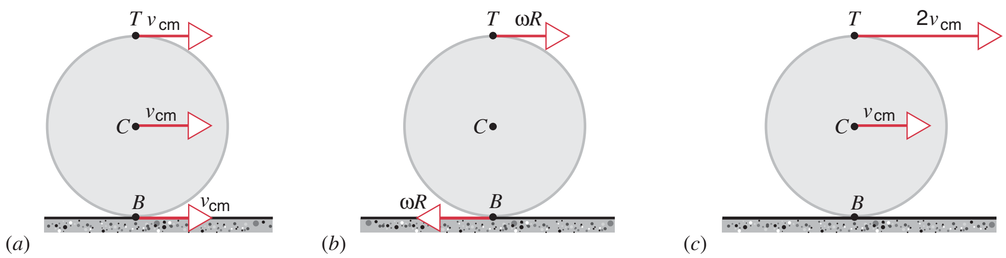


FIGURE 9-30. Rolling can be viewed as a superposition of pure translation and rotation about the center of mass. (*a*) The translational motion, in which all points move with the same linear velocity. (*b*) The rotational motion, in which all points move with the same angular velocity about the central axis. (*c*) The superposition of (*a*) and (*b*), in which the velocities at T, C, and B have been obtained by vector addition of the translational and rotational components.

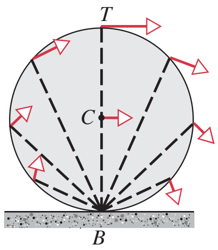


FIGURE 9-31. A rolling body can be considered to be rotating about an instantaneous axis at the point of contact B. The vectors show the instantaneous linear velocities of selected points.

SAMPLE PROBLEM 9-11. A solid cylinder of mass M and radius R starts from rest and rolls without slipping down an inclined plane of length L and height h (Fig. 9-32). Find the speed of its center of mass when the cylinder reaches the bottom.

Solution The free-body diagram of Fig. 9-32*b* shows the forces acting on the cylinder: the weight $M\vec{\mathbf{g}}$, the normal force $\vec{\mathbf{N}}$, and the frictional force $\vec{\mathbf{f}}$. Based on the choice of x and y axes shown in the figure, the components of the net force on the cylinder are $\Sigma F_x = Mg \sin\theta - f$ and $\Sigma F_y = N - Mg\cos\theta$. If we apply Newton's second law with $a_x = a_{cm}$ and $a_y = 0$, we obtain for the x and y equations

$$Mg\sin\theta - f = Ma_{cm} \qquad \text{and} \qquad N - Mg\cos\theta = 0.$$

To find the net torque about the center of mass, we note that the lines of action of both $\vec{\mathbf{N}}$ and $M\vec{\mathbf{g}}$ pass through the center of mass and so their moment arms are zero. Only the frictional force contributes to the torque, and so $\Sigma\tau_z = -fR$. Newton's second law for rotation then gives

$$-fR = I_{cm}\alpha_z.$$

In Fig. 9-32, the z axis is out of the page and so α_z is indeed negative. The condition for rolling without slipping is $v_{cm} = \omega R$; differentiating this expression gives $a_{cm} = \alpha R$, which relates the magnitudes of a_{cm} and α. Substituting $\alpha_z = -a_{cm}/R$ and $I_{cm} = \frac{1}{2}MR^2$ (for a cylinder), we find

$$f = -\frac{I_{cm}\alpha_z}{R} = -\frac{(\frac{1}{2}MR^2)(-a_{cm}/R)}{R} = \frac{1}{2}Ma_{cm}.$$

Substituting this into the first translational equation, we find

$$a_{cm} = \tfrac{2}{3}g\sin\theta.$$

That is, the acceleration of the center of mass for the rolling cylinder ($\frac{2}{3}g\sin\theta$) is less than its acceleration would be if the cylinder were sliding down the incline ($g\sin\theta$). This result holds at any instant, regardless of the position of the cylinder along the incline.

Because the acceleration is constant, we can use the equations of Chapter 2 to find the velocity. With $v_{0x} = 0$ and taking $x - x_0 = L$ (where the x axis lies along the plane), Eqs. 2-26 and 2-28 respectively become $v_{cm} = a_{cm}t$ and $L = \frac{1}{2}a_{cm}t^2$. Solving the second equation for the time t, we find $t = \sqrt{2L/a_{cm}}$. With this result the first equation gives

$$v_{cm} = a_{cm}t$$
$$= a_{cm}\sqrt{\frac{2L}{a_{cm}}} = \sqrt{2La_{cm}} = \sqrt{2L(\tfrac{2}{3}g\sin\theta)} = \sqrt{\tfrac{4}{3}Lg\sin\theta}$$

This method also determines the force of static friction needed for rolling:

$$f = \tfrac{1}{2}Ma_{cm} = (\tfrac{1}{2}M)(\tfrac{2}{3}g\sin\theta) = \tfrac{1}{3}Mg\sin\theta.$$

What would happen if the force of static friction between the surfaces were less than this value?

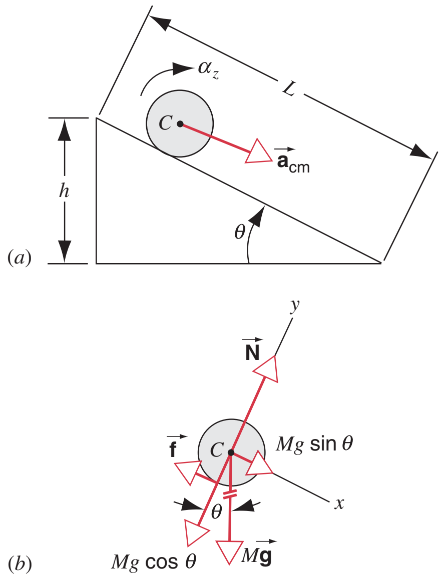


FIGURE 9-32. Sample Problem 9-11. (*a*) A cylinder rolls without slipping down the incline. (*b*) The free-body diagram of the cylinder.

SAMPLE PROBLEM 9-12. A uniform solid cylinder of radius R (= 12 cm) and mass M (= 3.2 km) is given an initial (clockwise) angular velocity ω_0 of 15 rev/s and then lowered on to a uniform horizontal surface (Fig. 9-33). The coefficient of kinetic friction between the surface and the cylinder is $\mu_k = 0.21$. Initially, the cylinder slips as it moves along the surface, but after a time t, pure rolling without slipping begins. (*a*) What is the velocity v_{cm} of the center of mass at the time t? (*b*) What is the value of t?

Solution (*a*) Figure 9-33*b* shows the forces that act on the cylinder. The x and y components of the net force are $\Sigma F_x = f$ and $\Sigma F_y = N - Mg$. During the interval from time 0 to time t while slipping occurs, the forces are constant and so the acceleration

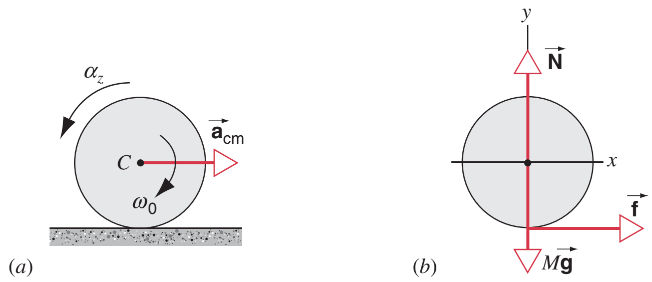


FIGURE 9-33. Sample Problem 9-12. (*a*) The rotating cylinder initially slips as it rolls. (*b*) The free-body diagram of the cylinder.

must be constant. In this time interval, $v_{fx} = v_{cm}$ and $v_{ix} = 0$. The acceleration is then

$$a_x = \frac{\Delta v_x}{\Delta t} = \frac{v_{fx} - v_{ix}}{t} = \frac{v_{cm} - 0}{t} = \frac{v_{cm}}{t}.$$

The x component of Newton's second law then gives

$$f = Ma_x = \frac{Mv_{cm}}{t}$$

Only the frictional force gives a torque about the center of mass, so the net torque is $\sum \tau_z = fR$. With $\omega_i = -\omega_0$ and $\omega_f = -v_{cm}/R$ at the instant when rolling without slipping begins (the minus signs indicating that the cylinder is spinning clockwise), the angular acceleration is

$$\alpha_z = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{t} = \frac{-v_{cm}/R + \omega_0}{t}.$$

Newton's second law for rotation gives $fR = I_{cm}\alpha_z$. Substituting for f and α_z from the above two equations, we obtain

$$\left(\frac{Mv_{cm}}{t}\right)R = \frac{\frac{1}{2}MR^2(-v_{cm}/R + \omega_0)}{t}$$

using $I_{cm} = \frac{1}{2}MR^2$ from Fig. 9-15. After eliminating common factors we can solve for v_{cm} to find

$$v_{cm} = \tfrac{1}{3}\omega_0 R = \tfrac{1}{3}(15\text{ rev/s})(2\pi\text{ rad/rev})(0.12\text{ m}) = 3.8\text{ m/s}.$$

Note that v_{cm} does not depend on the values of M, g, or μ_k. What, however, would occur if any of these quantities were zero?

(*b*) With $f = Mv_{cm}/t$ and also $f = \mu_k N = \mu_k Mg$, we can eliminate f and solve for t:

$$t = \frac{v_{cm}}{\mu_k g} = \frac{3.8\text{ m/s}}{(0.21)(9.80\text{ m/s}^2)} = 1.8\text{ s}.$$

SAMPLE PROBLEM 9-13. A toy yo-yo* of total mass $M = 0.24$ kg consists of two disks of radius $R = 2.8$ cm connected by a thin shaft of radius $R_0 = 0.25$ cm (Fig. 9-34*a*). A string of length $L = 1.2$ m is wrapped around the shaft. If the yo-yo is thrown downward with an initial velocity of $v_0 = 1.4$ m/s, what is its rotational velocity when it reaches the end of the string?

Solution The free-body diagram for the yo-yo is shown in Fig. 9-34*b*. The net force is $\sum F_y = Mg - T$ (taking the downward direction to be positive), and the net torque about the center of mass is $\sum \tau_z = TR_0$ (taking counterclockwise torques to be positive). The translational and rotational forms of Newton's second law then give

$$Mg - T = Ma_y \quad \text{and} \quad TR_0 = I\alpha_z.$$

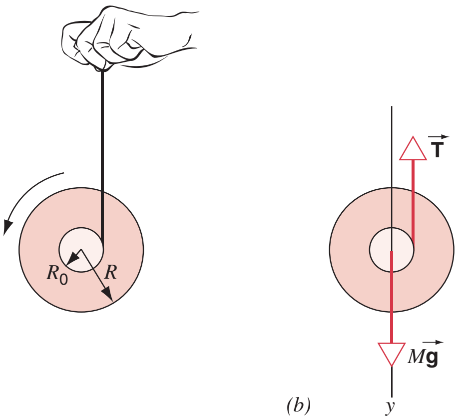


FIGURE 9-34. Sample Problem 9-13. (*a*) A yo-yo falls as the string unwinds from the axle. (*b*) The force diagram.

We consider the string to be of negligible thickness and assume that it does not slip as it is unwinding. The point where the string contacts the shaft is instantaneously at rest, just like the point B in Figs. 9-30 and 9-31. With $v_{cm} = \omega R_0$, it follows that (in magnitudes only) $a_{cm} = \alpha R_0$. In our notation for this problem, $a_{cm} = a_y$ (a positive quantity) and $\alpha = \alpha_z$ (also a positive quantity). Thus, taking $a_y = \alpha_z R_0$ and combining the force and torque equations to eliminate the tension, we solve for the angular acceleration:

$$\alpha_z = \frac{g}{R_0}\frac{1}{1 + I/MR_0^2}.$$

To complete the solution, we need the rotational inertia, which is not given. Let us assume that the thin shaft makes a negligible contribution to I (the mass and radius of the shaft are both small compared to the disks). Then the rotational inertia is $I = \frac{1}{2}MR^2$ and

$$\alpha_z = \frac{g}{R_0}\frac{1}{1 + R^2/2R_0^2}$$

$$= \frac{980\text{ cm/s}^2}{0.25\text{ cm} + (2.8\text{ cm})^2/2(0.25\text{ cm})} = 61.5\text{ rad/s}^2.$$

To find the final angular velocity from this acceleration, we can use Eq. 8-6, $\omega_z = \omega_{0z} + \alpha_z t$, if we know the time t for the yo-yo to unwind. This time can be found from Eq. 8-7, $\phi = \phi_0 + \omega_{0z}t + \frac{1}{2}\alpha_z t^2$. The angle through which the yo-yo rotates as the string unwinds is $\phi - \phi_0 = L/R_0 = 480$ rad, and the initial angular velocity is $\omega_{0z} = v_0/R_0 = (1.4\text{ m/s})(0.0025\text{ m}) = 560$ rad/s. With these substitutions, Eq. 8-7 then gives

$$(30.75\text{ rad/s}^2)t^2 + (560\text{ rad/s})t - 480\text{ rad} = 0.$$

Solving this quadratic equation, we find $t = 0.82$ s or -19 s. The positive value is the physically meaningful one, and so

$$\omega_z = \omega_{0z} + \alpha_z t = 560\text{ rad/s} + (61.5\text{ rad/s}^2)(0.82\text{ s}) = 610\text{ rad/s}.$$

* See "The Yo-Yo: A Toy Flywheel," by Wolfgang Burger, *American Scientist,* March–April 1984, p. 137.

MULTIPLE CHOICE

9-1 Torque

1. Consider the object in Fig. 9-2. Invert the coordinate system so that $x \to -x$, $y \to -y$ and $z \to -z$. Clearly $\vec{\mathbf{r}} \to -\vec{\mathbf{r}}$ under this transformation. What happens to $\vec{\tau}$ and $\vec{\mathbf{F}}$?
- (A) $\vec{\tau} \to \vec{\tau}$ and $\vec{\mathbf{F}} \to \vec{\mathbf{F}}$.
- (B) $\vec{\tau} \to \vec{\tau}$ and $\vec{\mathbf{F}} \to -\vec{\mathbf{F}}$.
- (C) $\vec{\tau} \to -\vec{\tau}$ and $\vec{\mathbf{F}} \to \vec{\mathbf{F}}$.
- (D) $\vec{\tau} \to -\vec{\tau}$ and $\vec{\mathbf{F}} \to -\vec{\mathbf{F}}$.

2. A particle is located at $\vec{\mathbf{r}} = 0\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$, in meters. A constant force $\vec{\mathbf{F}} = 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ (in newtons) begins to act on the particle. As the particle accelerates under the action of this force the torque, as measured about the origin,
- (A) increases. (B) decreases. (C) is zero.
- (D) is a nonzero constant.

3. In one of his many action movies Jackie Chan jumped off a building by wrapping a rope around his waist and then allowed it to unwind as he fell to the ground, much the same as a yo-yo. Assuming his acceleration toward the ground was a constant much less than g, the tension in the rope would be
- (A) almost equal to his weight.
- (B) exactly equal to his weight.
- (C) much less than his weight.
- (D) exactly zero.

(See *Who Am I,* starring Jackie Chan.)

9-2 Rotational Inertia and Newton's Second Law

9-3 Rotational Inertia of Solid Bodies

4. About what axis would a uniform cube have its minimum rotational inertia?
- (A) Any axis passing through the center of the cube and the center of one face
- (B) Any axis passing through the center of the cube and the center of one edge
- (C) Any axis passing through the center of the cube and one vertex (a diagonal)
- (D) A uniform cube has the same rotational inertia for any axis of rotation through its center.

9-4 Torque Due to Gravity

9-5 Equilibrium Applications of Newton's Laws for Rotation

5. A long straight rod experiences several forces, each acting at a different location on the rod. All forces are perpendicular to the rod. The rod might be in translational equilibrium, rotational equilibrium, both, or neither.

(*a*) If a calculation reveals that the net torque about the left end is zero, then one can conclude that the rod
- (A) is definitely in rotational equilibrium.
- (B) is in rotational equilibrium only if the net force on the rod is also zero.
- (C) might not be in rotational equilibrium even if the net force on the rod is also zero.
- (D) might be in rotational equilibrium even if the net force is not zero.

(*b*) If a calculation reveals that the net force on the rod is zero, then one can conclude that the rod
- (A) is definitely in rotational equilibrium.
- (B) is in rotational equilibrium only if the net torque about every axis through any one point is found to be zero.
- (C) might be in rotational equilibrium if the net torque about every axis through any one point is found to be zero.
- (D) might be in rotational equilibrium even if the net torque about any axis through any one point is not zero.

6. A parent pushes a balanced frictionless playground merry-go-round. The parent exerts a force $\vec{\mathbf{F}}$ tangent to the merry-go-round resulting in a torque of 240 N·m; the distance between the center of the merry-go-round and the point of application of the force is 1.6 m.

(*a*) Is the merry-go-round in equilibrium?
- (A) Yes, for both translational and rotational motion
- (B) Only for translational motion
- (C) Only for rotational motion
- (D) No, not for translational or rotational motion

(*b*) What, if any, is the magnitude of the horizontal force exerted by the merry-go-round axle on the merry-go-round?
- (A) 384 N (B) 240 N
- (C) 150 N (D) There is no force.

7. A ladder is at rest with its upper end against a wall and its lower end on the ground. A worker is about to climb it. When is it more likely to slip?
- (A) Before the worker is on it.
- (B) When the worker is on the lowest rung.
- (C) When the worker is halfway up the ladder.
- (D) When the worker is on the top rung.

9-6 Nonequilibrium Applications of Newton's Laws for Rotation

8. Newton's second law for translational motion in the xy plane is $\sum \vec{\mathbf{F}} = m\vec{\mathbf{a}}$; Newton's second law for rotation is $\sum \tau_z = I\alpha_z$. Consider the case of a particle moving in the xy plane under the influence of a single force.
- (A) Both $\sum \vec{\mathbf{F}} = m\vec{\mathbf{a}}$ and $\sum \tau_z = I\alpha_z$ must be used to analyze the motion of this particle.
- (B) Either $\sum \vec{\mathbf{F}} = m\vec{\mathbf{a}}$ or $\sum \tau_z = I\alpha_z$ can be used to analyze the motion of this particle.
- (C) Only $\sum \vec{\mathbf{F}} = m\vec{\mathbf{a}}$ needs to be used to analyze the motion of this particle.
- (D) Only $\sum \tau_z = I\alpha_z$ can be used to analyze the motion of this particle.

9-7 Combined Rotational and Translational Motion

9. Consider four objects, all solid spheres. Sphere (A) has radius r and mass m, (B) has radius $2r$ and mass m, (C) has radius r and mass $2m$, and (D) has radius r and mass $3m$. All can be placed at the same point on the same inclined plane where they will roll without slipping to the bottom. The answer to the following questions might also be (E), all are the same.

(*a*) Which object has the largest rotational inertia?

(*b*) If released from rest, which object will experience the largest net torque?

(*c*) If released from rest, which object will experience the largest linear acceleration?

(*d*) If allowed to roll down the incline, which object will have the largest speed at the bottom of the incline?
(*e*) If allowed to roll down the incline, which object will reach the bottom of the incline in the shortest time?

10. Consider four objects: (A), a solid sphere; (B), a spherical shell; (C), a solid disk; and (D), a metal hoop. All have the same mass and radius; all can be placed at the same point on the same inclined plane where they will roll without slipping to the bottom. The answer to the following questions might also be (E), all are the same.

(*a*) Which object has the largest rotational inertia about its axis of symmetry?
(*b*) If released from rest, which object will experience the largest net torque?
(*c*) If released from rest, which object will experience the largest linear acceleration?
(*d*) If allowed to roll down the incline, which object will have the largest speed at the bottom of the incline?
(*e*) If allowed to roll down the incline, which object will reach the bottom of the incline in the shortest time?

QUESTIONS

1. Explain why the wheel is such an important invention.

2. A yo-yo falls to the bottom of its cord and then climbs back up. (*a*) Does it reverse its direction of rotation at the bottom? Explain your answer. (*b*) What "pulls" the yo-yo back to the top?

3. A yo-yo is resting on a horizontal table and is free to roll (see Fig. 9-35). If the string is pulled by a horizontal force such as F_1, which way will the yo-yo roll? What happens when the force F_2 is applied (its line of action passes through the point of contact of the yo-yo and table)? If the string is pulled vertically with the force F_3, what happens?

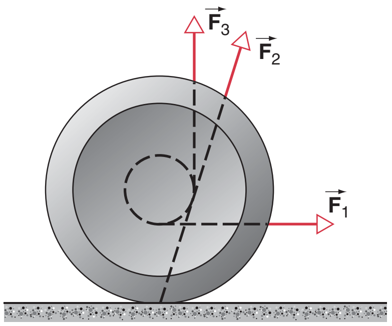


FIGURE 9-35. Question 3.

4. Do the center of mass and the center of gravity coincide for a building? for a lake? Under what conditions does the difference between the center of mass and the center of gravity become significant? Give an example.

5. If a rigid body is thrown into the air without spinning, it does not spin during its flight, provided air resistance can be ignored. What does this simple result imply about the location of the center of gravity?

6. The Olympic gymnast Mary Lou Retton did some amazing things on the uneven parallel bars. A friend tells you that careful analysis of films of Retton's exploits shows that, no matter what she does, her center of mass is above her point(s) of support at all times, as required by the laws of physics. Comment on your friend's statement.

7. Stand facing the edge of an open door, one foot on each side of the door. You will find that you are not able to stand on your toes. Why?

8. Sit in a straight-backed chair and try to stand up without leaning forward. Why can't you do it?

9. Long balancing poles help a tightrope walker to maintain balance. How?

10. Is there such a thing as a truly rigid body? If so, give an example. If not, explain why.

11. You are sitting in the driver's seat of a parked automobile. You are told that the forces exerted upward by the ground on each of the four tires are different. Discuss the factors that enter into a consideration of whether this statement is true or not.

12. In Sample Problem 9-7, if the wall were not frictionless, would the empirical laws of friction supply us with the extra condition needed to determine the extra (vertical) force exerted by the wall on the ladder?

13. Can the mass of an object be considered as concentrated at its center of mass for purposes of computing its rotational inertia? If yes, explain why. If no, offer a counterexample.

14. About what axis is the rotational inertia of your body the least? About what axis through your center of mass is your rotational inertia the greatest?

15. If two circular disks of the same weight and thickness are made from metals having different densities, which disk, if either, will have the larger rotational inertia about its symmetry axis?

16. The rotational inertia of a body of rather complicated shape is to be determined. The shape makes a mathematical calculation from $\int r^2\,dm$ exceedingly difficult. Suggest ways in which the rotational inertia about a particular axis could be measured experimentally.

17. Five solids are shown in cross section in Fig. 9-36. The cross sections have equal heights and equal maximum widths. The solids have equal masses. Which one has the largest rotational inertia about a perpendicular axis through the center of mass? Which has the smallest?

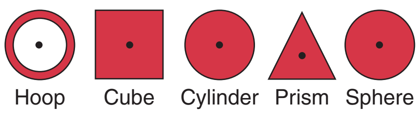


FIGURE 9-36. Question 17.

18. Does Eq. 9-17 still hold if the slab is not "thin"—that is, if its thickness is comparable to (or even greater than) a or b?

19. You can distinguish between a raw egg and a hardboiled one by spinning each one on a table. Explain how. Also, if you stop a spinning raw egg with your fingers and release it very quickly, it will resume spinning. Why?

20. For storing wind energy or solar energy, flywheels have been suggested. The amount of energy that can be stored in a flywheel depends on the density and tensile strength of the material making up the flywheel and for a given weight one wants the lowest density strong material available. Can you make this plausible? (See "Flywheels," by R. F. Post and S. F. Post, *Scientific American,* December 1973, p. 17.)

21. Apart from appearance, why do sports cars have wire wheels?

22. Fig. 9-37*a* shows a meter stick, half of which is wood and half of which is steel, that is pivoted at the wooden end at O. A force is applied to the steel end at a. In Fig. 9-37*b*, the stick is pivoted at the steel end at O' and the same force is applied at the wooden end at a'. Does one get the same angular acceleration in each case? If not, in which case is the angular acceleration greater?

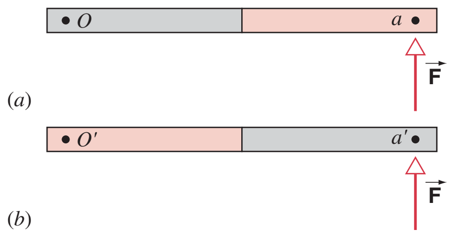


FIGURE 9-37. Question 22.

23. Describe qualitatively what happens to the system of Fig. 9-26 if the disk is given an initial clockwise angular velocity as it is released. What changes, if any, occur in the linear acceleration of the block, or the angular acceleration of the disk? See Sample Problem 9-10.

24. A cannonball and a marble roll from rest down an incline. Which gets to the bottom first?

25. A cylindrical can filled with corned beef and an identical can filled with apple juice both roll down an incline. Compare their angular and linear accelerations. Explain the difference.

26. A solid wooden cylinder rolls down two different inclined planes of the same height but with different angles of inclination. Will it reach the bottom with the same speed in each case? Will it take longer to roll down one incline than the other? Explain your answers.

27. A solid brass cylinder and a solid wooden cylinder have the same radius and mass, the wooden cylinder being longer. You release them together at the top of an incline. Which will beat the other to the bottom? Suppose that the cylinders are now made to be the same length (and radius) and that the masses are made to be equal by boring a hole along the axis of the brass cylinder. Which cylinder will win the race now? Explain your answers. Assume that the cylinders roll without slipping.

28. State Newton's three laws of motion in words suitable for rotating bodies.

29. Two heavy disks are connected by a short rod of much smaller radius. The system is placed on a ramp so that the disks hang over the sides as in Fig. 9-38. The system rolls down the ramp without slipping. (*a*) Near the bottom of the ramp the disks touch the horizontal table and the system takes off with greatly increased translational speed. Explain why. (*b*) If this system raced a hoop (of any radius) down the ramp, which would reach the bottom first?

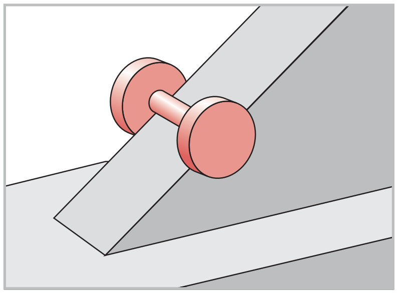


FIGURE 9-38. Question 29.

30. In cutting down a tree, a logger makes a cut on the side facing the direction in which the tree is to fall. Explain why. Would it be safe to stand directly behind the tree on the opposite side of the fall?

31. Comment on each of the following assertions about skiing. (*a*) In downhill racing, one wants skis that do not turn easily. (*b*) In slalom racing, one wants skis that turn easily. (*c*) Therefore, the rotational inertia of downhill skis should be larger than that of slalom skis. (*d*) Considering that there is low friction between skis and snow, how does a skier exert torques to turn or stop a turn? (See "The Physics of Ski Turns," by J. I. Shonie and D. L. Mordick, *The Physics Teacher,* December 1972, p. 491.)

32. Consider a straight stick standing on end on (frictionless) ice. What would be the path of its center of mass if it falls?

33. Explain why a wheel rolling on a flat horizontal surface cannot be slowed down by static friction. Assuming no slipping, what does slow the wheel down?

34. Ruth and Roger are cycling along a path at the same speed. The wheels of Ruth's bike are a little larger in diameter than the wheels of Roger's bike, How do the angular speeds of their wheels compare? What about the speeds of the top portions of their wheels?

35. A cylindrical drum, pushed along by a board from an initial position shown in Fig. 9-39, rolls forward on the ground a distance $L/2$, equal to half the length of the board. There is no slipping at any contact. Where is the board then? How far has the man walked?

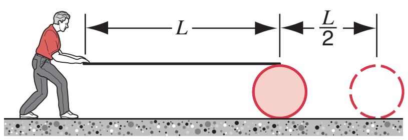


FIGURE 9-39. Question 35.

EXERCISES

9-1 Torque

1. A particle is located at coordinates $x = 2.0$ m, $y = 3.0$ m. What is the magnitude of the torque about the origin when the particle is acted upon by a force of magnitude 5.0 N in (*a*) the positive x direction, (*b*) the positive y direction, and (*c*) the negative x direction?

2. Figure 9-40 shows the lines of action and the points of application of two forces about the origin O, all vectors being in the plane of the figure. Imagine these forces to be acting on a rigid body pivoted about an axis through O and perpendicular to the plane of the figure. (*a*) Find an expression for the magnitude of the resultant torque on the body. (*b*) If $r_1 = 1.30$ m, $r_2 = 2.15$ m, $F_1 = 4.20$ N, $F_2 = 4.90$ N, $\theta_1 = 75.0°$, and $\theta_2 = 58.0°$, what are the magnitude and direction of the resultant torque?

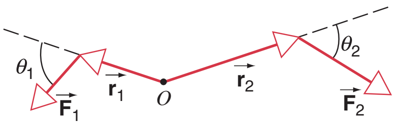


FIGURE 9-40. Exercise 2.

3. Redraw Fig. 9-40 under the following transformations: (*a*) $\vec{F} \to -\vec{F}$, (*b*) $\vec{r} \to -\vec{r}$, and (*c*) $\vec{F} \to -\vec{F}$ and $\vec{r} \to -\vec{r}$, in each case showing the new direction of the torque. Check for consistency with the right-hand rule.

4. The object shown in Fig. 9-41 is pivoted at O about an axis perpendicular to the plane of the page. Three forces act on it in the directions shown on the figure: $F_A = 10$ N at point A, 8.0 m from O; $F_B = 16$ N at point B, 4.0 m from O; and $F_C = 19$ N at point C, 3.0 m from O. What are the magnitude and direction of the resultant torque about O?

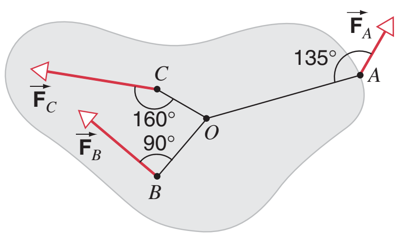


FIGURE 9-41. Exercise 4.

5. Two vectors $\vec{r}$ and $\vec{s}$ lie in the xy plane. Their magnitudes are $r = 4.5$ units and $s = 7.3$ units. Their directions are, respectively, 320° and 85° measured counterclockwise from the positive x axis. Find the magnitude and the direction of $\vec{r} \times \vec{s}$.

6. Vector $\vec{a}$ has magnitude 3.20 units and lies in the yz plane 63.0° from the $+y$ axis with a positive z component. Vector $\vec{b}$ has magnitude 1.40 units and lies in the xz plane 48.0° from the $+x$ axis with a positive z component. Find $\vec{a} \times \vec{b}$.

7. Vectors $\vec{a}$ and $\vec{b}$ lie in the xy plane. The angle between $\vec{a}$ and $\vec{b}$ is ϕ, which is less than 90°. Let $\vec{c} = \vec{a} \times (\vec{b} \times \vec{a})$. Find the magnitude of $\vec{c}$ and the angle between $\vec{b}$ and $\vec{c}$.

8. Let $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{b} = 4\hat{i} - 2\hat{j} - 3\hat{k}$. Let $\vec{c} = \vec{a} \times \vec{b}$. (*a*) Find **c**, expressed in unit vector notation. (*b*) Find the angle between $\vec{a}$ and $\vec{b}$.

9. What is the torque about the origin on a particle located at $x = 1.5$ m, $y = -2.0$ m, $z = 1.6$ m due to a force $\vec{F} = (3.5\text{ N})\hat{i} - (2.4\text{ N})\hat{j} + (4.3\text{ N})\hat{k}$? Express your result in unit vector notation.

10. A particle is located at $\vec{r} = (0.54\text{ m})\hat{i} + (-0.36\text{ m})\hat{j} + (0.85\text{ m})\hat{k}$. A constant force of magnitude 2.6 N acts on the particle. Find the components of the torque about the origin when the force acts in (*a*) the positive x direction and (*b*) the negative z direction.

9-2 Rotational Inertia and Newton's Second Law

11. A small lead sphere of mass 25 g is attached to the origin by a thin rod of length 74 cm and negligible mass. The rod pivots about the z axis in the xy plane. A constant force of 22 N in the y direction acts on the sphere. (*a*) Considering the sphere to be a particle, what is the rotational inertia about the origin? (*b*) If the rod makes an angle of 40° with the positive x axis, find the angular acceleration of the rod.

12. Three particles are attached to a thin rod of length 1.00 m and negligible mass that pivots about the origin in the xy plane. Particle 1 (mass 52 g) is attached a distance of 27 cm from the origin, particle 2 (35 g) is at 45 cm, and particle 3 (24 g) at 65 cm. (*a*) What is the rotational inertia of the assembly? (*b*) If the rod were instead pivoted about the center of mass of the assembly, what would be the rotational inertia?

13. Two thin rods of negligible mass are rigidly attached at their ends to form a 90° angle. The rods rotate in the xy plane with the joined ends forming the pivot at the origin. A particle of mass 75 g is attached to one rod a distance of 42 cm from the origin, and a particle of mass 30 g is attached to the other rod a distance of 65 cm from the origin. (*a*) What is the rotational inertia of the assembly? (*b*) How would the rotational inertia change if the particles were both attached to one rod at the given distances from the origin?

14. Consider the assembly of Exercise 13 when the first rod lies along the positive x axis and the second rod along the positive y axis. A force $\vec{F} = (3.6\text{ N})\hat{i} + (2.5\text{ N})\hat{j}$ acts on both particles. Find the resulting angular acceleration.

9-3 Rotational Inertia of Solid Bodies

15. A helicopter rotor blade is 7.80 m long and has a mass of 110 kg. (*a*) What force is exerted on the bolt attaching the blade to the rotor axle when the rotor is turning at 320 rev/min? (Hint: For this calculation the blade can be considered to be a point mass at the center of mass. Why?) (*b*) Calculate the torque that must be applied to the rotor to bring it to full speed from rest in 6.70 s. Ignore air resistance. (The blade cannot be considered to be a point mass for this calculation. Why not? Assume the distribution of a uniform rod.)

16. Each of three helicopter rotor blades shown in Fig. 9-42 is 5.20 m long and has a mass of 240 kg. The rotor is rotating at

350 rev/min. What is the rotational inertia of the rotor assembly about the axis of rotation? (Each blade can be considered a thin rod.)

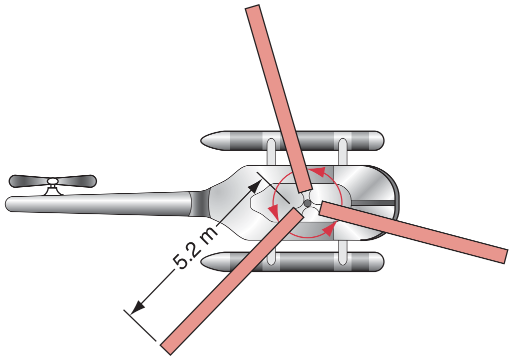


FIGURE 9-42. Exercise 16.

17. Fig. 9-43 shows a uniform block of mass, M and edge lengths a, b, and c. Calculate its rotational inertia about an axis through one corner and perpendicular to the large face of the block. (Hint: See Fig. 9-15.)

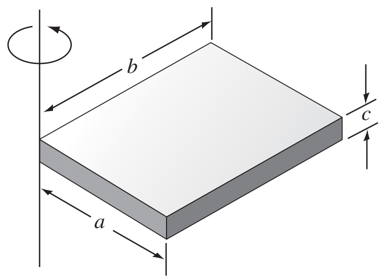


FIGURE 9-43. Exercise 17.

18. Calculate the rotational inertia of a meter stick, with mass 0.56 kg, about an axis perpendicular to the stick and located at the 20-cm mark.

19. Two particles, each with mass m, are fastened to each other and to a rotation axis by two rods, each with length L and mass M, as shown in Fig. 9-44. The combination rotates around the rotation axis with angular velocity ω. Obtain an algebraic expression for the rotational inertia of the combination about this axis.

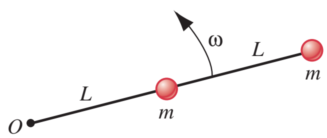


FIGURE 9-44. Exercise 19.

20. (*a*) Show that a solid cylinder of mass M and radius R is equivalent to a thin hoop of mass M and radius $R/\sqrt{2}$, for rotation about a central axis. (*b*) The radial distance from a given axis at which the mass of a body could be concentrated without altering the rotational inertia of the body about that axis is called the *radius of gyration.* Let k represent the radius of gyration and show that

$$k = \sqrt{I/M}.$$

This gives the radius of the "equivalent hoop" in the general case.

21. Fig. 9-45 shows the solid rod considered in Section 9-3 (see also Fig. 9-12) divided into an arbitrary number N of pieces. (*a*) What is the mass m_n of each piece? (*b*) Show that the distance of each piece from the axis of rotation can be written $r_n = (n - 1)L/N + (\frac{1}{2})L/N = (n - \frac{1}{2})L/N$. (*c*) Use Eq. 9-13 to evaluate the rotational inertia of this rod, and show that it reduces to Eq. 9-14. You may need the following sums:

$$\sum_{n=1}^{N} 1 = N,$$

$$\sum_{n=1}^{N} n = N(N + 1)/2,$$

$$\sum_{n=1}^{N} n^2 = N(N + 1)(2N + 1)/6.$$

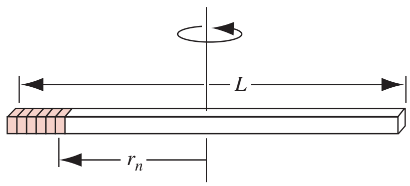


FIGURE 9-45. Exercise 21.

9-4 Torque Due to Gravity

9-5 Equilibrium Applications of Newton's Laws for Rotation

22. A certain nut is known to require forces of 46 N exerted on it from both sides to crack it. What forces F will be required when it is placed in the nutcracker shown in Fig. 9-46?

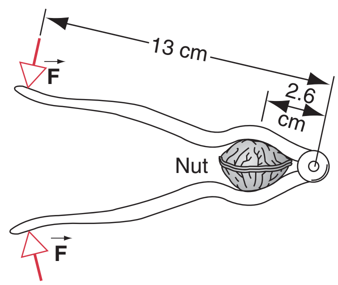


FIGURE 9-46. Exercise 22.

23. The leaning Tower of Pisa (see Fig. 9-47) is 55 m high and 7.0 m in diameter. The top of the tower is displaced 4.5 m from the vertical. Treating the tower as a uniform, circular cylinder, (*a*) what additional displacement, measured at the top, will bring the tower to the verge of toppling? (*b*) What angle with the vertical will the tower make at that moment? (The current rate of movement of the top is 1 mm/year.)

FIGURE 9-47. Exercise 23.

24. A cube stays at rest on a horizontal table when a small horizontal force is applied perpendicular to and at the center of an upper edge. The force is now steadily increased. Does the cube slide or topple first? The coefficient of static friction between the surfaces is equal to 0.46.

25. In Sample Problem 9-7 the coefficient of static friction μ_s between the ladder and the ground is 0.54. How far up the ladder can the firefighter go before the ladder starts to slip?

26. A parked automobile of mass 1360 kg has a wheel base (distance between front and rear axles) of 305 cm. Its center of gravity is located 178 cm behind the front axle. Determine (*a*) the upward force exerted by the level ground on each of the front wheels (assumed the same) and (*b*) the upward force exerted by the level ground on each of the rear wheels (assumed the same).

27. A 160-lb person is walking across a level bridge and stops three-fourths of the way from one end. The bridge is uniform and weighs 600 lb. What are the values of the vertical forces exerted on each end of the bridge by its supports?

28. A diver of weight 582 N stands at the end of a uniform 4.48-m diving board of weight 142 N. The board is attached by two pedestals 1.55 m apart, as shown in Fig. 9-48. Find the tension (or compression) in each of the two pedestals.

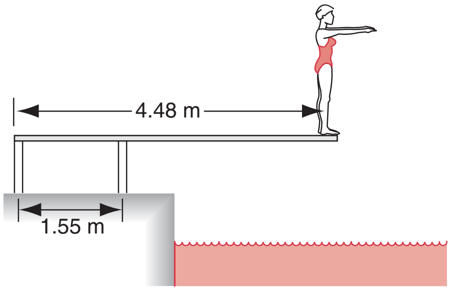


FIGURE 9-48. Exercise 28.

29. What minimum force F applied horizontally at the axle of the wheel in Fig. 9-49 is necessary to raise the wheel over an obstacle of height h? Take r as the radius of the wheel and W as its weight.

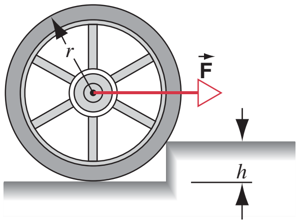


FIGURE 9-49. Exercise 29.

30. A 52.3-kg uniform square sign, 1.93 m on a side, is hung from a 2.88-m rod of negligible mass. A cable is attached to the end of the rod and to a point on the wall 4.12 m above the point where the rod is fixed to the wall, as shown in Fig. 9-50. (*a*) Find the tension in the cable. (*b*) Calculate the horizontal and vertical components of the force exerted by the wall on the rod.

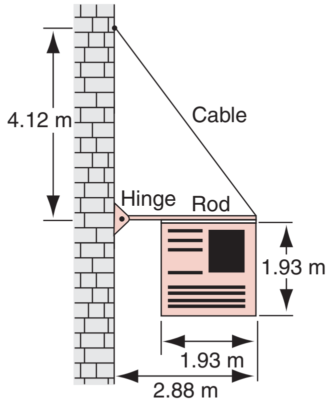


FIGURE 9-50. Exercise 30.

31. One end of a uniform beam weighing 52.7 lb and 3.12 ft long is attached to a wall with a hinge. The other end is supported by a wire making equal angles of 27.0° with the beam and wall (see Fig. 9-51). (*a*) Find the tension in the wire. (*b*) Compute the horizontal and vertical components of the force on the hinge.

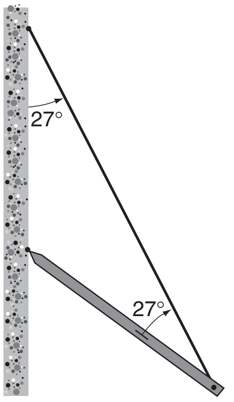


FIGURE 9-51. Exercise 31.

32. A 274-N plank, of length $L = 6.23$ m, rests on the ground and on a frictionless roller at the top of a wall of height

$h = 2.87$ m (see Fig. 9-52). The center of gravity of the plank is at its center. The plank remains in equilibrium for any value of $\theta \geq 68.0°$ but slips if $\theta < 68.0°$. Find the coefficient of static friction between the plank and the ground.

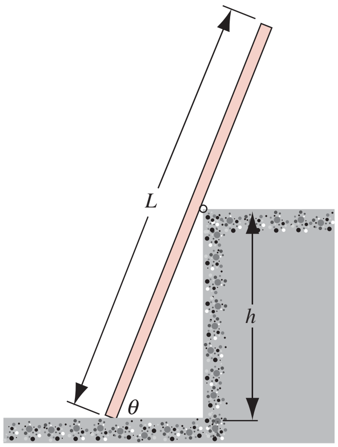


FIGURE 9-52. Exercise 32.

9-6 Nonequilibrium Applications of Newton's Laws for Rotation

33. A cylinder having a mass of 1.92 kg rotates about its axis of symmetry. Forces are applied as shown in Fig. 9-53: $F_1 = 5.88$ N, $F_2 = 4.13$ N, and $F_3 = 2.12$ N. Also, $R_1 = 4.93$ cm and $R_2 = 11.8$ cm. Find the magnitude and direction of the angular acceleration of the cylinder.

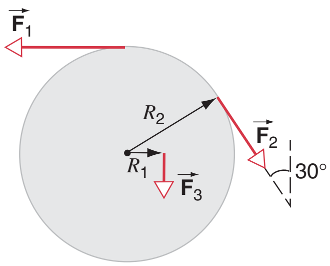


FIGURE 9-53. Exercise 33.

34. A thin spherical shell has a radius of 1.88 m. An applied torque of 960 N·m imparts an angular acceleration equal to 6.23 rad/s^2 about an axis through the center of the shell. Calculate (*a*) the rotational inertia of the shell about the axis of rotation and (*b*) the mass of the shell.

35. In the act of jumping off a diving board, a diver changed his angular velocity from zero to 6.20 rad/s in 220 ms. The diver's rotational inertia is 12.0 kg · m^2. (*a*) Find the angular acceleration during the jump. (*b*) What external torque acted on the diver during the jump?

36. Figure 9-54 shows the massive shield door at a neutron test facility at Lawrence Livermore Laboratory; this is the world's heaviest hinged door. The door has a mass of 44,000 kg, a rotational inertia about its hinge line of 8.7×10^4 kg·m^2, and a width of 2.4 m. What steady force, applied at its outer edge at right angles to the door, can move it from rest through an angle of 90° in 30 s?

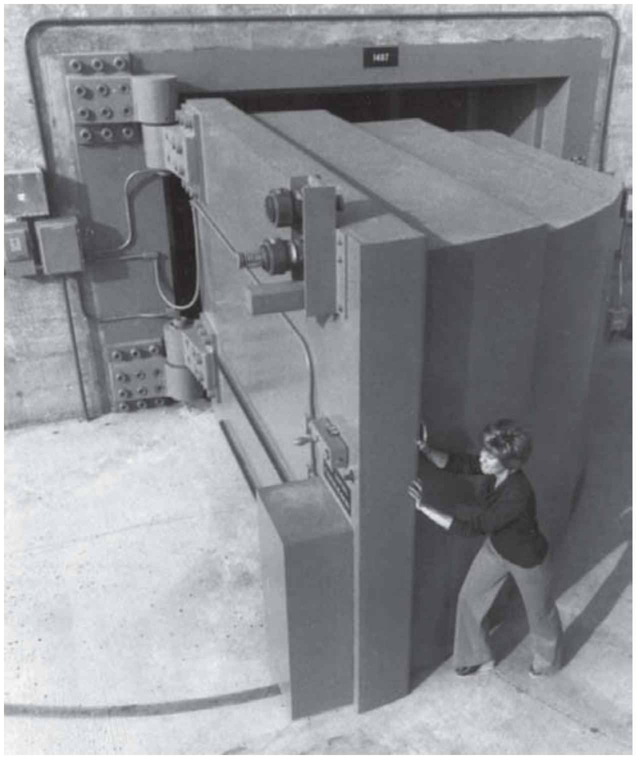


FIGURE 9-54. Exercise 36.

37. A pulley having a rotational inertia of 1.14×10^{-3} kg·m^2 and a radius of 9.88 cm is acted on by a force, applied tangentially at its rim, that varies in time as $F = At + Bt^2$, where $A = 0.496$ N/s and $B = 0.305$ N/s^2. If the pulley was initially at rest, find its angular speed after 3.60 s.

38. Two identical blocks, each of mass M, are connected by a light string over a frictionless pulley of radius R and rotational inertia I (Fig. 9-55). The string does not slip on the pulley, and it is not known whether or not there is friction between the plane and the sliding block. When this system is released, it is found that the pulley turns through an angle θ in time t and the acceleration of the blocks is constant. (*a*) What is the angular acceleration of the pulley? (*b*) What is the acceleration of the two blocks? (*c*) What are the tensions in the upper and lower sections of the string? All answers are to be expressed in terms of M, I, R, θ, g, and t.

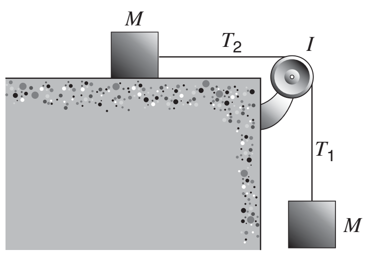


FIGURE 9-55. Exercise 38.

39. In an Atwood's machine one block has a mass of 512 g and the other a mass of 463 g. The pulley, which is mounted in horizontal frictionless bearings, has a radius of 4.90 cm. When released from rest, the heavier block is observed to fall 76.5 cm in 5.11 s. Calculate the rotational inertia of the pulley.

40. A wheel in the form of a uniform disk of radius 23.0 cm and mass 1.40 kg is turning at 840 rev/min in frictionless bearings. To stop the wheel, a brake pad is pressed against the rim of the wheel with a radially directed force of 130 N. The wheel makes 2.80 revolutions in coming to a stop. Find the coefficient of friction between the brake pad and the rim of the wheel.

9-7 Combined Rotational and Translational Motion

41. An automobile traveling 78.3 km/h has tires of 77.0-cm diameter. (*a*) What is the angular speed of the tires about the axle? (*b*) If the car is brought to a stop uniformly in 28.6 turns of the tires (no skidding), what is the angular acceleration of the wheels? (*c*) How far does the car advance during this braking period?

42. A yo-yo (see Sample Problem 9-13) has a rotational inertia of 950 $g \cdot cm^2$ and a mass of 120 g. Its axle radius is 3.20 mm and its string is 134 cm long. The yo-yo rolls from rest down to the end of the string. (*a*) What is its acceleration? (*b*) How long does it take to reach the end of the string? (*c*) If the yo-yo "sleeps" at the bottom of the string in pure rotary motion, what is its angular speed, in rev/s? (*d*) Repeat (*c*), but this time assume that the yo-yo was thrown down with an initial speed of 1.30 m/s.

43. An apparatus for testing the skid resistance of automobile tires is constructed as shown in Fig. 9-56. The tire is initially motionless and is held in a light framework that is freely pivoted at points *A* and *B*. The rotational inertia of the wheel about its axis is 0.750 $kg \cdot m^2$, its mass is 15.0 kg, and its radius is 30.0 cm. The tire is placed on the surface of a conveyor belt that is moving with a surface speed of 12.0 m/s, such that *AB* is horizontal. (*a*) If the coefficient of kinetic friction between the tire and the conveyor belt is 0.600, what time will be required for the wheel to achieve its final angular velocity? (*b*) What will be the length of the skid mark on the conveyor surface?

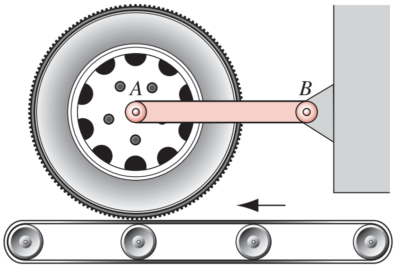


FIGURE 9-56. Exercise 43.

PROBLEMS

1. A crate in the form of a 1.12-m cube contains a piece of machinery whose design is such that the center of gravity of the crate and its contents is located 0.28 m above its geometrical center. The crate rests on a ramp that makes an angle θ with the horizontal. As θ is increased from zero, an angle will be reached at which the crate will either start to slide down the ramp or tip over. Which event will occur if the coefficient of static friction is (*a*) 0.60? (*b*) 0.70? In each case give the angle at which the event occurs.

2. A flexible chain of weight *W* hangs between two fixed points, *A* and *B*, at the same level, as shown in Fig. 9-57. Find (*a*) the force exerted by the chain on each endpoint and (*b*) the tension in the chain at the lowest point.

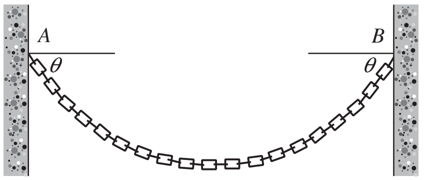


FIGURE 9-57. Problem 2.

3. A uniform sphere of weight *W* and radius *r* is being held by a rope attached to a frictionless wall a distance *L* above the center of the sphere, as in Fig. 9-58. Find (*a*) the tension in the rope and (*b*) the force exerted on the sphere by the wall.

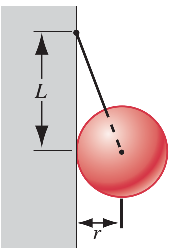


FIGURE 9-58. Problem 3.

4. A beam is carried by three workers, one worker at one end and the other two supporting the beam between them on a crosspiece so placed that the load is equally divided among the three. Find where the crosspiece is placed. Neglect the mass of the crosspiece.

5. A 74.6-kg window cleaner uses a 10.3-kg ladder that is 5.12 m long. He places one end 2.45 m from a wall and rests the upper end against a cracked window and climbs the ladder. He climbs 3.10 m up the ladder when the window breaks. Neglecting friction between the ladder and the window and assuming that the base of the ladder does not slip, find (*a*) the force exerted on the window by the ladder just before the window breaks and (*b*) the magnitude and direction of the force exerted on the ladder by the ground just before the window breaks.

6. Two identical uniform frictionless spheres, each of weight *W*, rest as shown in Fig. 9-59 at the bottom of a fixed, rectangular container. The line of centers of the spheres makes an angle θ with the horizontal. Find the forces exerted on the spheres (*a*) by the container bottom, (*b*) by the container sides, and (*c*) by one another.

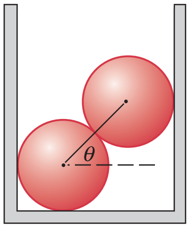


FIGURE 9-59. Problem 6.

7. A uniform sphere of weight *W* lies at rest wedged between two inclined planes of inclination angles θ_1 and θ_2 (Fig. 9-60).

(*a*) Assume that no friction is involved and determine the forces (directions and magnitudes) that the planes exert on the sphere. (*b*) What change would it make, in principle, if friction were taken into account?

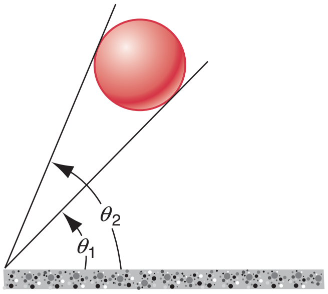


FIGURE 9-60. Problem 7.

8. A thin horizontal bar *AB* of negligible weight and length *L* is pinned to a vertical wall at *A* and supported at *B* by a thin wire *BC* that makes an angle θ with the horizontal. A weight *W* can be moved anywhere along the bar as defined by the distance *x* from the wall (Fig. 9-61). (*a*) Find the tension *T* in the thin wire as a function of *x*. (*b*) Find the horizontal and the vertical components of the force exerted on the bar by the pin at *A*. (*c*) With $W = 315$ N, $L = 2.76$ m, and $\theta = 32.0°$, find the maximum distance *x* before the wire breaks if the wire can withstand a maximum tension of 520 N.

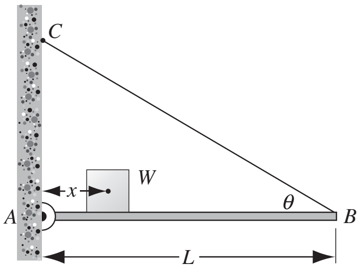


FIGURE 9-61. Problem 8.

9. A well-known problem is the following (see, for example, *Scientific American,* November 1964, p. 128): Uniform bricks are placed one upon another in such a manner as to have the maximum offset. This is accomplished by having the center of gravity of the top brick directly above the edge of the brick below it, the center of gravity of the two top bricks combined directly above the edge of the third brick from the top, and so on. (*a*) Justify this criterion for maximum offset; find the largest equilibrium offsets for four bricks. (*b*) Show that, if the process is continued downward, one can obtain as large an offset as one wants. (Martin Gardner, in the article referred to above, states: "With 52 playing cards, the first placed so that its end is flush with a table edge, the maximum overhang is a little more than $2\frac{1}{4}$ cardlengths . . . ") (*c*) Suppose now, instead, one piles up uniform bricks so that the end of one brick is offset from the one below it by a constant fraction, $1/n$, of a brick length *L*. How many bricks, *N*, can one use in this process before the pile will fall over? Check the plausibility of your answer for $n = 1$, $n = 2$, $n = \infty$.

10. (*a*) Show that the sum of the rotational inertias of a plane laminar body about any two perpendicular axes in the plane of the body is equal to the rotational inertia of the body about an axis through their point of intersection perpendicular to the plane. (*b*) Apply this to a circular disk to find its rotational inertia about a diameter as axis.

11. Prove that the rotational inertia of a flat square about a line drawn through the diagonal is equal to the rotational inertia about a line drawn through the center and crossing two opposite edges as a perpendicular bisector. (Hint: See Problem 10.)

12. Nine square holes have been cut in a flat square plate, as shown in Fig. 9-62. The plate has edge length *L*, and the holes have edge length *a*. The holes are located at the centers of the small squares formed by dividing each side of the square into three equal sections. Find the rotational inertia for rotations about an axis perpendicular to the plate passing through its center.

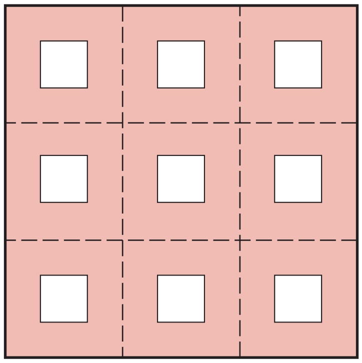


FIGURE 9-62. Problem 12.

13. (*a*) Show that for an object that can rotate about the *x*, *y*, or *z* axis

$$I_x + I_y + I_z = 2\int r^2\,dm,$$

where *r* is measured from the origin, *not* from the axis of rotation. (*b*) Is $I_x + I_y + I_z$ invariant under rotations of the coordinate system?

14. Use the results from Problem 13 to show that (*a*) the rotational inertia of a spherical shell of radius *R* is given by $I = \frac{2}{3}MR^2$ and (*b*) the rotational inertia of a solid sphere is given by $I = \frac{2}{5}MR^2$. Hint: Part (*a*) requires no significant integration. Part (*b*) uses

$$\frac{dm}{4\pi r^2 dr} = \frac{M}{(4/3)\pi R^3}.$$

15. In this problem we seek to compute the rotational inertia of a disk of mass *M* and radius *R* about an axis through its center and perpendicular to its surface. Consider a mass element *dm* in the shape of a ring of radius *r* and width *dr* (see Fig. 9-63). (*a*) What is the mass *dm* of this element, expressed as a fraction of the total mass *M* of the disk? (*b*) What is the rotational inertia *dI* of this element? (*c*) Integrate the result of part (*b*) to find the rotational inertia of the entire disk.

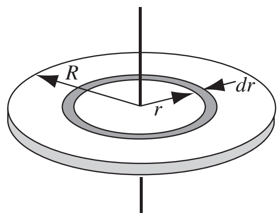


FIGURE 9-63. Problem 15.

16. In this problem, we use the result of the previous problem for the rotational inertia of a disk to compute the rotational inertia of a uniform solid sphere of mass M and radius R about an axis through its center. Consider an element dm of the sphere in the form of a disk of thickness dz at a height z above the center (see Fig. 9-64). (*a*) Expressed as a fraction of the total mass M, what is the mass dm of the element? (*b*) Considering the element as a disk, what is its rotational inertia dI? (*c*) Integrate the result of (*b*) over the entire sphere to find the rotational inertia of the sphere.

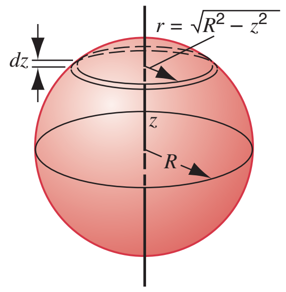


FIGURE 9-64. Problem 16.

17. Figure 9-65 shows two blocks each of mass m suspended from the ends of a rigid weightless rod of length $L_1 + L_2$, with $L_1 = 20.0$ cm and $L_2 = 80.0$ cm. The rod is held in the horizontal position shown in the figure and then released. Calculate the linear accelerations of the two blocks as they start to move.

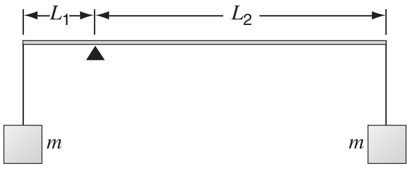


FIGURE 9-65. Problem 17.

18. A wheel of mass M and radius of gyration k (see Exercise 20) spins on a fixed horizontal axle passing through its hub. Assume that the hub rubs the axle of radius a at only the topmost point, the coefficient of kinetic friction being μ_k. The wheel is given an initial angular velocity ω_0. Assume uniform deceleration and find (*a*) the elapsed time and (*b*) the number of revolutions before the wheel comes to a stop.

19. A uniform disk of radius R and mass M is spinning with angular speed ω_0. It is placed on a flat horizontal surface; the coefficient of kinetic friction between disk and surface is μ_k. (*a*) Find the frictional torque on the disk. (*b*) How long will it take for the disk to come to rest?

20. A hoop rolling down an inclined plane of inclination angle θ keeps pace with a block sliding down the same plane. Show that the coefficient of kinetic friction between block and plane is given by $\mu_k = \frac{1}{2} \tan \theta$.

21. A uniform sphere rolls down an incline. (*a*) What must be the incline angle if the linear acceleration of the center of the sphere is to be $0.133g$? (*b*) For this angle, what would be the acceleration of a frictionless block sliding down the incline?

22. A solid cylinder of length L and radius R has a weight W. Two cords are wrapped around the cylinder, one near each end, and the cord ends are attached to hooks on the ceiling. The cylinder is held horizontally with the two cords exactly vertical and is then released (Fig. 9-66). Find (*a*) the tension in each cord as they unwind and (*b*) the linear acceleration of the cylinder as it falls.

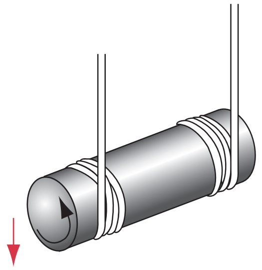


FIGURE 9-66. Problem 22.

23. Show that a cylinder will slip on an inclined plane of inclination angle θ if the coefficient of static friction between plane and cylinder is less than $\frac{1}{3} \tan \theta$.

24. A uniform disk, of mass M and radius R, lies on one side initially at rest on a frictionless horizontal surface. A constant force F is than applied tangentially at its perimeter by means of a string wrapped around its edge. Describe the subsequent (rotational and translational) motion of the disk.

25. A sphere, a cylinder, and a hoop (each of radius R and mass M) start from rest and roll down the same incline. (*a*) Which object gets to the bottom first? (*b*) Does your answer depend on the mass or radius of the objects? Explain.

CHAPTER 10

ANGULAR MOMENTUM

In Chapter 9 we discussed the dynamics of the rotational motion of a rigid body about an axis that is fixed in an inertial reference frame. We saw that the one-dimensional relation $\Sigma \tau_z = I\alpha_z$, in which only external torque components along the axis of rotation were considered, was sufficient to solve dynamical problems in this special case.

In this chapter we continue this analysis and extend it to situations in which the axis of rotation may not *be fixed in an inertial reference frame. To solve these dynamical problems we develop and use a three-dimensional vector relation for rotational motion, which is analogous to the vector form of Newton's second law,* $\vec{\mathbf{F}} = d\vec{\mathbf{P}}/dt$. *We also introduce* angular momentum *and show its importance as a dynamical property of rotations.*

Finally, we show that, for systems on which no net external torque acts, the important law of conservation of angular momentum *can be applied.*

10-1 ANGULAR MOMENTUM OF A PARTICLE

We have found *linear momentum* to be useful in dealing with the translational motion of single particles or of systems of particles, including rigid bodies. For example, linear momentum is conserved in collisions. For a single particle the linear momentum is $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$ (Eq. 6-1); for a system of particles it is $\vec{\mathbf{P}} = M\vec{\mathbf{v}}_{\text{cm}}$ (Eq. 7-21), in which M is the total system mass and $\vec{\mathbf{v}}_{\text{cm}}$ is the velocity of the center of mass. In rotational motion, the analogue of linear momentum is called *angular momentum,* which we define below for the special case of a single particle. Later, we broaden the definition to include systems of particles, and we show that angular momentum is as useful a concept in rotational motion as linear momentum is in translational motion.

Consider a particle of mass m and linear momentum $\vec{\mathbf{p}}$ at a position $\vec{\mathbf{r}}$ relative to the origin O of an inertial reference frame; for convenience (see Fig. 10-1) we have chosen the plane defined by the vectors $\vec{\mathbf{p}}$ and $\vec{\mathbf{r}}$ to be the xy plane. We define the *angular momentum* $\vec{\mathbf{l}}$ of the particle *with respect to the origin O* to be

$$\vec{\mathbf{l}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}. \tag{10-1}$$

As in the case of torque, angular momentum is defined in terms of a vector product or cross product (see Appendix H). Note that we must specify the origin O in order to define the position vector $\vec{\mathbf{r}}$ in the definition of angular momentum.

Angular momentum is a vector. Its magnitude is given by

$$l = rp \sin \theta, \tag{10-2}$$

where θ is the smaller angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{p}}$; its direction is perpendicular to the plane formed by $\vec{\mathbf{r}}$ and $\vec{\mathbf{p}}$. The sense is given by the right-hand rule: swing the fingers of the right hand from the direction of $\vec{\mathbf{r}}$ into the direction of $\vec{\mathbf{p}}$, through the smaller angle between them; the extended right thumb then points in the direction of $\vec{\mathbf{l}}$ (parallel to the z axis in Fig. 10-1).

We also can write the magnitude of $\vec{\mathbf{l}}$ either as

$$l = (r \sin \theta)p = pr_{\perp} \tag{10-3a}$$

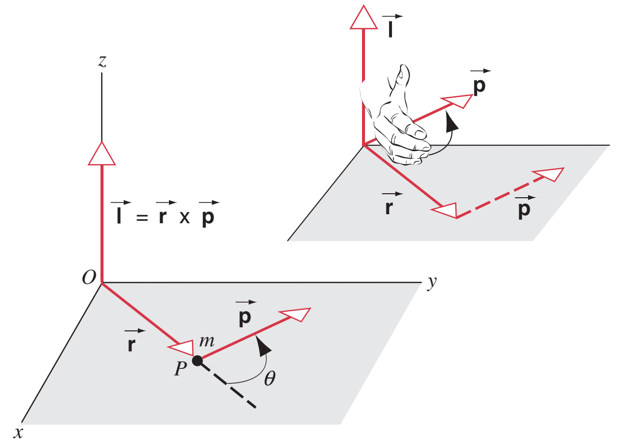


FIGURE 10-1. A particle of mass m, located at point P by the position vector $\vec{r}$, has a linear momentum $\vec{p} = m\vec{v}$. (For simplicity both $\vec{r}$ and $\vec{p}$ are assumed to lie in the xy plane.) Relative to the origin O, the particle has an angular momentum of $\vec{l} = \vec{r} \times \vec{p}$, which is parallel to the z axis in this case. The inset shows the use of the right-hand rule to find the direction of $\vec{l}$. Note that we can slide $\vec{p}$ without changing its direction until $\vec{r}$ and $\vec{p}$ are tail to tail.

or as

$$l = r(p \sin \theta) = rp_{\perp}, \tag{10-3b}$$

in which $r_{\perp}$ $(= r \sin \theta)$ is the component of $\vec{r}$ at right angles to the line of action of $\vec{p}$, and $p_{\perp}(= p \sin \theta)$ is the component of $\vec{p}$ at right angles to $\vec{r}$. Equation 10-3b shows that only the component of $\vec{p}$ perpendicular to $\vec{r}$ contributes to the angular momentum. When the angle θ between $\vec{r}$ and $\vec{p}$ is 0° or 180°, there is no perpendicular component ($p_{\perp} = p \sin \theta = 0$); then the line of action of $\vec{p}$ passes through the origin, and $r_{\perp}$ is also zero. In this case both Eqs. 10-3a and 10-3b show that the angular momentum l is zero.

We now derive an important relation between torque and angular momentum for a single particle. First, we differentiate Eq. 10-1 and obtain

$$\frac{d\vec{l}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}). \tag{10-4}$$

The derivative of a vector product is taken in the same way as the derivative of an ordinary product, except that we must not change the order of the terms. We have

$$\frac{d\vec{l}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}.$$

Here $d\vec{r}/dt$ is the instantaneous velocity $\vec{v}$ of the particle, and $\vec{p}$ equals $m\vec{v}$. Making these substitutions into the first product on the right, we obtain

$$\frac{d\vec{l}}{dt} = (\vec{v} \times m\vec{v}) + \vec{r} \times \frac{d\vec{p}}{dt}. \tag{10-5}$$

Now $\vec{v} \times m\vec{v} = 0$, because the vector product of two parallel vectors is zero. Replacing $d\vec{p}/dt$ in the second product by the net force $\sum \vec{F}$ acting on the particle, we have

$$\frac{d\vec{l}}{dt} = \vec{r} \times \sum \vec{F}.$$

The right side of this equation is just the net torque $\sum \vec{\tau}$. We therefore obtain

$$\sum \vec{\tau} = \frac{d\vec{l}}{dt}, \tag{10-6}$$

which states that *the net torque acting on a particle is equal to the time rate of change of its angular momentum.* Both the torque $\vec{\tau}$ and the angular momentum $\vec{l}$ in this equation must be defined with respect to the same origin. Equation 10-6 is the rotational analogue of Eq. 6-2, $\sum \vec{F} = d\vec{p}/dt$, which states that the net *force* acting on a particle is equal to the time rate of change of its *linear* momentum.

Equation 10-6, like all three-dimensional vector equations, is equivalent to three one-dimensional equations—namely,

$$\sum \tau_x = \frac{dl_x}{dt}, \qquad \sum \tau_y = \frac{dl_y}{dt}, \qquad \sum \tau_z = \frac{dl_z}{dt}. \tag{10-7}$$

Hence, the x component of the net external torque is given by the change with time of the x component of the angular momentum. Similar results hold for the y and z directions.

SAMPLE PROBLEM 10-1. A particle of mass m is released from rest at point P in Fig. 10-2, falling parallel to the (vertical) y axis. (a) Find the torque acting on m at any time t, with respect to origin O. (b) Find the angular momentum of m at any time t, with respect to this same origin. (c) Show that the relation $\sum \vec{\tau} = d\vec{l}/dt$ (Eq. 10-6) yields a correct result when applied to this familiar problem.

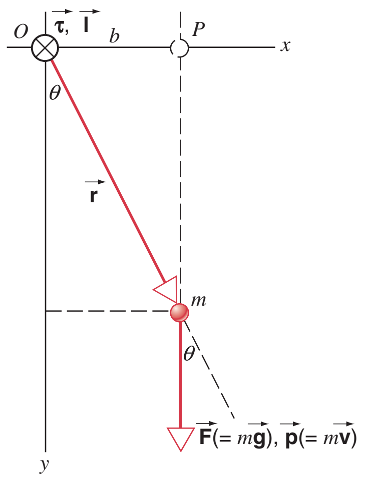


FIGURE 10-2. Sample Problem 10-1. A particle of mass m drops vertically from point P. The torque $\vec{\tau}$ and the angular momentum $\vec{l}$ with respect to the origin O are directed perpendicularly into the figure, as indicated by the symbol ⊗ at point O. This is the direction of the positive z axis.

Solution (*a*) The torque is given by $\vec{\tau} = \vec{r} \times \vec{F}$, and its magnitude is

$$\tau = rF \sin \theta.$$

In this example $r \sin \theta = b$ and $F = mg$, so that

$$\tau = mgb = \text{a constant}.$$

Note that the torque is simply the product of the force mg times the moment arm b. The right-hand rule shows that $\vec{\tau}$ is directed perpendicularly into the figure (along the positive z axis).
(*b*) The angular momentum is given by Eq. 10-1, $\vec{l} = \vec{r} \times \vec{p}$. Its magnitude is, from Eq. 10-2,

$$l = rp \sin \theta.$$

In this example $r \sin \theta = b$ and $p = mv = m(gt)$, so that

$$l = mgbt.$$

The right-hand rule shows that $\vec{l}$ is directed perpendicularly into the figure, which means that $\vec{l}$ and $\vec{\tau}$ are parallel vectors. The vector $\vec{l}$ changes with time in magnitude only, its direction always remaining the same in this case.
(*c*) Writing Eq. 10-6 in terms of z components, we have

$$\sum \tau_z = \frac{dl_z}{dt}.$$

Substituting the expression for τ_z and l_z from (*a*) and (*b*) above gives

$$mgb = \frac{d}{dt}(mgbt) = mgb,$$

which is an identity. Thus the relation $\vec{\tau} = d\vec{l}/dt$ yields correct results in this simple case. Indeed, if we cancel the constant b out of the first two terms above and if we substitute for gt the equivalent quantity v_y, we have

$$mg = \frac{d}{dt}(mv_y).$$

Since $mg = F_y$ and $mv_y = p_y$, this is the familiar result $F_y = dp_y/dt$. Thus, as we indicated earlier, relations such as $\vec{\tau} = d\vec{l}/dt$, though often vastly useful, are not new basic postulates or classical mechanics but are rather the reformulation of the Newtonian laws in the case of rotational motion.

Note that the magnitudes of τ and l depend on our choice of origin—that is, on b. In particular, if $b = 0$, then $\tau = 0$ and $l = 0$.

10-2 SYSTEMS OF PARTICLES

So far we have discussed only single particles. To calculate the total angular momentum $\vec{L}$ of a *system of particles* about a given point, we must add vectorially the angular momenta of all the individual particles about this point. For a system containing N particles we then have

$$\vec{L} = \vec{l}_1 + \vec{l}_2 + \cdots + \vec{l}_N = \sum_{n=1}^{N} \vec{l}_n, \qquad (10\text{-}8)$$

in which the (vector) sum is taken over all particles in the system.

As time goes on, the total angular momentum $\vec{L}$ of the system about a fixed reference point (which we choose, as in our basic definition of $\vec{l}$ in Eq. 10-1, to be the origin of an inertial reference frame) may change. That is,

$$\frac{d\vec{L}}{dt} = \frac{d\vec{l}_1}{dt} + \frac{d\vec{l}_2}{dt} + \cdots = \sum_{n=1}^{N} \frac{d\vec{l}_n}{dt}.$$

For each particle, $d\vec{l}_n/dt = \vec{\tau}_n$, and making this substitution we obtain

$$\frac{d\vec{L}}{dt} = \sum \vec{\tau}_n.$$

That is, the time rate of change of the *total* angular momentum of a system of particles equals the net torque due to the forces acting on the particles in the system.

Among the torques acting on the system will be (1) torques exerted on the particles of the system by internal forces between the particles and (2) torques exerted on the particles of the system by external forces. If Newton's third law holds in its so-called strong form—that is, if the forces between any two particles not only are equal and opposite but are also directed along the line joining the two particles—then the total internal torque is zero because the torque resulting from each internal action–reaction force pair is zero. (We proved this result in Section 9-2 for a two-particle system; by considering the particles in an N-particle system two at a time, we can show that it holds true in more complex systems as well.)

Hence the first source, the torque from internal forces, contributes nothing to the change in $\vec{L}$. Only the second source (the torque from external forces) remains, and we can write

$$\sum \vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt}, \qquad (10\text{-}9)$$

where $\Sigma\, \vec{\tau}_{\text{ext}}$ is the sum of the *external* torques acting on the system. In words, *the net external torque acting on a system of particles is equal to the time rate of change of the total angular momentum of the system.* The torque and the angular momentum must be calculated with respect to the same origin of an inertial reference frame. In situations in which no confusion is likely to arise, we drop the subscript on $\vec{\tau}_{\text{ext}}$ for convenience.

Equation 10-9 is the generalization of Eq. 10-6 to many particles. It holds whether the particles that make up the system are in motion relative to each other or whether they have fixed spatial relationships, as in a rigid body.

Equation 10-9 is the rotational analogue of Eq. 7-23; $\Sigma\, \vec{F}_{\text{ext}} = d\vec{P}/dt$, which tells us that for a system of particles (rigid body or not) the net external force acting on the system is equal to the time rate of change of its total linear momentum.

Let us extend further the analogy between the way a force changes linear momentum and the way a torque changes angular momentum. Suppose a force $\vec{F}$ acts on a particle moving with linear momentum $\vec{p}$. We can resolve

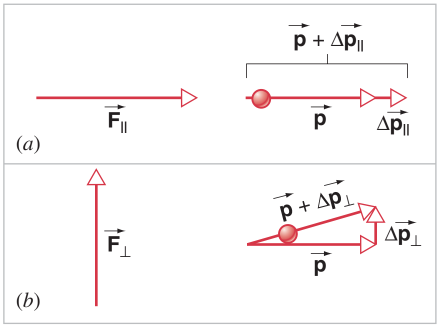


FIGURE 10-3. (*a*) When a force component $\vec{F}_{\parallel}$ acts parallel to the linear momentum $\vec{p}$ of a particle, the linear momentum changes by $\Delta\vec{p}_{\parallel}$, which is parallel to $\vec{p}$. (*b*) When a force component $\vec{F}_{\perp}$ acts perpendicular to the linear momentum $\vec{p}$ of a particle, the linear momentum changes by $\Delta\vec{p}_{\perp}$, which is perpendicular to $\vec{p}$. The particle now moves in the direction of the vector sum $\vec{p} + \Delta\vec{p}_{\perp}$.

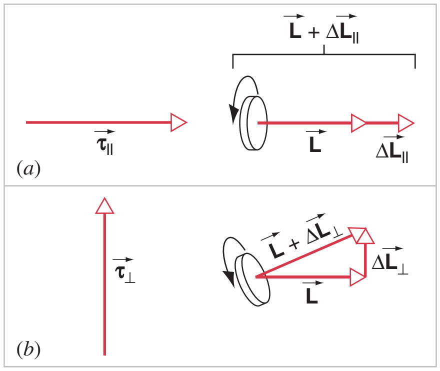


FIGURE 10-4. (*a*) When a torque component $\vec{\tau}_{\parallel}$ acts parallel to the angular momentum **L** of a system, the angular momentum changes by $\Delta\vec{L}_{\parallel}$, which is parallel to $\vec{L}$. (*b*) When a torque component $\vec{\tau}_{\perp}$ acts perpendicular to the angular momentum $\vec{L}$ of a system, the angular momentum changes by $\Delta\vec{L}_{\perp}$, which is perpendicular to $\vec{L}$. The axis of rotation now points in the direction corresponding to the vector sum $\vec{L} + \Delta\vec{L}_{\perp}$.

$\vec{F}$ into two components, as shown in Fig. 10-3: one component ($\vec{F}_{\parallel}$) is parallel to the (instantaneous) direction of $\vec{p}$ and another ($\vec{F}_{\perp}$) is perpendicular to $\vec{p}$. In a small interval of time Δt, the force produces a change in momentum $\Delta\vec{p}$ determined according to $\vec{F} = \Delta\vec{p}/\Delta t$. Thus $\Delta\vec{p}$ is parallel to $\vec{F}$. The component $\vec{F}_{\parallel}$ gives a change in momentum $\Delta\vec{p}_{\parallel}$ parallel to $\vec{p}$, which adds to $\vec{p}$ and changes its magnitude but not its direction (see Fig. 10-3*a*). The perpendicular component $\vec{F}_{\perp}$, on the other hand, gives an increment $\Delta\vec{p}_{\perp}$ that changes the direction of $\vec{p}$ but, when $\Delta\vec{p}_{\perp}$ is small compared with $\vec{p}$, leaves the magnitude of $\vec{p}$ unchanged (see Fig. 10-3*b*). An example of the latter is a particle moving in a circle at constant speed subject only to a centripetal force, which is always perpendicular to the tangential velocity.

The same analysis holds for the action of a torque, as shown in Fig. 10-4. In this case $\vec{\tau} = \Delta\vec{L}/\Delta t$, and $\Delta\vec{L}$ must be parallel to $\vec{\tau}$. We once again resolve $\vec{\tau}$ into two components, $\vec{\tau}_{\parallel}$ parallel to $\vec{L}$ and $\vec{\tau}_{\perp}$ perpendicular to $\vec{L}$. The component of $\vec{\tau}$ parallel to $\vec{L}$ changes the angular momentum in magnitude but not in direction (Fig. 10-4*a*). The component of $\vec{\tau}$ perpendicular to $\vec{L}$ gives an increment $\Delta\vec{L}_{\perp}$ perpendicular to $\vec{L}$, which changes the direction of $\vec{L}$ but not its magnitude (Fig. 10-4*b*). This latter condition is responsible for the motion of tops and gyroscopes, as we discuss in Section 10-5. Comparing Figs. 10-3 and 10-4, you can see the similarities between translational and rotational dynamics.

An example of the application of Eq. 10-9 for rotational dynamics is shown in Fig. 10-5. In Fig. 10-5*a*, one end of the axle of a spinning bicycle wheel rests freely on a post, and the other end is supported by a student's hand. The student pushes tangentially on the wheel with a force $\vec{f}$ at its rim, in order to make it spin faster. Taken about the center of the wheel, the torque exerted by the student is parallel to the angular momentum of the wheel, both vectors ($\vec{\tau}$ and $\vec{L}$) pointing toward the student. The result of this torque is an increase in the angular momentum of the wheel.

In Fig. 10-5*b*, the student has released one support of the axle. Now we consider the torques about the remaining point of support. There are two forces acting: a normal force at the point of support, which gives no torque about that point, and the wheel's weight acting downward at the center of mass. The torque about point O due to the weight is perpendicular to $\vec{L}$, and its effect is therefore to change the direction of $\vec{L}$, as in Fig. 10-4*b*. However, since the direction of $\vec{L}$ is also the direction of the axle,* the effect of the (downward) force

* This holds only if the axis of rotation is also an axis of symmetry of the body; see Section 10-3.

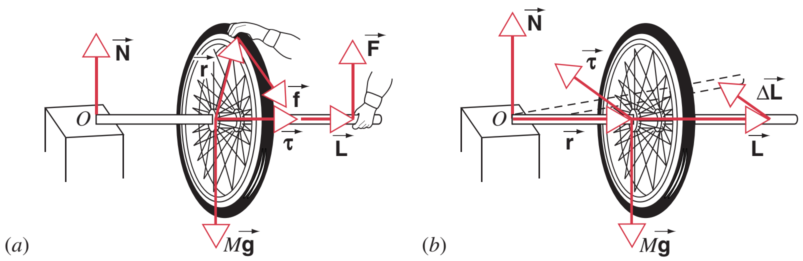


FIGURE 10-5. (*a*) A tangential force $\vec{f}$ on the rim of the wheel gives a torque $\vec{\tau}$ (about the center of the wheel) along the axis of rotation, increasing the magnitude of the angular velocity of the wheel but leaving its direction unchanged. (*b*) When the end of the axle is released, the gravitational torque about the point O points into the paper—that is, perpendicular to the rotational axis—as in Fig. 10-4*b*. This torque changes the direction of the rotational axis, and the shaft of the wheel moves in the horizontal plane toward the position shown by the dashed line.

of gravity is to turn the axle sideways in the horizontal plane. The wheel will pivot sideways about the point of support. Try it! (If you don't have a freely mounted bicycle wheel handy, a toy gyroscope works as well.)

As we have derived it, Eq. 10-9 holds when $\vec{\tau}$ and $\vec{L}$ are measured with respect to the origin of an inertial reference frame. We may well ask whether it still holds if we measure these two vectors with respect to an arbitrary point (a particular particle, say) in the moving system. In general, such a point would move in a complicated way as the body or system of particles translated, tumbled, and changed its configuration, and Eq. 10-9 would not apply to such a reference point. However, if the reference point is chosen to be the center of mass of the system, even though this point may be accelerating in our inertial reference frame, then Eq. 10-9 does hold. (See Exercise 7.) This is another remarkable property of the center of mass. Thus we can separate the general motion of a system of particles into the translational motion of its center of mass (Eq. 7-23) and the rotational motion about its center of mass (Eq. 10-9).

10-3 ANGULAR MOMENTUM AND ANGULAR VELOCITY

To introduce cases in which it is absolutely necessary to consider the three-dimensional vector nature of angular velocity, torque, and angular momentum, we first consider a simple example of a rotating particle that illustrates an instance in which the angular velocity and angular momentum are not parallel.

Figure 10-6*a* shows a single particle of mass *m* attached to a rigid, massless shaft by a rigid, massless arm of length r' perpendicular to the shaft. The particle moves in a circle of radius r', and we assume it does so at constant speed v. We imagine this experiment to be done in a region of negligible gravity, so that we need not consider the force of gravity acting on the particle. The only force that acts on the particle is the centripetal force exerted by the arm connecting the particle to the shaft.

The shaft is confined to the z axis by two thin ideal (frictionless) bearings. We let the lower bearing define the origin O of our coordinate system. The upper bearing, as we shall see, is necessary to prevent the shaft from wobbling about the z axis, which occurs when the angular velocity is not parallel to the angular momentum.

The angular velocity $\vec{\omega}$ of the particle points upward along (or, equivalently, parallel to) the z axis, as shown in Fig. 10-6*b*. No matter where the origin is chosen along the z axis, the angular velocity vector will be parallel to the axis. Its magnitude is similarly independent of the location of the origin, being given by $v/(r \sin \theta) = v/r'$.

The angular momentum $\vec{l}$ of the particle *with respect to the origin O* of the reference frame is given by Eq. 10-1, or

$$\vec{l} = \vec{r} \times \vec{p},$$

where $\vec{r}$ and $\vec{p}$ $(= m\vec{v})$ are shown in Fig. 10-6*b*. The vector $\vec{l}$ is perpendicular to the plane formed by $\vec{r}$ and $\vec{p}$, which means that $\vec{l}$ is *not* parallel to $\vec{\omega}$. Note (see Fig. 10-6*c*) that $\vec{l}$ has a (vector) component $\vec{l}_z$, that is parallel to $\vec{\omega}$, but it has another (vector) component $\vec{l}_\perp$ that is perpendicular to $\vec{\omega}$. Here is a case in which our analogy between linear and circular motion is not valid: $\vec{p}$ is always parallel to $\vec{v}$, but $\vec{l}$ is *not* always parallel to $\vec{\omega}$. If we choose our origin to lie in the plane of the circulating particle, then $\vec{l}$ *is* parallel to $\vec{\omega}$; otherwise, it is not.

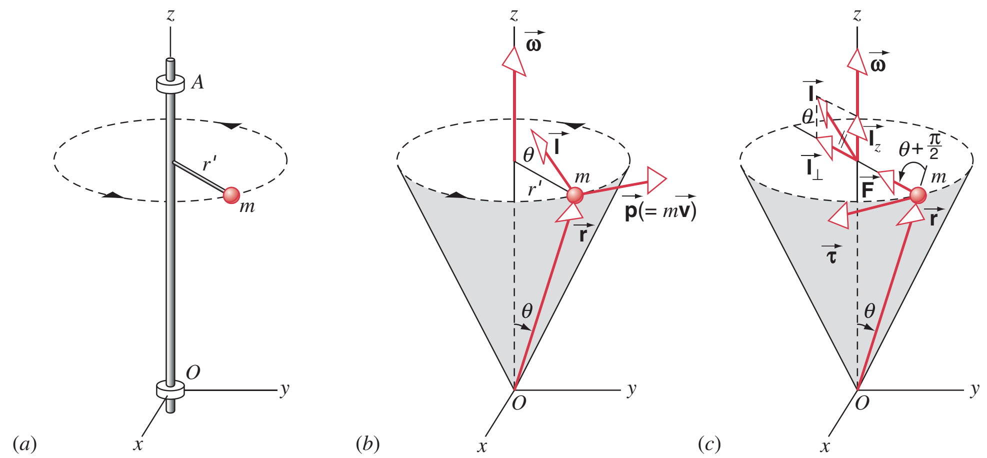


FIGURE 10-6. (*a*) A particle of mass *m* is attached through an arm of length r' to a shaft fixed by two bearings (at O and A) to rotate about the z axis. (*b*) The particle rotates with tangential speed v in a circle of radius r' about the z axis (the rods and bearings being omitted to simplify the drawing). The angular momentum $\vec{l} = \vec{r} \times \vec{p}$ about the origin O is shown. (*c*) For the particle to move in a circle, there must be a centripetal force $\vec{F}$ acting as shown, resulting in a torque $\vec{\tau}$ about O. For convenience, the angular momentum vector $\vec{l}$ and its components along and perpendicular to z are shown at the center of the circle.

Let us now consider the relationship between $\vec{l}_z$ and $\vec{\omega}$ for the rotating particle. From Fig. 10-6*c*, in which we have translated $\vec{l}$ to the center of the circle, we obtain

$$l_z = l \sin\theta = rp\sin\theta = r(mv)\sin\theta = r(mr'\omega)\sin\theta,$$

using $v = r'\omega$, where ω represents the magnitude of $\vec{\omega}$, which points in the z direction. Substituting r' (the radius of the circle in which the particle moves) for the product $r\sin\theta$ gives

$$l_z = mr'^2\omega. \tag{10-10}$$

Now mr'^2 is the rotational inertia I of the particle with respect to the z axis. Thus

$$l_z = I\omega. \tag{10-11}$$

Note that the vector relation $\vec{l} = I\vec{\omega}$ (which is analogous to the linear relation $\vec{p} = m\vec{v}$) is *not* correct in this case, because $\vec{l}$ and $\vec{\omega}$ do not point in the same direction.

Under what circumstances will the angular momentum and angular velocity point in the same direction? To illustrate, let us add another particle of the same mass m to the system, as shown in Fig. 10-7, by attaching another arm to the central shaft of Fig. 10-6*a* in the same location as the first arm but pointing in the opposite direction. The component $\vec{l}_\perp$ due to this second particle will be equal and opposite to that of the first particle, and the two $\vec{l}_\perp$ vectors sum to zero. The two $\vec{l}_z$ vectors point in the same direction, however, and add. Thus for this two-particle system, the total angular momentum $\vec{L}$ is parallel to $\vec{\omega}$.

We can now extend our system to a rigid body, made up of many particles. If the body is symmetric about the axis of rotation, by which we mean that for every mass element in the body there must be an identical mass element diametrically opposite the first element and at the same distance from the axis of rotation, then the body can be regarded as made up of sets of particle pairs of the kind we have been discussing. Since $\vec{L}$ and $\vec{\omega}$ are parallel for all such pairs, they are also parallel for rigid bodies that possess this kind of symmetry, which is called *axial* symmetry.

For such symmetrical rigid bodies $\vec{L}$ and $\vec{\omega}$ are parallel and we can write in vector form

$$\vec{L} = I\vec{\omega}. \tag{10-12}$$

Do not forget, however, that if $\vec{L}$ stands for the *total* angular momentum, then Eq. 10-12 applies *only* to bodies that have symmetry about the rotational axis. If $\vec{L}$ stands for the vector component of angular momentum along the rotational axis (that is, for $\vec{L}_z$), then Eq. 10-12 holds for *any* rigid body, symmetrical or not, that is rotating about a fixed axis.

For symmetrical bodies (such as the two-particle system of Fig. 10-7), the upper bearing in Fig. 10-6*a* may be removed, and the shaft will remain parallel to the z axis. You can verify this by noting how easy it is to spin a symmetrical object such as a small top held only between the thumb and forefinger of one hand. Any small asymmetry in the object requires the second bearing to keep the shaft in a fixed direction; the bearing must exert a torque on the shaft, which, otherwise would wobble as the object rotates, as we discuss at the end of this section. The issue of wobble is particularly serious for objects that rotate at high speeds, such as turbine rotors. Although designed to be symmetrical, such rotors, because of small errors of blade placement, for example, may be slightly asymmetrical. They may be restored to symmetry by the addition or removal of metal at appropriate places; this is done by spinning the wheel in a special device such that the wobble can be measured quantitatively and the appropriate corrective measure computed and indicated automatically. In a similar manner, lead weights are placed at strategic points on automobile tire rims to reduce wobble at high speeds. In "balancing" a wheel of your car, your mechanic is really just verifying that the angular momentum and angular velocity vectors of the wheel are parallel, thereby reducing the strain on the wheel bearings.

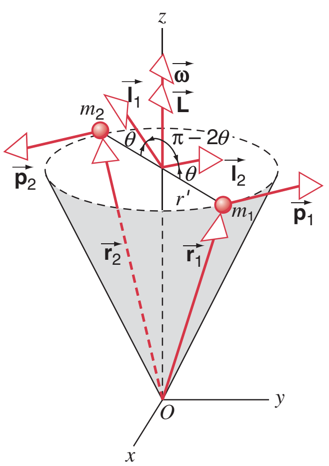


FIGURE 10-7. Two particles of mass m rotating as in Fig. 10-6 but at opposite ends of a diameter. The total angular momentum $\vec{L}$ of the two particles is in this case parallel to the angular velocity $\vec{\omega}$.

SAMPLE PROBLEM 10-2. Which has greater magnitude, the angular momentum of the Earth (relative to its center) associated with its rotation on its axis or the angular momentum of the Earth (relative to the center of its orbit) associated with its orbital motion around the Sun?

Solution For rotation on its axis, we treat the Earth as a uniform sphere ($I = \frac{2}{5}MR_E^2$). The angular speed is $\omega = 2\pi/T$, where T is the rotational period (24 h = 8.64×10^4 s). The magnitude of the rotational angular momentum about an axis through the center of the Earth is then

$$L_{rot} = I\omega = \tfrac{2}{5}MR_E^2\,\frac{2\pi}{T}$$

$$= \tfrac{2}{5}(5.98 \times 10^{24}\ \text{kg})(6.37 \times 10^{6}\ \text{m})^2\,\frac{2\pi}{8.64 \times 10^4\ \text{s}}$$

$$= 7.06 \times 10^{33}\ \text{kg}\cdot\text{m}^2/\text{s}.$$

To calculate the orbital angular momentum, we need the rotational inertia of the Earth about an axis through the Sun. For this we can

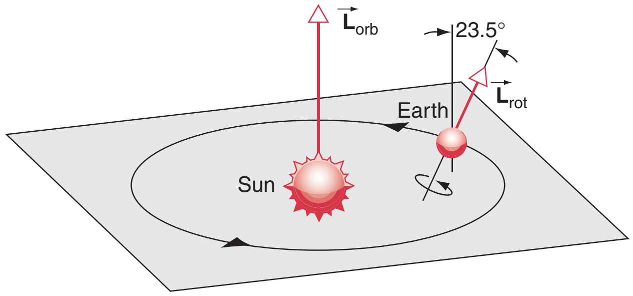


FIGURE 10-8. Sample Problem 10-2. The Earth rotates in an (assumed circular) orbit around the Sun, and it also rotates about its axis. The two angular momentum vectors are not parallel, because the Earth's rotational axis is inclined at an angle of 23.5° with respect to the normal to the plane of the orbit. The lengths of the vectors are not drawn to scale; L_{orb} should be greater than L_{rot} by a factor of about 4×10^6.

treat the Earth as a "particle," with angular momentum $L = R_{orb}\,p$, where R_{orb} is the radius of the orbit and p is the linear momentum of the Earth. The angular velocity is again given by $\omega = 2\pi/T$, where now T is the orbital period (1 y $= 3.16 \times 10^7$ s). The magnitude of the orbital angular momentum about an axis through the Sun is

$$L_{orb} = R_{orb}p = R_{orb}Mv = R_{orb}M(\omega R_{orb}) = MR^2_{orb}\frac{2\pi}{T}$$

$$= (5.98 \times 10^{24}\text{ kg})(1.50 \times 10^{11}\text{ m})^2 \frac{2\pi}{3.16 \times 10^7\text{ s}}$$

$$= 2.67 \times 10^{40}\text{ kg}\cdot\text{m}^2/\text{s}.$$

The orbital angular momentum is thus far greater than the rotational angular momentum.

The orbital angular momentum vector points at right angles to the plane of the Earth's orbit (Fig. 10-8), while the rotational angular momentum is inclined at an angle of 23.5° to the normal to the plane. Neglecting the very slow precession of the rotational axis, the two vectors remain constant in both magnitude and direction as the Earth moves in its orbit.

SAMPLE PROBLEM 10-3. In Sample Problem 9-10, find the acceleration of the falling block by direct application of Eq. 10-9 ($\Sigma\vec{\tau} = d\vec{\mathbf{L}}/dt$).

Solution The system shown in Fig. 10-9, consisting of the pulley (assumed to be a uniform disk of mass M and radius R) and the block of mass m, is acted on by two external forces, the downward pull of gravity $m\vec{\mathbf{g}}$ acting on m and the upward force exerted by the bearings of the shaft of the disk, which we take as our origin. (The tension in the cord is an internal force and does not act from the outside on the system of disk + block.) Only the first of these external forces exerts a torque about the origin; its magnitude is $(mg)R$ and its direction is along the positive z axis in Fig. 10-9.

The z component of the angular momentum of the system about the origin O at any instant is

$$L_z = I\omega + (mv)R,$$

in which $I\omega$ is the angular momentum of the (symmetrical) disk and $(mv)R$ is the angular momentum (= linear momentum × moment arm) of the falling block about the origin. Both these contributions to $\vec{\mathbf{L}}$ have positive z components.

Applying $\Sigma\,\vec{\tau}_z = dL_z/dt$ yields

$$(mg)R = \frac{d}{dt}(I\omega + mvR)$$

$$= I\left(\frac{d\omega}{dt}\right) + mR\left(\frac{dv}{dt}\right)$$

$$= I\alpha + mRa.$$

Since $a = \alpha R$ and $I = \frac{1}{2}MR^2$, this reduces to

$$mgR = (\tfrac{1}{2}MR^2)(a/R) + mRa$$

or

$$a = \frac{2mg}{M + 2m}.$$

This result is identical with the result of Sample Problem 9-10 as expected, because $\Sigma\,\tau_z = I\alpha_z$ and $\Sigma\,\tau_z = dL_z/dt$ are merely different ways of stating Newton's second law.

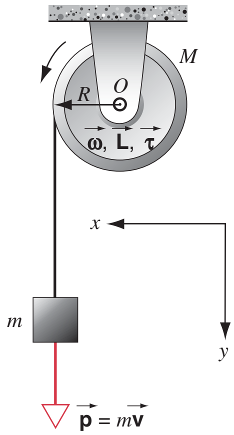


FIGURE 10-9. Sample Problem 10-3. The angular velocity, angular momentum, and net torque all point out of the page (in the positive z direction), as indicated by the symbol ⊙ at O.

The Torque on a Particle Moving in a Circular Path (Optional)

The perhaps unexpected result that $\vec{\mathbf{l}}$ and $\vec{\boldsymbol{\omega}}$ are not parallel in the simple case shown in Fig. 10-6 may cause some concern. However, this result is consistent with the general relationship $\vec{\tau} = d\vec{\mathbf{l}}/dt$ for the torque acting on a single particle. The vector $\vec{\mathbf{l}}$ is changing with time in that example as the particle moves; the change is entirely in direction and not in magnitude. As the particle revolves, $\vec{\mathbf{l}}_z$ remains constant in both magnitude and direction, but $\vec{\mathbf{l}}_\perp$ changes its direction. This change in $\vec{\mathbf{l}}_\perp$ must arise from the application of a torque. What is the source of this torque?

For the particle to move in a circle, it must be acted on by a centripetal force, as in Fig. 10-6c, provided by the supporting arm that connects the particle to the shaft. (We have

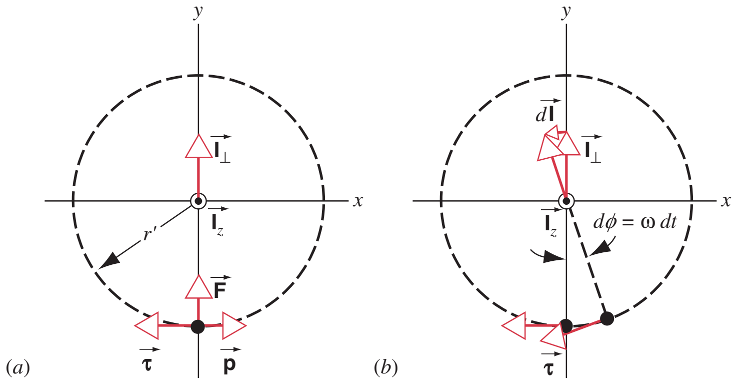


FIGURE 10-10. (*a*) A two-dimensional view of the plane of the rotating particle of Fig. 10-6. The z component of the angular momentum points out of the paper. (*b*) When the particle rotates through an angle $d\phi$, the vector component $\vec{l}_\perp$ in the plane changes by $d\vec{l}$. Note that $d\vec{l}$ is parallel to $\vec{\tau}$.

neglected other external forces, such as gravity.) The only torque about O is provided by $\vec{F}$ and is given by

$$\vec{\tau} = \vec{r} \times \vec{F}.$$

The torque $\vec{\tau}$ is tangent to the circle (perpendicular to the plane formed by $\vec{r}$ and $\vec{F}$) and in the direction shown in Fig. 10-6*c*, as you may verify from the right-hand rule.

Let us show that this torque satisfies the rotational form of Newton's second law, $\vec{\tau} = d\vec{l}/dt$. Figure 10-10*a* shows a two-dimensional view of the rotating particle, looking down along the z axis toward the xy plane. As the particle moves through the small angle $d\phi = \omega\, dt$ (Fig. 10-10*b*), the vector $\vec{l}_\perp$ changes by the small increment $d\vec{l}$. You can see from Fig. 10-10*b* that $d\vec{l}$ will always be parallel to $\vec{\tau}$, and so the directions of $d\vec{l}$ and $\vec{\tau}$ are consistent with $\vec{\tau} = d\vec{l}/dt$. We can also show that the magnitudes agree. The torque about O is, referring again to Fig. 10-6*c*,

$$\tau = rF \sin(\tfrac{1}{2}\pi + \theta) = rF \cos\theta.$$

In this case, $\vec{F}$ is the centripetal force and has magnitude $F = mv^2/r' = m\omega^2 r'$, where r' is the radius of the circular path ($r' = r\sin\theta$), so $F = m\omega^2 r \sin\theta$. Thus

$$\tau = m\omega^2 r^2 \sin\theta \cos\theta. \tag{10-13}$$

From Fig. 10-10*b*, $dl = l_\perp\, d\phi = l_\perp \omega\, dt$, from which we obtain

$$\frac{dl}{dt} = \omega l_\perp .$$

With $l = mvr$, then $l_\perp = mvr\cos\theta$. The tangential velocity v is $\omega r' = \omega r \sin\theta$, so

$$l_\perp = m\omega r^2 \sin\theta\cos\theta$$

and

$$\frac{dl}{dt} = \omega l_\perp = m\omega^2 r^2 \sin\theta\cos\theta. \tag{10-14}$$

Comparing Eqs. 10-13 and 10-14, we see that $\tau = dl/dt$, as expected. ■

Symmetrical Versus Asymmetrical Bodies (Optional)

How does the situation differ for symmetrical and asymmetrical rotating bodies? Suppose the rod connecting the two particles in the symmetrical body of Fig. 10-7 were inclined at an arbitrary angle β with respect to the central shaft. Figure 10-11 shows the connecting rod, the shaft, and the two bearings (assumed frictionless) that holds the shaft along the z axis. The shaft rotates at a constant angular speed ω about this axis, the vector $\vec{\omega}$ thus pointing along this axis. Experience tells us that such a system is "unbalanced" or "lopsided," and if the connecting rod were not rigidly fastened to the vertical shaft near O, it would tend to move until the angle β became 90°, in which position the system would then be symmetrical about the shaft.

At the instant shown in Fig. 10-11, the upper particle is moving into the page at right angles to it, and the lower particle is moving out of the page at right angles to it. The linear momentum vectors of the two particles are therefore equal but opposite, and so are their position vectors with respect to O. Hence, by application of the right-hand rule in $\vec{r} \times \vec{p}$, we find that $\vec{l}$ is the same for each particle and that their sum, the total angular momentum vector $\vec{L}$ of the system, is as shown in the figure, at right angles to the connecting rod and in the plane of the page. Hence $\vec{L}$ and $\vec{\omega}$ are not parallel at this instant. As the system rotates, the an-

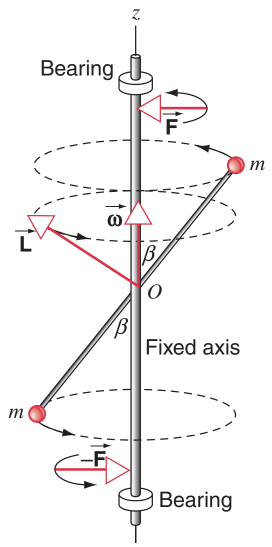


FIGURE 10-11. A rotating two-particle system, similar to Fig. 10-7, but with the axis of rotation making an angle β with the connecting rod. The angular momentum vector $\vec{L}$ rotates with the system, as do the forces $\vec{F}$ and $-\vec{F}$ exerted by the bearings.

gular momentum vector, while constant in magnitude, rotates around the fixed axis of rotation.

The rotation of $\vec{\mathbf{L}}$ about the fixed axis of Fig. 10-11 is perfectly consistent with the fundamental relation $\vec{\boldsymbol{\tau}} = d\vec{\mathbf{L}}/dt$. The external torque on the entire system arises from the unbalanced sideways forces exerted by the bearings on the shaft and transmitted by the shaft to the connecting rod. At the instant shown in the figure, the upper particle would tend to move outward to the right. The shaft would be pulled to the right against the upper bearing, which in turn exerts a force $\vec{\mathbf{F}}$ on the shaft that points to the left. Similarly, the lower particle tends to move outward to the left. The shaft would be pulled to the left against the lower bearing, which in turn exerts a force $-\vec{\mathbf{F}}$ on the shaft that points to the right. The torque $\vec{\boldsymbol{\tau}}$ about O as a result of these forces points perpendicularly out of the page, at right angles to the plane formed by $\vec{\mathbf{L}}$ and $\vec{\boldsymbol{\omega}}$, and in the right direction to account for the rotary motion of $\vec{\mathbf{L}}$. (Compare with Fig. 10-6*c*, in which $\vec{\boldsymbol{\tau}}$ was also perpendicular to the plane formed by $\vec{\mathbf{l}}$ and $\vec{\boldsymbol{\omega}}$.) Note that because $\vec{\boldsymbol{\tau}}$ is perpendicular to $\vec{\boldsymbol{\omega}}$, there is no component of the angular acceleration $\vec{\boldsymbol{\alpha}}$ in the direction of $\vec{\boldsymbol{\omega}}$, and so the angular velocity remains constant. In the absence of friction, the system will spin forever. Friction in the bearings would give rise to a torque directed along the shaft (parallel to $\vec{\boldsymbol{\omega}}$), which *would* have an angular acceleration component along $\vec{\boldsymbol{\omega}}$ and thus would change the angular velocity.

The forces $\vec{\mathbf{F}}$ and $-\vec{\mathbf{F}}$ lie in the plane of Fig. 10-11 at the instant shown. As the system rotates, these forces, and therefore the torque $\boldsymbol{\tau}$, rotate with it, so that $\vec{\boldsymbol{\tau}}$ always remains at right angles to the plane formed by $\vec{\boldsymbol{\omega}}$ and $\vec{\mathbf{L}}$. The rotating forces $\vec{\mathbf{F}}$ and $-\vec{\mathbf{F}}$ cause a wobble in the upper and lower bearings. The bearings and their supports must be made strong enough to provide these forces. For a symmetrical rotating body there is no bearing wobble, and the shaft rotates smoothly. ■

10-4 CONSERVATION OF ANGULAR MOMENTUM

In Eq. 10-9, we found that the time rate of change of the total angular momentum of a system of particles about a point fixed in an inertial reference frame (or about the center of mass) is equal to the net *external* torque acting on the system; that is

$$\sum \vec{\boldsymbol{\tau}}_{\text{ext}} = \frac{d\vec{\mathbf{L}}}{dt}. \qquad (10\text{-}9)$$

If the net external torque acting on the system is zero, then the angular momentum of the system does not change with time ($d\vec{\mathbf{L}}/dt = 0$). Thus

$$\vec{\mathbf{L}} = \text{a constant} \qquad \text{or} \qquad \vec{\mathbf{L}}_{\text{i}} = \vec{\mathbf{L}}_{\text{f}}. \qquad (10\text{-}15)$$

In this case the initial angular momentum is equal to the final angular momentum. Equation 10-15 is the mathematical statement of the principle of *conservation of angular momentum:*

If the net external torque acting on a system is zero, the total vector angular momentum of the system remains constant.

This is the second of the major conservation laws we have discussed. Along with conservation of linear momentum, conservation of angular momentum is a general result that is valid for a wide range of systems. It holds true in both the relativistic limit and in the quantum limit. No exceptions have ever been found.

Like conservation of linear momentum in a system on which the net external *force* is zero, conservation of angular momentum applies to the total angular momentum of a system of particles on which the net external *torque* is zero. The angular momentum of individual particles in a system may change due to *internal* torques (just as the linear momentum of each particle in a collision may change due to *internal* forces), but the total remains constant.

Angular momentum is (like linear momentum) a *vector* quantity so that Eqs. 10-15 is equivalent to three one-dimensional equations, one for each coordinate direction through the reference point. Conservation of angular momentum therefore supplies us with three conditions on the motion of a system to which it applies. *Any component of the angular momentum will be constant if the corresponding component of the torque is zero;* it might be the case that only one of the three components of torque is zero, which means that only one component of the angular momentum will be constant, the other components changing as determined by the corresponding torque components.

For a system consisting of a rigid body rotating with angular speed ω about an axis (the z axis, say) that is fixed in an inertial reference frame, we have

$$L_z = I\omega, \qquad (10\text{-}16)$$

where L_z is the component of the angular momentum along the rotation axis and I is the rotational inertia for this same axis. *If no net external torque acts, then L_z must remain constant.* If the rotational inertia I of the body changes (from I_{i} to I_{f})—for example, by a change in the distance of parts of the body from the axis of rotation—there must be a compensating change in ω from ω_{i} to ω_{f}. The principle of conservation of angular momentum in this case is expressed as $L_{\text{i}z} = L_{\text{f}z}$ or

$$I_{\text{i}}\omega_{\text{i}} = I_{\text{f}}\omega_{\text{f}}. \qquad (10\text{-}17)$$

Equation 10-17 holds not only for rotation about a fixed axis but also for rotation about an axis through the center of mass of a system that moves so that the axis always remains parallel to itself (see the discussion at the beginning of Section 9-7).

Conservation of angular momentum is a principle that regulates a wide variety of physical processes, from the subatomic world to the motion of acrobats, divers, and ballet dancers, to the contraction of stars that have run out of fuel, and to the condensation of galaxies. The following examples show some of these applications.

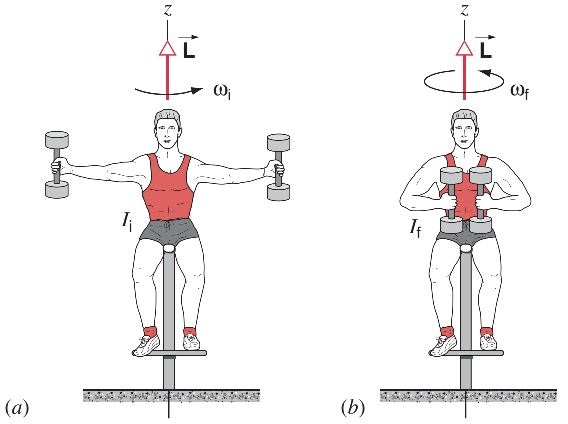


FIGURE 10-12. (*a*) In this configuration, the system (student + weights) has a larger rotational inertia and a smaller angular velocity. (*b*) Here the student has pulled the weights inward, giving a smaller rotational inertia and hence a larger angular velocity. The angular momentum $\vec{L}$ has the same value in both situations.

The Spinning Skater

A spinning ice skater pulls her arms close to her body to spin faster and extends them to spin slower. When she does this, she is applying Eq. 10-17. Another application of this principle is illustrated in Fig. 10-12, which shows a student sitting on a stool that can rotate freely about a vertical axis. Let the student extend his arms holding the weights, and we will set him into rotation at an angular velocity ω_i. His angular momentum vector $\vec{L}$ lies along the vertical axis (z axis) in the figure.

The system consisting of student + stool + weights is an isolated system on which no external vertical torque acts. The vertical component of angular momentum must therefore be conserved.

When the student pulls his arms (and the weights) closer to his body, the rotational inertia of his system is reduced from its initial value I_i to a smaller value I_f, because the weights are now closer to the axis of rotation. His final angular speed, from Eq. 10-17, is $\omega_f = \omega_i(I_i/I_f)$, which is greater than his initial angular velocity (because $I_f < I_i$), and the student rotates faster. To slow down, he need only extend his arms again.

The Springboard Diver*

Figure 10-13*a* shows a diver leaving the springboard. As she jumps, she pushes herself slightly forward so that she acquires a small rotational speed, just enough to carry her head-first into the water as her body rotates through one-half revolution during the arc.

While she is in the air, no external torques act on her to change her angular momentum about her center of mass.

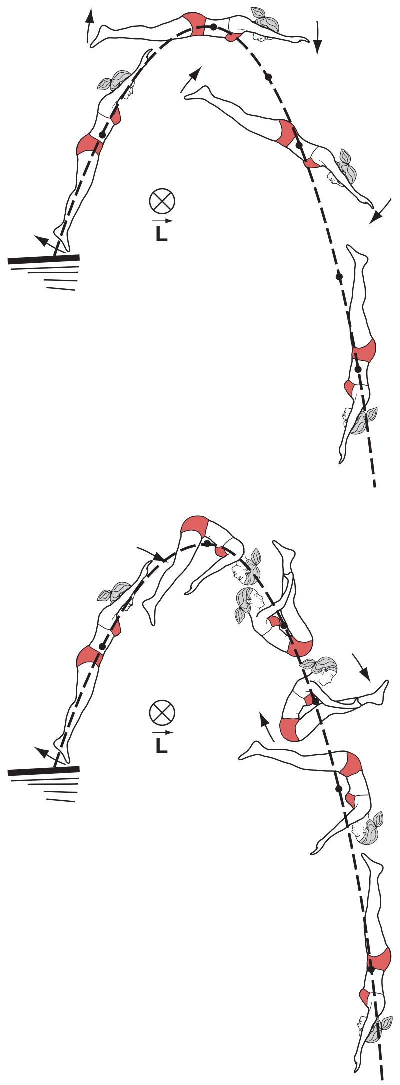


FIGURE 10-13. (*a*) A diver leaves the springboard in such a way that the springboard imparts to her an angular momentum $\vec{L}$. She rotates about her center of mass (indicated by the dot) by one-half revolution as the center of mass follows the parabolic trajectory. (*b*) By entering the tuck position, she reduces her rotational inertia and thus increases her angular velocity, enabling her to make $1\frac{1}{2}$ revolutions. The external forces and torques on her are the same in (*a*) and (*b*), as indicated by the constant value of the angular momentum $\vec{L}$.

(The only external force, gravity, acts *through* her center of mass and thus produces no torque about that point. We neglect air resistance, which could produce a net torque and change her angular momentum.) When she pulls her body into the *tuck position,* she lowers her rotational inertia, and therefore according to Eq. 10-17 her angular velocity must increase. The increased angular velocity enables her to complete $1\frac{1}{2}$ revolutions where she had previously completed only one-half revolution (Fig. 10-13*b*). At the end of the dive, she pulls back out into the *layout position* and slows her angular speed as she enters the water.

* See "The Mechanics of Swimming and Diving," by R. L. Page, *The Physics Teacher,* February 1976, p. 72; "The Physics of Somersaulting and Twisting," by Cliff Frohlich, *Scientific American,* March 1980, p. 155.

The Rotating Bicycle Wheel

Figure 10-14*a* shows a student seated on a stool that is free to rotate about a vertical axis. The student holds a bicycle wheel that has been set spinning. When the student inverts the spinning wheel, the stool begins to rotate (Fig. 10-14*b*).

No net vertical torque acts on the system consisting of student + stool + wheel, and therefore the vertical (z) component of the total angular momentum of the system must remain constant. Initially, the z component of the angular momentum of the rotating wheel is $+L_w$. The total initial angular momentum of the system is then $L_{iz} = +L_w$. When the wheel is turned over (as a result of an *internal* torque in the system), the z component of the total angular momentum must remain constant. The z component of the final total angular momentum is $L_{fz} = L_s + (-L_w)$, where L_s is the angular momentum of student + stool and $-L_w$ is the angular momentum of the inverted wheel. Conservation of angular momentum (in the absence of external torque) requires that $L_{iz} = L_{fz}$, and so the student and stool will rotate with angular momentum $L_s = +2L_w$.

We can also consider this situation from the standpoint of two separate systems, one being the wheel and the other being the student + stool. Neither of these systems is now isolated: the student's hand forms the connection between them. When the student attempts to invert the wheel, she must apply a torque to change the wheel's angular momentum. The force she exerts on the wheel to produce that torque is returned by the wheel as a reaction force on her, by Newton's third law. This external force on the system of student + stool causes that system to rotate. In this view the student exerts an external torque on the wheel to change its angular momentum, while the wheel exerts a torque on the student to change her angular momentum. If we consider the complete system consisting of student + stool + wheel, as we did above, this torque is an internal torque that did not enter into out calculations. Whether we consider the torque as internal or external depends on how we define our system.

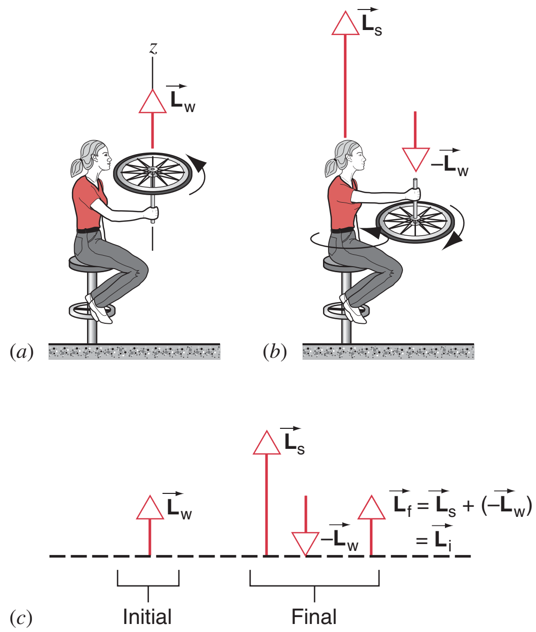


FIGURE 10-14. (*a*) A student holds a rotating bicycle wheel. The total angular momentum of the system is $\vec{L}_w$. (*b*) When the wheel is inverted, the student begins to rotate. (*c*) The total final angular momentum must be equal to the initial angular momentum.

The Stability of Spinning Objects

Consider again Fig. 10-3*b*. An object moving with linear momentum $\vec{p} = M\vec{v}$ has a *directional stability;* a deflecting force provides the impulse, corresponding to a sideways momentum increment $\Delta\vec{p}_\perp$, and as a result the direction of motion is changed by an angle $\theta = \tan^{-1}(\Delta p_\perp/p)$. The larger is the momentum p, the smaller is the angle θ. The same deflecting force is less effective in diverting an object with large linear momentum than it is in diverting an object with small linear momentum.

Angular momentum provides an object with *orientational stability* in much the same way. A rapidly spinning object (as in Fig. 10-4*b*) has a certain angular momentum $\vec{L}$. A torque $\vec{\tau}$ perpendicular to $\vec{L}$ changes the direction of $\vec{L}$, and therefore the direction of the axis of rotation, by an angle $\theta = \tan^{-1}(\Delta L_\perp/L)$. Once again, the larger is the angular momentum L, the less successful a given torque will be in changing the direction of the axis of the spinning object.

When we give an object rotational angular momentum about a symmetry axis, we in effect stabilize its orientation and make it more difficult for external forces to change its orientation. There are many common examples of this effect. A riderless bicycle given a slight push is able to remain upright for a far longer distance than we might expect. In this case it is the angular momentum of the spinning wheels that gives the stability. Minor bumps and curves of the roadway, which might otherwise topple or deflect a nonrotating object balanced on so narrow a base as a bicycle tire, have less effect in this case because of the tendency of the angular momentum of the wheels to fix their orientation.*

A football is thrown for a long forward pass such that it rotates about an axis that is roughly parallel to its translational velocity. This stabilizes the orientation of the football and keeps it from tumbling, which makes it possible to throw more accurately and catch more effectively. It also keeps the smallest profile of the football in the forward direction, thereby minimizing air resistance and increasing the range.

It is important to stabilize the orientation of a satellite, particularly if it is using its thrusters to move to a specific orbital position (Fig. 10-15). The orientation might be changed, for example, by friction from the thin residual atmosphere at orbital altitudes, by the solar wind (a beam of charged particles from the Sun), or by impacts from tiny

* See "The Stability of the Bicycle," by David E. H. Jones, *Physics Today,* April 1970, p. 34.

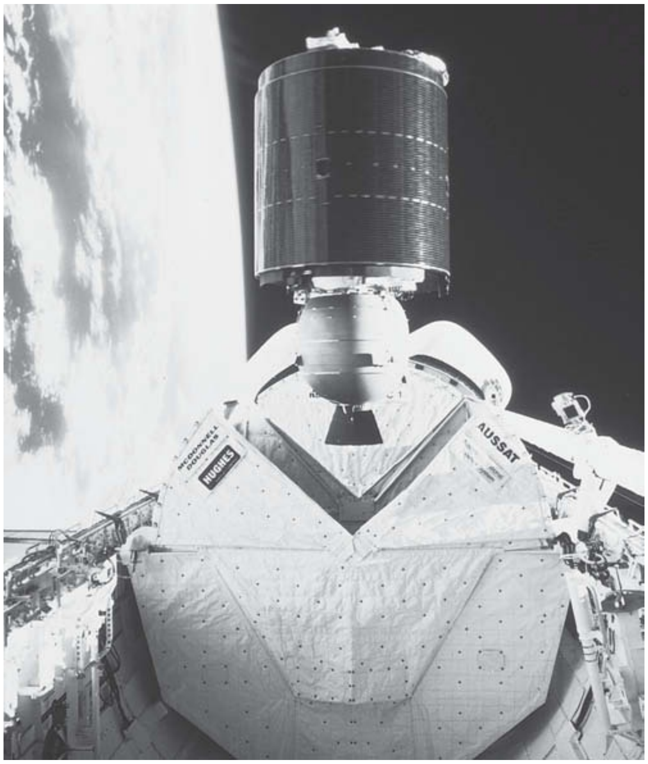


FIGURE 10-15. Deployment of a communications satellite from the bay of the space shuttle. The satellite is made to spin about its central axis (the vertical axis in this photo) to stabilize its orientation in space as it makes its way upward to its geosynchronous orbit.

meteoroids. To reduce the effects of such encounters, the craft is made to spin about an axis thereby stabilizing its orientation.

Collapsing Stars

Most stars rotate, as our Sun does. It turns once on its axis every month or so. (The Sun is a ball of gas and does not rotate quite like a rigid body; the regions near the poles have a rotational period of about 37 days, but the equator rotates once every 26 days.) The Sun is kept from collapsing by *radiation pressure,* in essence the effect of impulsive collisions of the emerging radiation with the atoms of the Sun. When the Sun's nuclear fuel is used up, the radiation pressure will vanish, and the Sun will begin to collapse, its density correspondingly increasing. At some point the density will become so great that the atoms simply cannot be crowded any closer together, and the collapse will be halted. This is the *white dwarf* stage, where the Sun will end its life.

In stars more than about 1.4 times as massive as the Sun, however, the gravitational force is so strong that the atoms cannot prevent further collapse. The atoms are in effect crushed by gravity, and the collapse continues until the nuclei are touching one another. The star has in effect become one giant atomic nucleus; it is called a *neutron star.* The radius of a neutron star of about 1.5 solar masses is about 11 km.

Suppose the star began its collapse like our Sun, rotating once every month. The forces during the collapse are clearly internal forces, which cannot change the angular momentum. The final angular speed is therefore related to the initial angular speed by Eq. 10-17: $\omega_f = \omega_i(I_i/I_f)$. The ratio of the rotational inertias will be the same as the ratio of the squares of the radii: $I_i/I_f = r_i^2/r_f^2$. If the initial radius were about the same as the Sun's (about 7×10^5 km), then

$$\frac{I_i}{I_f} = \frac{r_i^2}{r_f^2} = \frac{(7 \times 10^5\ \text{km})^2}{(11\ \text{km})^2} = 4 \times 10^9.$$

That is, its rotational speed goes from once per month to 4×10^9 per month, or more than 1000 revolutions per second!

Neutron stars can be observed from Earth, because (again like the Sun) they have magnetic fields that trap electrons, and the electrons are accelerated to very high tangential speeds as the star rotates. Such accelerated electrons emit radiation, which we see on Earth somewhat like a searchlight beacon as the star rotates. These sharp pulses of radiation earned rotating neutron stars the name *pulsars.* A sample of the radiation observed from a pulsar is shown in Fig. 10-16.

Conservation of angular momentum applies to a wide variety of astrophysical phenomena. The rotation of our galaxy, for example, is a result of the much slower initial rotation of the gas cloud from which the galaxy condensed; the rotation of the Sun and the orbits of the planets were determined by the original rotation of the material that formed our solar system.

SAMPLE PROBLEM 10-4. A 120-kg astronaut, carrying out a "space walk," is tethered to a spaceship by a fully extended cord 180 m long. An unintended operation of the propellant pack causes the astronaut to acquire a small tangential velocity of 2.5 m/s. To return to the spacecraft, the astronaut begins pulling along the tether at a slow and constant rate. With what force must the astronaut pull at distances of (*a*) 50 m and (*b*) 5 m from the spacecraft? What will be the astronaut's tangential speed at these points?

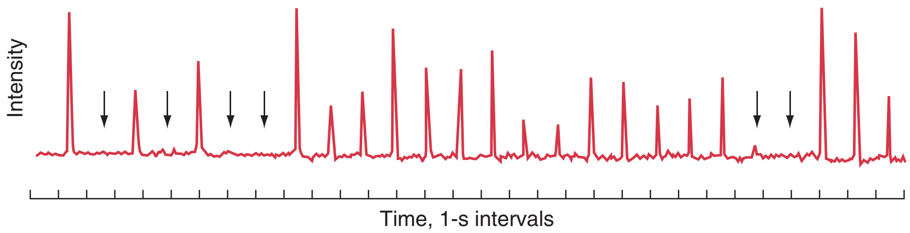


FIGURE 10-16. Electromagnetic pulses received on Earth from a rapidly rotating neutron star. The vertical arrows suggest pulses too weak to be detected. The interval between pulses is remarkably constant, being equal to 1.187,911,164 s.

Solution No external torques act on the astronaut, so that conservation of angular momentum holds. That is, the astronaut's initial angular momentum relative to the spaceship as origin ($Mv_i r_i$) when beginning to pull on the tether must equal the angular momentum (Mvr) at any point in the motion. Thus

$$Mvr = Mv_i r_i$$

or

$$v = \frac{v_i r_i}{r}.$$

The centripetal force at any stage is given by

$$F = \frac{Mv^2}{r} = \frac{Mv_i^2 r_i^2}{r^3}.$$

Initially, the required centripetal force is

$$F = \frac{(120\text{ kg})(2.5\text{ m/s})^2}{180\text{ m}} = 4.2\text{ N} \quad \text{(about 1 lb)}.$$

(*a*) When the astronaut is 50 m from the spacecraft, the tangential speed is

$$v = \frac{(2.5\text{ m/s})(180\text{ m})}{50\text{ m}} = 9.0\text{ m/s},$$

and the centripetal force is

$$F = \frac{(120\text{ kg})(2.5\text{ m/s})^2(180\text{ m})^2}{(50\text{ m})^3} = 194\text{ N} \quad \text{(about 44 lb)}.$$

(*b*) At 5 m from the ship, the speed goes up by a factor of 10 to 90 m/s, while the force increases by a factor of 10^3 to 1.94×10^5 N, or about 22 tons! It is clear that the astronaut cannot exert such a large force to return to the spacecraft. Even if the astronaut were being pulled toward the ship by a winch from within the spacecraft, the tether could not withstand such a large tension; at some point it would break and the astronaut would go shooting into space with whatever the tangential speed was at the time the tether broke. Moral: Space-walking astronauts should avoid acquiring tangential velocity. What could the astronaut do to move safely back to the ship?

SAMPLE PROBLEM 10-5. A turntable consisting of a disk of mass 125 g and radius 7.2 cm is spinning with an angular speed of 0.84 rev/s about a vertical axis (Fig. 10-17*a*). An identical, initially nonrotating disk is suddenly dropped onto the first. The friction between the two disks causes them eventually to rotate at the same speed. A third identical nonrotating disk is then dropped onto the combination, and eventually all three are rotating together (Fig. 10-17*b*). What is the angular speed of the combination?

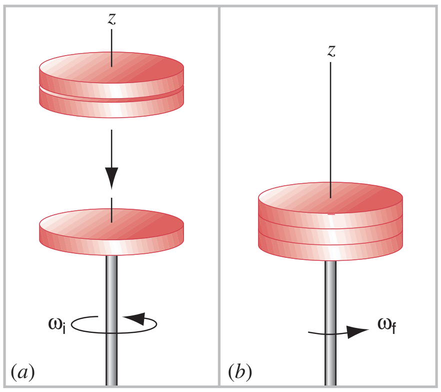


FIGURE 10-17. Sample Problem 10-5. (*a*) A disk is spinning with initial angular velocity ω_i. (*b*) Two identical disks, neither of which is initially rotating, are dropped onto the first, and the entire system then rotates with angular velocity ω_f.

Solution This problem is the rotational analogue of the completely inelastic collision, in which objects stick together (see Section 6-5). There is no net vertical external torque, so the vertical (z) component of angular momentum is constant. The frictional force between the disks is an internal force, which cannot change the angular momentum. Thus Eq. 10-17 applies, and we can write $I_i\omega_i = I_f\omega_f$, or

$$\omega_f = \omega_i \frac{I_i}{I_f}.$$

Without doing any detailed calculations, we know that the rotational inertia of three identical disks about their common axis will be three times the rotational inertia of a single disk. Thus $I_i/I_f = \frac{1}{3}$ and

$$\omega_f = (0.84\text{ rev/s})(\tfrac{1}{3}) = 0.28\text{ rev/s}.$$

10-5 THE SPINNING TOP*

A spinning top provides us with what is perhaps the most familiar example of the phenomenon shown in Fig. 10-4*b*, in which a lateral torque changes the direction but not the magnitude of an angular momentum. Figure 10-18*a* shows a top spinning about its axis. The bottom point of the top is assumed to be fixed at the origin O of our inertial reference frame. We know from experience that the axis of this rapidly spinning top will move slowly about the vertical axis. This motion is called *precession,* and it arises from the configuration illustrated in Fig. 10-4*b*, with gravity supplying the external torque.

Figure 10-18*b* shows a simplified diagram, with the top replaced by a particle of mass M located at the top's center of mass. The gravitational force Mg gives a torque about O of magnitude

$$\tau = Mgr \sin\theta. \tag{10-18}$$

The torque, which is perpendicular to the axis of the top and therefore perpendicular to $\vec{\mathbf{L}}$ (Fig. 10-18*c*), can change the direction of $\vec{\mathbf{L}}$ but not its magnitude. The change in $\vec{\mathbf{L}}$ in a small increment of time dt is given by

$$d\vec{\mathbf{L}} = \vec{\boldsymbol{\tau}}\, dt \tag{10-19}$$

and is in the same direction as $\vec{\boldsymbol{\tau}}$—that is, perpendicular to $\vec{\mathbf{L}}$. The effect of $\vec{\boldsymbol{\tau}}$ is therefore to change $\vec{\mathbf{L}}$ to $\vec{\mathbf{L}} + d\vec{\mathbf{L}}$, a vector of the same length as $\vec{\mathbf{L}}$ but pointing in a slightly different direction.

If the top has axial symmetry, and if it rotates about its axis at high speed, then the angular momentum will be along the axis of rotation of the top. As $\vec{\mathbf{L}}$ changes direc-

* See "The Amateur Scientist: The Physics of Spinning Tops, Including Some Far-Out Ones," by Jearl Walker, *Scientific American,* March 1981, p. 185.

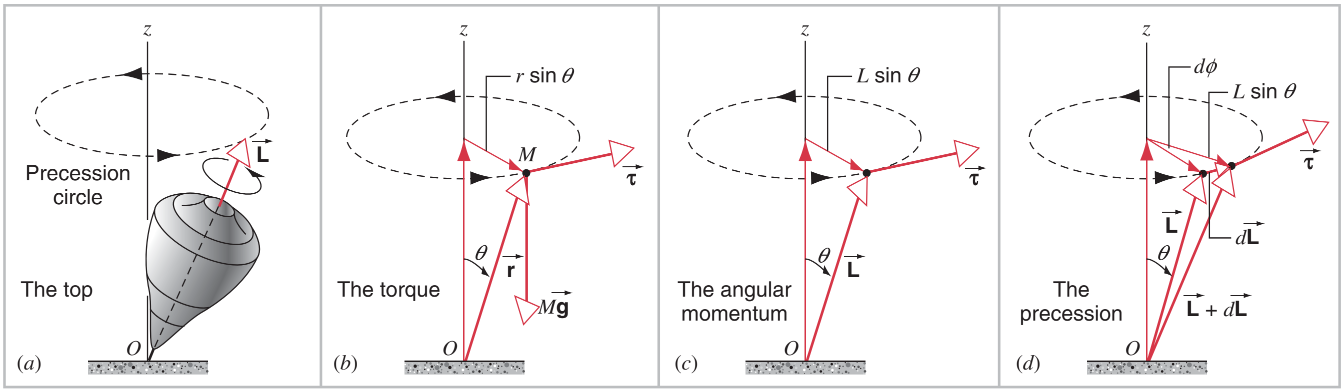


FIGURE 10-18. (*a*) A spinning top precesses about a vertical axis. (*b*) The weight of the top exerts a torque about the point of contact with the floor. (*c*) The torque is perpendicular to the angular momentum vector. (*d*) The torque changes the direction of the angular momentum vector, causing precession.

tion, the axis changes direction too. The tip of the $\vec{L}$ vector and the axis of the top trace out a circle about the z axis, as shown in Fig. 10-18*a*. This motion is the precession of the top.

In a time dt, the axis rotates through an angle $d\phi$ (see Fig. 10-18*d*), and thus the angular speed of precession ω_P is

$$\omega_P = \frac{d\phi}{dt}. \tag{10-20}$$

From Fig. 10-18*d*, we see that

$$d\phi = \frac{dL}{L\sin\theta} = \frac{\tau\, dt}{L\sin\theta}. \tag{10-21}$$

Thus

$$\omega_P = \frac{d\phi}{dt} = \frac{\tau}{L\sin\theta} = \frac{Mgr\sin\theta}{L\sin\theta} = \frac{Mgr}{L}. \tag{10-22}$$

The precessional speed is inversely proportional to the angular momentum and thus to the rotational angular speed; the faster the top is spinning, the slower it will precess. Conversely, as friction slows down the rotational angular speed, the precessional angular speed increases.

Equation 10-22 gives the relationship between the magnitudes of $\vec{\omega}_P$, $\vec{L}$, and $\vec{\tau}$. These quantities are vectors, and the vector relationship among them is

$$\vec{\tau} = \vec{\omega}_P \times \vec{L}. \tag{10-23}$$

You should be able to show that this relationship is consistent with the relationship between the magnitudes (Eq. 10-22) and also with the directions of the vectors given in Fig. 10-18. Note that for precessional motion about the z axis, the vector $\vec{\omega}_P$ is in the z direction.

Precession is commonly observed for spinning tops and gyroscopes. Even the Earth can be considered to be a spinning top, and the gravitational pull of the Sun and Moon on the tidal bulges near the equator causes a precession (called in astronomy the "precession of the equinoxes") in which the Earth's rotational axis traces out the surface of a cone (as in Fig. 10-18) with half-angle $\theta = 23.5°$, taking about 26,000 years to complete a full cycle.

There are two components to the angular momentum of the top: its rotational angular momentum about its symmetry axis and the precessional angular momentum. The total angular momentum is the sum of these two vectors, which in general does not lie along the symmetry axis of the top. Therefore our assumption that the symmetry axis of the top follows the direction of the angular momentum vector is not quite correct. If, however, the precessional angular momentum is much smaller than the rotational angular momentum of the top, there is only a very small deviation between the direction of the symmetry axis and the direction of the angular momentum. This small deviation causes a slight oscillation, called a *nutation,* of the axis of the top back and forth about the precessional circle.

10-6 REVIEW OF ROTATIONAL DYNAMICS

In physics we can often learn about a new subject by comparison or analogy with a subject we have already understood. For example, later in the text we will find that magnetic phenomena have much in common with electric phenomena, and so we can learn about magnetism by extension of our previous knowledge of electricity.

In the past three chapters we have introduced many new rotational quantities and pointed out their similarities with the corresponding translational quantities. It is useful to keep in mind these similarities, but it is also important to recall the differences between translational and rotational quantities and the special cases or limitations of applicability of the rotational equations. For example, some rotational equations apply only to rotation about an axis that is fixed in space.

Table 10-1 shows a comparison between translational and rotational quantities in dynamics.

TABLE 10-1 Review and Comparison of Translational and Rotational Dynamics*

Translational Quantity		*Equation Number*	*Rotational Quantity*		*Equation Number*
Velocity	$\vec{\mathbf{v}} = d\vec{\mathbf{r}}/dt$	2-9	Angular velocity	$\vec{\boldsymbol{\omega}} = d\vec{\boldsymbol{\phi}}/dt$	8-3
Acceleration	$\vec{\mathbf{a}} = d\vec{\mathbf{v}}/dt$	2-16	Angular acceleration	$\vec{\boldsymbol{\alpha}} = d\vec{\boldsymbol{\omega}}/dt$	8-5
Mass	m		Rotational inertia	$I = \sum mr^2$	9-10
Force	$\vec{\mathbf{F}}$		Torque	$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$	9-3
Newton's second law	$\sum \vec{\mathbf{F}}_{\text{ext}} = m\vec{\mathbf{a}}$	4-3	Newton's second law for rotations about a fixed axis	$\sum \tau_{\text{ext},z} = I\alpha_z$	9-11
Equilibrium condition	$\sum \vec{\mathbf{F}}_{\text{ext}} = 0$	9-22	Equilibrium condition	$\sum \vec{\boldsymbol{\tau}}_{\text{ext}} = 0$	9-23
Momentum of a particle	$\vec{\mathbf{p}} = m\vec{\mathbf{v}}$	6-1	Angular momentum of a particle	$\vec{\mathbf{l}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$	10-1
Momentum of a system of particles	$\vec{\mathbf{P}} = M\vec{\mathbf{v}}_{\text{cm}}$	7-21	Angular momentum of a system of particles	$\vec{\mathbf{L}} = I\vec{\boldsymbol{\omega}}$	10-12
General form of Newton's second law	$\sum \vec{\mathbf{F}}_{\text{ext}} = d\vec{\mathbf{P}}/dt$	7-23	General form of Newton's second law of rotations	$\sum \vec{\boldsymbol{\tau}}_{\text{ext}} = d\vec{\mathbf{L}}/dt$	10-9
Conservation of momentum in a system of particles for which $\sum \vec{\mathbf{F}}_{\text{ext}} = 0$	$\vec{\mathbf{P}} = \sum \vec{\mathbf{p}}_n$ = constant	6-12	Conservation of angular momentum in a system of particles for which $\sum \vec{\boldsymbol{\tau}}_{\text{ext}} = 0$	$\vec{\mathbf{L}} = \sum \vec{\mathbf{l}}_n$ = constant	10-15

* Some of these equations apply only under certain special conditions. Be sure you understand the conditions before using these equations. Equations that

MULTIPLE CHOICE

10-1 Angular Momentum of a Particle

1. A particle moves with position given by $\vec{\mathbf{r}} = 3t\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$, where $\vec{\mathbf{r}}$ is measured in meters when t is measured in seconds. For each of the following, consider only $t > 0$.

(*a*) The magnitude of the linear velocity of this particle is

(A) increasing in time. (B) constant in time.
(C) decreasing in time. (D) undefined.

(*b*) The magnitude of the linear momentum of this particle is

(A) increasing in time. (B) constant in time.
(C) decreasing in time. (D) undefined.

(*c*) The magnitude of the angular velocity of this particle about the origin is

(A) increasing in time. (B) constant in time.
(C) decreasing in time. (D) undefined.

(*d*) The magnitude of the angular momentum of this particle about the origin is

(A) increasing in time. (B) constant in time.
(C) decreasing in time. (D) undefined.

2. A particle moves with constant velocity $\vec{\mathbf{v}}$. The angular momentum of this particle about the origin is zero

(A) always.
(B) at exactly one time only.
(C) only if the trajectory of the particle passes through the origin.
(D) never.

3. A particle moves with constant momentum $\vec{\mathbf{p}} = (10\ \text{kg}\cdot\text{m/s})\hat{\mathbf{i}}$. The particle has an angular momentum about the origin of $\vec{\mathbf{l}} = (20\ \text{kg}\cdot\text{m}^2/\text{s})\hat{\mathbf{k}}$ when $t = 0$ s.

(*a*) The magnitude of the angular momentum of this particle is

(A) decreasing. (B) constant. (C) increasing.
(D) possibly but not necessarily constant.

(*b*) The trajectory of this particle

(A) definitely passes through the origin.
(B) might pass through the origin.
(C) will not pass through the origin, but it is uncertain how close it will pass to the origin.
(D) will not pass through the origin, but one can calculate *exactly* how close it will pass to the origin.

10-2 Systems of Particles

4. Two particles have angular momenta $|\vec{\mathbf{l}}_1| = 30\ \text{kg}\cdot\text{m}^2/\text{s}$ and $|\vec{\mathbf{l}}_2| = 40\ \text{kg}\cdot\text{m}^2/\text{s}$ as measured about the origin. Originally particle 1 moves in the xy plane and particle 2 moves in the yz

plane. If there are no external torques, then the total angular momentum is a constant of magnitude

(A) $|\vec{L}| = 10\ \text{kg}\cdot\text{m}^2/\text{s}$ (B) $|\vec{L}| = 50\ \text{kg}\cdot\text{m}^2/\text{s}$
(C) $|\vec{L}| = 70\ \text{kg}\cdot\text{m}^2/\text{s}$
(D) $10\ \text{kg}\cdot\text{m}^2/\text{s} \le |\vec{L}| \le 50\ \text{kg}\cdot\text{m}^2/\text{s}$

5. Two independent particles are originally moving with angular momenta $\vec{l}_1$ and $\vec{l}_2$ in a region of space with no external torques. A constant external torque $\vec{\tau}$ then acts on particle one, but *not* on particle two, for a time Δt. What is the change in the total angular momentum of the two particles?

(A) $\Delta\vec{L} = \vec{l}_1 - \vec{l}_2$ (B) $\Delta\vec{L} = \frac{1}{2}(\vec{l}_1 + \vec{l}_2)$
(C) $\Delta\vec{L} = \vec{\tau}\Delta t$
(D) $\Delta\vec{L}$ for the system is poorly defined because the two particles are not connected.

10-3 Angular Momentum and Angular Velocity

6. The linear velocity $\vec{v}$ and linear momentum $\vec{p}$ of a body

(A) are always parallel.
(B) are never parallel.
(C) are parallel only if $\vec{v}$ is constant.
(D) are parallel only if $\vec{v}$ is pointing in certain directions with respect to the body.

7. The angular velocity $\vec{\omega}$ and angular momentum $\vec{l}$ of a body with axial symmetry

(A) are always parallel.
(B) are never parallel.
(C) are parallel only if $\vec{\omega}$ is constant.
(D) are parallel only if $\vec{\omega}$ is pointing in certain directions with respect to the body.

8. A body, not necessarily rigid, is originally rotating with angular velocity of magnitude ω_0 and angular momentum of magnitude L_0. Something happens to the body to cause ω_0 to slowly decrease. Consequently

(A) L_0 must also be decreasing.
(B) L_0 could be constant or decreasing, but not increasing.
(C) L_0 could be constant, decreasing, or increasing.
(D) L_0 could be constant or increasing, but not decreasing.

10-4 Conservation of Angular Momentum

9. A solid object is rotating freely without experiencing any external torques. In this case

(A) both the angular momentum and angular velocity have constant directions.
(B) the direction of the angular momentum is constant but the direction of the angular velocity might not be constant.
(C) the direction of the angular velocity is constant but the direction of the angular momentum might not be constant.
(D) neither the angular momentum nor the angular velocity necessarily has a constant direction.

10. A physics professor is sitting on a rotating chair with her arms outstretched, each holding a medium sized barbell. The *frictionless* chair is originally rotating at a constant angular speed. She then pulls her arms closer to her body.

(*a*) When she brings her arms in, her angular velocity

(A) increases. (B) remains constant. (C) decreases.
(D) changes, but whether it increases or decreases depends on how she brings her arms in.

(*b*) When she brings her arms in her angular momentum

(A) increases. (B) remains constant. (C) decreases.
(D) changes, but whether it increases or decreases depends on how she brings her arms in.

10-5 The Spinning Top

11. Two wires are attached to the ends of the axle of a bicycle wheel so that the wheel is suspended, free to rotate in a vertical plane. The wheel is spun about the axle at a high speed. One of the wires supporting the axle is cut; the wheel as viewed from this side of the axle is rotating in a clockwise direction.

(*a*) Which way, as viewed from above, will the axis of the wheel precess?

(A) Clockwise (B) Counterclockwise
(C) The wheel will not precess, because it is not a spinning top.

(*b*) Before one of the two wires is cut, each wire has a tension of $W/2$ where W is the weight of the wheel. After cutting one of the wires, the magnitude of the tension in the wire that is still connected will be

(A) $W/2$. (B) slightly more than $W/2$.
(C) approximately W. (D) exactly W.

10-6 Review of Rotational Dynamics

QUESTIONS

1. We have encountered many vector quantities so far, including position, displacement, velocity, acceleration, force, momentum, and angular momentum. Which of these are defined independent of the choice of the origin in the reference frame?

2. A cylinder rotates with angular speed ω about an axis through one end, as in Fig. 10-19. Choose an appropriate origin and show qualitatively the vectors $\vec{L}$ and $\vec{\omega}$. Are these vectors parallel? Do symmetry considerations enter here?

3. When the angular speed ω of an object increases, its angular momentum may or may not also increase. Give an example in which it does and one in which it does not.

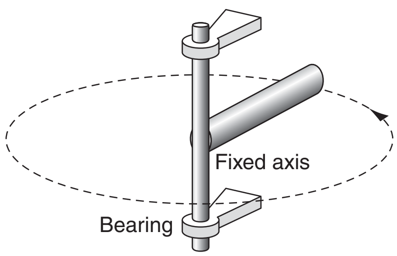


FIGURE 10-19. Question 2.

4. Is it possible for the angular momentum of an object to be zero if the angular velocity is nonzero? Is it possible for the angular velocity of an object to be zero if the angular momentum is nonzero? Explain.

5. A student stands on a table rotating with an angular speed ω while holding two equal dumbbells at arm's length. Without moving anything else, the two dumbbells are dropped to the floor. What change, if any, is there in the student's angular speed? Is angular momentum conserved? Explain your answers.

6. A circular turntable rotates at constant angular speed about a vertical axis. There is no friction and no driving torque. A circular pan rests on the turntable and rotates with it; see Fig. 10-20. The bottom of the pan is covered with a layer of ice of uniform thickness, which is, of course, also rotating with the pan. The ice melts but none of the water escapes from the pan. Is the angular speed now greater than, the same as, or less than the original speed? Give reasons for your answer.

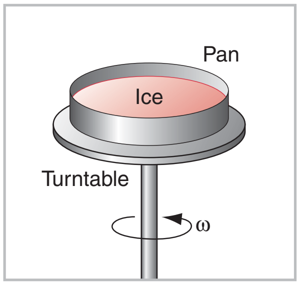


FIGURE 10-20. Question 6.

7. A circular turntable is rotating freely about a vertical axis. There is no friction at the axis of rotation. (*a*) A bug, initially at the center of the turntable, walks out to the rim and stops. How will the angular momentum of the system (turntable plus bug) change? How will the angular velocity of the turntable change? (*b*) If the bug falls off the edge of the turntable (without jumping), how will the angular velocity of the turntable change?

8. A famous physicist (R. W. Wood), who was fond of practical jokes, mounted a rapidly spinning flywheel in a suitcase, which he gave to a porter with instructions to follow him. What happens when the porter is led quickly around a corner? Explain in terms of $\vec{\tau} = d\vec{\mathbf{L}}/dt$.

9. An arrow turns in flight so as to be tangent to its flight path at all times. However, a football (thrown with considerable spin about its long axis) does not do this. Why this difference in behavior?

10. A passer throws a spiraling football to a receiver. Is its angular momentum constant, or nearly so? Distinguish between the cases in which the football wobbles and when it does not.

11. Can you suggest a simple theory to explain the stability of a moving bicycle? You must explain why it is much more difficult to balance yourself on a bicycle that is at rest than on one that is rolling. (See "The Stability of the Bicycle," by David E. H. Jones, *Physics Today,* April 1970, p. 34.)

12. Why does a long bar help a tightrope walker to keep his or her balance?

13. You are walking along a narrow rail and you start to lose your balance. If you start falling to the right, which way do you turn your body to regain balance? Explain.

14. The mounting bolts that fasten the engine of a jet plane to the structural framework of the plane are designed to snap apart if the (rapidly rotating) engine suddenly seizes up because of some mishap. Why are such "structural fuses" used?

15. A helicopter flies off, its propellers rotating. Why doesn't the body of the helicopter rotate in the opposite direction?

16. A single-engine airplane must be "trimmed" to fly level. (Trimming consists of raising one aileron and lowering the opposite one.) Why is this necessary? Is it necessary on a twin-engine plane under normal circumstances?

17. The propeller of an aircraft rotates clockwise as seen from the rear. When the pilot pulls upward out of a steep dive, she finds it necessary to apply left rudder at the bottom of the dive if she is to maintain her heading. Explain.

18. Many great rivers flow toward the equator. What effect does the sediment they carry to the sea have on the rotation of the Earth?

19. If the entire population of the world moved to Antarctica, would it affect the length of the day? If so, in what way?

20. Fig. 10-21*a* shows an acrobat propelled upward by a trampoline with zero angular momentum. Can the acrobat, by maneuvering his body, manage to land on his back as in Fig. 10-21*b*? Interestingly, 38% of questioned diving coaches and 34% of a sample of physicists gave the wrong answer. What do you think? (See "Do Springboard Divers Violate Angular Momentum Conservation?", by Cliff Frohlich, *American Journal of Physics,* July 1979, p. 583, for a full discussion.)

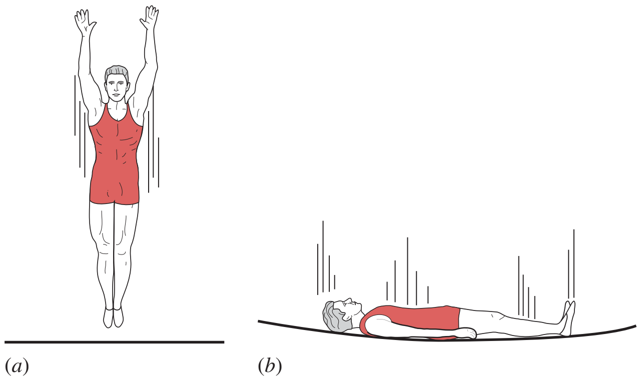


FIGURE 10-21. Question 20.

21. Explain, in terms of angular momentum and rotational inertia, exactly how one "pumps up" a swing in the sitting position. (See "How to Get the Playground Swing Going: A First Lesson in the Mechanics of Rotation," by Jearl Walker, *Scientific American,* March 1989, p. 106.)

22. Can you "pump" a swing so that it turns in a complete circle, moving completely around its support? Assume (if you wish) that the seat of the swing is connected to its support by a rigid rod rather than a rope or chain. Explain your answer.

23. Cats usually land on their feet if dropped, even if dropped upside down. How?

24. A massive spinning wheel can be used for a stabilizing effect on a ship. If mounted with its axis of rotation at right angles to the ship deck, what is its effect when the ship tends to roll from side to side?

25. If the top of Fig. 10-18 were not spinning, it would tip over. If its spin angular momentum is large compared to the change caused by the applied torque, the top precesses. What happens in between when the top spins slowly?

26. A Tippy-Top, having a section of a spherical surface of large radius on one end and a stem for spinning it on the opposite end, will rest on its spherical surface with no spin but slips over when spun, so as to stand on its stem. Explain. (See "The Tippy-Top," by George D. Freier, *The Physics Teacher,* January 1967, p. 36.) If you cannot find a Tippy-Top, use a hard-boiled egg; the "standing-on-end" behavior of the spinning egg is most easily followed if you put an ink mark on the "pointed" end of the egg.

27. A bicycle wheel spinning in a vertical plane can be supported from one end of the axle; the axle simply precesses. What "holds up" the other end of the axle? In other words, why doesn't the bicycle wheel fall?

28. Assume that a uniform rod rests in a vertical position on a surface of negligible friction. The rod is then given a horizontal blow at its lower end. Describe the motion of the center of mass of the rod and of its upper endpoint.

EXERCISES

10-1 Angular Momentum of a Particle

1. A particle of mass 13.7 g is moving with a constant velocity of magnitude 380 m/s. The particle, moving in a straight line, passes within 12 cm of the origin. Calculate the angular momentum of the particle about the origin.

2. If we are given r, p, and θ, we can calculate the angular momentum of a particle from Eq. 10-2. Sometimes, however, we are given the components (x, y, z) of $\vec{\mathbf{r}}$ and (v_x, v_y, v_z) of $\vec{\mathbf{v}}$ instead. (*a*) Show that the components of $\vec{\mathbf{l}}$ along the x, y, and z axes are then given by

$$l_x = m(yv_z - zv_y),$$
$$l_y = m(zv_x - xv_z),$$
$$l_z = m(xv_y - yv_x).$$

(*b*) Show that if the particle moves only in the xy plane, the resultant angular momentum vector has only a z component.

3. Show that the angular momentum about any point of a single particle moving with constant velocity remains constant throughout the motion.

4. (*a*) Use the data given in the appendices to compute the total angular momentum of all the planets due to their revolution about the Sun. (*b*) What fraction of this is associated with the planet Jupiter?

5. Calculate the angular momentum, about the Earth's center, of an 84.3-kg person on the equator of the rotating Earth.

10-2 Systems of Particles

6. The total angular momentum of a system of particles relative to the origin O of an inertial reference frame is given by $\vec{\mathbf{L}} = \Sigma\,(\vec{\mathbf{r}}_i \times \vec{\mathbf{p}}_i)$, where $\vec{\mathbf{r}}_i$ and $\vec{\mathbf{p}}_i$ are measured with respect to O. (*a*) Use the relations $\vec{\mathbf{r}}_i = \vec{\mathbf{r}}_{\text{cm}} + \vec{\mathbf{r}}_i'$ and $\vec{\mathbf{p}}_i = m_i\vec{\mathbf{v}}_{\text{cm}} + \vec{\mathbf{p}}_i'$ to express $\vec{\mathbf{L}}$ in terms of the positions $\vec{\mathbf{r}}_i'$ and momenta $\vec{\mathbf{p}}_i'$ relative to the center of mass C; see Fig. 10-22. (*b*) Use the definition of center of mass and the definition of angular momentum $\vec{\mathbf{L}}'$ with respect to the center of mass to obtain $\vec{\mathbf{L}} = \vec{\mathbf{L}}' + \vec{\mathbf{r}}_{\text{cm}} \times M\vec{\mathbf{v}}_{\text{cm}}$. (*c*) Show how this result can be interpreted as regarding the total angular momentum to be the sum of spin angular momentum (angular momentum relative to the center of mass) and orbital angular momentum (angular momentum of the motion of the center of mass C with respect to O if all the system's mass were concentrated at C).

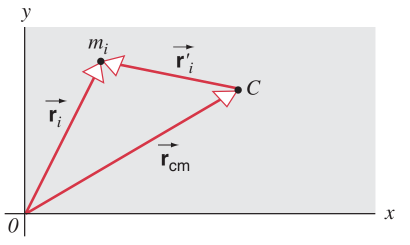


FIGURE 10-22. Exercises 6 and 7.

7. Let $\vec{\mathbf{r}}_{\text{cm}}$ be the position vector of the center of mass C of a system of particles with respect to the origin O of an inertial reference frame, and let $\vec{\mathbf{r}}_i'$ be the position vector of the ith particle, of mass m_i, with respect to the center of mass C. Hence $\vec{\mathbf{r}}_i = \vec{\mathbf{r}}_{\text{cm}} + \vec{\mathbf{r}}_i'$ (see Fig. 10-22). Now define the total angular momentum of the system of particles relative to the center of mass C to be $\vec{\mathbf{L}}' = \Sigma\,(\vec{\mathbf{r}}_i' \times \vec{\mathbf{p}}_i')$, where $\vec{\mathbf{p}}_i' = m_i\,d\vec{\mathbf{r}}_i'/dt$. (*a*) Show that $\vec{\mathbf{p}}_i' = m_i\,d\vec{\mathbf{r}}_i/dt - m_i\,d\vec{\mathbf{r}}_{\text{cm}}/dt = \vec{\mathbf{p}}_i - m_i\vec{\mathbf{v}}_{\text{cm}}$. (*b*) Show next that $d\vec{\mathbf{L}}'/dt = \Sigma\,(\vec{\mathbf{r}}_i' \times d\vec{\mathbf{p}}_i'/dt)$. (*c*) Combine the results of (*a*) and (*b*) and, using the definition of center of mass and Newton's third law, show that $\vec{\boldsymbol{\tau}}'_{\text{ext}} = d\vec{\mathbf{L}}'/dt$, where $\vec{\boldsymbol{\tau}}'_{\text{ext}}$ is the sum of all the external torques acting on the system about its center of mass.

10-3 Angular Momentum and Angular Velocity

8. The time integral of a torque is called the angular impulse. (*a*) Starting from $\vec{\boldsymbol{\tau}} = d\vec{\mathbf{L}}/dt$, show that the resultant angular impulse equals the change in angular momentum. This is the rotational analogue of the linear impulse-momentum relation. (*b*) For rotation around a fixed axis, show that

$$\int \tau\,dt = F_{\text{av}}r(\Delta t) = I(\omega_{\text{f}} - \omega_{\text{i}}),$$

where r is the moment arm of the force, F_{av} is the average value of the force during the time it acts on the object, and ω_{i} and ω_{f} are the angular velocities of the object just before and just after the force acts.

9. A sanding disk with rotational inertia $1.22 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$ is attached to an electric drill whose motor delivers a torque of 15.8 N · m. Find (*a*) the angular momentum and (*b*) the angular speed of the disk 33.0 ms after the motor is turned on.

10. A wheel of radius 24.7 cm, moving initially at 43.3 m/s, rolls to a stop in 225 m. Calculate (*a*) its linear acceleration and (*b*) its angular acceleration. (*c*) The wheel's rotational inertia is $0.155\ \text{kg}\cdot\text{m}^2$. Calculate the torque exerted by rolling friction on the wheel.

11. Show that $\vec{\mathbf{L}} = I\vec{\boldsymbol{\omega}}$ for the two-particle system of Fig. 10-7.

12. Fig. 10-23 shows a symmetrical rigid body rotating about a fixed axis. The origin of coordinates is fixed for convenience at the center of mass. Prove, by summing over the contributions made to the angular momentum by all the mass elements m_i into which the body is divided, that $\vec{\mathbf{L}} = I\vec{\boldsymbol{\omega}}$, where $\vec{\mathbf{L}}$ is the total angular momentum.

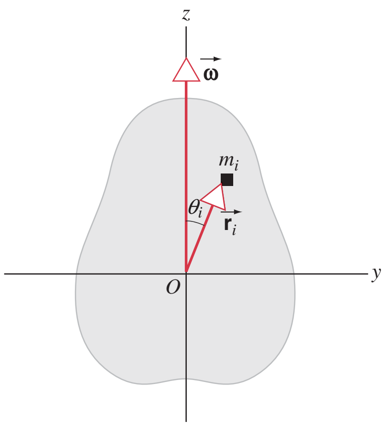


FIGURE 10-23. Exercise 12.

13. A uniform stick has a mass of 4.42 kg and a length of 1.23 m. It is initially lying flat at rest on a frictionless horizontal surface and is struck perpendicularly by a puck imparting a horizontal impulsive force of impulse 12.8 N·s at a distance of 46.4 cm from the center. Determine the subsequent motion of the stick.

14. A cylinder rolls down an inclined plane of angle θ. Show, by direct application of Eq. 10-9 ($\Sigma\, \vec{\boldsymbol{\tau}}_{\rm ext} = d\vec{\mathbf{L}}/dt$) that the acceleration of its center of mass is $\frac{2}{3}g \sin\theta$. Compare this method with that used in Sample Problem 9-11.

15. Two cylinders having radii R_1 and R_2 and rotational inertias I_1 and I_2, respectively, are supported by axes perpendicular to the plane of Fig. 10-24. The large cylinder is initially rotating with angular velocity ω_0. The small cylinder is moved to the right until it touches the large cylinder and is caused to rotate by the frictional force between the two. Eventually, slipping ceases, and the two cylinders rotate at constant rates in opposite directions. Find the final angular velocity ω_2 of the small cylinder in terms of I_1, I_2, R_1, R_2, and ω_0. (Hint: Angular momentum is not conserved. Apply the angular impulse equation to each cylinder. See Exercise 8.)

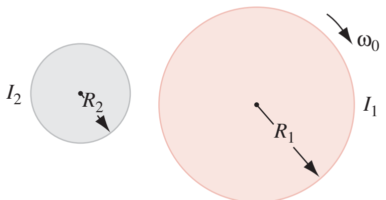


FIGURE 10-24. Exercise 15.

10-4 Conservation of Angular Momentum

16. Astronomical observations show that from 1870 to 1900 the length of the day increased by about 6.0×10^{-3} s. (*a*) What corresponding fractional change in the Earth's angular velocity resulted? (*b*) Suppose that the cause of this change was a shift of molten material in the Earth's core. What resulting fractional change in the Earth's rotational inertia could account for the answer to part (*a*)?

17. Suppose that the Sun runs out of nuclear fuel and suddenly collapses to form a so-called white dwarf star, with a diameter equal to that of the Earth. Assuming no mass loss, what would then be the new rotation period of the Sun, which currently is about 25 days? Assume that the Sun and the white dwarf are uniform spheres.

18. In a lecture demonstration, a toy train track is mounted on a large wheel that is free to turn with negligible friction about a vertical axis; see Fig. 10-25. A toy train of mass m is placed on the track and, with the system initially at rest, the electrical power is turned on. The train reaches a steady speed v with respect to the track. What is the angular velocity ω of the wheel, if its mass is M and its radius R? (Neglect the mass of the spokes of the wheel.)

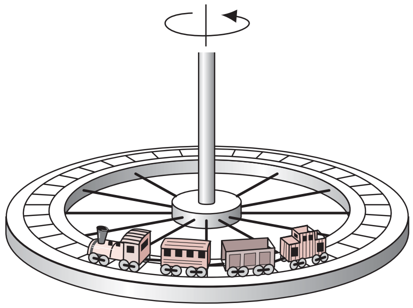


FIGURE 10-25. Exercise 18.

19. The rotor of an electric motor has a rotational inertia $I_m = 2.47 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$ about its central axis. The motor is mounted parallel to the axis of a space probe having a rotational inertia $I_p = 12.6\ \text{kg}\cdot\text{m}^2$ about its axis. Calculate the number of revolutions of the motor required to turn the probe through 25.0° about its axis.

20. A man stands on a frictionless platform that is rotating with an angular speed of 1.22 rev/s; his arms are outstretched and he holds a weight in each hand. With his hands in this position the total rotational inertia of the man, the weights, and the platform is $6.13\ \text{kg}\cdot\text{m}^2$. If by moving the weights the man decreases the rotational inertia to $1.97\ \text{kg}\cdot\text{m}^2$, what is the resulting angular speed of the platform?

21. A wheel with rotational inertia $1.27\ \text{kg}\cdot\text{m}^2$ is rotating with an angular speed of 824 rev/min on a shaft whose rotational inertia is negligible. A second wheel, initially at rest and with rotational inertia $4.85\ \text{kg}\cdot\text{m}^2$, is suddenly coupled to the same shaft. What is the angular speed of the resultant combination of the shaft and two wheels?

22. Show that the value of $l_\perp$ in Fig. 10-6*c* is given by mvh, where h is the distance along the z axis from O to the point of connection of the radial arm to the vertical shaft.

23. With center and spokes of negligible mass, a certain bicycle wheel has a thin rim of radius 36.3 cm and mass 3.66 kg; it can turn on its axle with negligible friction. A man holds the wheel above his head with the axis vertical while he stands on a turntable free to rotate without friction; the wheel rotates clockwise, as seen from above, with an angular speed of 57.7 rad/s, and the turntable is initially at rest. The rotational inertia of wheel-plus-man-plus-turntable about the common axis of rotation is $2.88\ \text{kg}\cdot\text{m}^2$. (*a*) The man's hand suddenly stops the rotation of the wheel (relative to the turntable). Determine the resulting angular velocity (magnitude and direction) of the system. (*b*) The experiment is repeated with noticeable friction introduced into the axle of the wheel, which, starting from the same initial angular speed (57.7 rad/s), gradually comes to rest (relative to the turntable) while the man holds the wheel as described above. (The turntable is still free to rotate without friction.) Describe what happens to the system, giving as much quantitative information as the data permit.

24. A girl of mass 50.6 kg stands on the edge of a frictionless merry-go-round of mass 827 kg and radius 3.72 m that is not moving. She throws a 1.13-kg rock in a horizontal direction that is tangent to the outer edge of the merry-go-round. The speed of the rock, relative to the ground, is 7.82 m/s. Calculate (*a*) the angular speed of the merry-go-round and (*b*) the linear speed of the girl after the rock is thrown. Assume that the merry-go-round is a uniform disk.

25. In a playground there is a small merry-go-round of radius 1.22 m and mass 176 kg. The radius of gyration (see Exercise 9-20) is 91.6 cm. A child of mass 44.3 kg runs at a speed of 2.92 m/s tangent to the rim of the merry-go-round when it is at rest and then jumps on. Neglect friction between the bearings and the shaft of the merry-go-round and find the angular speed of the merry-go-round and child.

10-5 The Spinning Top

26. A top is spinning at 28.6 rev/s about an axis making an angle of 34.0° with the vertical. Its mass is 492 g and its rotational inertia is $5.12 \times 10^{-4}\ \text{kg}\cdot\text{m}^2$. The center of mass is 3.88 cm from the pivot point. The spin is clockwise as seen from above. Find the magnitude (in rev/s) and direction of the angular velocity of precession.

27. A gyroscope consists of a rotating disk with a 48.7-cm radius suitably mounted at the midpoint of a 12.2-cm-long axle so that it can spin and precess freely. Its spin rate is 975 rev/min. The mass of the disk is 1.14 kg and the mass of the axle is 130 g. Find the time required for one precession if the axle is supported at one end and is horizontal.

10-6 Review of Rotational Dynamics

PROBLEMS

1. A particle P with mass 2.13 kg has position $\vec{\mathbf{r}}$ and velocity $\vec{\mathbf{v}}$ as shown in Fig. 10-26. It is acted on by the force $\vec{\mathbf{F}}$. All three vectors lie in a common plane. Presume that $r = 2.91$ m, $v = 4.18$ m/s, and $F = 1.88$ N. Compute (*a*) the angular momentum of the particle and (*b*) the torque, about the origin, acting on the particle. What are the directions of these two vectors?

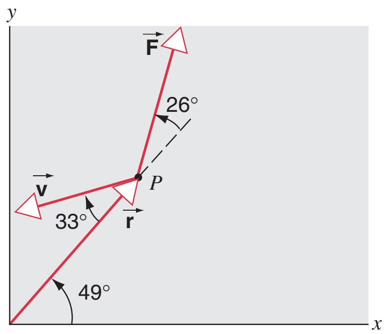


FIGURE 10-26. Problem 1.

2. Two particles, each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d. Find an expression for the total angular momentum of the system about any origin.

3. To get a billiard ball to roll without sliding from the start, the cue must hit the ball not at the center (that is, a height above the table equal to the ball's radius R) but exactly at a height $2R/5$ above the center. Prove this result. [See Arnold Sommerfeld, *Mechanics, Volume 2 of Lectures on Theoretical Physics,* Academic Press, Orlando (1964 paperback edition), pp. 158–161, for a supplement on the mechanics of billiards.]

4. The axis of the cylinder in Fig. 10-27 is fixed. The cylinder is initially at rest. The block of mass M is initially moving to the right without friction and with speed v_1. It passes over the cylinder to the dashed position. When it first makes contact with the cylinder, it slips on the cylinder, but the friction is large enough so that slipping ceases before M loses contact with the cylinder. The cylinder has a radius R and a rotational inertia I. Find the final speed v_2, in terms of v_1, M, I, and R. This can be done most easily by using the relation between impulse and change in momentum.

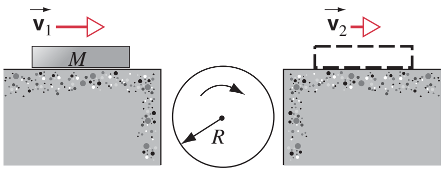


FIGURE 10-27. Problem 4.

5. A billiard ball, initially at rest, is given a sharp impulse by a cue. The cue is held horizontally a distance h above the centerline as in Fig. 10-28. The ball leaves the cue with a speed v_0, and, because of its "forward English," eventually acquires a final speed of $9v_0/7$. Show that $h = 4R/5$, where R is the radius of the ball.

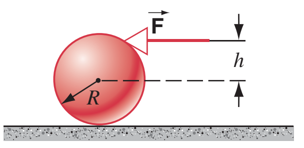


FIGURE 10-28. Problem 5.

6. In Problem 5, imagine $\vec{\mathbf{F}}$ to be applied below the centerline. (*a*) Show that it is impossible, with this "reverse English," to reduce the forward speed to zero, without rolling having set in, unless $h = R$. (*b*) Show that it is impossible to give the ball a backward velocity unless $\vec{\mathbf{F}}$ has a downward vertical component.

7. A bowler throws a bowling ball of radius $R = 11.0$ cm down the lane with initial speed $v_0 = 8.50$ m/s. The ball is thrown in such a way that it skids for a certain distance before it starts to roll. It is not rotating at all when it first hits the lane, its motion being pure translation. The coefficient of kinetic friction between the ball and the lane is 0.210. (*a*) For what length of time does the ball skid? (Hint: As the ball skids, its speed v decreases and its angular speed ω increases; skidding ceases when $v = R\omega$.) (*b*) How far down the lane does it skid? (*c*) How many revolutions does it make before it starts to roll? (*d*) How fast is it moving when it starts to roll?

8. A uniform flat disk of mass M and radius R rotates about a horizontal axis through its center with angular speed ω_0. (*a*) What is its angular momentum? (*b*) A chip of mass m breaks off the edge of the disk at an instant such that the chip rises vertically above the point at which it broke off (Fig. 10-29). How high above the point does it rise before starting to fall? (*c*) What is the final angular speed of the broken disk?

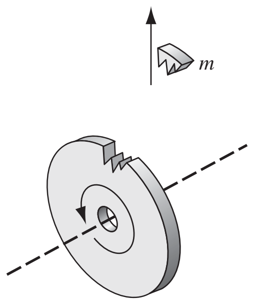


FIGURE 10-29. Problem 8.

9. If the polar ice caps of the Earth were to melt and the water returned to the oceans, the oceans would be made deeper by about 30 m. What effect would this have on the Earth's rotation? Make an estimate of the resulting change in the length of the day. (Concern has been expressed that warming of the atmosphere resulting from industrial pollution could cause the ice caps to melt.)

10. The Earth was formed about 4.5 billion years ago, probably as a sphere of roughly uniform density. Shortly thereafter, heat from the decay of radioactive elements caused much of the Earth to melt. This allowed the heavier material to sink toward the center of the Earth, forming the core. Today, we can picture the Earth as made up of a core of radius 3570 km and density 10.3 g/cm^3 surrounded by a mantle of density 4.50 g/cm^3 extending to the surface of the Earth (radius 6370 km). We ignore the crust of the Earth. Calculate the fractional change in the length of the day due to the formation of the core.

11. A cockroach, mass m, runs counterclockwise around the rim of a lazy Susan (a circular dish mounted on a vertical axle) of radius R and rotational inertia I with frictionless bearings. The cockroach's speed (relative to the Earth) is v, whereas the lazy Susan turns clockwise with angular speed ω. The cockroach finds a bread crumb on the rim and, of course, stops. Find the angular speed of the lazy Susan after the cockroach stops.

12. Two skaters, each of mass 51.2 kg, approach each other along parallel paths separated by 2.92 m. They have equal and opposite velocities of 1.38 m/s. The first skater carries a long light pole 2.92 m long, and the second skater grabs the end of it as he passes; see Fig. 10-30. Assume frictionless ice. (*a*) Describe quantitatively the motion of the skaters after they are connected by the pole. (*b*) By pulling on the pole, the skaters reduce their separation to 0.940 m. Find their angular speed then.

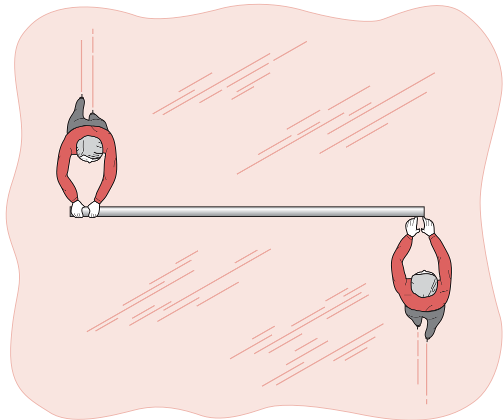


FIGURE 10-30. Problem 12.

CHAPTER 11

ENERGY 1: WORK AND KINETIC ENERGY

We have seen how Newton's laws are useful in understanding and analyzing a wide variety of problems in mechanics. In this and the following two chapters we consider a different approach based on one of the truly fundamental and universal concepts in physics: energy.

There are many kinds of energy. In this chapter we consider one particular form—kinetic energy, the energy associated with a body because of its motion. We also introduce the concept of work, which is related to kinetic energy through the work–energy theorem. This theorem, derived from Newton's laws, provides new and different insight into the behavior of mechanical systems. In Chapter 12 we introduce a second kind of energy—potential energy—and begin developing a conservation law for energy. In Chapter 13 we discuss energy in a more comprehensive way and generalize the law of conservation of energy, which is one of the most useful laws of physics.

11-1 WORK AND ENERGY

Figure 11-1 shows a wheelchair rider pushing his chair uphill. As he pushes down on the wheel with a force $\vec{\mathbf{F}}$, a torque $\vec{\mathbf{r}} \times \vec{\mathbf{F}}$ is exerted about the instantaneous point of contact between the wheel and the ground. This torque causes the wheel to rotate forward. Another way of looking at the problem is to consider the frictional force $\vec{\mathbf{f}}$ exerted on the ground by the wheel (due to the rider's effort); the reaction force $-\vec{\mathbf{f}}$, exerted on the wheel by the ground, pushes the chair forward. A similar figure might be drawn for a person riding a bicycle.

Eventually the arms of the rider or the legs of the bicyclist become tired, and the rider is unable to maintain the original speed up the hill. Perhaps they become so tired that he stops completely. We can analyze the forces exerted in this problem based on Newton's laws, but those laws cannot explain why the rider's ability to exert a force to move forward becomes used up. That is, we *cannot* regard the rider's body as "containing" a quantity of force that is depleted by the effort.

For this analysis, we must introduce the new concepts of *work* and *energy.* As for so many other words that we use to describe physics concepts, we must be careful not to confuse their everyday meanings with the precise definitions we give them as physical quantities. The *physics* concept of work involves a force that is exerted as the point of

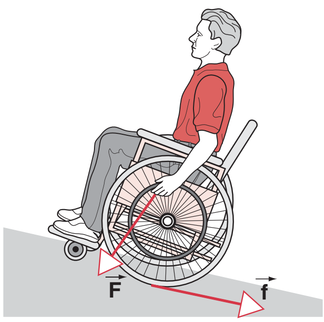


FIGURE 11-1. A wheelchair rider pushes the chair uphill. The force $\vec{\mathbf{F}}$, exerted on the wheel by the rider, gives a torque about the point where the wheel contacts the ground.

application moves through some distance, and one way to define the energy of a system is a measure of its capacity to do work. In the case of the wheelchair rider, he does work because he exerts a force as the wheelchair moves forward through some distance. For him to do work, he must expend some of his supply of energy—that is, the chemical energy stored in his muscle fibers—which can be replenished from his body's store of energy through resting and which ultimately comes from the food he eats.

The energy stored in a system may take many forms: for example, chemical, electrical, gravitational, or mechanical. In this chapter we study the relationship between work and one particular type of energy—the energy of motion of a body, which we call *kinetic energy.*

11-2 WORK DONE BY A CONSTANT FORCE

Figure 11-2*a* shows a block of mass m being lifted through a vertical distance h by a winch that is turned by a motor. The block is raised at a constant velocity; since its acceleration is equal to zero, the net force acting on it is, by Newton's second law, also equal to zero. The magnitude of the upward force $\vec{\mathbf{T}}$ exerted by the motor and winch must thus equal the magnitude of the downward force $m\vec{\mathbf{g}}$ due to gravity.

In Fig. 11-2*b*, a conveyor belt is operated by a motor to move an identical block a distance L up an incline that makes an angle θ with the horizontal. If the block moves at a constant velocity, the net force is again zero, and so the magnitude of the force $\vec{\mathbf{F}}$ up the incline exerted by the belt must equal the component of the weight $mg \sin \theta$ that acts down the incline.

In both cases, the final result is the same—the block has been raised a distance h. If we release the block and allow it to fall, it will reach the ground with a certain speed v. We could use the falling block to accomplish some objective, such as driving a spike into the ground or launching a projectile from a catapult. The outcome would be the same, no matter how the block was originally raised.

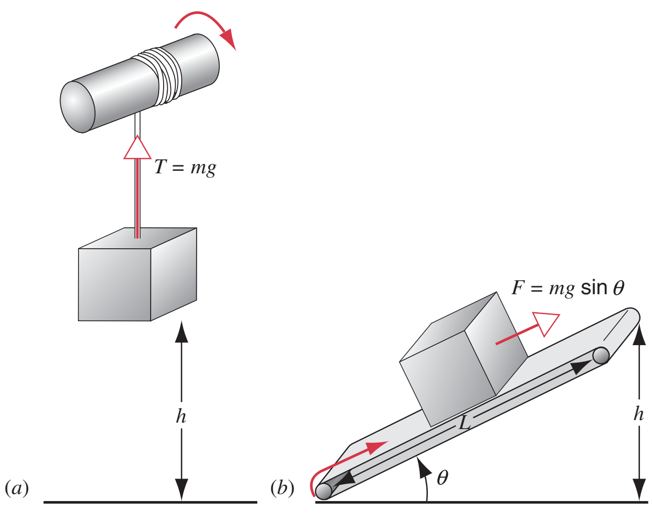


FIGURE 11-2. (*a*) A motor-driven winch raises a weight mg a distance h. (*b*) A motor turns a conveyor belt that moves an identical weight along an incline until it has been raised a distance h.

Once the block has been raised, we can turn the two motors off and the block will remain in place. That is, it costs some fuel or electrical power to run the motors *only* to lift the block, not to hold it in place. *The investment in this process is in the lifting, not in the holding.*

We define the work W done by a constant force $\vec{\mathbf{F}}$ that moves a body through a displacement $\vec{\mathbf{s}}$ *in the direction of the force* as the product of the magnitudes of the force and the displacement:

$$W = Fs \qquad (\text{constant force, } \vec{\mathbf{F}} \parallel \vec{\mathbf{s}}). \qquad (11\text{-}1)$$

In Fig. 11-2*a* the motor exerts a force of magnitude $T = mg$ in moving the block a distance h. Since the force is in the direction of the motion, the work done by the motor is, according to Eq. 11-1, $W = Th = mgh$. In Fig. 11-2*b* the motor exerts a force of magnitude $F = mg \sin \theta$ in moving the block a distance L, so the work done by the motor is $W = (mg \sin \theta)(L) = mgh$ with $h = L \sin \theta$. It is no accident that the same amount of work is done by the motor in both processes—in each case the motor invested the same amount of effort (work) in raising the block, as evidenced by the identical outcomes that resulted from using the falling block to perform some other task.

In both Figs. 11-2*a* and 11-2*b* the force was exerted parallel to the direction of motion of the block. Suppose that instead a worker exerts a horizontal force $\vec{\mathbf{F}}$ on the block to push it up the incline. Now the force and the motion are in different directions (Fig. 11-3). The force component $F \sin \theta$ perpendicular to the plane has no effect in raising the block. Only the component $F \cos \theta$ in the direction of motion does any work in raising the block.

Consider the arbitrary case illustrated in Fig. 11-4. A bead slides without friction along a thin horizontal rod. The bead moves from A to B, which we represent by the displacement vector $\vec{\mathbf{s}}$. A constant force $\vec{\mathbf{F}}$ is exerted on the bead by an external agent; $\vec{\mathbf{F}}$ makes an angle ϕ with the displacement vector. Only the component of the force $F \cos \phi$ along the displacement vector contributes to the work, so the work done by the force $\vec{\mathbf{F}}$ is

$$W = (F \cos \phi)s = Fs \cos \phi \qquad (\text{constant force}). \qquad (11\text{-}2)$$

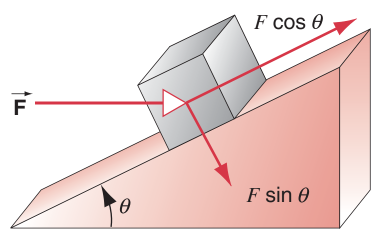


FIGURE 11-3. A worker (not shown) exerts a horizontal force $\vec{\mathbf{F}}$ on a block, pushing it along the incline.

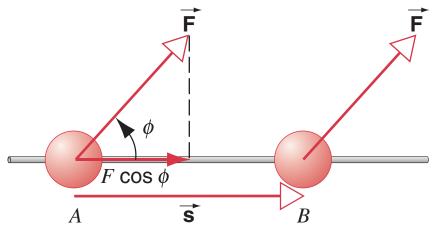


FIGURE 11-4. A bead slides along a thin rod from A to B. A constant force $\vec{\mathbf{F}}$, which makes an angle ϕ with the wire, acts on the bead at every point between A and B.

Equation 11-2 gives the work done by the particular force $\vec{\mathbf{F}}$. There may be several forces acting on the object; for example, in Fig. 11-3, in addition to the force $\vec{\mathbf{F}}$ there is the normal force $\vec{\mathbf{N}}$, the force of gravity $m\vec{\mathbf{g}}$, and perhaps also a frictional force $\vec{\mathbf{f}}$. We must calculate the work separately for each force that acts.

Note several features of Eq. 11-2:

1. If $F = 0$, then $W = 0$. For work to be done, a force must be exerted.

2. If $s = 0$, then $W = 0$. For work to be done by a force, there must be movement of the point of application of that force through some distance.

3. If $\phi = 90°$, then $W = 0$. For work to be done by a force, a component of the force must act in the direction of the displacement (or in the opposite direction). If a force is always perpendicular to the direction of motion, then the work done by that particular force is zero.

4. When $\phi = 0°$, then $W = Fs$. If the force and the displacement are in the same direction, Eq. 11-2 reduces to Eq. 11-1.

5. When $\phi = 180°$, then $W = -Fs$. If the force acts opposite to the direction of the displacement, then that force does *negative* work. In Fig. 11-2*a*, for example, a gravitational force mg (not shown) acts downward on the block. As the block moves upward a distance h, the work done by this force is $-mgh$.

As an example of these concepts, consider Fig. 11-5. In Fig. 11-5*a*, a block is sliding down a plane. The gravitational force $m\vec{\mathbf{g}}$ does positive work, the frictional force $\vec{\mathbf{f}}$ does negative work, and the normal force $\vec{\mathbf{N}}$ does zero work. In Fig. 11-5*b*, the tension in the cord $\vec{\mathbf{T}}$ is not a constant force, because its direction changes even if its magnitude remains constant. However, if we imagine the circular path to be divided into a series of infinitesimal displacements, then each small displacement (which is tangent to the circle) is perpendicular to $\vec{\mathbf{T}}$ (which acts in the radial direction). Thus the work done by the tension is zero.

Note that we can write Eq. 11-2 either as $(F\cos\phi)(s)$ or $(F)(s\cos\phi)$. This suggests that the work can be calculated in two different ways, which give the same result: either we multiply the magnitude of the displacement by the component of the force in the direction of the displacement, or we multiply the magnitude of the force by the component of the displacement in the direction of the force. Each way reminds us of an important part of the definition of work: there must be a component of $\vec{\mathbf{s}}$ in the direction of $\vec{\mathbf{F}}$, and there must be a component of $\vec{\mathbf{F}}$ in the direction of $\vec{\mathbf{s}}$ (Fig. 11-6).

As we have defined it (Eq. 11-2), work proves to be a very useful concept in physics. Our special definition of the word "work" does not correspond to the colloquial usage of the term. This may be confusing. A person holding a heavy weight at rest in the air (Fig. 11-7) may be working hard in

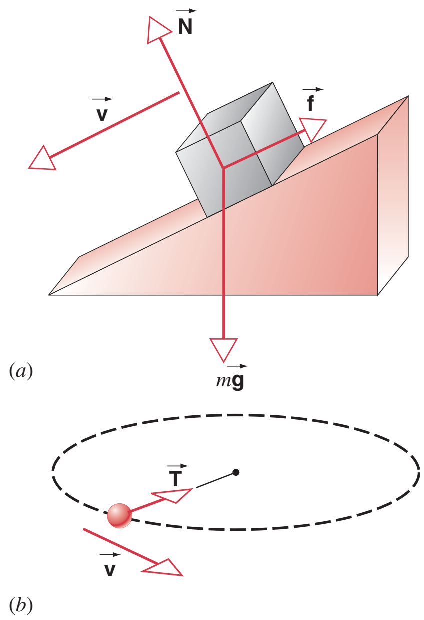


FIGURE 11-5. (*a*) A block slides down a plane, acted upon by three forces: gravity ($m\vec{\mathbf{g}}$) due to the Earth, friction ($\vec{\mathbf{f}}$) due to the plane, and the normal force ($\vec{\mathbf{N}}$) also due to the plane. (*b*) A body attached to a cord revolves in a horizontal circle, acted upon only by the tension ($\vec{\mathbf{T}}$) due to the cord.

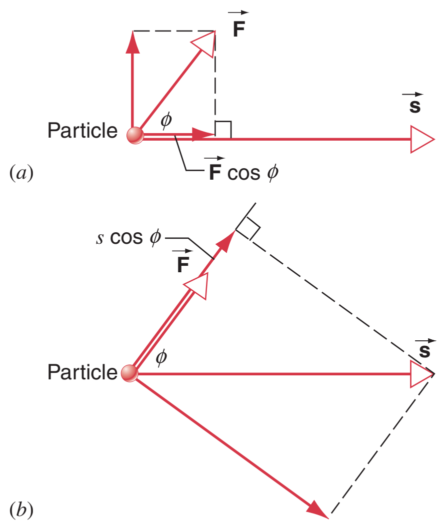


FIGURE 11-6. (*a*) The work W done on the particle by the force $\vec{\mathbf{F}}$ interpreted as $W = (F\cos\phi)(s)$. (*b*) The work W interpreted as $W = (F)(s\cos\phi)$.

FIGURE 11-7. A weightlifter is holding a weight above his head. In this configuration, the weightlifter is doing no work, as we have defined it.

the physiological sense, but from the point of view of physics that person is not doing any work on the weight. We say this because the weight does not move.

Why, then, does the weightlifter become tired and eventually lose his ability to support the weights? If we examine his muscles, we find that work is being done microscopically even when the weight does not move. A muscle is not a solid support and cannot sustain a load in a static manner. The individual muscle fibers repeatedly relax and contract, and work is done in each contraction. This microscopic work depletes his internal supply of energy, and gradually he becomes too tired to hold the weights. In this chapter we do not consider this "internal" form of work. We use *work* only in the strict sense of Eq. 11-2, so that it does indeed vanish when there is no motion of the body on which the force acts.

Note that work, unlike properties such as mass, volume, or temperature, is not an intrinsic property of a body. We cannot say, for example, that a body gains, loses, or contains a certain amount of work when it moves through a distance as a force acts on it. Work is associated with the force that acts on the body, or with the agent that exerts that force.

The unit of work is determined from the work done by a unit force in moving a body a unit distance in the direction of the force. The SI unit of work is the *newton-meter,* called the *joule* (abbreviation J). In the British system the unit of work is the foot-pound. In cgs systems the unit of work is the dyne-centimeter, called the *erg.* Using the relations between the newton, dyne, and pound, and between the meter, centimeter, and foot, we obtain 1 joule $= 10^7$ ergs $=$ 0.7376 ft·lb.

A convenient unit of work when dealing with atomic or subatomic particles is the *electron-volt* (abbreviation eV), where 1 eV $= 1.60 \times 10^{-19}$ J. The work required to remove an outer electron from an atom has a typical magnitude of several eV. The work required to remove a proton or a neutron from a nucleus has a typical magnitude of several MeV (10^6 eV). The work required to accelerate an electron in the 2-mile-long linear accelerator at Stanford is several GeV (10^9 eV). The work required to accelerate a proton in the Fermilab accelerator is about 10^{12} eV (1 TeV).

SAMPLE PROBLEM 11-1. A block of mass $m = 11.7$ kg is to be pushed a distance of $s = 4.65$ m along an incline so that it is raised a distance of $h = 2.86$ m in the process (Fig. 11-8*a*). Assuming frictionless surfaces, calculate how much work you would do on the block if you applied a force parallel to the incline to push the block up at constant speed.

Solution A free-body diagram of the block is given in Fig. 11-8*b*. We must first find F, the magnitude of the force pushing the block up the incline. Because the motion is not accelerated (we are given that the speed is constant), the net force parallel to the plane must be zero. We choose our x axis parallel to the plane, with its positive direction up the plane. The net force along the plane is then $\Sigma F_x = F - mg \sin \theta$. With $a_x = 0$, Newton's second law gives $F - mg \sin \theta = 0$, or

$$F = mg \sin \theta = (11.7 \text{ kg})(9.80 \text{ m/s}^2)\left(\frac{2.86 \text{ m}}{4.65 \text{ m}}\right) = 70.5 \text{ N}.$$

Then the work done by $\vec{\mathbf{F}}$, from Eq. 11-2 with $\phi = 0°$, is

$$W = Fs \cos 0° = (70.5 \text{ N})(4.65 \text{ m}) = 328 \text{ J}.$$

Note that the angle ϕ $(=0°)$ used in this expression is the angle between the applied force and the displacement of the block, both of which are parallel to the incline. The angle ϕ must not be confused with the angle θ of the incline.

If you were to raise the block vertically at constant speed without using the incline, the work you do would be the vertical component of the force you exert on the block, which is equal to mg, times the vertical distance h, or

$$W = mgh = (11.7 \text{ kg})(9.80 \text{ m/s}^2)(2.86 \text{ m}) = 328 \text{ J},$$

the same as before. The only difference is that the incline permits a smaller force ($F = 70.5$ N) to raise the block than would be required without the incline ($mg = 115$ N). On the other hand, the

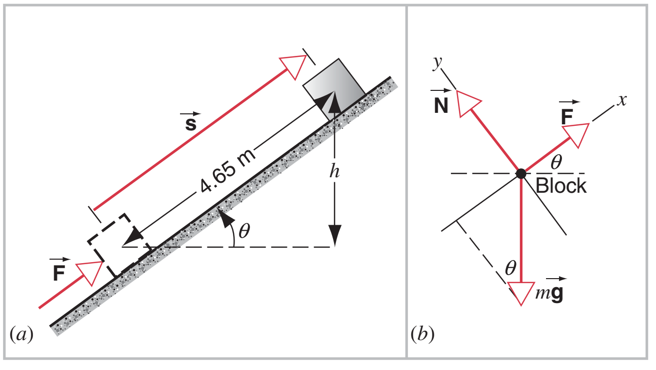


FIGURE 11-8. Sample Problem 11-1. (*a*) A force $\vec{\mathbf{F}}$ moves a block up a plane through a displacement $\vec{\mathbf{s}}$. (*b*) A free-body diagram for the block.

distance you must push the block up the incline (4.65 m) is greater than the distance you would move it if you raised it directly (2.86 m).

SAMPLE PROBLEM 11-2. A child pulls a 5.6-kg sled a distance of $s = 12$ m along a horizontal surface at a constant speed. What work does the child do on the sled if the coefficient of kinetic friction μ_k is 0.20 and the rope makes an angle of $\phi = 45°$ with the horizontal?

Solution The situation is shown in Fig. 11-9*a* and the forces acting on the sled are shown in the free-body diagram of Fig. 11-9*b*. $\vec{F}$ is the child's pull, $m\vec{g}$ the sled's weight, $\vec{f}$ the frictional force, and $\vec{N}$ the normal force exerted by the surface on the sled. To evaluate the work, we must first find the magnitude of force F. With the choice of x and y axes shown on the free-body diagram of Fig. 11-9*b*, the components of the net force are $\Sigma F_x = F\cos\phi - f$ and $\Sigma F_y = F\sin\phi + N - mg$. With both $a_x = 0$ and $a_y = 0$, Newton's second law gives

$$F\cos\phi - f = 0 \quad \text{and} \quad F\sin\phi + N - mg = 0.$$

The frictional force is related to the normal force by $f = \mu_k N$. Combining these three equations, we can eliminate f and N to find an expression for F:

$$F = \frac{\mu_k mg}{\cos\phi + \mu_k \sin\phi}.$$

With $\mu_k = 0.20$, $mg = (5.6\text{ kg})(9.8\text{ m/s}^2) = 55$ N, and $\phi = 45°$ we obtain

$$F = \frac{(0.20)(55\text{ N})}{\cos 45° + (0.20)(\sin 45°)} = 13\text{ N}.$$

Then with $s = 12$ m, the work done by the child on the sled is, using Eq. 11-2,

$$W = Fs\cos\phi = (13\text{ N})(12\text{ m})(\cos 45°) = 110\text{ J}.$$

The vertical component of the pull $\vec{F}$ does no work on the sled. Note, however, that it reduces the normal force between the sled and the surface ($N = mg - F\sin\phi$) and thereby reduces the magnitude of the force of friction ($f = \mu_k N$).

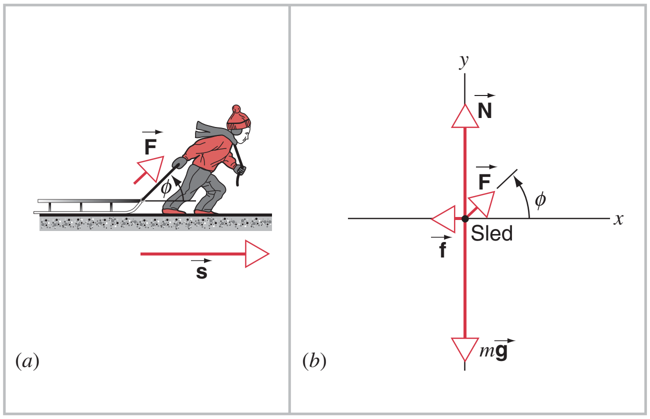


FIGURE 11-9. Sample Problem 11-2. (*a*) A child displaces a sled an amount $\vec{s}$ by pulling with a force $\vec{F}$ on a rope that makes an angle ϕ with the horizontal. (*b*) A free-body diagram for the sled.

Would the child do more work, less work, or the same amount of work on the sled if $\vec{F}$ were applied horizontally instead of at 45° from the horizontal? Do any of the other forces acting on the sled do work on it?

Work as a Dot Product

Work is a scalar quantity; it is characterized only by a magnitude and a sign. It is computed, however, by combining two vectors ($\vec{F}$ and $\vec{s}$). In Chapters 8–10, we often found the need to multiply two vectors to obtain another vector, which we expressed in compact form as the vector or cross product (for example, $\vec{\tau} = \vec{r} \times \vec{F}$ or $\vec{l} = \vec{r} \times \vec{p}$). Here we are multiplying two vectors to obtain a scalar. A compact way of writing this is in terms of the *scalar* or *dot* product of the two vectors.

Consider two vectors $\vec{A}$ and $\vec{B}$ (Fig. 11-10) separated by an angle ϕ. The dot product of $\vec{A}$ and $\vec{B}$ is defined in terms of the magnitudes A and B as

$$\vec{A}\cdot\vec{B} = AB\cos\phi, \tag{11-3}$$

which is read as "A dot B." Clearly we can write this either as $A(B\cos\phi)$ or as $B(A\cos\phi)$, which suggests that the dot product can be regarded as the product of the magnitude of one vector and the component of the other in the direction of the first, as Fig. 11-10 suggests. The magnitudes A and B are always positive, but the dot product may be positive, negative, or zero depending on the value of the angle ϕ. If $\vec{A}$ and $\vec{B}$ are perpendicular to one another ($\phi = 90°$), the dot product is zero. Unlike the cross product, the order of the vectors in the dot product is unimportant; that is, $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$. Also, note that the dot product of a vector with itself is just the squared magnitude of the vector: $\vec{A}\cdot\vec{A} = A^2$.

These properties of the dot product exactly match the properties of the work, as we have defined it in terms of the vectors $\vec{F}$ and $\vec{s}$. This suggests that we may write Eq. 11-2 as

$$W = \vec{F}\cdot\vec{s} \qquad \text{(constant force).} \tag{11-4}$$

If we write the vectors $\vec{A}$ and $\vec{B}$ in terms of their components ($\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$), then the dot product is

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z. \tag{11-5}$$

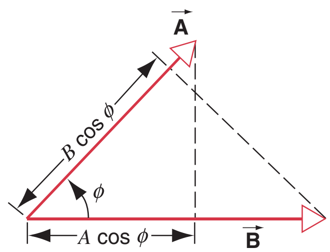


FIGURE 11-10. The dot product of two vectors $\vec{A}$ and $\vec{B}$ can be regarded as the product of the magnitude of one vector and the component of the other vector in the direction of the first.

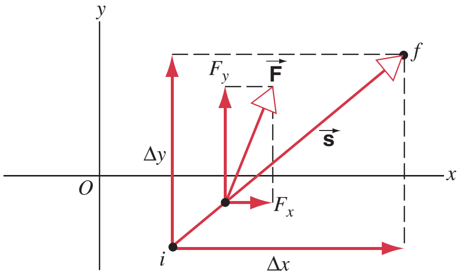


FIGURE 11-11. Here a particle moves from initial location i to final location f through the displacement $\vec{s}$ as the constant force $\vec{F}$ acts on the particle. When the force $\vec{F}$ and the displacement $\vec{s}$ are in arbitrary directions, we can find the work by resolving $\vec{F}$ and $\vec{s}$ into their x and y components.

To derive this expression, we use Eq. 11-3 to find the dot products of the unit vectors: $\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1$ and $\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{i}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = 0$. If the force and displacement vectors lie in the xy plane (Fig. 11-11), we can write the work in the form of Eq. 11-5; with $\vec{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}}$ and $\vec{s} = \Delta x\hat{\mathbf{i}} + \Delta y\hat{\mathbf{j}}$, we have

$$W = F_x\Delta x + F_y\Delta y \qquad \text{(constant force)}. \qquad (11\text{-}6)$$

The two terms on the right side of this equation *cannot* be interpreted as the components of work. *Work is a scalar, and scalars do not have components.* It may appear from Eq. 11-6 that the value of the work depends on where we draw the coordinate axes; however, Eq. 11-2 shows that this is not true. In general, *the value of the dot product is independent of the choice of coordinate axes.*

Although the force $\vec{F}$ is an invariant (it has the same magnitude and direction for any choice of inertial reference frame), the displacement $\vec{s}$ of a particle over a given time interval is not invariant. Observers in different inertial reference frames all measure the same $\vec{F}$ but measure different values for the magnitude and direction of the displacement $\vec{s}$. As a result, the value determined for the work will depend on the inertial reference frame of the observer. Different observers might find the work to be positive, negative, or zero. We discuss this point further in Section 11-6.

11-3 POWER

In designing a mechanical system, it is often necessary to consider not only how much work must be done but also how rapidly the work is to be done. The same amount of work is done in raising a given body through a given height whether it takes 1 second or 1 year to do so. However, the *rate at which work is done* is very different in the two cases.

We define *power* as the rate at which work is done. (Here we consider only *mechanical* power, which results from mechanical work. A more general view of power as energy delivered per unit time permits us to broaden the concept of power to include electrical power, solar power, and so on.) If a certain force performs work W on a body in a time t, the *average power* due to the force is

$$P_{\text{av}} = \frac{W}{t}. \qquad (11\text{-}7)$$

The *instantaneous power* P is

$$P = \frac{dW}{dt}, \qquad (11\text{-}8)$$

where dW is the small amount of work done in the infinitesimal time interval dt. If the power is constant in time, then $P = P_{\text{av}}$ and

$$W = Pt. \qquad (11\text{-}9)$$

The SI unit of power is the joule per second, which is called the *watt* (abbreviation W):

$$1\text{ W} = 1\text{ J/s}.$$

This unit is named in honor of James Watt (1736–1819), who made major improvements to the steam engines of his day. In the British system, the unit of power is 1 ft · lb/s, although a more common practical unit, the *horsepower* (hp), is generally used to describe the power of such devices as electric motors or automobile engines. One horsepower is defined to be 550 ft · lb/s, which is equivalent to about 746 W.

Work can also be expressed in units of power × time. This is the origin of the term *kilowatt-hour,* which the electrical company uses to measure how much work (in the form of electrical energy) it has delivered to your house. One kilowatt-hour is the work done in 1 hour by an agent working at a constant rate of 1 kW.

We can also express the power delivered to a body in terms of the velocity of the body and the force that acts on it. In a short interval of time dt, the body moves a distance $d\vec{s}$ and the work done on the body is $dW = \vec{F}\cdot d\vec{s}$. We can rewrite Eq. 11-8 as

$$P = \frac{dW}{dt} = \frac{\vec{F}\cdot d\vec{s}}{dt} = \vec{F}\cdot\frac{d\vec{s}}{dt},$$

which becomes, after substituting the velocity $\vec{v}$ for $d\vec{s}/dt$,

$$P = \vec{F}\cdot\vec{v}. \qquad (11\text{-}10)$$

If $\vec{F}$ and $\vec{v}$ are parallel to one another, this can be written

$$P = Fv. \qquad (11\text{-}11)$$

Note that the power can be negative if $\vec{F}$ and $\vec{v}$ are antiparallel. Delivering negative power to a body means doing negative work on it: the force exerted on the body by the external agent is in a direction opposite to its displacement $d\vec{s}$ and therefore opposite to $\vec{v}$.

SAMPLE PROBLEM 11-3. An empty elevator has a weight of 5160 N (1160 lb). It is designed to carry a maximum load of 20 passengers from the ground floor to the 25th floor of a building in a time of 18 seconds. Assuming the average weight of a passenger to be 710 N (160 lb) and the distance between floors to be 3.5 m (11 ft), what is the average power that must be supplied by the elevator motor? (Assume that all the work that lifts the elevator comes from the motor and that the elevator has no counterweight.)

Solution We assume that the elevator ascends at constant velocity, and that the distances traveled during acceleration and deceleration can be neglected. At constant velocity, the net force is zero and so the upward force exerted by the motor is in magnitude equal to the total weight of the elevator and passengers: $F = 5160\ \text{N} + 20(710)\ \text{N} = 19{,}400\ \text{N}$. The work that must be done is

$$W = Fs = (19{,}400\ \text{N})(25 \times 3.5\ \text{m}) = 1.7 \times 10^6\ \text{J}.$$

The average power is therefore

$$P_{av} = \frac{W}{t} = \frac{1.7 \times 10^6\ \text{J}}{18\ \text{s}} = 94\ \text{kW}.$$

This is the same as 126 hp, roughly the power delivered by the engine of an automobile. Of course, frictional losses and other inefficiencies will increase the power that the motor must provide to lift the elevator.

In practice, an elevator usually has a counterweight that falls as the elevator cab rises. Gravity does *positive* work on the falling counterweight and *negative* work on the rising elevator. The work that must be provided by the motor, which is equal to the magnitude of the *net* work done by gravity, is therefore greatly reduced.

11-4 WORK DONE BY A VARIABLE FORCE

So far we have considered only the work done by a *constant* force. Many of the forces we have previously considered do not change in magnitude or direction as a body moves; gravity near the Earth's surface is a good example. However, many other forces do change in magnitude with the displacement of the body, and so we must consider how to evaluate the work done by such forces. We assume a one-dimensional situation: the force has only an x component, and the particle moves only in the (positive or negative) x direction. We will first discuss the general procedure for analyzing the work done by a variable force, and then we will apply that method to the analysis of an important type of force that we have not yet considered—namely, the force exerted by a spring when it is stretched or compressed.

Let a body move along the x axis from x_i to x_f as a force $F_x(x)$ is applied to it. By writing the force as $F_x(x)$ we indicate that the force varies in magnitude (and possibly in direction) as the displacement of the body changes. Our strategy for this analysis will be to divide the interval from x_i to x_f into a large number of smaller intervals. Within each small interval we regard the force as being approximately constant (although the force may be different for different intervals), so that the work in any interval can be calculated using the methods for constant forces developed previously in this chapter. Eventually we will make the intervals infinitely numerous and vanishingly small, which leads us to the methods of calculus.

The smooth curve in Fig. 11-12 shows an arbitrary force $F_x(x)$ that acts on a body that moves from x_i to x_f. We divide the total displacement into a number N of small intervals of equal width δx (Fig. 11-12*a*). Consider the first interval, in which there is a small displacement δx from x_i to $x_i + \delta x$. We make this interval so small that the x component of the force is approximately constant at the value F_1. We can then use Eq. 11-6 to find the work δW_1 done by the force in that interval: $\delta W_1 = F_1\,\delta x$. Similarly, in the second interval, in which the body moves from $x_i + \delta x$ to $x_i + 2\delta x$, the force is nearly constant with x component F_2, and the work done by the force in that interval is $\delta W_2 = F_2\,\delta x$. Continuing on for all the N intervals, we can find the total work as the sum of all such terms:

$$W = \delta W_1 + \delta W_2 + \delta W_3 + \cdots$$
$$= F_1\,\delta x + F_2\,\delta x + F_3\,\delta x + \cdots$$

or

$$W = \sum_{n=1}^{N} F_n\,\delta x. \qquad (11\text{-}12)$$

To make a better approximation we can divide the total displacement from x_i to x_f into a larger number of intervals, as in Fig. 11-12*b*, so that δx is smaller and the value of F_n

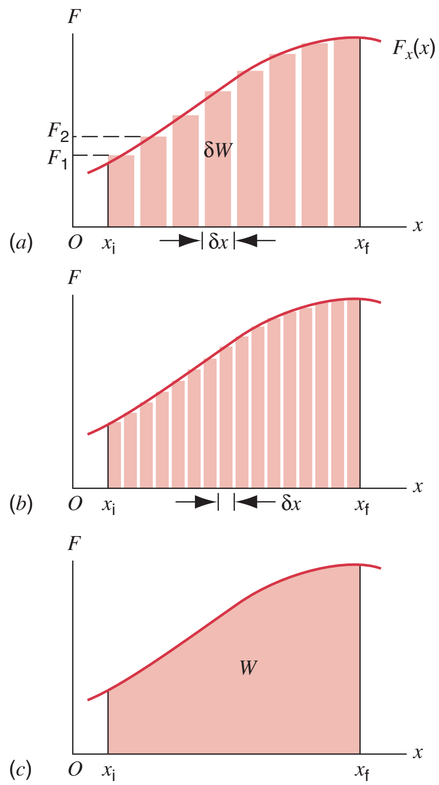


FIGURE 11-12. (*a*) The area under the curve of the variable one-dimensional force $F_x(x)$ is approximated by dividing the region between the limits x_i and x_f into a number of intervals of width δx. The sum of the areas of the rectangular strips is approximately equal to the area under the curve. (*b*) A better approximation is obtained using a larger number of narrower strips. (*c*) In the limit $\delta x \to 0$, the actual area is obtained.

in each interval is more typical of the force within the interval. It is clear that we can obtain better and better approximations by taking δx smaller and smaller so as to have a larger and larger number of intervals. We can obtain an exact result for the work done by F_x if we let δx go to zero and the number of intervals N go to infinity. Hence the exact result is

$$W = \lim_{\delta x \to 0} \sum_{n=1}^{N} F_n \, \delta x. \tag{11-13}$$

The relation

$$\lim_{\delta x \to 0} \sum_{n=1}^{N} F_n \, \delta x = \int_{x_i}^{x_f} F_x(x) \, dx$$

defines the integral of F_x with respect to x from x_i to x_f. Numerically, this quantity is exactly equal to the area between the force curve and the x axis between the limits x_i and x_f (Fig. 11-12*c*). Hence, an integral can be interpreted graphically as an area. We can write the total work done by F_x in displacing a body from x_i to x_f as

$$W = \int_{x_i}^{x_f} F_x(x) \, dx. \tag{11-14}$$

The sign of W is automatically determined in Eq. 11-14 by the sign of F_x and by the endpoints of the interval, x_i and x_f. For example, if F_x is always positive and if the particle moves in the positive x direction ($x_f > x_i$), then W will be positive.

Work Done by the Spring Force

As an example of a one-dimensional variable force, we consider the force exerted by a spring when it is stretched or compressed. Figure 11-13 shows a body attached to a spring. With no force applied, the spring is unstretched and the body is located at $x = 0$. We call this the *relaxed* state of the spring. Suppose an external force $\vec{\mathbf{F}}_{ext}$ is applied to the body, which either stretches (Fig. 11-13*a*) or compresses (Fig. 11-13*b*) the spring. The spring exerts a force $\vec{\mathbf{F}}_s$ that opposes the applied force. The spring force is sometimes called a *restoring force,* because it always acts in a direction to restore the body to its location at $x = 0$. We will assume that the body moves slowly, so that we can regard it as being in equilibrium at all times. In this case, $\vec{\mathbf{F}}_{ext} = -\vec{\mathbf{F}}_s$.

What is the nature of the force that the spring exerts on the body as the spring is stretched or compressed? From experiments we learn that this force is not constant—the more we change the length of a spring, the greater the force exerted by the spring (equivalently, we might say the greater the external force that must be exerted to change its length). We also learn that, for most springs, to a good approximation the magnitude of this force varies *linearly* with the distance the spring is stretched or compressed from its relaxed length. In one dimension, we can write the x component of the force exerted by the spring on the body attached to it as

$$F_s = -kx, \tag{11-15}$$

which is known as *Hooke's law.* The constant k in Eq. 11-15 is called the *force constant* of the spring (or sometimes the *spring constant*) and its SI unit is newtons per meter (N/m). It is a measure of the force necessary to stretch a spring by a given amount; stiffer springs have larger values of k. Equation 11-15 is valid as long as we do not stretch the string beyond a limited range.

The minus sign in Eq. 11-15 reminds us that the direction of the force exerted by the spring is always opposite to

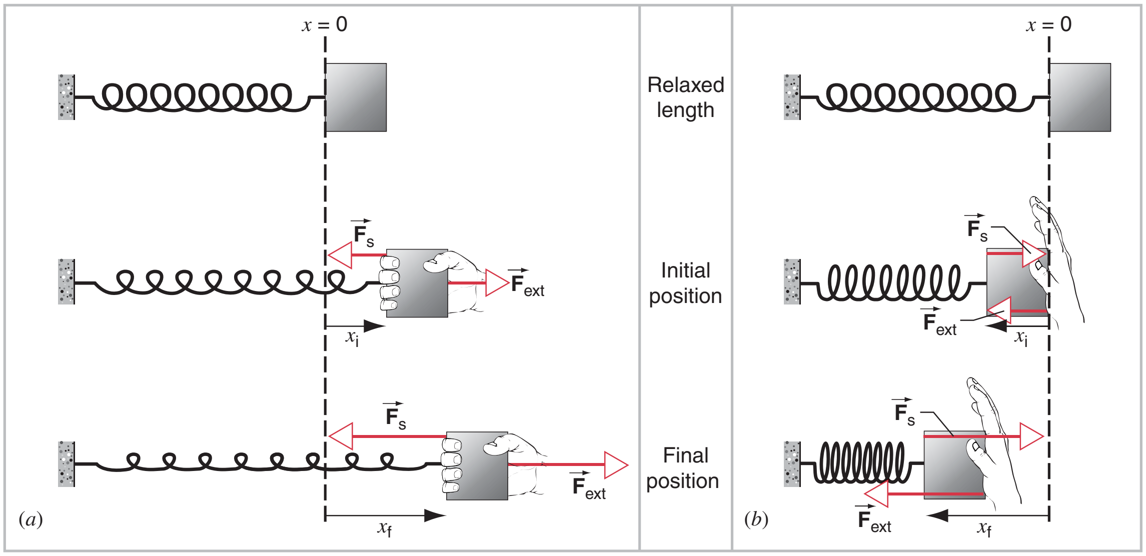


FIGURE 11-13. A body attached to a spring is at $x = 0$ when the spring is relaxed. An external force moves the body from initial displacement x_i to final displacement x_f. The x axis is positive to the right. (*a*) Stretching. (*b*) Compression.

the displacement of the body from its position when the spring is in its relaxed state (which we define as $x = 0$). When the spring is stretched, using the coordinate system of Fig. 11-13*a*, then $x > 0$ and so F_s is negative, indicating that the spring force acts to the left. When the spring is compressed, as in Fig. 11-13*b*, then $x < 0$ and $F_s > 0$.

Equation 11-14 can be applied to calculate the work done by the spring force in Fig. 11-13*a*. Let us stretch the spring from its initial state (where $x = x_i$) to its final state (where $x = x_f$). The work done on the body by the spring force during this displacement is:

$$W_s = \int_{x_i}^{x_f} F_s\,dx = \int_{x_i}^{x_f} (-kx)\,dx = -\tfrac{1}{2}k(x_f^2 - x_i^2). \qquad (11\text{-}16)$$

Equation 11-16 shows that the work done by the spring is negative when $x_f > x_i$, as is the case in Fig. 11-13*a*; the direction of $\vec{\mathbf{F}}_s$ is opposite to the displacement, so the negative value of W agrees with the discussion following Eq. 11-2.

If the external force acts to compress the spring, as in Fig. 11-13*b*, both x_i and x_f are negative. However, $|x_f| > |x_i|$ and again Eq. 11-16 shows that the work done on the body by the spring is negative. Thus Eq. 11-16 remains valid no matter how the body moves under the action of the spring force. Note that, according to Eq. 11-16, the work done by the spring force is zero if the body moves from a positive displacement $+x$ to a negative displacement $-x$ of equal magnitude. Can you explain this in terms of the force exerted by the spring?

If we begin stretching or compressing at the relaxed position ($x_i = 0$) and move the body through a distance x, then

$$W_s = -\tfrac{1}{2}kx^2. \qquad (11\text{-}17)$$

Because x is squared in Eq. 11-17, the work done by the spring on the body is the same in both magnitude and sign for stretching and compression by the same distance x.

With $\vec{\mathbf{F}}_{ext} = -\vec{\mathbf{F}}_s$, the work done on the body by the external force is positive when the work done by the spring force is negative. Thus $W_{ext} > 0$ for both cases shown in Fig. 11-13.

Figure 11-14 shows how Fig. 11-12 would look for the spring force. The shaded regions represent the negative work done on the body by the spring force for the two cases in Fig. 11-13. By a geometrical calculation, you should be able to show that the shaded areas correspond to the work given in Eq. 11-16 and that the signs are also given correctly.

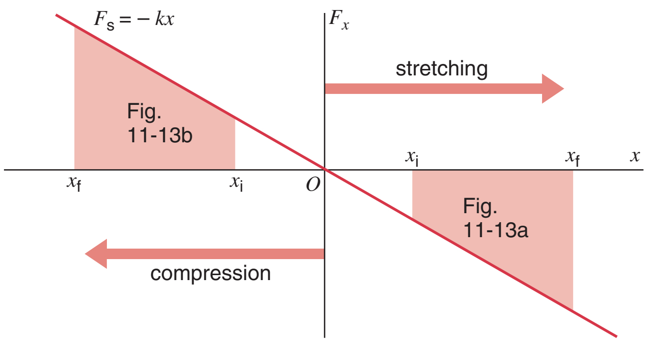


FIGURE 11-14. The work done by the spring force on the body as it moves from x_i to x_f is equal to the area under the graph of $F_s = -kx$ between x_i and x_f. The shaded areas represent the negative work done by the spring in Figs. 11-13*a* and 11-13*b*.

SAMPLE PROBLEM 11-4. A spring hangs vertically in its relaxed state. A block of mass $m = 6.40$ kg is attached to the spring, but the block is held in place so that the spring at first does not stretch. Now the hand holding the block is slowly lowered (Fig. 11-15*a*), so that the block descends at constant speed until it reaches the point at which it hangs at equilibrium with the hand removed. At this point the spring is measured to have stretched a distance $d = 0.124$ m from its previous relaxed length. Find the work done on the block in this process by (*a*) gravity, (*b*) the spring, and (*c*) the hand.

Solution (*a*) We can find the force constant of the spring, which is not given in this problem, from the condition at equilibrium. Taking the y axis to be positive upward, the net force in the y direction at equilibrium (Fig. 11-15*b*) is $\sum F_y = kd - mg$. At equilibrium, $\sum F_y = 0$ so $kd = mg$, or

$$k = mg/d = (6.40\text{ kg})(9.80\text{ m/s}^2)/(0.124\text{ m}) = 506\text{ N/m}.$$

To find the work done by gravity, W_g, we note that gravity is a constant force, and the force and the displacement are parallel, so we can use Eq. 11-1:

$$W_g = Fs = mgd = (6.40\text{ kg})(9.80\text{ m/s}^2)(0.124\text{ m}) = +7.78\text{ J}.$$

This is positive, because the force and displacement are in the same direction.

(*b*) To find the work W_s done by the spring, we use Eq. 11-17 with $x = -d$:

$$W_s = -\tfrac{1}{2}kd^2 = -\tfrac{1}{2}(506\text{ N/m})(0.124\text{ m})^2 = -3.89\text{ J}.$$

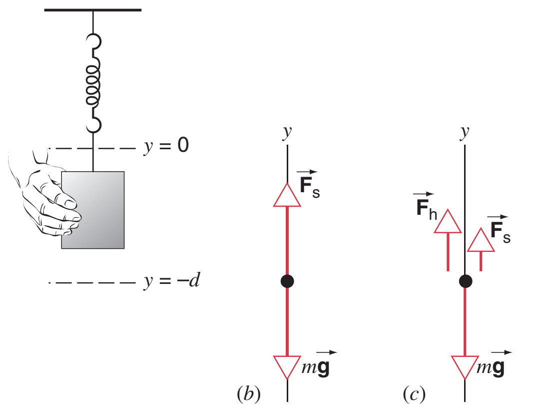


FIGURE 11-15. Sample Problem 11-4. (*a*) A hand lowers a block attached to a spring. (*b*) The free-body diagram of the block at its equilibrium position. (*c*) The free-body diagram of the block as it is lowered.

This is negative, because the force and displacement are in opposite directions.

(c) To find the work done by the hand, we need the upward force F_h exerted by the hand. As the block is lowered at constant speed, $a_y = 0$. Based on the free-body diagram of Fig. 11-15c, the net force during this process is $\Sigma F_y = F_s + F_h - mg$, so $F_h = mg - F_s$. Note that, until the block reaches its equilibrium position, $mg > F_s$ so that $F_h > 0$ as we should expect (F_h has a positive y component, because it acts upward). We can find the work done by the hand from an integral of the form of Eq. 11-14: $W_h = \int F_h(y)\,dy$, with $F_h = mg - (-ky)$:

$$W_h = \int_0^{-d} (mg + ky)\,dy = mg(-d) + \tfrac{1}{2}k(-d)^2 = -3.89 \text{ J}.$$

Note that $W_s + W_g + W_h = 0$. Can you explain this?

11-5 WORK DONE BY A VARIABLE FORCE: TWO-DIMENSIONAL CASE (Optional)

The force $\vec{\mathbf{F}}$ acting on a particle may vary in direction as well as in magnitude, and the particle may move along a curved path. To compute the work in this general case we divide the path into a large number of small displacements $\delta\vec{\mathbf{s}}$, each tangent to the path and pointing in the direction of motion. Figure 11-16 shows two selected displacements for a particular situation; it also shows the force $\vec{\mathbf{F}}$ and the angle ϕ between $\vec{\mathbf{F}}$ and $\delta\vec{\mathbf{s}}$ at each location. We can find the amount of work δW done on the particle during a displacement $\delta\vec{\mathbf{s}}$ from

$$\delta W = \vec{\mathbf{F}} \cdot \delta\vec{\mathbf{s}} = F \cos\phi\, \delta s. \qquad (11\text{-}18)$$

Here $\vec{\mathbf{F}}$ is the force at the location of $\delta\vec{\mathbf{s}}$. The work done by the variable force $\vec{\mathbf{F}}$ on the particle as the particle moves from i to f in Fig. 11-16 is found approximately by adding up (summing) the elements of work done over each of the line segments that make up the path from i to f. If the line segments $\delta\vec{\mathbf{s}}$ become infinitesimally small, they may be replaced by differentials $d\vec{\mathbf{s}}$ and the sum over the line segments may be replaced by an integral, as in Eq. 11-14. The work is then found from

$$W = \int_i^f \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \int_i^f F\cos\phi\, ds. \qquad (11\text{-}19)$$

We cannot evaluate this integral until we are able to say how F and ϕ in Eq. 11-19 vary from point to point along the path; both are functions of the x and y coordinates of the particle in Fig. 11-16.

We can obtain an expression equivalent to Eq. 11-19 by writing $\vec{\mathbf{F}}$ and $d\vec{\mathbf{s}}$ in terms of their components. Thus $\vec{\mathbf{F}} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}}$ and $d\vec{\mathbf{s}} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}$, so that $\vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = F_x\,dx + F_y\,dy$. In this evaluation recall that $\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = 1$ and $\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{i}} = 0$. Substituting this result into Eq. 11-19, we obtain

$$W = \int_i^f (F_x\,dx + F_y\,dy). \qquad (11\text{-}20)$$

This result is similar to Eq. 11-6, which was derived for constant forces. Equation 11-20 reduces to Eq. 11-6 when the force is constant. Integrals such as those in Eqs. 11-19 and 11-20 are called *line integrals;* to evaluate them we must know how $F\cos\phi$ or F_x and F_y vary as the particle moves along a particular line (or curve) of a specified functional form $y(x)$. The extension of Eq. 11-20 to three dimensions is straightforward.

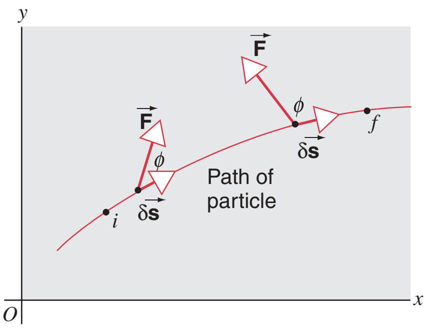


FIGURE 11-16. A particle moves from point i to point f along the path shown. During its motion it is acted on by a force $\vec{\mathbf{F}}$ that varies in both magnitude and direction. As $\delta\vec{\mathbf{s}} \to 0$, we replace the interval by $d\vec{\mathbf{s}}$, which is in the direction of the instantaneous velocity and therefore tangent to the path. The path is divided into many small intervals $\delta\vec{\mathbf{s}}$.

SAMPLE PROBLEM 11-5. A small object of mass m is suspended from a string of length L. The object is pulled sideways by a force F that is always horizontal, until the string finally makes an angle ϕ_m with the vertical (Fig. 11-17a). The displacement is accomplished at a small constant speed. Find the work done by all the forces that act on the object.

Solution The motion is along an arc of radius L, and the displacement $d\vec{\mathbf{s}}$ is always along the arc. At an intermediate point in the motion, the string makes an angle ϕ with the vertical, and from

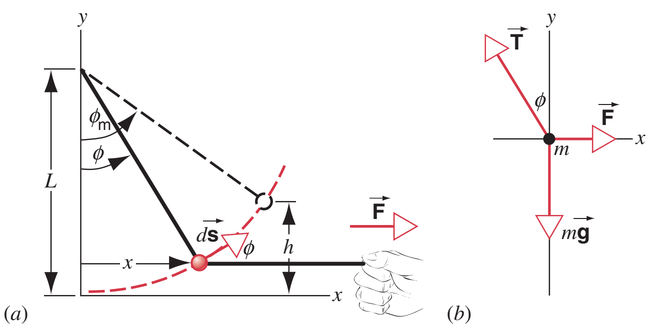


FIGURE 11-17. Sample Problem 11-5. (a) A particle is suspended from a string of length L and is pulled aside by a horizontal force $\vec{\mathbf{F}}$. The maximum angle reached is ϕ_m. (b) A free-body diagram for the particle.

the free-body diagram of Fig. 11-17*b* we see by applying Newton's second law, with $a_x = 0$ and $a_y = 0$, that

$$x \text{ component:} \qquad F - T\sin\phi = 0,$$

$$y \text{ component:} \qquad T\cos\phi - mg = 0.$$

Combining these two equations to eliminate T, we find

$$F = mg\tan\phi.$$

Since F acts only in the x direction, we can use Eq. 11-20 with $F_x = F$ and $F_y = 0$ to find the work done by F. Thus

$$W_F = \int F\,dx = \int_0^{\phi_m} mg\tan\phi\,dx.$$

To carry out the integral over ϕ, we must have a single integration variable; we choose to define x in terms of ϕ. At an arbitrary intermediate position, when the horizontal coordinate is x, we see that $x = L\sin\phi$ and thus $dx = L\cos\phi\,d\phi$. Substituting for dx, we can now carry out the integration:

$$\begin{aligned} W_F &= \int_0^{\phi_m} mg\tan\phi\,(L\cos\phi\,d\phi) \\ &= mgL\int_0^{\phi_m}\sin\phi\,d\phi = mgL(-\cos\phi)\Big|_0^{\phi_m} \\ &= mgL(1-\cos\phi_m). \end{aligned}$$

From Fig. 11-17*a*, we can see that $h = L(1-\cos\phi_m)$, and thus

$$W_F = mgh.$$

The work W_g done by the (constant) gravitational force mg can be evaluated using a similar technique based on Eq. 11-20 (taking $F_x = 0$, $F_y = -mg$) to give $W_g = -mgh$ (see Exercise 25). The minus sign enters because the direction of the vertical displacement is opposite to the direction of the gravitational force. The work W_T done by the tension in the string is zero, because $\vec{\mathbf{T}}$ is perpendicular to the displacement $d\vec{\mathbf{s}}$ at every point of the motion. Now you can see that the total work is zero: $W_{net} = W_F + W_g + W_T = mgh - mgh + 0 = 0$. Can you explain this?

Note that in this problem the (positive) work done by the horizontal force $\vec{\mathbf{F}}$ in effect cancels the (negative) work done by the vertical force $m\vec{\mathbf{g}}$. This can occur because work is a *scalar:* it has no direction or components. The motion of the particle depends on the *total* work done on it, which is the scalar sum of the values of the work associated with each of the individual forces.

11-6 KINETIC ENERGY AND THE WORK–ENERGY THEOREM

As we learned in Chapter 3, when we apply a net external force to a body, the body accelerates according to Newton's second law. If we apply that force over a measured interval of distance or time, the velocity of the body changes from its initial value $\vec{\mathbf{v}}_i$ to its final value $\vec{\mathbf{v}}_f$.

In this chapter we develop a different way of describing the same situation using the language of work and energy instead of force and acceleration. We have already introduced work and discussed how to calculate the work done by a force in a variety of situations. We now complete this analysis by introducing one form of energy, the *kinetic energy* or energy of motion, and showing how the kinetic energy of a body is related to the work done on it.

So far we have been discussing the work done by any single force that may act on a body. Now we want to consider the combined effect of *all* the forces that act on the body. For the time being we will make one simplifying assumption—*we assume that only constant forces act on the body.* Later in this section we show that the same result can be obtained when variable forces act.

Our first goal is to find the *net work* due to all the forces that act on the body. We can find the net work in either of two ways: (1) find the net force $\vec{\mathbf{F}}_{net} = \Sigma\vec{\mathbf{F}}$ and then compute the work $W_{net} = \vec{\mathbf{F}}_{net}\cdot\vec{\mathbf{s}}$ done by that force on the body as it moves through displacement $\vec{\mathbf{s}}$, or (2) find the work done on the body by each individual force ($W_1 = \vec{\mathbf{F}}_1\cdot\vec{\mathbf{s}}$, $W_2 = \vec{\mathbf{F}}_2\cdot\vec{\mathbf{s}}$, etc.) and then add to find the net work: $W_{net} = W_1 + W_2 + \cdots$. The two methods give identical results, and the choice between them is a matter of convenience.

According to Newton's second law, $\vec{\mathbf{F}}_{net} = m\vec{\mathbf{a}}$. As the body moves through displacement $\vec{\mathbf{s}}$, this net force causes its velocity to change from $\vec{\mathbf{v}}_i$ to $\vec{\mathbf{v}}_f$. For constant forces, the acceleration is constant, and so we can use the relationships of Section 4-1 between velocity and acceleration. From Eq. 4-1 we can obtain $\vec{\mathbf{a}} = (\vec{\mathbf{v}}_f - \vec{\mathbf{v}}_i)/\Delta t$, where Δt is the time interval for the body to move through the displacement $\vec{\mathbf{s}}$. Combining Eqs. 4-1 and 4-2, we obtain $\vec{\mathbf{s}} = \frac{1}{2}(\vec{\mathbf{v}}_i + \vec{\mathbf{v}}_f)\Delta t$, which can also be obtained from Eq. 2-7 and the three-dimensional generalization of Eq. 2-27: $\vec{\mathbf{v}}_{av} = \vec{\mathbf{s}}/\Delta t = \frac{1}{2}(\vec{\mathbf{v}}_i + \vec{\mathbf{v}}_f)$. We therefore have

$$W_{net} = \vec{\mathbf{F}}_{net}\cdot\vec{\mathbf{s}} = m\vec{\mathbf{a}}\cdot\vec{\mathbf{s}} = m\,\frac{(\vec{\mathbf{v}}_f - \vec{\mathbf{v}}_i)}{\Delta t}\cdot\frac{(\vec{\mathbf{v}}_i + \vec{\mathbf{v}}_f)\Delta t}{2}. \tag{11-21}$$

Multiplying the dot products, we have $(\vec{\mathbf{v}}_f - \vec{\mathbf{v}}_i)\cdot(\vec{\mathbf{v}}_i + \vec{\mathbf{v}}_f) = \vec{\mathbf{v}}_f\cdot\vec{\mathbf{v}}_i + \vec{\mathbf{v}}_f\cdot\vec{\mathbf{v}}_f - \vec{\mathbf{v}}_i\cdot\vec{\mathbf{v}}_i - \vec{\mathbf{v}}_i\cdot\vec{\mathbf{v}}_f$. One of the properties of the dot product of any two vectors is that the order of the vectors does not matter; that is, $\vec{\mathbf{A}}\cdot\vec{\mathbf{B}} = \vec{\mathbf{B}}\cdot\vec{\mathbf{A}}$. (This is *not* true for the *cross* product.) Thus the first and fourth terms in the sum cancel. Furthermore, the dot product of any vector with itself is simply the square of the magnitude of the vector, as Eq. 11-3 shows, and so $\vec{\mathbf{v}}_f\cdot\vec{\mathbf{v}}_f = v_f^2$ and $\vec{\mathbf{v}}_i\cdot\vec{\mathbf{v}}_i = v_i^2$. Making these substitutions in Eq. 11-21, we obtain

$$W_{net} = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2. \tag{11-22}$$

We define the quantity $\frac{1}{2}mv^2$ as the *kinetic energy K* of a body of mass m moving with speed v:

$$K = \tfrac{1}{2}mv^2. \tag{11-23}$$

Kinetic energy has the same dimensions as work, and we measure it in the same units as work (joules, ergs, foot-pounds, electron-volts). Like work, kinetic energy is a scalar quantity. In fact, we can represent it as a dot product between two vectors: $K = \frac{1}{2}m\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}$, just as we have represented the scalars work and power as dot products (see Eqs.

11-4 and 11-10). Using Eq. 11-5, we can also write the dot product in terms of the components of the vectors, so $K = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}mv_z^2$. However, the individual terms on the right-hand side do *not* represent components of kinetic energy. Because kinetic energy is a scalar quantity, *there is no direction associated with kinetic energy and it has no components.* Note also that, unlike work, *kinetic energy can never be negative.*

In terms of the initial and final kinetic energies $K_i = \frac{1}{2}mv_i^2$ and $K_f = \frac{1}{2}mv_f^2$, we can rewrite Eq. 11-23 as

$$W_{net} = \Delta K = K_f - K_i. \qquad (11\text{-}24)$$

Equation 11-24 is the mathematical representation of an important result called the *work–energy theorem:*

> *The net work done by the forces acting on a body is equal to the change in the kinetic energy of the body.*

Although we have derived the work–energy theorem for constant forces, it holds in general for nonconstant forces as well. Later in this section we give a more general proof of this theorem. Like Newton's second law, which we used in its derivation, the work–energy theorem applies only to particles or to bodies that behave like particles. This restriction is discussed in greater detail at the end of this section.

The work–energy theorem is similar in form to the impulse–momentum theorem (Eq. 6-5), $\vec{\mathbf{J}}_{net} = \Delta\vec{\mathbf{p}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i$, even though one deals with scalar quantities (W and K) and the other with vectors ($\vec{\mathbf{J}}$ and $\vec{\mathbf{p}}$). Each is based on Newton's second law, and each is just another way of stating that a property of the body related to its velocity (kinetic energy or momentum) changes as a result of a net force acting on the body. Each also leads to an important conservation law: the momentum of a system of particles remains constant if the net impulse is zero, and the kinetic energy of a system of particles remains constant if the net work is zero.

Kinetic energy is but one of many forms of energy that can be associated with a body. Usually a form of energy is associated with a state or condition of a body: its state of motion, its location (for example, its height in the Earth's gravity), its temperature, the electrical current flowing through it, and so forth. Later in the text we will discuss these and other forms of energy, along with a law for conservation of energy that is more general than Eq. 11-24.

Energy can be transferred from one body to another or transformed from one form to another. One way of transferring or transforming energy is by doing work. When we do work on a body, we may increase its kinetic energy. Where does this energy come from? If we push on it with our hand, it comes from the internal store of energy in our body; if we use a motor, the energy comes from electrical energy, which in turn comes from fuel at the power plant. So we have an alternative definition of work:

> *Work is a way of transferring energy to or from a body due to a force that acts on it.*

There is one other way of transferring energy between objects, which arises from a temperature difference between them. This energy transfer is called *heat* and is discussed in Chapter 13.

When the magnitude of the velocity of a body is constant, there is no change in kinetic energy, and therefore the resultant force does no work. In uniform circular motion, for example, the resultant force acts toward the center of the circle and is always at right angles to the direction of motion. Such a force does no work on the body: it changes the direction of the velocity of the body but not its magnitude. Only when the resultant force has a component in the direction of motion does it do work on the particle and change its kinetic energy.

The work–energy theorem does *not* represent a new, independent law of classical mechanics. We have simply *defined* work (Eq. 11-2, for instance) and kinetic energy (Eq. 11-23) and *derived* the relation between them from Newton's second law. The work–energy theorem is useful, however, for solving problems in which the net work done on a body by external forces is easily computed and in which we are interested in finding the body's speed at certain positions. Of even more significance is the work–energy theorem as a starting point for a broad generalization of the concept of energy and how energy can be stored or shared among the parts of a complex system. The principle of conservation of energy is the subject of the next two chapters.

General Proof of the Work–Energy Theorem

The following calculation gives a proof of Eq. 11-24 in the case of nonconstant forces in one dimension, which we take to be the x direction. We let $F_{net,x}$ represent the net force acting on the body. The net work done by all the external forces that act on the body is $W_{net} = \int F_{net,x}\,dx$. Because the velocity changes with location and the location changes with time, we can use the chain rule of calculus to write $dv_x/dt = (dv_x/dx)(dx/dt)$. The net force can then be written as

$$F_{net,x} = ma_x = m\frac{dv_x}{dt} = m\frac{dv_x}{dx}\frac{dx}{dt}$$

$$= m\frac{dv_x}{dx}v_x = mv_x\frac{dv_x}{dx}.$$

Thus

$$W_{net} = \int F_{net,x}\,dx = \int mv_x\frac{dv_x}{dx}\,dx = \int mv_x\,dv_x.$$

The variable of integration is now the velocity v_x. Let us integrate from initial velocity v_{ix} to final velocity v_{fx}:

$$W_{net} = \int_{v_{ix}}^{v_{fx}} mv_x\,dv_x = m\int_{v_{ix}}^{v_{fx}} v_x\,dv_x = \tfrac{1}{2}m(v_{fx}^2 - v_{ix}^2)$$

$$= \tfrac{1}{2}mv_{fx}^2 - \tfrac{1}{2}mv_{ix}^2.$$

This is identical with Eq. 11-24 when the motion is only in the x direction and shows that the work–energy theorem holds even for nonconstant forces. The same result, $W_{net} = \Delta K$, follows in a straightforward way for nonconstant forces in two or three dimensions.

SAMPLE PROBLEM 11-6. One method of determining the kinetic energy of neutrons in a beam, such as from a nuclear reactor, is to measure how long it takes a particle in the beam to pass two fixed points a known distance apart. This technique is known as the *time-of-flight* method. Suppose a neutron travels a distance of $d = 6.2$ m in a time of $t = 160\ \mu$s. What is its kinetic energy? The mass of a neutron is 1.67×10^{-27} kg.

Solution We find the speed from

$$v = \frac{d}{t} = \frac{6.2\ \text{m}}{160 \times 10^{-6}\ \text{s}} = 3.88 \times 10^4\ \text{m/s}.$$

From Eq. 11-23, the kinetic energy is

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(1.67 \times 10^{-27}\ \text{kg})(3.88 \times 10^4\ \text{m/s})^2$$
$$= 1.26 \times 10^{-18}\ \text{J} = 7.9\ \text{eV}.$$

In nuclear reactors, neutrons are produced in nuclear fission with typical kinetic energies of a few MeV. Negative work has been done on the neutrons in this example by an external agent (called a moderator), thereby reducing their kinetic energies by a considerable factor from a few MeV to a few eV.

SAMPLE PROBLEM 11-7. A body of mass $m = 4.5$ g is dropped from rest at a height $h = 10.5$ m above the Earth's surface. Neglecting air resistance, what will its speed be just before it strikes the ground?

Solution We assume that the body can be treated as a particle. We could solve this problem using a method based on Newton's laws, such as we considered in Chapter 3. We choose instead to solve it here using the work–energy theorem. The gain in kinetic energy is equal to the work done by the resultant force, which here is the force of gravity. This force is constant and directed along the line of motion, so that the work done by gravity is

$$W = \vec{\mathbf{F}} \cdot \vec{\mathbf{s}} = mgh.$$

Initially, the body has a speed $v_0 = 0$ and finally a speed v. The gain in kinetic energy of the body is

$$\Delta K = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = \tfrac{1}{2}mv^2 - 0.$$

According to the work–energy theorem, $W = \Delta K$ and so

$$mgh = \tfrac{1}{2}mv^2.$$

The speed of the body is then

$$v = \sqrt{2gh} = \sqrt{2(9.80\ \text{m/s}^2)(10.5\ \text{m})} = 14.3\ \text{m/s}.$$

Note that this result is independent of the mass of the object, as we have previously deduced using Newton's laws.

SAMPLE PROBLEM 11-8. A block of mass $m = 3.63$ kg slides on a horizontal frictionless table with a speed of $v = 1.22$ m/s. It is brought to rest in compressing a spring in its path. By how much is the spring compressed if its force constant k is 135 N/m?

Solution The change in kinetic energy of the block is

$$\Delta K = K_f - K_i = 0 - \tfrac{1}{2}mv^2.$$

The work W done by the spring on the block when the spring is compressed from its relaxed length through a distance d is, according to Eq. 11-17,

$$W = -\tfrac{1}{2}kd^2.$$

Using the work–energy theorem, $W = \Delta K$, we obtain

$$-\tfrac{1}{2}kd^2 = -\tfrac{1}{2}mv^2$$

or

$$d = v\sqrt{\frac{m}{k}} = (1.22\ \text{m/s})\sqrt{\frac{3.63\ \text{kg}}{135\ \text{N/m}}} = 0.200\ \text{m} = 20.0\ \text{cm}.$$

The Work–Energy Theorem and Reference Frames

Newton's laws are valid only in inertial frames of reference. (In fact, Newton's first law helps us to test whether a frame is inertial or not.) If we find Newton's second law to hold in one frame of reference, then it holds in *all* inertial frames. If two observers in different inertial frames move at constant velocity $\vec{\mathbf{v}}$ relative to one another and observe the same experiment, they measure identical values for the forces, masses, and accelerations, and so they agree completely in their analysis using Newton's second law.

Because we derived the work–energy theorem from Newton's second law, we might suspect that, as in the case of Newton's second law, observers in different inertial frames will agree on the results of applying the work–energy theorem. However, unlike forces and accelerations, displacements and velocities measured by observers in different inertial frames will in general be different, and so they will deduce different values for the work and kinetic energies in the experiment.

Even though the two observers obtain different numerical values for the work and kinetic energy in their respective reference frames, they both agree that $W = \Delta K$. The work–energy theorem is an example of an *invariant* law of physics. An invariant law is one that has the same form in all inertial reference frames. The measured values of the physical quantities, such as W and K, might be different in the two reference frames, but the laws involving those quantities have the same form for both observers (and for observers in all other inertial frames).

SAMPLE PROBLEM 11-9. A worker is exerting a force $F = 5.63$ N in pushing a crate of mass 12.0 kg that moves without friction on a flatbed railroad car (Fig. 11-18*a*). The train is moving at a constant speed of 15.0 m/s in the same direction that the worker is pushing the crate. According to observer O, who is also riding on the flatbed car, the crate starts from rest and is pushed by the worker for a distance of $s = 2.4$ m. (*a*) Find the final speed of the crate according to observer O. (*b*) Find the work W' and the change in kinetic energy $\Delta K'$ according to observer O' who is at

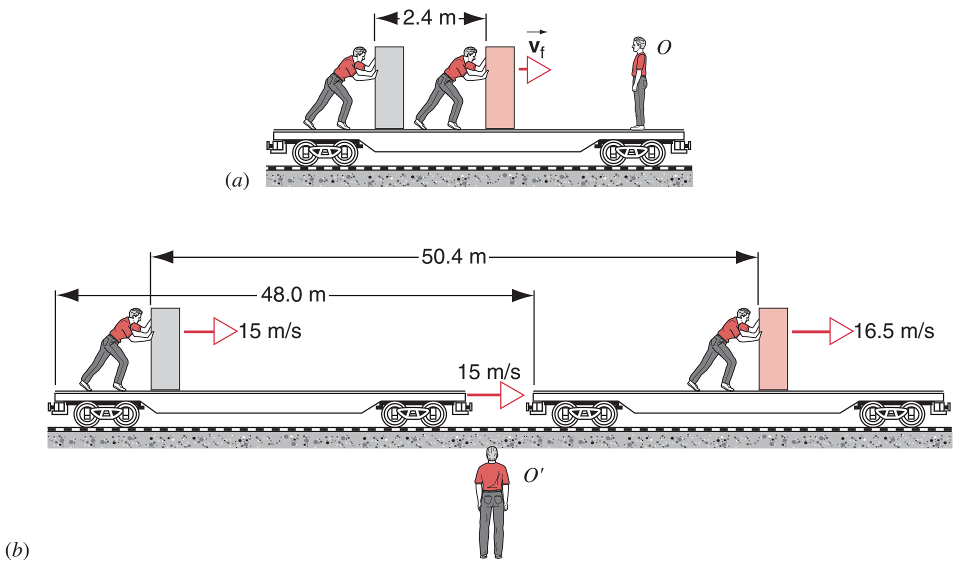


FIGURE 11-18. Sample Problem 11-9. A worker on a flatbed railroad car pushing a crate forward, as viewed by (*a*) an observer O on the train and (*b*) an observer O' on the ground.

rest on the ground, and show that the work–energy theorem is valid for this observer.

Solution (*a*) All displacements, velocities, and forces are to the right in Fig. 11-18, which we take to be the positive x direction. According to O, the work done is $W = Fs = (5.63\ \text{N})(2.4\ \text{m}) = 13.5\ \text{J}$. The work–energy theorem then gives $K_f - K_i = W = 13.5\ \text{J}$. Since $K_i = 0$ according to the observer on the car, $K_f = 13.5\ \text{J}$, and so

$$v_f = \sqrt{\frac{2K_f}{m}} = \sqrt{\frac{2(13.5\ \text{J})}{12.0\ \text{kg}}} = 1.50\ \text{m/s}.$$

(*b*) The situation according to O' is shown in Fig. 11-18*b*. We first calculate how far the railroad car moves in the time it takes the worker to push the crate forward. From the impulse–momentum theorem ($J_x = \Delta p_x$, written in terms of x components; see Eq. 6-5), as applied by O, we have

$$\Delta t = \frac{\Delta p_x}{F_x} = \frac{mv_x}{F_x} = \frac{(12.0\ \text{kg})(1.50\ \text{m/s})}{5.63\ \text{N}} = 3.20\ \text{s}.$$

Both observers agree on the measurement of this time interval. In 3.20 s, the railroad car moves forward a distance of $(15.0\ \text{m/s})(3.20\ \text{s}) = 48.0\ \text{m}$, so according to O' the crate moves a total distance of $s' = 48.0\ \text{m} + 2.4\ \text{m} = 50.4\ \text{m}$. Both observers agree on the value of the force exerted by the worker, so according to O' the work is

$$W' = F's' = (5.63\ \text{N})(50.4\ \text{m}) = 284\ \text{J}.$$

According to O', the initial speed of the crate is $v_i' = 15.0\ \text{m/s}$ (the speed of the railroad car) and its final speed is $v_f' = 15.0\ \text{m/s} + 1.5\ \text{m/s} = 16.5\ \text{m/s}$, so the change in kinetic energy according to O' is

$$\begin{aligned}\Delta K' &= K_f' - K_i' = \tfrac{1}{2}mv_f'^2 - \tfrac{1}{2}mv_i'^2 \\ &= \tfrac{1}{2}(12.0\ \text{kg})(16.5\ \text{m/s})^2 - \tfrac{1}{2}(12.0\ \text{kg})(15.0\ \text{m/s})^2 \\ &= 284\ \text{J}.\end{aligned}$$

Thus $W' = \Delta K'$, according to observer O'. Note that O and O' measure different values for the work and the change in kinetic energy, but both agree that the work equals the change in kinetic energy. For these two inertial observers, the work–energy theorem has the same form.

Limitation of the Work–Energy Theorem

We derived the work–energy theorem, Eq. 11-24, directly from Newton's second law, which, in the form in which we have stated it, applies *only to particles.* Hence the work–energy theorem, as we have presented it so far, likewise applies only to bodies that can be regarded as particles. Previously, we considered an object to behave like a particle if all parts of the object move in exactly the same way. In the use of the work–energy theorem, we can treat an extended object as a particle if the only kind of energy it has is kinetic energy.

Consider, for example, a test car that is crashed head-on into a heavy, rigid concrete barrier. The kinetic energy of the car certainly decreases as the car hits the barrier, crumples up, and comes to rest. However, there are forms of energy other than kinetic energy that enter into this situation. There is internal energy associated with the bending and crumpling of the body of the car; some of this internal energy may appear, for instance, as an increase in the temperature of the car, and some may be transferred to the surroundings as heat. Note that, even though the barrier may exert a large force on the car during the crash, the force does no work because *the point of application of the force on the car does not move.* (Recall our original definition of work—given by Eq. 11-1 and illustrated in Fig. 11-1—the force must act through some distance to do work.) Thus in this case $\Delta K \neq 0$, but $W = 0$; clearly, Eq. 11-24 does not hold. The car does *not* behave like a particle: every part of it does *not* move in exactly the same way.

For similar reasons, from the work–energy standpoint, we cannot treat a sliding block acted on by a frictional force as a particle (even though we *can* continue to treat it as a particle, as we did in Chapter 5, when analyzing its behavior using Newton's laws). The frictional force, which we represented as a constant force $\vec{\mathbf{f}}$, is in reality quite complicated, involving the making and breaking of many microscopic welds (see Section 5-3), which deform the surfaces and result in changes in internal energy of the surfaces (which may in part be revealed as an increase in the temperature of the surfaces). Because of the difficulty of accounting for these other forms of energy, and because the objects do not behave as particles, it is generally not correct to apply the particle form of the work–energy theorem to objects subject to frictional forces.

In these examples, we must view the crashing car and the sliding block not as particles but as systems containing large numbers of particles. Although it would be correct to apply the work–energy theorem to each individual particle in the system, it would be hopelessly complicated to do so. In Chapter 13 we begin to develop a simpler method for dealing with complex systems of particles, and we show how to extend the work–energy theorem so that we may apply it in such cases.

11-7 WORK AND KINETIC ENERGY IN ROTATIONAL MOTION

So far in this chapter we have considered only translational motion. In this section we extend our discussion of work and kinetic energy to rotating bodies.

We begin by calculating the work done on a rigid body that rotates about a fixed axis, just as we started out in this chapter by considering the work done on a body that moves in one dimension. Figure 11-19 shows an arbitrary rigid body to which an external agent applies a force $\vec{\mathbf{F}}$ at point P, a distance r from the rotational axis. As the body rotates through a small angle $d\theta$ about the axis, point P moves through a distance $ds = r\,d\theta$. The component of the force in the direction of motion of P is $F \sin \phi$, and so the work dW done by the force is

$$dW = (F \sin \phi)\,ds = (F \sin \phi)(r\,d\theta) = (rF \sin \phi)\,d\theta.$$

Noting that $rF \sin \phi$ is also the component of the torque of the force $\vec{\mathbf{F}}$ about the z axis, we have $dW = \tau_z\,d\theta$, and for a rotation from angle θ_i to angle θ_f the work is

$$W = \int_{\theta_i}^{\theta_f} \tau_z\,d\theta. \qquad (11\text{-}25)$$

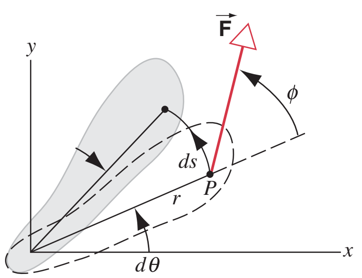


FIGURE 11-19. A rigid body rotates counterclockwise about an axis perpendicular to the page (the z axis). An external force $\vec{\mathbf{F}}$ (in the plane of the page) is applied to point P of the body, a distance r from the axis of rotation.

Note that Eq. 11-25 is the rotational analogue of Eq. 11-14, with the force replaced by the torque and the linear coordinate replaced by the angular coordinate.

If the torque is constant as the body rotates through an angle $\theta = \theta_f - \theta_i$, the work done on the body by this torque is

$$W = \tau_z \theta, \qquad (11\text{-}26)$$

which is analogous to Eq. 11-1 for the constant force.

The instantaneous power expended in rotational motion can be obtained from Eq. 11-8:

$$P = \frac{dW}{dt} = \frac{\tau_z\,d\theta}{dt} = \tau_z \omega_z \qquad (11\text{-}27)$$

where $\omega_z = d\theta/dt$ is the rotational velocity about the z axis. This is the rotational analogue of Eq. 11-11. Note that the directions of $\vec{\boldsymbol{\tau}}$ and $\vec{\boldsymbol{\omega}}$ are parallel in the geometry of Fig. 11-19 (both out of the page, along the z axis). The average power for rotational motion in which a total amount of work W is done in a time t is given by Eq. 11-7, $P_{av} = W/t$.

In Eqs. 11-25, 11-26, and 11-27, as in all equations that mix angular and nonangular quantities, *the angular quantities must be expressed in radian measure.*

Rotational Kinetic Energy

Figure 11-20 shows a rigid body rotating about a fixed axis with angular speed ω. We can consider the body as a collection of N particles $m_1, m_2, \ldots$ moving with tangential speeds $v_1, v_2, \ldots$. If r_n indicates the distance of particle m_n from the axis of rotation, then $v_n = r_n\omega$ and its kinetic energy is $\frac{1}{2}m_n v_n^2 = \frac{1}{2}m_n r_n^2 \omega^2$.

The total kinetic energy of the entire rotating body is the sum of the kinetic energies of all of the N particles:

$$K = \tfrac{1}{2}m_1 r_1^2 \omega^2 + \tfrac{1}{2}m_2 r_2^2 \omega^2 + \cdots = \tfrac{1}{2}\left(\sum m_n r_n^2\right)\omega^2 \qquad (11\text{-}28)$$

or, in terms of the rotational inertia $I = \Sigma m_n r_n^2$,

$$K = \tfrac{1}{2}I\omega^2. \qquad (11\text{-}29)$$

This expression, which gives the kinetic energy of a rigid body of rotational inertia I rotating with angular speed ω, is exactly analogous to Eq. 11-23 for the translational kinetic

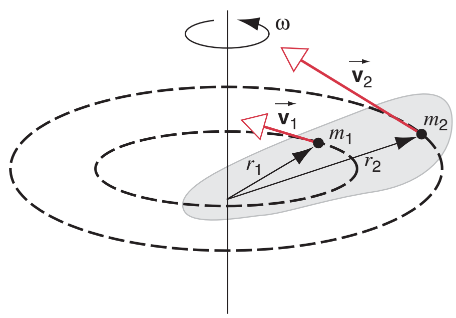


FIGURE 11-20. A rigid body rotates about a fixed axis. Every particle has the same angular speed ω, but the tangential speeds vary with the distance r from the axis of rotation.

TABLE 11-1 Comparison of Energy-Related Translational and Rotational Quantities

Translational Quantity		*Equation Number*	*Rotational Quantity*		*Equation Number*
Work*	$W = \int F_x \, dx$	11-14	Work	$W = \int \tau_z \, d\theta$	11-25
Power*	$P = F_x v_x$	11-11	Power	$P = \tau_z \omega_z$	11-27
Kinetic energy	$K = \frac{1}{2}mv^2$	11-23	Rotational kinetic energy	$K = \frac{1}{2}I\omega^2$	11-29
Work–energy theorem	$W = \Delta K$	11-24	Work–energy theorem	$W = \Delta K$	11-24

* To emphasize the symmetry between the translational and rotational quantities, these equations are written in one-dimensional form.

energy, $K = \frac{1}{2}mv^2$. The mass in Eq. 11-23 is replaced by the rotational inertia and the linear speed by the angular speed.

The rotational kinetic energy given by Eq. 11-29 is not a new kind of kinetic energy. It is simply the sum of the ordinary translational kinetic energies of all the particles of the body. Even though the entire body may not be in translational motion, each particle has a tangential velocity, and thus each particle has a kinetic energy. The instantaneous direction of each particle's velocity changes as the body rotates, but kinetic energy depends on v^2 and is a scalar, so there is no direction associated with it. It is therefore quite proper to add the kinetic energies of the particles of the rotating body. The rotational kinetic energy $\frac{1}{2}I\omega^2$ is merely a convenient way of expressing the total kinetic energy of all the particles in the rigid body.

The rotational form of the work–energy theorem is exactly the same as the translational form: $W = \Delta K$, with the rotational work given by Eq. 11-25 or 11-26 and the rotational kinetic energy by Eq. 11-29. In general, the work done on a body could be accompanied by both rotational and translational motion. In this case W represents the total work done on the body, and ΔK must include the sum of the translational and rotational terms. We consider the kinetic energy in combined translational and rotational motion in Chapter 12.

In Table 10-1 we presented a comparison between translational and rotational quantities for kinematics and dynamics. Table 11-1 shows an additional comparison of the energy-related quantities for translational and rotational motion.

SAMPLE PROBLEM 11-10. A space probe coasting in a region of negligible gravity is rotating with an angular speed of 2.4 rev/s about an axis that points in its direction of motion (Fig. 11-21). The spacecraft is in the form of a thin spherical shell of radius 1.7 m and mass 245 kg. It is necessary to reduce the rotational speed to 1.8 rev/s by firing tangential thrusters along the "equator" of the probe. What constant force must the thrusters exert if the change in angular speed is to be accomplished as the probe rotates through 3.0 revolutions? Assume the fuel ejected by the thrusters is a negligible fraction of the mass of the probe.

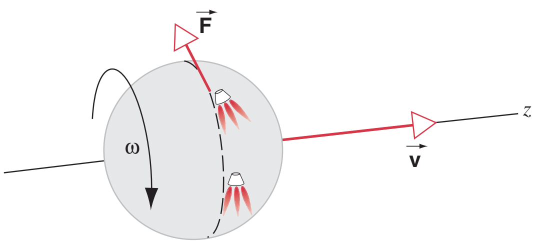


FIGURE 11-21. Sample Problem 11-10.

Solution For a thin spherical shell we find the rotational inertia about a central axis from Fig. 9-15:

$$I = \tfrac{2}{3}MR^2 = \tfrac{2}{3}(245\ \text{kg})(1.7\ \text{m})^2 = 472\ \text{kg}\cdot\text{m}^2.$$

The change in rotational kinetic energy is

$$\begin{aligned}\Delta K &= \tfrac{1}{2}I\omega_f^2 - \tfrac{1}{2}I\omega_i^2 \\ &= \tfrac{1}{2}(472\ \text{kg}\cdot\text{m}^2)[(2\pi\ \text{rad/rev})(1.7\ \text{rev/s})]^2 \\ &\quad -\tfrac{1}{2}(472\ \text{kg}\cdot\text{m}^2)[(2\pi\ \text{rad/rev})(2.4\ \text{rev/s})]^2 \\ &= -2.67 \times 10^4\ \text{J}.\end{aligned}$$

According to Eq. 11-26, the rotational work for a constant torque is $W = \tau_z\theta$, where $\tau_z = -RF$ if the thruster force F is applied tangentially. The minus sign indicates that the torque points in the negative z direction. Using the work–energy theorem $W = \Delta K$ with $W = -RF\theta$, we solve for the thruster force F:

$$\begin{aligned}F &= \frac{W}{-R\theta} = \frac{\Delta K}{-R\theta} = \frac{-2.67 \times 10^4\ \text{J}}{-(1.7\ \text{m})[(2\pi\ \text{rad/rev})(3.0\ \text{rev})]} \\ &= 833\ \text{N}.\end{aligned}$$

This problem could also be solved by using the formulas for rotational kinematics to find the (constant) angular acceleration, followed by $\tau_z = I\alpha_z$ to find the force.

11-8 KINETIC ENERGY IN COLLISIONS

In Chapter 6 we analyzed collisions between two bodies by applying the law of conservation of linear momentum. It is also instructive to consider the kinetic energies of the colliding bodies.

We consider a collision between two bodies that move along the x axis. Line 1 of Fig. 11-22*a* shows the velocities before the collision in the laboratory frame of reference, and Fig. 11-22*b* shows the same collision as viewed from the center-of-mass frame of reference.

We first discuss an *elastic* collision, which we defined in Section 6-5 as a collision in which, in the center-of-mass reference frame, the momenta of the colliding bodies are

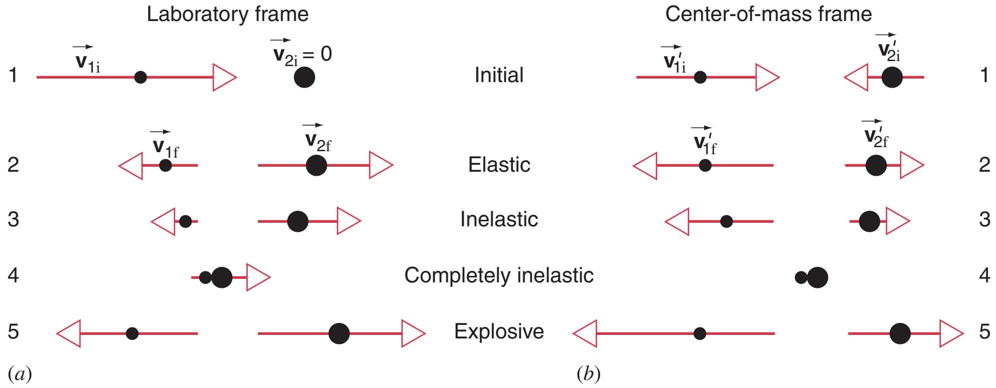


FIGURE 11-22. A one-dimensional collision between two objects as viewed from (*a*) the laboratory frame and (*b*) the center-of-mass frame. In the laboratory frame, m_2 is initially at rest.

simply reversed in direction. If the momenta are reversed, the directions of the velocities of the colliding bodies must also be reversed (see Fig. 11-22*b*, lines 1 and 2). Because the velocities for each body before and after the collision are equal in magnitude ($v'_{1i} = v'_{1f}$ and $v'_{2i} = v'_{2f}$), it is clear that in the center-of-mass frame we must have $K'_{1i} = K'_{1f}$ for m_1 and $K'_{2i} = K'_{2f}$ for m_2. The *total* initial kinetic energy, $K'_i = K'_{1i} + K'_{2i}$, is thus equal to the total final kinetic energy, $K'_f = K'_{1f} + K'_{2f}$, in this reference frame.

In the laboratory frame of reference (which we describe using unprimed coordinates), it is *not* true that the individual kinetic energies are unchanged in the collision; that is, in general $v_{1i} \neq v_{1f}$ and so $K_{1i} \neq K_{1f}$, and similarly for m_2. What about the *total* kinetic energy of m_1 and m_2 in this frame? Before the collision, the *total* initial kinetic energy is $K_i = \frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2$, and after the collision, the *total* final kinetic energy is $K_f = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$. If we use Eqs. 6-24 and 6-25 for the final velocities in the expression for K_f, after doing the necessary algebra we find

$$K_i = K_f \qquad \text{(elastic).} \qquad (11\text{-}30)$$

The individual kinetic energies of the colliding bodies may change; that is, in general $K_{1i} \neq K_{1f}$ and $K_{2i} \neq K_{2f}$, but their sum does remain constant ($K_{1i} + K_{2i} = K_{1f} + K_{2f}$).

We therefore have an alternative definition of an elastic collision:

In an elastic collision, the total kinetic energy of the two bodies remains constant; that is, the total kinetic energy before the collision equals the total kinetic energy after the collision.

We see that in at least two frames of reference (the center-of-mass frame and the laboratory frame) the total initial and final kinetic energies of the two-body system are equal. In fact, because the laboratory frame is an arbitrarily chosen frame, *the total kinetic energy remains constant in all inertial frames of reference.* We can understand this result by imagining that there is a spring at its relaxed length between the two bodies. As the bodies collide, they compress the spring, and some of their kinetic energy is lost due to the work done by the spring. When the spring expands again, it does an equal amount of work on the bodies, which *increases* their kinetic energy. If the spring returns to its relaxed length, there is no net work done on the system consisting of the two bodies, and so the total *final* kinetic energy of the system must equal the total *initial* kinetic energy.

Of course, there are no springs in collisions between real bodies—*it is the colliding objects themselves that behave elastically,* just like springs. The interatomic forces of the objects can be regarded as elastic; the objects do work on one another in changing each other's kinetic energy, but the net work done by the entire system of the two objects is zero, so the change in kinetic energy of the system is zero.

On the other hand, imagine a spring between the two bodies in an *inelastic* collision (compare lines 1 and 3 in Fig. 11-22*b*). The spring will be compressed in the collision, but it does not return to its full relaxed length after the collision. (Perhaps there is a ratchet mechanism that keeps the spring somewhat compressed.) The two bodies do work on the spring in compressing it, but the spring does less work on the bodies when it expands again, so the final kinetic energy is less than the initial kinetic energy. All observers, no matter what their inertial frame of reference, will agree that the spring remains somewhat compressed after the collision, so all observers will agree that some kinetic energy has been lost (though the amount of the loss will vary with the reference frame of the observer). So we can characterize an inelastic collision in terms of the kinetic energy:

In an inelastic collision, the total final kinetic energy is less than the total initial kinetic energy.

Even though the total kinetic energy decreases, the total linear momentum remains constant.

All collisions between extended bodies are to some extent inelastic. If you drop a golf ball or a tennis ball on a hard surface, it does not quite bounce to its original height. The height difference between successive bounces is a measure of the loss in kinetic energy in each collision with the Earth.

In collisions between real bodies (without springs), where does this kinetic energy go? It may go into the work done in deforming one of the bodies or changing its shape, as for example in a collision involving a ball of clay. Real objects do not compress like ideal springs—often there are dissipative forces similar to friction. Some of the energy might go into creating a shock wave or raising the temperature of the objects.

If the two bodies stick together, we have a *completely inelastic collision* (compare lines 1 and 4 of Fig. 11-22*b*). This type of collision loses the maximum amount of kinetic energy, consistent with the conservation of momentum.

Finally, imagine a collision in which the spring between the two bodies is compressed before the collision, but is released as the bodies collide. The colliding bodies may further compress the spring, but as the spring expands to its relaxed length it delivers more kinetic energy to the bodies than they started with. The two bodies may do work on the spring in compressing it, but the spring does a *greater* amount of work on the two bodies as it expands. This is an *explosive* or energy-releasing collision.

In an explosive collision, the total final kinetic energy is greater than the total initial kinetic energy.

Once again, linear momentum remains constant even though the kinetic energy increases.

Energy-releasing collisions often occur in nuclear reactions when internal energy stored in the colliding nuclei is converted into kinetic energy. The resulting nuclei after the collision have less internal nuclear energy but greater total kinetic energy than the original nuclei.

SAMPLE PROBLEM 11-11. In a nuclear reactor, neutrons lose energy by making collisions with nuclei of atoms of the materials that may be present in the core of the reactor. If a neutron of mass m_n has initial kinetic energy of 5.0 MeV, how much kinetic energy will it lose if it makes a head-on elastic collision with a nucleus of lead ($m_{Pb} = 206m_n$), carbon ($m_C = 12m_n$), or hydrogen ($m_H = m_n$)?

Solution We may take the struck atoms to be initially at rest (actually, they have small "thermal" speeds that are negligible compared with the speed of the neutron). The final speed of the incident neutron in a head-on elastic collision with a nucleus at rest is given by Eq. 6-24 with $v_{2i} = 0$: $v_{1f} = [(m_1 - m_2)/(m_1 + m_2)]v_{1i}$. The final kinetic energy of the neutron is

$$K_{1f} = \tfrac{1}{2}m_1 v_{1f}^2 = \tfrac{1}{2}m_1\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 v_{1i}^2 = K_{1i}\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2.$$

For a collision with lead, the final neutron kinetic energy is

$$K_{1f} = (5.0\text{ MeV})\left(\frac{m_n - m_{Pb}}{m_n + m_{Pb}}\right)^2$$

$$= (5.0\text{ MeV})\left(\frac{m_n - 206m_n}{m_n + 206m_n}\right)^2 = 4.9\text{ MeV},$$

corresponding to a loss of 5.0 MeV − 4.9 MeV = 0.1 MeV. A similar calculation for carbon gives $K_{1f} = 3.6$ MeV (a loss of 1.4 MeV) and for hydrogen, $K_{1f} = 0$ (a loss of 5.0 MeV, all of its initial energy). A neutron thus loses the most energy in a collision with a hydrogen nucleus, whose mass is closest to the neutron mass.

These results show why a hydrogen-rich material, such as water or paraffin, is far more effective in slowing down or "moderating" neutrons than a heavy material such as lead. Even though we have oversimplified the problem by assuming a one-dimensional "head-on" collision, the same basic conclusion follows if we consider a two-dimensional glancing collision: a neutron will lose more energy in hydrogen-rich materials.

Neutrons released in the fission of uranium in reactors typically have kinetic energies in the MeV range. However, reactor operation requires those neutrons to initiate new fission events, which occurs with high probability only if the neutrons are slowed down to kinetic energies in the eV range. For this reason, the uranium fuel elements must be mixed with the lighter material that will serve as the moderator for the neutrons.

SAMPLE PROBLEM 11-12. A ballistic pendulum (Fig. 11-23) is a device that was used to measure the speeds of bullets before electronic timing devices were available. It consists of a large block of wood of mass M, hanging from two long pairs of cords. A bullet of mass m is fired into the block, and the block + bullet combination swings upward, its center of mass rising a vertical distance h before the pendulum comes momentarily to rest at the end of its arc. Take the mass of the block to be $M = 5.4$ kg and the mass of the bullet to be $m = 9.5$ g. (*a*) What is the initial speed of the bullet if the block rises to a height of $h = 6.3$ cm? (*b*) What fraction of the initial kinetic energy is lost in this collision?

Solution (*a*) Let us divide the problem into two parts: (1) The bullet moving with speed v_i enters the block and comes to rest relative to the block, after which the bullet + block combination moves with a common speed v_f. We assume this happens very quickly. (2) The combination, now moving with speed v_f, swings upward until it comes to rest. Part 1 is an example of a completely inelastic collision, in which the two colliding objects stick together after the collision. Momentum is conserved in the collision, so Eq. 6-20 gives, with $v_{2i} = 0$ (the block is initially at rest),

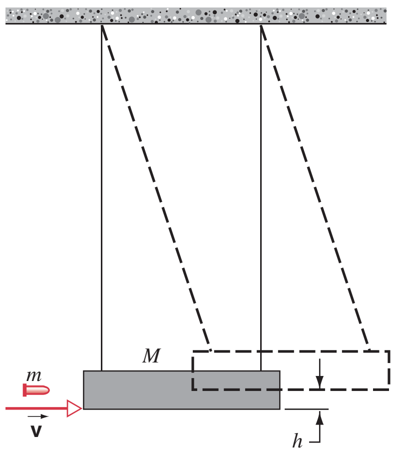


FIGURE 11-23. Sample Problem 11-12. A ballistic pendulum is used to measure the speed of a bullet.

$mv_i = (m + M)v_f$. Part 2 of the problem can be analyzed using the work–energy theorem. The net work on the block + bullet combination is that done by gravity: $W_{net} = W_g = -(m + M)gh$, and as it swings upward and comes to rest the change in the kinetic energy of the combination is $\Delta K = 0 - \frac{1}{2}(m + M)v_f^2$. The work–energy theorem, $W_{net} = \Delta K$, then gives

$$-(m + M)gh = -\tfrac{1}{2}(m + M)v_f^2 = -\tfrac{1}{2}(m + M)\left(\frac{mv_i}{m + M}\right)^2,$$

where the last result follows from substituting for v_f from the momentum conservation result of part 1. Solving for v_i, we find

$$v_i = \left(\frac{M + m}{m}\right)\sqrt{2gh}$$

$$= \left(\frac{5.4\text{ kg} + 0.0095\text{ kg}}{0.0095\text{ kg}}\right)\sqrt{(2)(9.8\text{ m/s}^2)(0.063\text{ m})} = 630\text{ m/s}.$$

We can look at the ballistic pendulum as a kind of transformer, exchanging the high speed of a light object (the bullet) for the low—and thus more easily measureable—speed of a massive object (the block).

(*b*) We can write the final kinetic energy as

$$K_f = \tfrac{1}{2}(m + M)v_f^2 = \tfrac{1}{2}(m + M)\left(\frac{mv_i}{m + M}\right)^2 = \tfrac{1}{2}mv_i^2\left(\frac{m}{m + M}\right).$$

The ratio between the initial and final kinetic energies is

$$\frac{K_f}{K_i} = \frac{m}{m + M} = \frac{9.5\text{ g}}{9.5\text{ g} + 5.4\text{ kg}} = 0.0018.$$

Only 0.18% of the initial kinetic energy remains after the collision. The remaining 99.82% is stored inside the pendulum as internal energy (perhaps in part as a temperature increase) or transferred to the environment—for example, as heat or sound waves.

MULTIPLE CHOICE

11-1 Work and Energy

11-2 Work Done by a Constant Force

1. A student picks a box off the table and puts it on the floor. Let the total work done by the student be W. One can conclude
(A) $W > 0$. (B) $W = 0$. (c) $W < 0$.
(D) nothing about the sign of W.

2. An object of mass 2.0 kg moves in uniform circular motion on a horizontal frictionless table. The radius of the circle is 0.75 m and the centripetal force is 10.0 N.
(*a*) The work done by this force when the object moves through one-half of a complete revolution is
(A) 0 J. (B) 3.75 J.
(C) 10.0 J. (D) 7.5π J.
(*b*) The work done by this force when the object moves through one complete revolution is
(A) 0 J. (B) 7.5 J.
(C) 20.0 J. (D) 15π J.

3. Which of the following quantities are independent of the choice of inertial frame? (There may be more than one correct answer.)
(A) Velocity (B) Acceleration
(C) Force (D) Work

4. Naval guns on battleships are sometimes rated in energy units of *ton-feet*. What (approximately) is this in metric units?
(A) 3×10^1 J (B) 3×10^2 J
(C) 3×10^3 J (D) 3×10^4 J

11-3 Power

5. An engine with constant power output drives an automobile. When the auto approaches a hill the driver shifts from high gear to low gear. The driver does this
(A) to increase the force pushing the car forward.
(B) to increase the power output from the tires.
(C) Both (A) and (B) are correct.
(D) Neither (A) nor (B) is correct.

6. Assume the aerodynamic drag force on a car is proportional to the speed. If the power output from the engine is doubled, then the maximum speed of the car
(A) is unchanged.
(B) increases by a factor of $\sqrt{2}$.
(C) is also doubled.
(D) increases by a factor of four.

7. An engineer wants to design an improved elevator for a building. The original design used a motor that could lift 1000 kg through a distance of 20 meters in 30 seconds. The engineer wants a motor that can lift 800 kg through a distance of 30 meters in 20 seconds. Compared to the old motor, the new motor
(A) should exert a force of the same magnitude, but must provide a larger power output.
(B) should exert a force of a larger magnitude, and must provide a larger power output.
(C) can exert a force of a smaller magnitude, and can provide a smaller power output.
(D) can exert a force of a smaller magnitude, but must provide the same power output.
(E) can exert a force of a smaller magnitude, but must provide a larger power output.

11-4 Work Done by a Variable Force

8. The force exerted by a special compression device is given by $F_x(x) = kx(x - l)$ for $0 \le x \le l$, where l is the maximum possible compression and k is a constant.
(*a*) The force required to compress the device a distance d is a maximum when
(A) $d = 0$. (B) $d = l/4$. (C) $d = l/\sqrt{2}$.
(D) $d = l/2$. (E) $d = l$.
(*b*) The work required to compress the device a distance d is a maximum when
(A) $d = 0$. (B) $d = l/4$. (C) $d = l/\sqrt{2}$.
(D) $d = l/2$. (E) $d = l$.

11-5 Work Done by a Variable Force: Two-Dimensional Case

11-6 Kinetic Energy and the Work–Energy Theorem

9. A particle has a constant kinetic energy K. Which of the following quantities *must* also be constant?
(A) Position (B) Speed
(C) Velocity (D) Momentum

10. A 0.20-kg puck slides across a frictionless floor with a speed of 10 m/s. The puck strikes a soft wall and stops.
(*a*) The magnitude of the impulse on the puck is
(A) $0\ \text{kg}\cdot\text{m/s}$. (B) $1\ \text{kg}\cdot\text{m/s}$.
(C) $2\ \text{kg}\cdot\text{m/s}$. (D) $4\ \text{kg}\cdot\text{m/s}$.
(*b*) The net work done on the puck is
(A) -20 J. (B) -10 J.
(C) 0 J. (D) 20 J.

11. A 0.20-kg puck slides across a frictionless floor with a speed of 10 m/s. The puck strikes a wall and bounces off with a speed of 10 m/s in the opposite direction.
(*a*) The magnitude of the impulse on the puck is
(A) $0\ \text{kg}\cdot\text{m/s}$. (B) $1\ \text{kg}\cdot\text{m/s}$.
(C) $2\ \text{kg}\cdot\text{m/s}$. (D) $4\ \text{kg}\cdot\text{m/s}$.
(*b*) The net work done on the puck is
(A) -20 J. (B) -10 J.
(C) 0 J. (D) 20 J.

12. Two cars are at a stop light. When the light turns green the car of mass m begins to move with acceleration a; the car of mass $2m$ moves in the same direction with acceleration $a/2$. Which car engine delivers the most power?
(A) The car of mass m
(B) The car of mass $2m$
(C) The power is the same for both cars.

11-7 Work and Kinetic Energy in Rotational Motion

13. Four solid objects, each with the same mass and radius, are spinning freely with the same angular speed. Which object requires the most work to *stop* it?
(A) A solid sphere spinning about a diameter
(B) A hollow sphere spinning about a diameter
(C) A solid disk spinning about an axis perpendicular to the plane of the disk and through the center
(D) A hoop spinning about an axis along a diameter
(E) The work required is the same for all four objects.

14. Four solid objects, each with the same mass and radius, are spinning freely with the same angular momentum. Which object requires the most work to *stop* it?
(A) A solid sphere spinning about a diameter
(B) A hollow sphere spinning about a diameter
(C) A solid disk spinning about an axis perpendicular to the plane of the disk and through the center
(D) A hoop spinning about an axis along a diameter
(E) The work required is the same for all four objects.

15. Four solid objects, each with the same mass, are spinning freely with the same angular momentum and the same angular speed. Which object requires the most work to *stop* it?
(A) A solid sphere spinning about a diameter
(B) A hollow sphere spinning about a diameter
(C) A solid disk spinning about an axis perpendicular to the plane of the disk and through the center
(D) A hoop spinning about an axis along a diameter
(E) The work required is the same for all four objects.

11-8 Kinetic Energy in Collisions

16. A considerable amount of the initial kinetic energy is "lost" in the ballistic pendulum (Sample Problem 11-12). Taking this into consideration we can conclude
(A) that the calculated speed of the bullet is probably too low.
(B) that the calculated speed of the bullet is probably too high.
(C) that the calculated speed of the bullet is probably correct only if the collision was elastic.
(D) that the calculated speed of the bullet is probably correct because the collision conserved momentum.

QUESTIONS

1. Can you think of other words like work whose colloquial meanings are often different from their scientific meanings?

2. Explain why you become physically tired when you push against a wall, fail to move it, and therefore do no work on the wall.

3. Suppose that three constant forces act on a particle as it moves from one position to another. Prove that the work done on the particle by the resultant of these three forces is equal to the sum of the work done by each of the three forces calculated separately.

4. The inclined plane (Sample Problem 11-1) is a simple "machine" that enables us to do work with the application of a smaller force than is otherwise necessary. The same statement applies to a wedge, a lever, a screw, a gear wheel, and a pulley combination (Problem 3). Far from saving us work, however, such machines in practice require that we do a little more work with them than without them. Why is this so? Why do we use such machines?

5. In a tug of war, one team is slowly giving way to the other. What work is being done and by whom?

6. Why can you much more easily ride a bicycle for a mile on level ground than run the same distance? In each case, you transport your own weight for a mile and in the first you must also transport the bicycle and, moreover, do so in a shorter time! (See *The Physics Teacher,* March 1981, p. 194.)

7. Suppose that the Earth revolves around the Sun in a perfectly circular orbit. Does the Sun do any work on the Earth?

8. You slowly lift a bowling ball from the floor and put it on a table. Two forces act on the ball: its weight, of magnitude mg, and your upward force, also of magnitude mg. These two forces add to zero so that it would seem that no work is done. On the other hand, you know that you have done some work. What is wrong?

9. Why can a car so easily pass a loaded truck when going uphill? The truck is heavier, of course, but its engine is more powerful in proportion (or is it?). What considerations enter into choosing the design power of a truck engine and of a car engine?

10. Does the power needed to raise a box onto a platform depend on how fast it is raised?

11. You lift some library books from a lower to a higher shelf in time Δt. Does the work that you do depend on (*a*) the mass of the books, (*b*) the weight of the books, (*c*) the height of the upper shelf above the floor, (*d*) the time Δt, and (*e*) whether you lift the books sideways or directly upward?

12. We hear a lot about the "energy crisis." Would it be more accurate to speak of a "power crisis"?

13. You cut a spring in half. What is the relation of the force constant k for the original spring to that for either of the half-springs?

14. Springs A and B are identical except that A is stiffer than B; that is, $k_A > k_B$. On which spring is more work expended if they are stretched (*a*) by the same amount and (*b*) by the same force?

15. In picking up a book from the floor and putting it on a table, you do work. However, the kinetic energy of the book does not change. Is there a violation of the work–energy theorem here? Explain why or why not.

16. Does the work–energy theorem hold if friction acts on an object? Explain your answer.

17. The work done by the net force on a particle is equal to the change in kinetic energy. Can it happen that the work done by one of the component forces alone will be greater than the change in kinetic energy? If so, give examples.

18. The world record for the pole vault is about 5.5 m. Could the record be raised to, say, 8 m by using a pole long enough? If not, why not? How high might an athlete get?

19. A light object and a heavy object have equal kinetic energies of translation. Which one has the larger momentum?

20. Can a body have kinetic energy without having momentum? Can a body have momentum without having kinetic energy?

21. An object of mass m is traveling with an initial speed v. The object is brought to a rest by a variable force that acts over a distance d for a time t. There are two ways to calculate the magnitude of the "average" force,

$$F_{av} = mv/t$$

or

$$F_{av} = mv^2/2d.$$

Are the two methods equivalent? Under what conditions, if any, will they yield the same average? Will one method tend to produce a larger result, and, if so, which method?

22. Comment on these statements: In a car collision, the force the car exerts on being stopped can be determined either from its momentum or its kinetic energy. In one case the time of stopping and in the other the distance of stopping also need to be known.

23. Steel is more elastic than rubber. Explain what this means.

24. Discuss the possibility that, if only we could take into account internal motions of atoms in objects, all collisions are elastic.

25. We have seen that the conservation of momentum may apply whether kinetic energy is conserved or not. What about the reverse; that is, does the conservation of kinetic energy imply the conservation of momentum in classical physics? (See "Connection Between Conservation of Energy and Conservation of Momentum," by Carl G. Adler, *American Journal of Physics,* May 1976, p. 483.)

26. The following statement was taken from an exam paper: "The collision between two helium atoms is perfectly elastic, so that momentum is conserved." What do you think of this statement?

27. Two clay balls of equal mass and speed strike each other head-on, stick together, and come to rest. Kinetic energy is certainly not conserved. What happened to it? How is momentum conserved?

28. Consider a one-dimensional elastic collision between a moving object A and an object B initially at rest. How would you choose the mass of B, in comparison to the mass of A, in order that B should recoil with (*a*) the greatest speed, (*b*) the greatest momentum, and (*c*) the greatest kinetic energy?

29. In commenting on the fact that kinetic energy is not conserved in a totally inelastic collision, a student observed that kinetic energy is not conserved in an explosion and that a totally inelastic collision is merely the reverse of an explosion. Is this a useful or valid observation?

30. Does kinetic energy depend on the direction of the motion involved? Can it be negative? Does its value depend on the reference frame of the observer?

31. Does the work done by the net force acting on a particle depend on the (inertial) reference frame of the observer? Does the change in kinetic energy so depend? If so, give examples.

32. A man rowing a boat upstream is at rest with respect to the shore. (*a*) Is he doing any work? (*b*) If he stops rowing and moves down with the stream, is any work being done on him?

EXERCISES

11-1 Work and Energy

11-2 Work Done by a Constant Force

1. To push a 52-kg crate across a floor, a worker applies a force of 190 N, directed 22° below the horizontal. As the crate moves 3.3 m, how much work is done on the crate by (*a*) the worker, (*b*) the force of gravity, and (*c*) the normal force of the floor on the crate?

2. A 106-kg object is initially moving in a straight line with a speed of 51.3 m/s. (*a*) If it is brought to a stop with a deceleration of 1.97 m/s^2, what force is required, what distance does the object travel, and how much work is done by the force? (*b*) Answer the same questions if the object's deceleration is 4.82 m/s^2.

3. To push a 25-kg crate up a 27° incline, a worker exerts a force of 120 N, parallel to the incline. As the crate slides 3.6 m, how much work is done on the crate by (*a*) the worker, (*b*) the force of gravity, and (*c*) the normal force of the incline?

4. A worker pushed a 58.7-lb block (m = 26.6 kg) a distance of 31.3 ft (= 9.54 m) along a level floor at constant speed with a force directed 32.0° below the horizontal. The coefficient of kinetic friction is 0.21. How much work did the worker do on the block?

5. A 52.3-kg trunk is pushed 5.95 m at constant speed up a 28.0° incline by a constant horizontal force. The coefficient of kinetic friction between the trunk and the incline is 0.19. Calculate the work done by (*a*) the applied force and (*b*) the force of gravity.

6. A 47.2-kg block of ice slides down an incline 1.62 m long and 0.902 m high. A worker pushes up on the ice parallel to the incline so that it slides down at constant speed. The coefficient of kinetic friction between the ice and the incline is 0.110. Find (*a*) the force exerted by the worker, (*b*) the work done by the worker on the block of ice, and (*c*) the work done by gravity on the ice.

7. Use Eqs. 11-3 and 11-5 to calculate the angle between the two vectors $\vec{a} = 3\hat{i} + 3\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + 3\hat{k}$.

8. A vector $\vec{a}$ of magnitude 12 units and another vector $\vec{b}$ of magnitude 5.8 units point in directions differing by 55°. Find the scalar product of the two vectors.

9. Two vectors, $\vec{r}$ and $\vec{s}$, lie in the *xy* plane. Their magnitudes are 4.5 and 7.3 units, respectively, whereas their directions are 320° and 85° measured counterclockwise from the positive *x* axis. What is the value of $\vec{r} \cdot \vec{s}$?

10. (*a*) Calculate $\vec{r} = \vec{a} - \vec{b} + \vec{c}$, where $\vec{a} = 5\hat{i} + 4\hat{j} - 6\hat{k}$, $\vec{b} = -2\hat{i} + 2\hat{j} + 3\hat{k}$, and $\vec{c} = 4\hat{i} + 3\hat{j} + 2\hat{k}$. (*b*) Calculate the angle between $\vec{r}$ and the $+z$ axis. (*c*) Find the angle between $\vec{a}$ and $\vec{b}$.

11-3 Power

11. A 57-kg woman runs up a flight of stairs having a rise of 4.5 m in 3.5 s. What average power must she supply?

12. In a 100-person ski lift, a machine raises passengers averaging 667 N in weight a height of 152 m in 55.0 s, at constant speed. Find the power output of the motor, assuming no frictional losses.

13. A swimmer moves through the water at a speed of 0.22 m/s. The drag force opposing this motion is 110 N. How much power is developed by the swimmer?

14. The hydrogen-filled airship *Hindenburg* (see Fig. 11-24) could cruise at 77 knots with the engines providing 4800 hp. Calculate the air drag force in newtons on the airship at this speed.

FIGURE 11-24. Exercise 14.

15. How much power, in horsepower, must be developed by the engine of a 1600-kg car moving at 26 m/s (= 94 km/h) on a level road if the forces of resistance total 720 N?

16. The motor on a water pump is rated at 6.6 hp. From how far down a well can water be pumped up at the rate of 220 gal/min?

17. Suppose that your car averages 30 mi/gal of gasoline. (*a*) How far could you travel on 1 kW·h of energy consumed? (*b*) If you are driving at 55 mi/h, at what rate are you expending energy? The heat of combustion of gasoline is 140 MJ/gal.

18. What power is developed by a grinding machine whose wheel has a radius of 20.7 cm and runs at 2.53 rev/s when the tool to be sharpened is held against the wheel with a force of 180 N? The coefficient of friction between the tool and the wheel is 0.32.

19. A fully loaded freight elevator has a total mass of 1220 kg. It is required to travel downward 54.5 m in 43.0 s. The counterweight has a mass of 1380 kg. Find the power output, in hp, of the elevator motor. Ignore the work required to start and stop the elevator; that is, assume that it travels at constant speed.

20. A jet airplane is traveling 184 m/s. In each second the engine takes in 68.2 m^3 of air having a mass of 70.2 kg. The air is used to burn 2.92 kg of fuel each second. The energy is used to compress the products of combustion and to eject them at the rear of the engine at 497 m/s relative to the plane. Find (*a*) the thrust of the jet engine and (*b*) the delivered power (horsepower).

11-4 Work Done by a Variable Force

21. A 10-kg object moves along the *x* axis. Its acceleration as a function of its position is shown in Fig. 11-25. What is the net work performed on the object as it moves from $x = 0$ to $x = 8.0$ m?

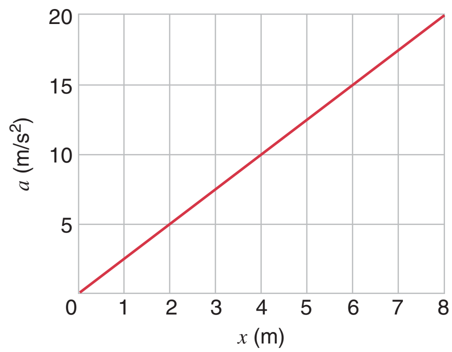


FIGURE 11-25. Exercise 21.

22. A 5.0-kg block moves in a straight line on a horizontal frictionless surface under the influence of a force that varies with position as shown in Fig. 11-26. How much work is done by the force as the block moves from the origin to $x = 8.0$ m?

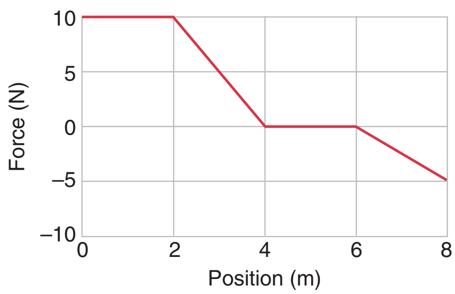


FIGURE 11-26. Exercise 22.

23. Figure 11-27 shows a spring with a pointer attached, hanging next to a scale graduated in millimeters. Three different weights are hung from the spring, in turn, as shown. (*a*) If all weight is removed from the spring, which mark on the scale will the pointer indicate? (*b*) Find the weight W.

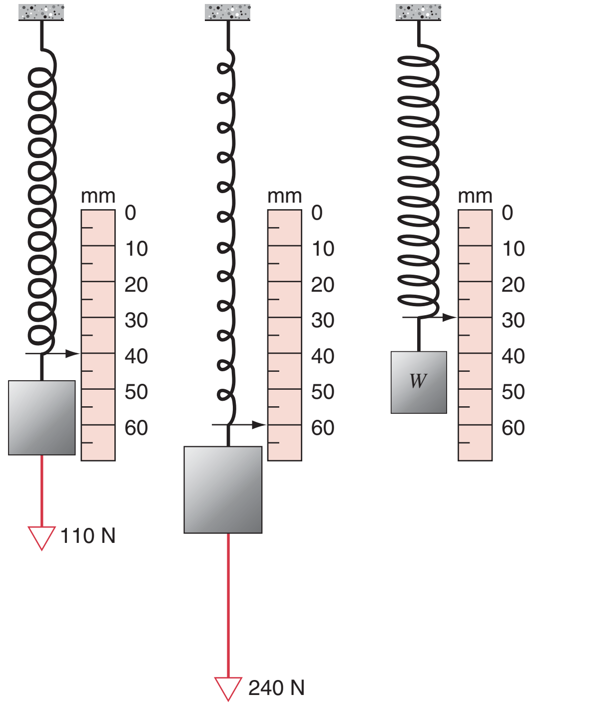


FIGURE 11-27. Exercise 23.

24. A spring has a force constant of 15.0 N/cm. (*a*) How much work is required to extend the spring 7.60 mm from its relaxed position? (*b*) How much work is needed to extend the spring an additional 7.60 mm?

11-5 Work Done by a Variable Force: Two-Dimensional Case

25. By integrating along the arc, show that the work done by gravity in Sample Problem 11-5 is equal to $-mgh$.

26. An object of mass 0.675 kg on a frictionless table is attached to a string that passes through a hole in the table at the center of the horizontal circle in which the object moves with constant speed. (*a*) If the radius of the circle is 0.500 m and the speed is 10.0 m/s, compute the tension in the string. (*b*) It is found that drawing an additional 0.200 m of the string down through the hole, thereby reducing the radius of the circle to 0.300 m, has the effect of multiplying the original tension in the string by 4.63. Compute the total work done by the string on the revolving object during the reduction of the radius.

11-6 Kinetic Energy and the Work–Energy Theorem

27. A conduction electron in copper near the absolute zero of temperature has a kinetic energy of 4.2 eV. What is the speed of the electron?

28. Calculate the kinetic energies of the following objects moving at the given speeds: (*a*) a 110-kg football linebacker running at 8.1 m/s; (*b*) a 4.2-g bullet at 950 m/s; (*c*) the aircraft carrier *Nimitz,* 91,400 tons at 32.0 knots.

29. A proton (nucleus of the hydrogen atom) is being accelerated in a linear accelerator. In each stage of such an accelerator the proton is accelerated along a straight line at 3.60×10^{15} m/s^2. If a proton enters such a stage moving initially with a speed of 2.40×10^7 m/s and the stage is 3.50 cm long, compute (*a*) its speed at the end of the stage and (*b*) the gain in kinetic energy resulting from the acceleration. The mass of the proton is 1.67×10^{-27} kg. Express the energy in electron-volts.

30. A single force acts on a particle in rectilinear motion. A plot of velocity versus time for the particle is shown in Fig. 11-28. Find the sign (positive or negative) of the work done by the force on the particle in each of the intervals AB, BC, CD, and DE.

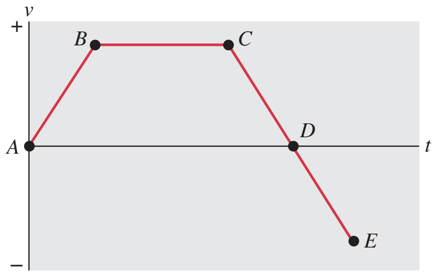


FIGURE 11-28. Exercise 30.

31. A force acts on a 2.80-kg particle in such a way that the position of the particle as a function of time is given by $x = (3.0 \text{ m/s})t - (4.0 \text{ m/s}^2)t^2 + (1.0 \text{ m/s}^3)t^3$. (*a*) Find the work done by the force during the first 4.0 s. (*b*) At what instantaneous rate is the force doing work on the particle at the instant $t = 3.0$ s?

32. The Earth circles the Sun once a year. How much work would have to be done on the Earth to bring it to rest relative to the Sun? See Appendix C for numerical data and ignore the rotation of the Earth about its own axis.

33. A 3700-lb automobile ($m = 1600$ kg) starts from rest on a level road and gains a speed of 45 mi/h (= 72 km/h) in 33 s. (*a*) What is the kinetic energy of the auto at the end of the 33 s? (*b*) What is the average net power delivered to the car during the 33-s interval? (*c*) What is the instantaneous power at the end of the 33-s interval assuming that the acceleration was constant?

11-7 Work and Kinetic Energy in Rotational Motion

34. A molecule has a rotational inertia of 14,000 u · pm^2 and is spinning at an angular speed of 4.30×10^{12} rad/s. (*a*) Express the rotational inertia in kg · m^2. (*b*) Calculate the rotational kinetic energy in eV.

35. The oxygen molecule has a total mass of 5.30×10^{-26} kg and a rotational inertia of 1.94×10^{-46} kg · m^2 about an axis through the center perpendicular to the line joining the atoms. Suppose that such a molecule in a gas has a mean speed of 500 m/s and that its rotational kinetic energy is two-thirds of its translational kinetic energy. Find its average angular velocity.

36. Delivery trucks that operate by making use of energy stored in a rotating flywheel have been used in Europe. The trucks are charged by using an electric motor to get the flywheel up to its top speed of 624 rad/s. One such flywheel is a solid, homogeneous cylinder with a mass of 512 kg and a radius of 97.6 cm. (*a*) What is the kinetic energy of the flywheel after charging? (*b*) If the truck operates with an average power requirement of 8.13 kW, for how many minutes can it operate between chargings?

37. A 31.4-kg wheel with radius 1.21 m is rotating at 283 rev/min. It must be brought to a stop in 14.8 s. Find the required average power. Assume the wheel to be a thin hoop.

38. Two wheels, A and B, are connected by a belt as in Fig. 11-29. The radius of B is three times the radius of A. What would be the ratio of the rotational inertias I_A/I_B, if (*a*) both wheels have the same angular momenta and (*b*) both wheels have the same rotational kinetic energy? Assume that the belt does not slip.

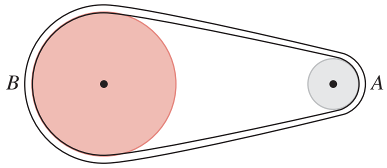


FIGURE 11-29. Exercise 38.

39. Assume the Earth to be a sphere of uniform density. (*a*) Calculate its rotational kinetic energy. (*b*) Suppose that this energy could be harnessed for our use. For how long could the Earth supply 1.00 kW of power to each of the 6.17×10^9 persons on the Earth?

11-8 Kinetic Energy in Collisions

40. The last stage of a rocket is traveling at a speed of 7600 m/s. This last stage is made up of two parts that are clamped together—namely, a rocket case with a mass of 290.0 kg and a payload capsule with a mass of 150.0 kg. When the clamp is released, a compressed spring causes the two parts to separate with a relative speed of 910.0 m/s. (*a*) What are the speeds of the two parts after they have separated? Assume that all velocities are along the same line. (*b*) Find the total kinetic energy of the two parts before and after they separate and account for the difference, if any.

41. A 35.0-ton railroad freight car collides with a stationary caboose car. They couple together and 27.0% of the initial kinetic energy is dissipated as heat, sound, vibrations, and so on. Find the weight of the caboose.

42. A body of mass 8.0 kg is traveling at 2.0 m/s under the influence of no external force. At a certain instant an internal explosion occurs, splitting the body into two chunks of 4.0 kg mass each; 16 J of translational kinetic energy are imparted to the two-chunk system by the explosion. Neither chunk leaves the line of the original motion. Determine the speed and direction of motion of each of the chunks after the explosion.

43. Show that a slow neutron (called a thermal neutron) that is scattered through 90° in an elastic collision with a deuteron, that is initially at rest, loses two-thirds of its initial kinetic energy to the deuteron. (The mass of a neutron is 1.01 u; the mass of a deuteron is 2.01 u.)

44. A certain nucleus, at rest, spontaneously disintegrates into three particles. Two of them are detected; their masses and velocities are as shown in Fig. 11-30. (*a*) What is the momentum of the third particle, which is known to have a mass of 11.7×10^{-27} kg? (*b*) How much kinetic energy in MeV appears in the disintegration process?

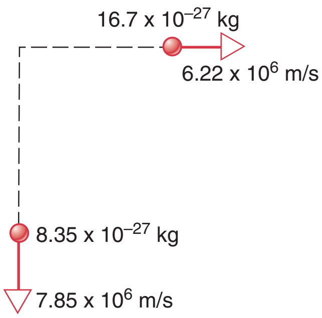


FIGURE 11-30. Exercise 44.

PROBLEMS

1. Electric fields can be used to pull electrons out of metals. To remove an electron from tungsten, the electric field must do 4.5 eV of work. Suppose that the distance over which the electric field acts is 3.4 nm. Calculate the minimum force that the field must exert on the electron being removed.

2. A cord is used to lower vertically a block of mass M a distance d at a constant downward acceleration of $g/4$. (*a*) Find the work done by the cord on the block. (*b*) Find the work done by the force of gravity.

3. Figure 11-31 shows an arrangement of pulleys designed to facilitate the lifting of a heavy load L. Assume that friction can be ignored everywhere and that the pulleys to which the load is attached weigh a total of 20.0 lb. An 840-lb load is to be raised 12.0 ft. (*a*) What is the minimum applied force F that can lift the load? (*b*) How much work must be done against gravity in lifting the 840-lb load 12.0 ft? (*c*) Through what distance must the applied force be exerted to lift the load 12.0 ft? (*d*) How much work must be done by the applied force F to accomplish this task?

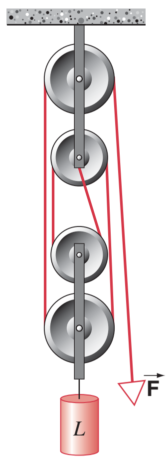


FIGURE 11-31. Problem 3.

4. A worker can lift a 75-kg block directly off the ground on to a loading dock, or can push the block up a frictionless incline from the ground to the loading dock. Lifting the block requires 680 J of work to be done. Pushing the block up the incline requires a minimum applied force of 320 N. Find the angle the incline makes with the horizontal.

5. A horse pulls a cart with a force of 42.0 lb at an angle of 27.0° with the horizontal and moves along at a speed of 6.20 mi/h. (*a*) How much work does the horse do in 12.0 min? (*b*) Find the power output of the horse, in hp of course.

6. A 1380-kg block of granite is dragged up an incline at a constant speed of 1.34 m/s by a steam winch (Fig. 11-32). The coefficient of kinetic friction between the block and the incline is 0.41. How much power must be supplied by the winch?

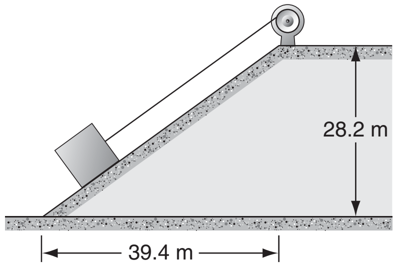


FIGURE 11-32. Problem 6.

7. Show that the speed v reached by a car of mass m that is driven with constant power P is given by

$$v = (3xP/m)^{1/3},$$

where x is the distance traveled from rest.

8. (*a*) Show that the power output of an airplane cruising at constant speed v in level flight is proportional to v^3. Assume that the aerodynamic drag force is given by $D = bv^2$. (*b*) By what factor must the engines' power be increased to increase the air speed by 25.0%?

9. An escalator joins one floor with another one 8.20 m above. The escalator is 13.3 m long and moves along its length at 62.0 cm/s. (*a*) What power must its motor deliver if it is required to carry a maximum of 100 persons per minute, of average mass 75.0 kg? (*b*) An 83.5-kg man walks up the escalator in 9.50 s. How much work does the motor do on him? (*c*) If this man turned around at the middle and walked down the escalator so as to stay at the same level in space, would the motor do work on him? If so, what power does it deliver for this purpose? (*d*) Is there any (other?) way the man could walk along the escalator without consuming power from the motor?

10. The power output from a motor on a trolley is a function of velocity and is given by $P(v) = av(b - v^2)$, where a and b are constants and $P = 0$ for $v^2 > b$. (*a*) At what speed is the maximum power output from the motor? (*b*) At what speed is maximum force exerted by the motor? (*c*) At $v = 0$ the power output is zero. Does this mean that the motor will be unable to move the trolley if it is originally at rest? Explain.

11. The force exerted on an object is $\vec{\mathbf{F}} = F_0(x/x_0 - 1)\hat{\mathbf{i}}$. Find the work done in moving the object from $x = 0$ to $x = 3x_0$ (*a*) by plotting $F_x(x)$ and finding the area under the curve, and (*b*) by evaluating the integral analytically.

12. (*a*) Estimate the work done by the force shown on the graph (Fig. 11-33) in displacing a particle from $x = 1$ m to $x =$ 3 m. Refine your method to see how close you can come to the exact answer of 6 J. (*b*) The curve is given analytically by $F_x = A/x^2$, where $A = 9\ \mathrm{N \cdot m^2}$. Show how to calculate the work by the rules of integration.

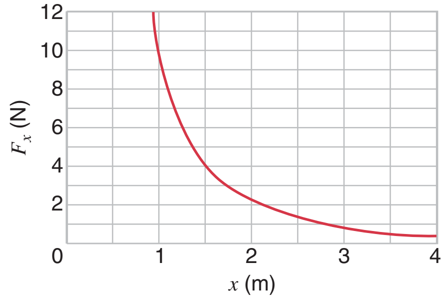


FIGURE 11-33. Problem 12.

13. A "stiff" spring has a force law given by $F = -kx^3$. The work required to stretch the spring from the relaxed state $x = 0$ to the stretched length $x = l$ is W_0. In terms of W_0, how much work is required to extend the spring from the stretched length l to the length $2l$?

14. Two springs, each with force constant k and unstretched length l_0, are connected in a straight line as shown in Fig. 11-34. (*a*) Find an expression for the work required to move the point of attachment between the two springs a perpendicular distance x from the equilibrium point. (*b*) Use the binomial expansion to find the first nonvanishing term in the expression for the work when $x \ll l_0$.

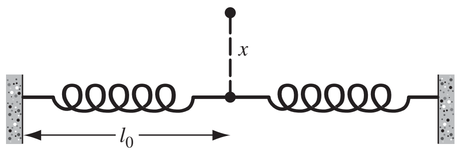


FIGURE 11-34. Problem 14.

15. Four springs, each with force constant k and unstretched length l_0, are connected as shown in Fig. 11-35. The springs obey Eq. 11-15 for both stretching and compression. Show

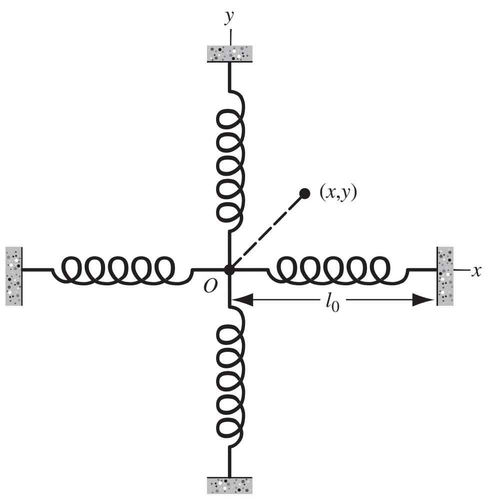


FIGURE 11-35. Problem 15.

that the work required to move the point of attachment from the equilibrium position in a straight line to the point x,y (with $x << l_0$ and $y << l_0$) is $W = kd^2$, where $d^2 = x^2 + y^2$.

16. An 1100-kg car is traveling at 46 km/h on a level road. The brakes are applied long enough to remove 51 kJ of kinetic energy. (*a*) What is the final speed of the car? (*b*) How much more kinetic energy must be removed by the brakes to stop the car?

17. A running man has half the kinetic energy that a boy of half his mass has. The man speeds up by 1.00 m/s and then has the same kinetic energy as the boy. What were the original speeds of man and boy?

18. A 0.550-kg projectile is launched from the edge of a cliff with an initial kinetic energy of 1550 J and at its highest point is 140 m above the launch point. (*a*) What is the horizontal component of its velocity? (*b*) What was the vertical component of its velocity just after launch? (*c*) At one instant during its flight the vertical component of its velocity is found to be 65.0 m/s. At that time, how far is it above or below the launch point?

19. A comet having a mass of 8.38×10^{11} kg strikes the Earth at a relative speed of 30 km/s. (*a*) Compute the kinetic energy of the comet in "megatons of TNT"; the detonation of 1 million tons of TNT releases 4.2×10^{15} J of energy. (*b*) The diameter of the crater blasted by a large explosion is proportional to the one-third power of the explosive energy released, with 1 megaton of TNT producing a crater about 1 km in diameter. What is the diameter of the crater produced by the impact of the comet? (In the past, atmospheric effects produced by impacts of comets may have been the cause of mass extinctions of many species of animals and plants; it is thought by many that dinosaurs became extinct by this mechanism.)

20. A 263-g block is dropped onto a vertical spring with force constant $k = 2.52$ N/cm (Fig. 11-36). The block sticks to the spring, and the spring compresses 11.8 cm before coming momentarily to rest. While the spring is being compressed, how much work is done (*a*) by the force of gravity and (*b*) by the spring? (*c*) What was the speed of the block just before it hit the spring? (*d*) If this initial speed of the block is doubled, what is the maximum compression of the spring? Ignore friction.

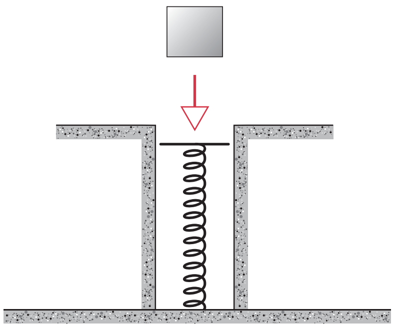


FIGURE 11-36. Problem 20.

21. An object of mass m accelerates uniformly from rest to a speed v_f in time t_f. (*a*) Show that the work done on the object as a function of time t, in terms of v_f and t_f, is

$$W = \frac{1}{2} m \frac{v_f^2}{t_f^2} t^2.$$

(*b*) As a function of time t, what is the instantaneous power delivered to the object?

22. A uniform steel rod of length 1.20 m and mass 6.40 kg has attached to each end a small ball of mass 1.06 kg. The rod is constrained to rotate in a horizontal plane about a vertical axis through its midpoint. At a certain instant, it is observed to be rotating with an angular speed of 39.0 rev/s. Because of axle friction, it comes to rest 32.0 s later. Compute, assuming a constant frictional torque, (*a*) the angular acceleration, (*b*) the retarding torque exerted by axle friction, (*c*) the kinetic energy lost due to the axle friction, and (*d*) the number of revolutions executed during the 32.0 s. (*e*) Now suppose that the frictional torque is known not to be constant. Which, if any, of the quantities (*a*), (*b*), (*c*), or (*d*) can still be computed without requiring any additional information? If any exists, give its value.

23. A 1040-kg car has four 11.3-kg wheels. What fraction of the total kinetic energy of the car is due to rotation of the wheels about their axles? Assume that the wheels have the same rotational inertia as disks of the same mass and size. Explain why you do not need to know the radius of the wheels.

24. A man stands on a platform that is rotating with an angular speed of 1.22 rev/s; his arms are outstretched and he holds a weight in each hand. With his hands in this position the total rotational inertia of the man, the weights, and the platform is $6.13\ \text{kg}\cdot\text{m}^2$. If by moving the weights the man decreases the rotational inertia to $1.97\ \text{kg}\cdot\text{m}^2$, (*a*) what is the resulting angular speed of the platform and (*b*) what is the ratio of the new kinetic energy to the original kinetic energy? Assume the platform rotates without friction.

25. In Chapter 10, Exercise 21, the final angular speed of two coupled wheels was found. What fraction of the original kinetic energy was lost when the wheels were coupled?

26. In Chapter 10, Problem 11, a cockroach running on a lazy Susan stops to eat a bread crumb. How much kinetic energy is lost?

27. In Chapter 10, Problem 12, two skaters holding onto a pole (*a*) originally skated in a circle of diameter 2.92 m, but (*b*) the diameter of the circle decreased to 0.940 m when the skaters pulled on the pole. Calculate the kinetic energy of the system in parts (*a*) and (*b*). From where does the change come?

28. A 2500-kg unmanned space probe is moving in a straight line at a constant speed of 300 m/s. A rocket engine on the space probe executes a burn in which a thrust of 3000 N acts for 65.0 s. What is the change in kinetic energy of the probe if the thrust is (*a*) backward, (*b*) forward, or (*c*) sideways? Assume that the mass of the ejected fuel is negligible compared to the mass of the space probe. (See also Exercise 13 of Chapter 6.)

29. A force exerts an impulse J on an object of mass m, changing its speed from v_i to v_f. The force and the object's motion are along the same straight line. Show that the work done by the force is $\frac{1}{2}J(v_i + v_f)$.

30. Suppose that the blades on a helicopter push vertically down the cylindrical column of air they sweep out as they rotate. The total mass of the helicopter is 1820 kg and the length of the blades is 4.88 m. Find the minimum power needed to keep the helicopter airborne. Assume that the density of air is $1.23\ \text{kg/m}^3$.

31. A ball of mass m is projected with speed v_i into the barrel of a spring gun of mass M initially at rest on a frictionless surface; see Fig. 11-37. The ball sticks in the barrel at the point of maximum compression of the spring. No energy is lost in friction. (*a*) What is the speed of the spring gun after the ball comes to rest in the barrel? (*b*) What fraction of the initial kinetic energy of the ball is lost through work done on the spring?

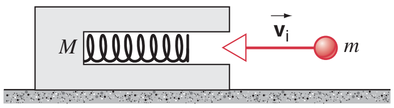


FIGURE 11-37. Problem 31.

32. A block of mass $m_1 = 1.88$ kg slides along a frictionless table with a speed of 10.3 m/s. Directly in front of it, and moving in the same direction, is a block of mass $m_2 = 4.92$ kg moving at 3.27 m/s. A massless spring with a force constant $k =$ 11.2 N/cm is attached to the backside of m_2, as shown in Fig. 11-38. When the blocks collide, what is the maximum compression of the spring? (Hint: At the moment of maximum compression of the spring, the two blocks move as one; find the velocity by noting that the collision is completely inelastic to this point.)

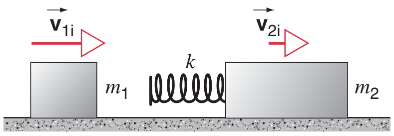


FIGURE 11-38. Problem 32.

33. Two objects, A and B, collide. A has mass 2.0 kg, and B has mass 3.0 kg. The velocities before the collisions are $\vec{\mathbf{v}}_{iA} = (15\text{ m/s})\hat{\mathbf{i}} + (30\text{ m/s})\hat{\mathbf{j}}$ and $\vec{\mathbf{v}}_{iB} = (-10\text{ m/s})\hat{\mathbf{i}} + (5.0\text{ m/s})\hat{\mathbf{j}}$. After the collision, $\vec{\mathbf{v}}_{fA} = (-6.0\text{ m/s})\hat{\mathbf{i}} + (30\text{ m/s})\hat{\mathbf{j}}$. How much kinetic energy was gained or lost in the collision? (See Chapter 6, Exercise 25.)

34. Consider two observers, one whose frame is attached to the ground and another whose frame is attached, say, to a train moving with uniform velocity u with respect to the ground. Each observes that a particle, initially at rest with respect to the train, is accelerated by a constant force applied to it for time t in the forward direction. (*a*) Show that for each observer the work done by the force is equal to the gain in kinetic energy of the particle, but that one observer measures these quantities to be $\frac{1}{2}ma^2t^2$, whereas the other observer measures them to be $\frac{1}{2}ma^2t^2 + maut$. Here a is the common acceleration of the particle of mass m. (*b*) Explain the differences in work done by the same force in terms of the different distances through which the observers measure the force to act during the time t. Explain the different final kinetic energies measured by each observer in terms of the work the particle could do in being brought to rest relative to each observer's frame.

35. A particle of mass m_1, moving with speed v_{1i}, collides head-on with a particle of mass m_2, initially at rest, in a completely inelastic collision. (*a*) What is the kinetic energy of the system before collision? (*b*) What is the kinetic energy of the system after collision? (*c*) What fraction of the original kinetic energy was lost? (*d*) Let v_{cm} be the velocity of the center of mass of the system. View the collision from a primed reference frame moving with the center of mass so that $v'_{1i} = v_{1i} - v_{cm}$ and $v_{2i} = -v_{cm}$. Repeat parts (*a*), (*b*), and (*c*), as seen by an observer in this reference frame. Is the kinetic energy lost the same in each case? Explain.

36. Consider a situation such as that in Chapter 6, Problem 16 (Fig. 6-32) but in which the collisions now may be either all elastic, all inelastic, or some elastic and some inelastic; also, the masses are now m, m', and M. Show that to transfer the maximum kinetic energy from m to M, the intermediate body should have a mass $m' = \sqrt{mM}$—that is, the geometric mean of the adjacent masses. (It is interesting to note that this same relationship exists between masses of successive layers of air in the exponential horn in acoustics. (See "Energy Transfer in One-Dimensional Collisions of Many Objects," by John B. Hart and Robert B. Herrmann, *American Journal of Physics,* January 1968, p. 46.)

COMPUTER PROBLEM

1. The power output from the motor on a 2.0-kg radio-controlled car is dependent on the velocity of the car and is given by

$$P = v(5 - v)/3$$

where P is measured in watts when v is measured in m/s. Assume the car starts from rest, and numerically generate a graph of position as a function of time and velocity as a function of time for the car.

CHAPTER 12

ENERGY 2: POTENTIAL ENERGY

In the last chapter we began our study of energy with an introduction to work and kinetic energy. In this chapter we introduce another kind of energy, potential energy, which is energy that can be stored in a system when certain types of forces act between its components.

Taking into account the kinetic and potential energies of a system, we have the law of conservation of mechanical energy, which provides a way of understanding mechanical problems that is based on Newton's laws but often provides new or different insights. Based on this conservation law, we can re-analyze a number of problems in translational and rotational motion that we have previously solved with Newton's laws. The next chapter continues to broaden and expand our concepts of energy into a more general form of energy conservation law.

12-1 CONSERVATIVE FORCES

Potential energy is defined only for a certain class of forces called *conservative forces.* Before we define what we mean by a conservative force, let us consider examples of the behavior of three different forces: the spring force, $F_x = -kx$; the gravitational force, $F_y = mg$; and the frictional force, $f = \mu N$. Our goal is to examine the work done by each force as a particle acted upon by that force moves along a path and returns to its starting point.

1. *The spring force.* Figure 12-1 shows a block of mass m attached to a spring of force constant k; the block slides without friction across a horizontal surface. Initially (Fig. 12-1*a*) an external agent has compressed the spring so that the block is displaced to $x = +d$ from its position at $x = 0$ when the spring is relaxed. The external agent is suddenly removed at $t = 0$, and the spring begins to do work on the block. As the block moves from $x = +d$ to $x = 0$, the spring does work $+\frac{1}{2}kd^2$ (Eq. 11-16). According to the work–energy theorem, this work appears as kinetic energy of the block.

As the block passes through $x = 0$ (see Fig. 12-1*b*), the direction of the spring force reverses, and the spring now acts to slow down the block, doing *negative* work on it. When the block has been brought momentarily to rest at $x = -d$, as in Fig. 12-1*c*, the amount of this negative work done by the spring force between $x = 0$ and $x = -d$ is $-\frac{1}{2}kd^2$. Similarly, from $x = -d$ to $x = 0$, the spring force does work $+\frac{1}{2}kd^2$, and from $x = 0$ back to $x = +d$, it does work $-\frac{1}{2}kd^2$. The block is now back in its original position (compare Figs. 12-1*a* and 12-1*e*), and we see from adding the four separate contributions that the total work done on the block by the spring force in the complete cycle is zero.

2. *The force of gravity.* Figure 12-2 shows an example of a system consisting of a ball acted on by the Earth's gravity. The ball is projected upward by an external agent that gives it an initial speed v_0 and thus an initial kinetic energy $\frac{1}{2}mv_0^2$. As the ball rises, the Earth does work on it and eventually brings it momentarily to rest at $y = h$. The work done by the Earth as the ball rises from $y = 0$ to $y = h$ is $-mgh$ (the constant force mg times the distance h, negative

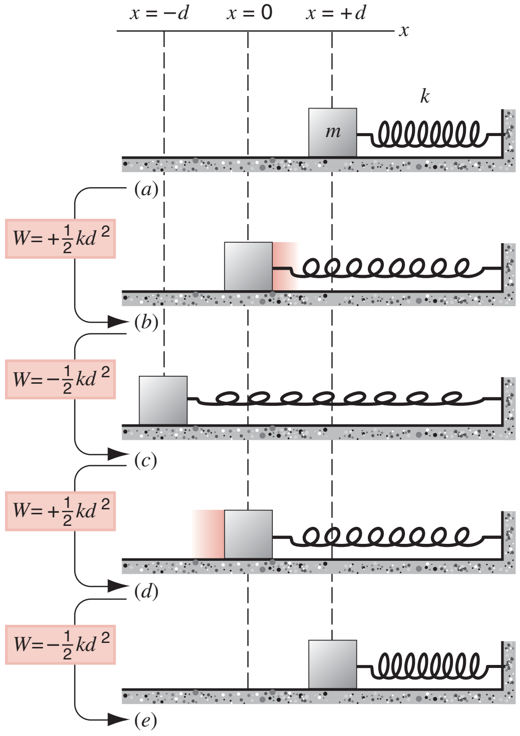


FIGURE 12-1. A block moves under the action of a spring force from (a) $x = +d$ to (b) $x = 0$, moving left, to (c) $x = -d$, to (d) $x = 0$, moving right, and (e) back to $x = +d$. The work done by the spring force between each pair of successive positions is shown in the boxes at the left. Note that the total work done by the spring force on the block is zero for the round trip.

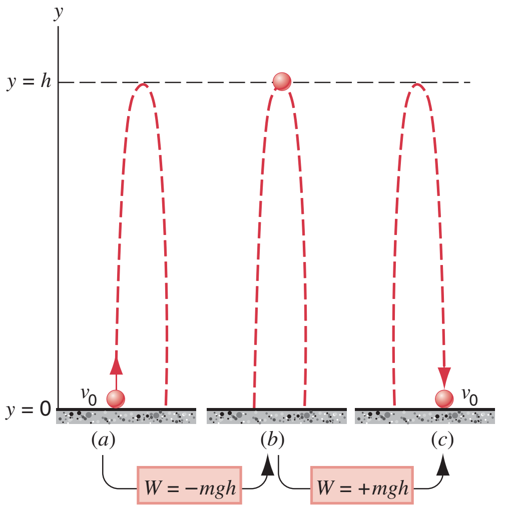


FIGURE 12-2. A ball is thrown upward against the Earth's gravity. In (a) it is just leaving its starting point, in (b) it has reached the top of its trajectory, and in (c) it has returned to its original height. The work done by the Earth's gravity between the pairs of successive positions is shown in the boxes at the bottom. Note that the total work done by the force of gravity on the ball is zero for the round trip.

because the force and displacement are in opposite directions as the ball rises). As the ball falls from $y = h$ to $y = 0$, the force of gravity does work $+mgh$. The total work done on the ball by the force of gravity during the round trip is zero.

3. *The frictional force.* For our third example, consider a disk of mass m on the end of a thin but rigid rod of length R. The disk is given an initial speed v_0, and the rod constrains it to move in a circle of radius R over a horizontal surface that exerts a frictional force on the disk (see Fig. 12-3). The only force that does work on the disk is the frictional force, exerted *by* the surface *on* the bottom of the disk. This force always acts in a direction opposite to the direction in which the disk is moving, so that the work done on the disk by the frictional force is always negative. After the disk has returned to its starting point, the work done by the frictional force is definitely not zero; the total work for the "round trip" is, in fact, a negative quantity.

Note the differences between these three examples. In the first two (the spring force and gravity), the object returned to its starting point after a round trip with no total work done on it. In the third example (the frictional force), there was total work done on the object during the round trip. We find it useful to attach labels to forces in order to identify this basic difference in behavior. Specifically,

Consider the total work done by a force that acts on a particle as the particle moves around a closed path and returns to its starting point. If this total work is zero, we call the force a conservative force. If the total work for the round trip is not zero, we call the force a nonconservative force.

The elastic restoring force (spring force) and gravity are two examples of conservative forces. Friction is an example of a nonconservative force.

A second way to identify a force as conservative or nonconservative is based on a comparison of the work done when the object on which the force acts moves from one location to another by several different paths. For example, suppose you are moving packages of mass m from the basement to the first floor in a building that has several floors, each of height h. If you move a package directly from the basement to the first floor, the (conservative) gravitational force acting on the package does work $W_g = -mgh$. If instead you first move it to the fifth floor ($W_g = -5mgh$) and then return it to the first floor ($W_g = +4mgh$), the total work done by gravity for the entire process is $W_g = -mgh$, exactly the same as if you had carried the package directly. No matter how many intermediate stopping points or how many times you go back and forth over the same path, when you finally deliver the package to the first floor, the total work done by gravity between the original location of the package (the basement) and its final location (the first floor) will be $-mgh$.

On the other hand, consider the behavior of the nonconservative frictional force for the system illustrated in Fig. 12-3 as the disk moves along two different paths from posi-

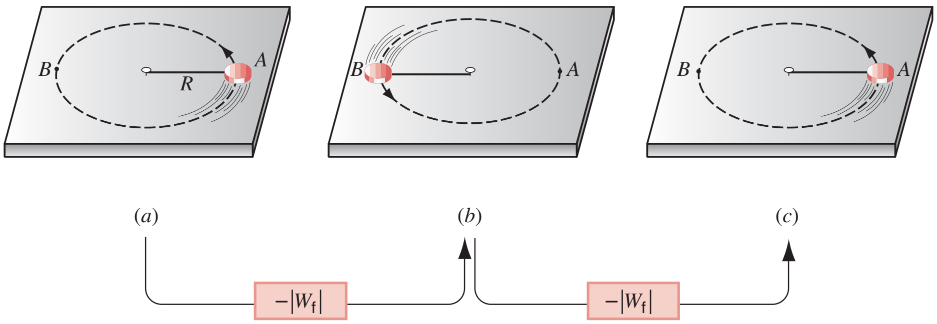


FIGURE 12-3. A disk moves with friction in a circle on a horizontal surface. The positions shown represent (*a*) an arbitrary starting point *A*, (*b*) one-half revolution later (at *B*), and (*c*) another half revolution later (back at *A*). The work done by friction between successive positions is indicated in the boxes at the bottom. Note that the total work done by the frictional force on the disk is *not* zero for the round trip, but instead has the negative value $-2|W_f|$.

tion *A* to position *B*. In one case, the disk moves through one-half revolution from *A* to *B*, and in the second case it moves through $1\frac{1}{2}$ revolutions. Although calculating the work done by the frictional force requires some care (see Section 13-3) it seems clear that the magnitude of the (negative) work done by friction is larger in the second case than in the first, because the frictional force acts over a larger distance. For the frictional force, the work done depends on the path taken between the initial and the final positions of the object on which that force acts.

This leads to our second way of distinguishing conservative forces.

Consider the work done by a force that acts on an object as the object moves from an initial position to a final position along any arbitrarily chosen path. If this work is the same for all such paths, we call the force a conservative force. If the work is not the same for all paths, we call the force a nonconservative force.

With the help of Fig. 12-4, we can show that the two criteria we have developed for identifying conservative forces are precisely equivalent. In Fig. 12-4*a*, a particle moves around a closed path from *a* to *b* and back again. If only a conservative force $\vec{\mathbf{F}}$ acts on the particle, the total work done on the particle by that force during the cycle must be zero. That is,

$$W_{ab,1} + W_{ba,2} = 0$$

or

$$\int_{a\ \text{(path 1)}}^{b} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} + \int_{b\ \text{(path 2)}}^{a} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = 0, \qquad (12\text{-}1)$$

where $W_{ab,1}$ means "the work done by the force when the particle moves from *a* to *b* along path 1" and $W_{ba,2}$ means

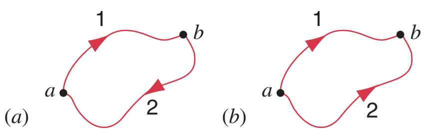


FIGURE 12-4. (*a*) A particle, acted on by a conservative force, moves in a round trip starting at point *a*, passing through point *b*, and returning to point *a*. (*b*) A particle starts from point *a* and travels to point *b* following either of two possible paths.

"the work done by the force when the particle moves from *b* to *a* along path 2." Equation 12-1 is the mathematical statement equivalent to the *first* criterion for a conservative force.

Reversing the direction in which we travel any particular path interchanges the limits of integration and changes the sign of the displacement; that is, the work in going from *a* to *b* is related to the work in going from *b* to *a*:

$$\int_a^b \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = -\int_b^a \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} \qquad \text{(any particular path)}$$

or, in the case of path 2,

$$W_{ab,2} = -W_{ba,2}. \qquad (12\text{-}2)$$

Combining Eqs. 12-1 and 12-2 gives

$$W_{ab,1} = W_{ab,2}$$

or

$$\int_{a\ \text{(path 1)}}^{b} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \int_{a\ \text{(path 2)}}^{b} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}}. \qquad (12\text{-}3)$$

This is the mathematical representation of the *second* definition of a conservative force: the work done by the force is the same for any arbitrary path between *a* and *b*. Thus the first definition leads directly to the second and (by a similar argument) the second leads to the first, so that the two definitions are equivalent.

12-2 POTENTIAL ENERGY

In the previous section we discussed two systems in which conservative forces act. These systems have some common characteristics: they consist of at least two objects (the block and the spring or the ball and the Earth) interacting through a force (the elastic force or gravity) that does work and transfers energy between the parts of the system as they move relative to one another.

In situations in which a conservative force acts between objects in a system, it is convenient and useful to define another kind of energy: the *potential energy.* The potential energy U is energy associated with the *configuration* of a system. Here "configuration" means how the parts of a system are located or arranged with respect to one another (for example, the compression or stretching of the spring in the block–spring system or the height of the ball in the ball–Earth system).

When work is done in a system by a conservative force, the configuration of its parts changes, and so the potential energy changes from its initial value U_i to its final value U_f. We define the change in potential energy associated with a single force as

$$\Delta U = U_f - U_i = -W, \tag{12-4}$$

in which W is the work done by that force as the system moves from a specified initial configuration to a specified final configuration.

It is very important to remember that potential energy is characteristic of the *system* and not of any of the individual objects within the system. We should properly speak of "the elastic potential energy of the block–spring system" or "the gravitational potential energy of the ball–Earth system" (*not* "the elastic potential energy of the spring" or "the gravitational potential energy of the ball"). However, the change in the configuration of the block–spring system comes about because of the stretching and compression of the spring. The block, taken to be rigid, does not change its shape as it moves. Thus we often associate the potential energy of the block–spring system with the spring alone. Similarly, the change in the configuration of the ball–Earth system comes about largely because of the motion of the ball, so we often associate the potential energy of this system with the ball alone. It is true that the Earth recoils when the ball is projected upward but, because of its much greater mass, the Earth's displacement is negligibly small compared with that of the ball.

Let us therefore consider the case in which it is necessary to consider the work done on only one object in the system. If the object moves only in the x direction, its x coordinate is all that is required to specify the configuration of the system. Using Eq. 11-14 for the work done by a force in one dimension, we obtain

$$\Delta U = U(x_f) - U(x_i) = -W = -\int_{x_i}^{x_f} F_x(x)\,dx. \tag{12-5}$$

Equation 12-5 allows us to calculate the difference in potential energy between any two locations x_i and x_f for a particle on which force $F_x(x)$ acts. However, we are often interested in knowing the potential energy associated with an arbitrary location or configuration x relative to a particular reference location x_0:

$$U(x) - U(x_0) = -\int_{x_0}^{x} F_x(x)\,dx. \tag{12-6}$$

Because only differences or changes in potential energy are significant, we are free to choose the reference point at any convenient location, and we are free to define the potential energy $U(x_0)$ at the reference point to have any convenient value. The function $U(x)$ can then be used to find the potential energy at any arbitrary location in the system—for example, at x_1, x_2, and so forth. If we choose a different reference point or a different value for $U(x_0)$, then $U(x_1)$ and $U(x_2)$ will both change, but the physically important quantities, such as $U(x_1) - U(x_2)$, are unchanged. The analysis of the dynamical behavior is thus independent of the choice of $U(x_0)$.

It may be that the initial and final states of the system are the same—that is, the force is acting on a particle that is making a "round trip." If potential energy is to have any meaning in such cases, we must have $\Delta U = U_f - U_i = 0$, because i and f represent the same location. Equation 12-4 then requires that W must be zero. As we have seen, this can be true only for conservative forces. Thus *we can associate potential energy only with conservative forces.* In particular, because $W \neq 0$ for a round trip, we cannot associate potential energy with the force of friction.

The inverse of Eq. 12-6 allows us to calculate the force from the potential energy:

$$F_x(x) = -\frac{dU(x)}{dx}. \tag{12-7}$$

Equation 12-7 gives another way of looking at the potential energy: *the potential energy is a function of position whose negative derivative gives the force.*

We now illustrate the calculation of potential energy with the two examples of conservative forces we considered in Section 12-1, the block–spring system and the ball–Earth system.

The Spring Force

We choose the reference position x_0 of the block in the block–spring system of Fig. 12-1 to be that in which the spring is in its relaxed state ($x_0 = 0$), and we declare the potential energy of the system to be zero when the block is at that location $[U(x_0) = 0]$. The potential energy of the block–spring system can be found by substituting these values into Eq. 12-6 and evaluating the integral for the spring force, $F_x(x) = -kx$:

$$U(x) - 0 = -\int_0^x (-kx)\,dx$$

or

$$U(x) = \tfrac{1}{2}kx^2. \tag{12-8}$$

Whenever the block is displaced a distance x from its reference position, the potential energy of the system is $\frac{1}{2}kx^2$. The same result is obtained whether x is positive or negative; that is, whether the spring is stretched or compressed by a given amount x, the stored energy is the same.

Differentiating Eq. 12-8, we see that Eq. 12-7 is satisfied:

$$-\frac{dU}{dx} = -\frac{d}{dx}\left(\tfrac{1}{2}kx^2\right) = -kx = F_x.$$

The Force of Gravity

For the ball–Earth system, we represent the vertical coordinate by y rather than x, and we take the upward direction to be positive. We choose the reference point $y_0 = 0$ at the surface of the Earth, and we define $U(y_0) = 0$ at that point. We can now evaluate the potential energy $U(y)$ of the system from Eq. 12-6 with $F_y(y) = -mg$:

$$U(y) - 0 = -\int_0^y (-mg)\,dy$$

$$U(y) = mgy. \tag{12-9}$$

Note that Eq. 12-7 is satisfied for this potential energy: $-dU/dy = -mg = F_y$.

SAMPLE PROBLEM 12-1. An elevator cab of mass $m = 920$ kg moves from street level to the top of the World Trade Center in New York, a height of $h = 412$ m above ground. What is the change in the gravitational potential energy of the cab–Earth system?

Solution From Eq. 12-9, we obtain

$$\begin{aligned}\Delta U &= mg\,\Delta y = mgh = (920\text{ kg})(9.80\text{ m/s}^2)(412\text{ m})\\ &= 3.7\times 10^6\text{ J} = 3.7\text{ MJ}.\end{aligned}$$

This is almost exactly 1 kW·h; the equivalent quantity of electrical energy costs a few cents.

Using Eq. 12-4, we see that the gravitational force acting on the cab does work in the amount of -3.7 MJ as the cab rises. The negative sign is appropriate, because the gravitational force acting on the cab and the displacement of the cab are in opposite directions.

SAMPLE PROBLEM 12-2. At the end of the track in a certain train terminal, trains are prevented from crashing into the platform by a bumper mounted on a stiff spring of force constant 1.25×10^8 N/m. One day a train hits the bumper and compresses the spring by a distance of 5.6 cm when it is brought to rest. What is the potential energy stored in the spring at that compression?

Solution We take $U = 0$ when the spring is relaxed ($x = 0$). Then from Eq. 12-8 we have

$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2}(1.25\times 10^8\text{ N/m})(0.056\text{ m})^2 = 1.96\times 10^5\text{ J}.$$

12-3 CONSERVATION OF MECHANICAL ENERGY

Now that we have introduced the concept of potential energy, we can combine it with the concept of kinetic energy and develop a law of *conservation of mechanical energy* that will permit fresh insight into mechanics problems.

Consider an isolated system—that is, one on which either no *external* forces are present, or, if such forces are present, they do no work on the system. For example, two pucks connected by a spring and free to slide on a horizontal, frictionless surface would qualify as an isolated system according to this definition. Gravity and the normal force—both external forces—act on this system but neither force does any work on the system.

Even though no external forces affect this isolated system, the particles *within* the system can exert forces on one another. These forces, which we call *internal* forces, may do work on the particles as the configuration of the system changes. We assume that the internal forces are conservative, so that we can associate a potential energy with each of them. In our puck–spring system, for example, the spring exerts a (conservative) force on each puck. If the spring increases or decreases its length as the system slides on the horizontal frictionless surface, the spring force does work on each puck and changes their kinetic energies.

To analyze a simple case, consider the block–spring system of Fig. 12-1*a* as the block moves from $x = +d$ to $x = 0$. As the spring expands, the kinetic energy of the block *increases* by ΔK, which, from the work–energy theorem (Eq. 11-24), is given by

$$\Delta K = W, \tag{12-10}$$

where W is the (positive) work done on the block by the spring force. Also, as the spring expands, the potential energy of the block–spring system *decreases* by ΔU, which, from the definition of potential energy (Eq. 12-4), is given by

$$\Delta U = -W. \tag{12-11}$$

Thus the increase in kinetic energy is exactly equal to the decrease in potential energy of this conservative block–spring system: $\Delta K = -\Delta U$.

We can extend this conclusion to the more general case of an isolated conservative system consisting of many particles that interact with each other by means of a number of conservative forces, such as elastic spring forces, gravitational forces, and electrical forces. The *total* change in kinetic energy of all of the particles that make up the system is equal in magnitude, but opposite in sign, to the *total* change in the potential energy of the system, or

$$\Delta K_{\text{total}} = -\Delta U_{\text{total}}.$$

We can recast this, perhaps more usefully, as

$$\Delta K_{\text{total}} + \Delta U_{\text{total}} = 0. \tag{12-12}$$

Equation 12-12 says that, in an isolated system in which only conservative forces act, any change in the total kinetic energy of the system must be balanced by an equal and opposite change in its potential energy, so that the sum of these changes is zero.

We can also interpret Eq. 12-12 as $\Delta(K_{\text{total}} + U_{\text{total}}) = 0$. That is, when only conservative forces act, the change in the quantity $K_{\text{total}} + U_{\text{total}}$ is zero. We define this quantity to be the *total mechanical energy* E_{total} of the system:

$$E_{\text{total}} = K_{\text{total}} + U_{\text{total}}. \tag{12-13}$$

Using this definition of the total mechanical energy, Eq. 12-12 becomes

$$\Delta(K_{\text{total}} + U_{\text{total}}) = \Delta E_{\text{total}} = 0. \qquad (12\text{-}14)$$

For convenience we drop the subscripts "total," with the understanding that when we apply the result $\Delta(K + U) = \Delta E = 0$ to a system of particles we must always use the *total* values of the various energies for the system.

If the change in any quantity is zero, then that quantity must remain constant, so we can rewrite Eq. 12-14 as

$$E_{\text{i}} = E_{\text{f}} \quad \text{or} \quad K_{\text{i}} + U_{\text{i}} = K_{\text{f}} + U_{\text{f}}, \qquad (12\text{-}15)$$

where the subscripts i and f refer to the initial and final states of the system. That is, the initial value of the total mechanical energy is equal to the final value.

Equation 12-15 is the mathematical statement of the *law of conservation of mechanical energy:*

> *In an isolated system in which only conservative forces act, the total mechanical energy remains constant.*

Forces that act within our system can change kinetic into potential or potential into kinetic energy, or even one form of potential energy into another, but the total mechanical energy remains constant. If nonconservative forces, such as friction, act in the system, then the total mechanical energy is *not* constant; we consider this case in Chapter 13.

As an example of the conservation of mechanical energy, let us consider again the block–spring system of Fig. 12-1. At an arbitrary point in the motion, the spring is extended or compressed by a distance x (relative to the $x = 0$ reference position) and the block is moving with speed v, so that the total mechanical energy is $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$. When the spring has its maximum extension or compression x_{m}, the block is instantaneously at rest; at that point the mechanical energy is all potential and $E = \frac{1}{2}kx_{\text{m}}^2$. As the spring then returns to its relaxed length and the block moves toward $x = 0$, the potential energy decreases and the kinetic energy increases, until at $x = 0$ the potential energy becomes zero and the kinetic energy reaches its maximum value of $\frac{1}{2}mv_{\text{m}}^2$, so $E = \frac{1}{2}mv_{\text{m}}^2$. Figure 12-5 illustrates the variation of the kinetic and potential energies as the system moves. Note that at every stage of the motion, the sum $K + U$ remains constant.

Similarly, as the ball at first rises in the ball–Earth system of Fig. 12-2, the gravitational potential energy increases as the kinetic energy decreases, but the total mechanical energy remains constant. Taking $U = 0$ at the point of release, the initial mechanical energy is $E = \frac{1}{2}mv_0^2$. (As we discussed above, the motion of the Earth is negligible in this system, so we can associate the kinetic energy entirely with the ball.) At an arbitrary height y, the total mechanical energy is the sum of the kinetic plus potential energies, $E = \frac{1}{2}mv^2 + mgy$. At its maximum height h, the speed is zero and so $E = mgh$. At each location, the total mechanical energy has the same value, although it may be shared differently between its kinetic and potential parts. As the ball falls,

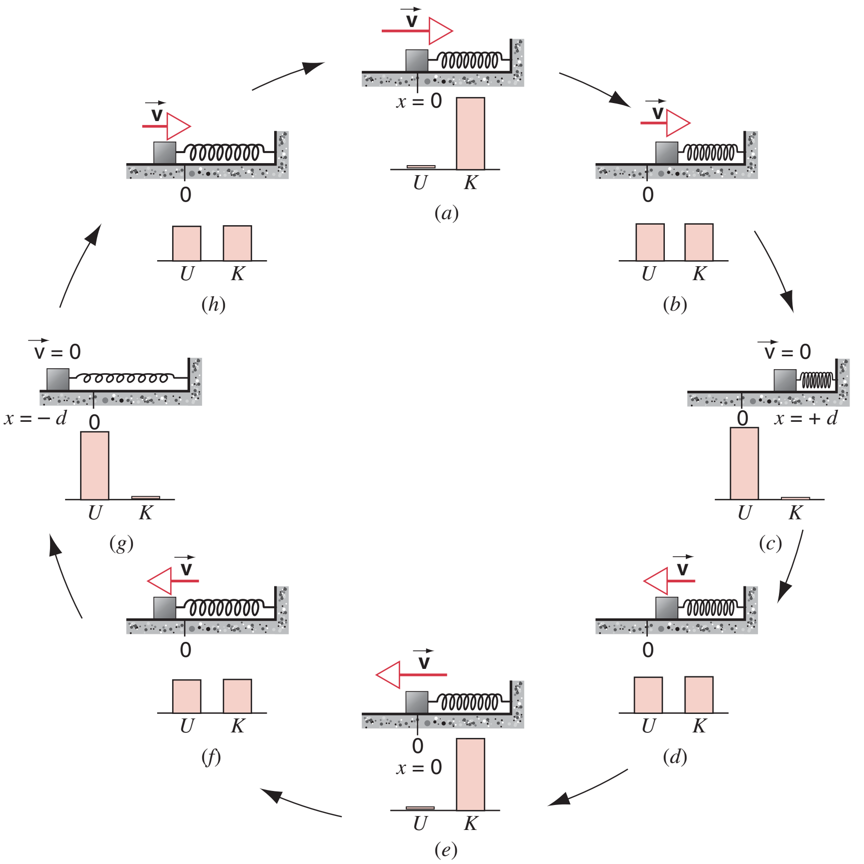


FIGURE 12-5. A block attached to a spring oscillates back and forth on a horizontal frictionless surface. The mechanical energy E of the system remains constant but is shared differently between kinetic and potential energy as the system moves. At certain times (a, e) the energy is all kinetic, at others (c, g) it is all potential, and at still others (b, d, f, h) it is shared equally between the two forms.

the system loses potential energy as the ball gains kinetic energy, again keeping the total mechanical energy constant.

Sample Problem 12-3. The spring of a spring gun is compressed a distance d of 3.2 cm from its relaxed state, and a ball of mass m (= 12 g) is put in the barrel. With what speed will the ball leave the barrel once the gun is fired? The force constant k of the spring is 7.5 N/cm. Assume no friction and a horizontal gun barrel.

Solution Our isolated system consists of the ball + spring, as in the case of Fig. 12-1. The initial configuration consists of the ball at rest against the compressed spring. Thus $E_i = K_i + U_i = 0 + \frac{1}{2}kd^2$, using Eq. 12-8 for the potential energy when the spring is compressed by a distance d. When the spring expands to its relaxed length ($x = 0$), the end of the spring (along with the ball) is moving with its maximum speed v_m; as the spring expands beyond its relaxed length, the end of the spring begins to slow down, but the ball continues to move with speed v_m and so it is no longer in contact with the spring. At this instant, $E_f = K_f + U_f = \frac{1}{2}mv_m^2 + 0$. Conservation of energy ($E_f = E_i$) then gives

$$\tfrac{1}{2}mv_m^2 + 0 = 0 + \tfrac{1}{2}kd^2.$$

Solving for v_m yields

$$v_m = d\sqrt{\frac{k}{m}} = (0.032\ \text{m})\sqrt{\frac{750\ \text{N/m}}{12 \times 10^{-3}\ \text{kg}}} = 8.0\ \text{m/s}.$$

Sample Problem 12-4. A roller coaster (Fig. 12-6) slowly lifts a car filled with passengers to a height of $y = 25$ m, from which it accelerates downhill. Neglecting friction, with what speed will the car reach the bottom?

Solution We take our system to consist of the car (with its passengers) plus the Earth. This system meets our criterion for an isolated system, because the track (which is not part of the system) does no work on the car (we assume no friction, and the normal force of the track on the car does no work because its direction is always perpendicular to the displacement of the car). When the car is at rest at the top of the track, the total mechanical energy is

$$E_i = U_i + K_i = mgy + 0,$$

where we have taken $y = 0$ at the bottom of the track. When the car reaches the bottom, the mechanical energy E_f is

$$E_f = U_f + K_f = 0 + \tfrac{1}{2}mv^2,$$

with the reference for U chosen so that $U = 0$ at $y = 0$. Conservation of energy means $E_i = E_f$, and thus

$$mgy = \tfrac{1}{2}mv^2.$$

Solving for v, we obtain

$$v = \sqrt{2gy} = \sqrt{(2)(9.8\ \text{m/s}^2)(25\ \text{m})} = 22\ \text{m/s}.$$

This is the same speed with which an object dropped vertically from a height of 25 m would hit the ground. The normal force of the track does not change the speed of the "falling" car; it merely changes the car's direction. Note that the result is independent of the mass of the car or of its occupants.

As the roller coaster car travels, its speed increases and decreases as it passes through the valleys and peaks of the track. As long as no peak is higher than the starting point, there is enough mechanical energy in the system to overcome any of the intermediate hills of potential energy and carry the system through to the finish.

Figure 12-6. A device for converting gravitational potential energy into kinetic energy.

You can readily appreciate the advantages of the energy technique from this problem. To use Newton's laws would require knowing the exact shape of the track, and then we would need to find the force components and the acceleration at every point. This could be quite a difficult procedure. On the other hand, the solution using Newton's laws would provide more information than the solution using the energy method—for instance, the time it takes the car to reach the bottom.

Applications of the Conservation of Mechanical Energy

Our law of the conservation of mechanical energy came from the definition of potential energy ($W = -\Delta U$) and from the work–energy theorem ($W = \Delta K$), which was in turn obtained from Newton's second law. We can therefore use the law of conservation of mechanical energy to analyze conservative systems to which we have previously applied Newton's laws. By way of illustration, we reconsider some problems that we have already solved using Newton's laws. We discuss only problems in linear mechanics in which the forces are conservative and the bodies behave like particles.

SAMPLE PROBLEM 12-5. Using conservation of mechanical energy, analyze the Atwood's machine (Sample Problem 5-5) to find the velocity and the acceleration of the blocks after they have moved a distance y from rest.

Solution Review the problem and the free-body diagram from Fig. 5-9. For our system we take the two blocks plus the Earth. For simplicity, we assume that both blocks start from rest at the same level, which we define as $y = 0$, the reference point for gravitational potential energy. The initial potential energy is therefore zero. The initial kinetic energy is also zero, and so $E_i = 0$. After the system is released, block 1 moves up to position $+y$, block 2 moves down to position $-y$, and both blocks are moving with speed v. The final total mechanical energies are therefore $\frac{1}{2}m_1v^2 + m_1gy$ for block 1 and $\frac{1}{2}m_2v^2 - m_2gy$ for block 2. Conservation of mechanical energy then gives $E_f = E_i$, or

$$\tfrac{1}{2}m_1v^2 + m_1gy + \tfrac{1}{2}m_2v^2 - m_2gy = 0.$$

Solving for the speed v, we obtain

$$v = \sqrt{2\frac{m_2 - m_1}{m_1 + m_2}gy}.$$

Block 1 is therefore moving upward with velocity $v_y = +v$ and acceleration $a_y = dv_y/dt$:

$$a_y = \frac{dv_y}{dt} = \frac{1}{2}\left(2\frac{m_2 - m_1}{m_1 + m_2}gy\right)^{-1/2}\left(2\frac{m_2 - m_1}{m_1 + m_2}g\right)\frac{dy}{dt}.$$

If we replace dy/dt with the expression for the velocity v_y and rearrange the terms, we obtain

$$a_y = \frac{m_2 - m_1}{m_1 + m_2}g.$$

This is the same result we obtained in Sample Problem 5-5, which demonstrates that methods based on Newton's laws and on energy conservation give identical results.

SAMPLE PROBLEM 12-6. Using conservation of mechanical energy, find the speed of the blocks in Sample Problem 5-6 after they have moved a distance L from rest.

Solution Review the problem and Fig. 5-10. Our system is both blocks plus the Earth. For this problem we will use the form of energy conservation given in Eq. 12-12. The initial kinetic energy is zero, so the change in kinetic energy is $\Delta K = K_f = \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2$, where v is the speed of the blocks after they have moved a distance L. There is no potential energy change for block 1 (which moves horizontally), so the net change in potential energy is that due to the change in the vertical position of block 2, or $\Delta U = m_2g\Delta y = m_2g(-L)$. Conservation of mechanical energy then gives

$$\Delta K + \Delta U = \tfrac{1}{2}m_1v^2 + \tfrac{1}{2}m_2v^2 - m_2gL = 0.$$

Solving for the speed v, we obtain

$$v = \sqrt{\frac{2m_2gL}{m_1 + m_2}}.$$

Once again, you should show that differentiating this expression with respect to time (treating dL/dt as an appropriate velocity component) leads to the expression for the acceleration found in the solution to Sample Problem 5-6.

12-4 ENERGY CONSERVATION IN ROTATIONAL MOTION

In Section 11-7 we discussed how to apply concepts of work and kinetic energy to problems involving rotational motion. We can also apply the conservation of mechanical energy to analyze the motion of systems involving objects that can rotate about an axis as well as move translationally. There is no *separate* conservation law for rotational motion; instead, the kinetic energies in Eq. 12-15 may contain both rotational and translational terms.

SAMPLE PROBLEM 12-7. Using conservation of mechanical energy, reconsider Sample Problem 9-10 to find the speed of the block after it falls from rest through a distance of 0.56 m.

Solution Review the problem and Fig. 9-26. Our system is the block, the disk, and the Earth. If the block falls from rest, then $K_i = 0$ for both the block and the disk. Let $y = 0$ be the initial position of the block, where $U_i = 0$; after the block falls to vertical coordinate $-y$, its potential energy is $U_f = mg(-y)$. The final kinetic energy of the falling block is $\frac{1}{2}mv^2$, and the final kinetic energy of the disk is $\frac{1}{2}I\omega^2$. Because the cord does not stretch, the speed of the falling block is the same as the tangential speed of the disk, and so $\omega = v/R$. Conservation of mechanical energy then gives $E_i = E_f$, or

$$0 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I(v/R)^2 - mgy$$

and solving for v (with $I = \frac{1}{2}MR^2$ for the disk) we find

$$v = \sqrt{\frac{4mgy}{M + 2m}}$$

$$= \sqrt{\frac{4(1.2\ \text{kg})(9.8\ \text{m/s}^2)(0.56\ \text{m})}{2.5\ \text{kg} + 2(1.2\ \text{kg})}} = 2.3\ \text{m/s}.$$

You should be able to show that the acceleration found in Sample Problem 9-10 leads to this vertical speed. Once again, we see that methods based on Newton's laws and on energy conservation give identical results.

Combined Rotational and Translational Motion

In Section 9-7, we considered the analysis of combined rotational and translational motion using Newton's laws. Now let us consider a different analysis based on work–energy methods. As we did in Section 9-7, we again restrict our analysis to the case in which the rotational axis remains in the same direction in space as the object moves.

Let us first show that the kinetic energy of an arbitrary body in this special case can be written as the sum of independent translational and rotational terms. Figure 12-7

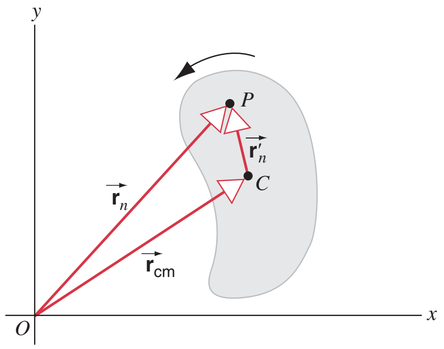


FIGURE 12-7. The center of mass C of a body undergoing both rotational and translational motion is located instantaneously at the position $\vec{\mathbf{r}}_{cm}$. An arbitrary particle P of the body is located at $\vec{\mathbf{r}}_n$ relative to the origin O and at $\vec{\mathbf{r}}'_n$ relative to the center of mass C.

shows an arbitrary body of mass M. The center of mass C is located instantaneously at the position $\vec{\mathbf{r}}_{cm}$ relative to the origin of the chosen inertial reference frame. A particle P of mass m_n is located at the position $\vec{\mathbf{r}}_n$ relative to the origin and at the position $\vec{\mathbf{r}}'_n$ relative to the center of mass of the body. The translational motion is restricted to the xy plane; that is, the vector $\vec{\mathbf{v}}_n$ describing the motion of m_n has only x and y components. The body also rotates with instantaneous angular velocity ω about an axis passing through the center of mass and perpendicular to the page. Relative to O, the kinetic energy of the particle of mass m_n is $\frac{1}{2}m_n v_n^2$ and the total kinetic energy of the body is found from the sum over all such particles:

$$K = \sum_{n=1}^{N} \tfrac{1}{2} m_n v_n^2. \qquad (12\text{-}16)$$

From Fig. 12-7, we see that $\vec{\mathbf{r}}_n = \vec{\mathbf{r}}_{cm} + \vec{\mathbf{r}}'_n$. Differentiating, we find the corresponding relationship between the velocities: $\vec{\mathbf{v}}_n = \vec{\mathbf{v}}_{cm} + \vec{\mathbf{v}}'_n$, where $\vec{\mathbf{v}}_n$ is the velocity of the particle relative to the origin O, $\vec{\mathbf{v}}_{cm}$ is the velocity of the center of mass, and $\vec{\mathbf{v}}'_n$ is the velocity of the particle relative to the center of mass. Observed from the reference frame of the center of mass, the motion is pure rotation about an axis through the center of mass; thus $\vec{\mathbf{v}}'_n$ has magnitude $\omega r'_n$.

The quantity v_n^2 that appears in Eq. 12-16 can be written as $\vec{\mathbf{v}}_n \cdot \vec{\mathbf{v}}_n$ or, using the velocity transformation equation $\vec{\mathbf{v}}_n = \vec{\mathbf{v}}_{cm} + \vec{\mathbf{v}}'_n$, as $(\vec{\mathbf{v}}_{cm} + \vec{\mathbf{v}}'_n)\cdot(\vec{\mathbf{v}}_{cm} + \vec{\mathbf{v}}'_n) = \vec{\mathbf{v}}_{cm}\cdot\vec{\mathbf{v}}_{cm} + 2\vec{\mathbf{v}}_{cm}\cdot\vec{\mathbf{v}}'_n + \vec{\mathbf{v}}'_n\cdot\vec{\mathbf{v}}'_n$. The kinetic energy from Eq. 12-16 can then be written as

$$K = \sum_{n=1}^{N} \tfrac{1}{2} m_n v_n^2$$

$$= \sum_{n=1}^{N} \tfrac{1}{2} m_n (v_{cm}^2 + 2\vec{\mathbf{v}}_{cm}\cdot\vec{\mathbf{v}}'_n + v_n'^2). \qquad (12\text{-}17)$$

Let us consider the three terms in this sum individually: (1) In the first term in Eq. 12-17, the only quantity that involves the summation index n is the particle mass m_n, and for this term $\Sigma\, m_n = M$, the total mass of the body. (Note that v_{cm} passes through the summation symbol because it does not depend on the index n.) This term then becomes $\Sigma\, \frac{1}{2} m_n v_{cm}^2 = \frac{1}{2} M v_{cm}^2$. (2) In the second term we have $\Sigma\, \frac{1}{2} m_n (2\vec{\mathbf{v}}_{cm}\cdot\vec{\mathbf{v}}'_n) = \vec{\mathbf{v}}_{cm}\cdot\Sigma\, m_n \vec{\mathbf{v}}'_n$. The quantity $\Sigma\, m_n \vec{\mathbf{v}}'_n$ is the total momentum of all the particles in the body, measured in the center-of-mass reference frame: $\vec{\mathbf{P}}' = \Sigma\, m_n \vec{\mathbf{v}}'_n$, which is zero as shown in Eq. 7-24. (3) The third term in Eq. 12-17 can be simplified if we recall that the motion in the primed (center-of-mass) reference frame is a pure rotation, and so $v'_n = \omega r'_n$. The third term then becomes $\Sigma\, \frac{1}{2} m_n v_n'^2 = \Sigma\, \frac{1}{2} m_n r_n'^2 \omega^2$. The summation here gives the rotational inertia in the center-of-mass frame $I_{cm} = \Sigma\, m_n r_n'^2$, and so $\Sigma\, \frac{1}{2} m_n v_n'^2 = \frac{1}{2} I_{cm}\omega^2$. With the middle term equal to zero, the remaining two terms of Eq. 12-17 then give

$$K = \tfrac{1}{2} M v_{cm}^2 + \tfrac{1}{2} I_{cm}\omega^2. \qquad (12\text{-}18)$$

Equation 12-18 indicates that the total kinetic energy of the moving object consists of two terms, one associated with the pure translation of the center of mass of the object at velocity $\vec{\mathbf{v}}_{cm}$, and the other associated with pure rotation about an axis through the center of mass. The two terms are quite independent: the rotation would be present even in the absence of translation (for example, as observed from a frame of reference moving at $\vec{\mathbf{v}}_{cm}$). The velocities $\vec{\mathbf{v}}_{cm}$ and $\vec{\boldsymbol{\omega}}$ are, in this general case, independent of one another: we can provide any amount of rotational kinetic energy and any amount of translational kinetic energy.

Rolling without Slipping. For the special case of rolling without slipping, which we also discussed in Section 9-7, the angular speed and the center-of-mass speed are not independent—they are related by $v_{cm} = \omega R$ for an object of radius R. The total kinetic energy is then completely determined by *either* the translational speed v_{cm} or the rotational speed ω, and we can find the corresponding expressions for the kinetic energy by substituting into Eq. 12-18:

$$K = \tfrac{1}{2} M v_{cm}^2 + \tfrac{1}{2} I_{cm} v_{cm}^2/R^2, \qquad (12\text{-}19a)$$

$$K = \tfrac{1}{2} M \omega^2 R^2 + \tfrac{1}{2} I_{cm}\omega^2. \qquad (12\text{-}19b)$$

In either case, only one parameter (v_{cm} or ω) is sufficient to determine the kinetic energy.

When an object rolls without slipping, there is a frictional force exerted at the instantaneous point of contact between the object and the surface on which it rolls (Fig. 9-33, for example). However, *this frictional force does no work on the moving object* because the point of application of the force does not move. That is, the force does not move a point on the object through some distance. Instead, the frictional force is applied first at one point on the object and then, as the object rotates, at a different point on the object. An ideal wheel can roll without slipping on a horizontal surface at constant translational and rotational velocity; if there were external work done on the wheel (for example by friction), its kinetic energy would change, which is not the case. If the wheel were instead *sliding* on the surface, then the frictional force would do work and would change the translational and rotational kinetic energies.

SAMPLE PROBLEM 12-8. Using energy conservation, find the final speed of the rolling cylinder in Fig. 9-32 when it reaches the bottom of the plane.

Solution Figure 9-32 shows the forces that act on the rolling cylinder. For our system we take the cylinder and the Earth. Even though there is a frictional force present, it does no work and so it cannot change the mechanical energy. The initial kinetic energy is zero and the initial potential energy is $Mgh = MgL\sin\theta$ relative to the base of the plane where $U = 0$; thus $E_i = K_i + U_i = 0 + MgL\sin\theta$. The final potential energy is zero (because that is our chosen reference point), and the kinetic energy is given by Eq. 12-19a in terms of the final translational speed of the center of mass; thus $E_f = K_f + U_f = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}v_{cm}^2/R^2 + 0$. Setting $E_f = E_i$, we obtain

$$\tfrac{1}{2}Mv_{cm}^2 + \tfrac{1}{2}I_{cm}v_{cm}^2/R^2 = MgL\sin\theta.$$

With $I_{cm} = \frac{1}{2}MR^2$, we can solve for v_{cm} to find

$$v_{cm} = \sqrt{\tfrac{4}{3}gL\sin\theta},$$

in agreement with the result of Sample Problem 9-11.

SAMPLE PROBLEM 12-9. Find the final angular speed of the yo-yo of Sample Problem 9-13 using energy conservation.

Solution The motion of the yo-yo as it unwinds down the string is another example of combined rotational and translational motion. The point of contact between the cord and the shaft plays the same role as the point of contact between the wheel and the ground in rolling without slipping. For our system we take the yo-yo plus the Earth. The yo-yo has initial translational speed v_0 and final angular speed ω, so its change in kinetic energy is (using Eq. 12-19b for K_f and Eq. 12-19a for K_i)

$$\Delta K = K_f - K_i = (\tfrac{1}{2}M\omega^2R_0^2 + \tfrac{1}{2}I_{cm}\omega^2) - \left(\tfrac{1}{2}Mv_0^2 + \tfrac{1}{2}I_{cm}\frac{v_0^2}{R_0^2}\right).$$

Let the yo-yo fall from initial location $y = 0$, where $U_i = 0$, to the vertical coordinate $-y$, where $U_f = Mg(-y)$. The change in potential energy as the yo-yo falls is then $\Delta U = -Mgy$. Setting $\Delta K + \Delta U = 0$ and solving for the final angular speed ω, we obtain

$$\omega = \sqrt{\left(\frac{v_0}{R_0}\right)^2 + \frac{2gy}{R_0^2 + R^2/2}}.$$

You should show that by differentiating the above expression for ω you obtain the expression for α derived in Sample Problem 9-13 using Newton's laws.

12-5 ONE-DIMENSIONAL CONSERVATIVE SYSTEMS: THE COMPLETE SOLUTION

Our goal in the analysis of a mechanical system is often to describe the motion of a particle as a function of the time. In Chapters 3 and 4 we showed how to solve this problem by applying Newton's laws, which, in one dimension, allow us to solve for the position and velocity as functions of the time. In this chapter we have solved many of the same problems using conservation of energy, which can yield the velocity of the bodies in a system in a final configuration that is different from the initial configuration. These two methods, the dynamical or force method and the energy method, give identical results, but the energy method as we have applied it so far does not give the position and velocity of bodies in the system as functions of the time. In this section, we show how the energy method can be extended to provide this information.

We assume a one-dimensional system with a force that depends only on position.* Associated with this force is the potential energy function $U(x)$, which also depends on the coordinates. Equation 12-13 for the definition of mechanical energy, $E = K + U$, gives a relationship between x and v_x:

$$U(x) + \tfrac{1}{2}mv_x^2 = E \qquad (12\text{-}20)$$

and solving for v_x, we obtain

$$v_x = \pm\sqrt{\frac{2}{m}[E - U(x)]}. \qquad (12\text{-}21)$$

For any particular value of the total mechanical energy E of the system, Eq. 12-21 tells us that the motion is restricted to regions of the x axis where $E > U(x)$, because we cannot allow a negative kinetic energy or an imaginary velocity.

If we plot $U(x)$ as a function of x, we can get a good qualitative description of the motion, based on Eq. 12-21. For example, consider the potential energy function shown in Fig. 12-8a. This represents the potential energy of a particle moving in one dimension along the x axis. The relationship between the potential energy and the force is determined by Eq. 12-7, $F_x = -dU/dx$. The force corresponding to this potential energy is shown in Fig. 12-8b. We consider several different possible choices for the total mechanical energy of the system. For any particular value of the energy (for example, E_4), the kinetic energy at any point (for example, x_4) is found from the difference between the total energy and the potential energy.

$\boldsymbol{E = E_0}$. This is the lowest possible energy of the system. At this point $E = U$, so $K = 0$. The particle must be at rest at the point x_0.

$\boldsymbol{E = E_1}$. With this energy, the particle can move in the region between x_1 and x_2. Since the kinetic energy is the difference between E and $U(x)$, we see from the graph that the particle has its maximum kinetic energy, and thus its maximum speed, at x_0. As it approaches x_1 or x_2, the speed decreases. At x_1 and x_2 the particle stops

*In one dimension, forces that depend only on position are *always* conservative; this is not necessarily true in two or three dimensions, as we discuss in Section 12-6. The (constant) gravitational force is conservative, even though it does not depend explicitly on position. However, the (constant) frictional force is *not* conservative, because its direction depends on the direction of motion and not on the position; it can thus be regarded as a velocity-dependent force.

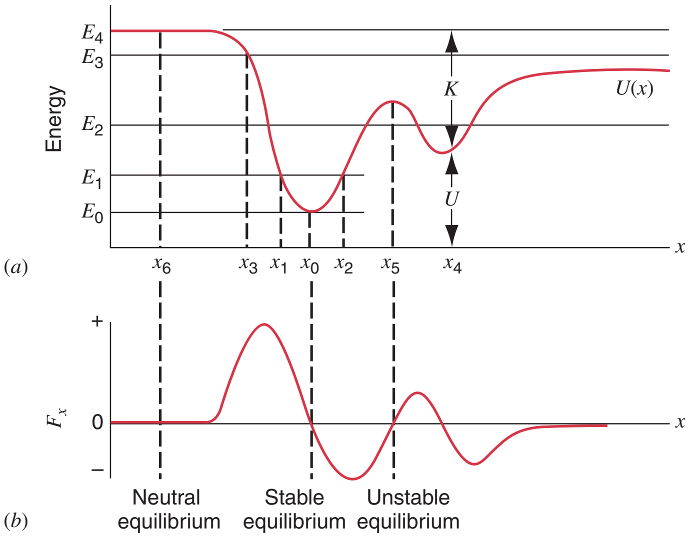


FIGURE 12-8. (*a*) A potential energy function $U(x)$. (*b*) The x component of the force corresponding to that potential energy.

and reverses its direction. The points x_1 and x_2 are called *turning points* of the motion.

$\boldsymbol{E = E_2}$**.** At the energy E_2 there are four turning points, and the particle can move back and forth in either of the two valleys of the potential energy function.

$\boldsymbol{E = E_3}$**.** At this energy there is only one turning point in the motion, at x_3. If the particle is initially moving in the negative x direction, it will stop at x_3 and then move in the positive x direction.

$\boldsymbol{E = E_4}$**.** At energies above E_4 there are no turning points, and the particle does not reverse direction. The speed changes according to Eq. 12-21 as the particle moves.

At a point where $U(x)$ has a minimum value, such as at $x = x_0$, the slope of the curve is zero, and therefore the force is zero; that is, $F_x(x_0) = -(dU/dx)_{x=x_0} = 0$. A particle at rest at this point will remain at rest. Furthermore, if the particle is displaced slightly in either direction, the force, $F_x(x) = -dU/dx$, will tend to return it, and it will oscillate about the equilibrium point. This equilibrium point is therefore called a point of *stable equilibrium.* If the particle moves slightly to the left of x_0 (that is, to smaller x), the force is positive and the particle is pushed toward larger x (that is, back toward x_0). If the particle moves to the right of x_0, it experiences a negative force that again moves it back toward x_0.

At a point where $U(x)$ has a maximum value, such as at $x = x_5$, the slope of the curve is zero so that the force is again zero; that is, $F_x(x_5) = -(dU/dx)_{x=x_5} = 0$. A particle at rest at this point will remain at rest. However, if the particle is displaced even the slightest distance from this point, the force $F_x(x)$ will tend to push it farther from the equilibrium position. Such an equilibrium point is therefore called a point of *unstable equilibrium.* At the point in Fig. 12-8*b* corresponding to x_5, moving away from x_5 to the right (toward larger x) results in a positive force that pushes the particle toward even larger x.

In an interval in which $U(x)$ is constant, such as near $x = x_6$, the slope of the curve is zero, and so the force is zero; that is, $F_x(x_6) = -(dU/dx)_{x=x_6} = 0$. Such a location is called one of *neutral equilibrium,* since a particle can be displaced slightly without experiencing either a repelling or a restoring force.

From this it is clear that if we know the potential energy function for the region of x in which the body moves, we know a great deal about the motion of the body.

SAMPLE PROBLEM 12-10. The potential energy function for the force between two atoms in a diatomic molecule can be expressed approximately as follows:

$$U(x) = \frac{a}{x^{12}} - \frac{b}{x^6},$$

where a and b are positive constants and x is the distance between atoms. Find (*a*) the equilibrium separation between the atoms, (*b*) the force between the atoms, and (*c*) the minimum energy necessary to break the molecule apart (that is, to separate the atoms from the equilibrium position to $x = \infty$).

Solution (*a*) In Fig. 12-9*a* we show $U(x)$ as a function of x. Equilibrium occurs at the coordinate x_m, where $U(x)$ is a minimum, which is found from

$$\left(\frac{dU}{dx}\right)_{x=x_m} = 0.$$

That is,

$$\frac{-12a}{x_m^{13}} + \frac{6b}{x_m^7} = 0$$

or

$$x_m = \left(\frac{2a}{b}\right)^{1/6}.$$

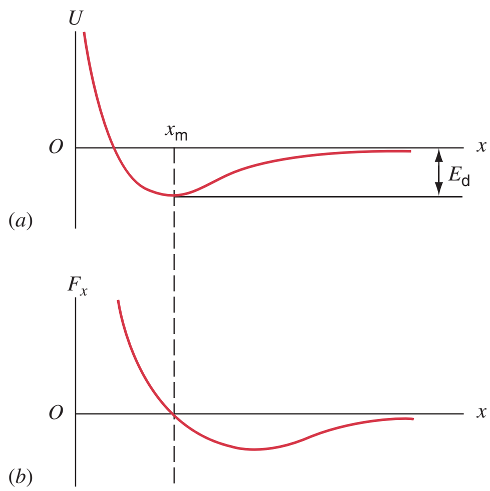


FIGURE 12-9. Sample Problem 12-10. (*a*) The potential energy and (*b*) the force between two atoms in a diatomic molecule as a function of the distance x separating the atoms. Note that the potential energy is taken as zero when the atoms are infinitely separated.

(*b*) From Eq. 12-7, we can find the force corresponding to this potential energy:

$$F_x(x) = -\frac{dU}{dx} = -\frac{d}{dx}\left(\frac{a}{x^{12}} - \frac{b}{x^6}\right) = \frac{12a}{x^{13}} - \frac{6b}{x^7}.$$

We plot the force as a function of the separation between the atoms in Fig. 12-9*b*. When the force is positive (from $x = 0$ to $x = x_m$), the atoms are repelled from one another (the force is directed toward increasing x). When the force is negative (from $x = x_m$ to $x = \infty$), the atoms are attracted to one another (the force is directed toward decreasing x). At $x = x_m$ the force is zero; this is the equilibrium point and is a point of stable equilibrium.

(*c*) The minimum energy needed to break up the molecule into separate atoms is called the *dissociation energy,* E_d. From the potential energy plotted in Fig. 12-9*a*, we see that we can separate the atoms to $x = \infty$, where $U = 0$, whenever $E \geq 0$. The *minimum* energy needed corresponds to $E = 0$, which means that the atoms will be infinitely separated ($U = 0$) and at rest ($K = 0$) in their final state. In the molecule's equilibrium state, however, its energy is all potential so that (see Fig. 12-9*a*) $E = U(x_m)$, a negative quantity. The energy that we must add to the molecule in its equilibrium state to raise its energy from this negative value to zero is what we have called its dissociation energy E_d. Thus

$$U(x_m) + E_d = 0,$$

or

$$E_d = -U(x_m) = -\frac{a}{x_m^{12}} + \frac{b}{x_m^6}.$$

Inserting the value for x_m, we find

$$E_d = \frac{b^2}{4a},$$

which is a positive quantity, as it must be. This energy could be supplied by doing external work on the molecule, perhaps using electric forces, or else by increasing the kinetic energy of one atom of the molecule relative to the other.

General Solution for $x(t)$

If we can find $x(t)$, we know all about the future behavior of the particle. Using Newton's laws, we can obtain this function by first finding the acceleration. Let us see how we can use energy methods to reach this same goal.

We begin with Eq. 12-21. With $v_x = dx/dt$, we can solve for dt to obtain

$$dt = \frac{dx}{\pm\sqrt{(2/m)[E - U(x)]}}. \qquad (12\text{-}22)$$

Note that the two variables in this equation are separated, t appearing only on the left and x on the right.

Suppose the particle is initially located at $x = x_0$ when $t = 0$, and it reaches its final position x at time t. We can therefore integrate Eq. 12-22. The integral on the left side, $\int_0^t dt$, gives simply t, so we have

$$t = \int_{x_0}^{x} \frac{dx}{\pm\sqrt{(2/m)[E - U(x)]}}. \qquad (12\text{-}23)$$

In applying this equation, we choose the + sign when v_x is in the positive x direction, and the − sign when v_x is in the negative x direction. If v_x changes direction during the motion, we must break the integral into separate + and − parts.

After carrying out the integration of Eq. 12-23, we would obtain t as a function of x. It is then usually possible to solve for x as a function of t either analytically or numerically.

As an example of this procedure, we will solve Eq. 12-23 for a particle acted on by a spring force, for which $U(x) = \frac{1}{2}kx^2$. At $t = 0$, the particle is located at x_0 and is at rest ($v_x = 0$). At that point its mechanical energy is $E = \frac{1}{2}kx_0^2$, and since the mechanical energy remains constant, its energy at every point has this value. In this case Eq. 12-23 becomes

$$t = \int_{x_0}^{x} \frac{dx}{\pm\sqrt{(2/m)[\frac{1}{2}kx_0^2 - \frac{1}{2}kx^2]}} = \pm\sqrt{\frac{m}{k}}\int_{x_0}^{x}\frac{dx}{\sqrt{x_0^2 - x^2}}.$$

The integral is a standard form that can be found in integral tables and is equal to $-\cos^{-1}(x/x_0)$:

$$t = \pm\sqrt{\frac{m}{k}}\left[-\cos^{-1}\left(\frac{x}{x_0}\right)\Big|_{x_0}^{x}\right] = \pm\sqrt{\frac{m}{k}}\left[-\cos^{-1}\left(\frac{x}{x_0}\right) + 0\right]$$

because $\cos^{-1}(x_0/x_0) = \cos^{-1} 1 = 0$.

With some manipulation we can solve for x to find

$$x(t) = x_0 \cos\sqrt{\frac{k}{m}}\, t.$$

Note that $\cos(\pm\theta) = \cos\theta$.

The one-dimensional motion of a particle acted on by a spring force is a sinusoidal oscillation. In Chapter 17 we will derive this same result using Newton's laws.

12-6 THREE-DIMENSIONAL CONSERVATIVE SYTEMS (Optional)

So far we have discussed potential energy and conservation of mechanical energy in one-dimensional systems in which the force is directed along the line of motion. We can easily generalize to systems in three dimensions, in which the force and displacement may have arbitrary and different directions.

Consider a system in which a particle moves along a path (Fig. 12-10) from an initial location $\vec{\mathbf{r}}_i = x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}} + z_i\hat{\mathbf{k}}$ to final location at $\vec{\mathbf{r}}_f = x_f\hat{\mathbf{i}} + y_f\hat{\mathbf{j}} + z_f\hat{\mathbf{k}}$. The particle is part of a system that exerts a conservative force $\vec{\mathbf{F}}$ on the particle. (For simplicity we again assume that we can focus our attention on this particle and that no work is done on the rest of the system.) Associated with this force is a potential energy function $U(x, y, z)$; as the particle

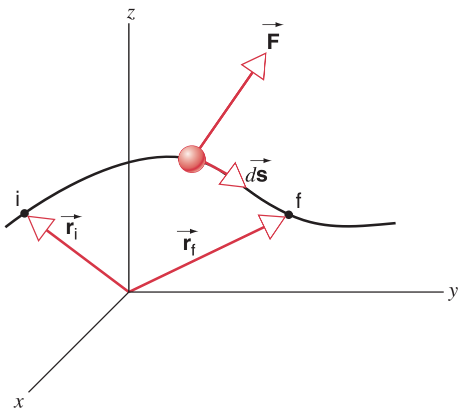


FIGURE 12-10. A particle moves along a path from i to f. A conservative force $\vec{\mathbf{F}}$ acts on the particle.

moves between the initial and final locations, the change in potential energy can be defined by analogy with Eq. 12-5:

$$\Delta U = U(x_f, y_f, z_f) - U(x_i, y_i, z_i)$$
$$= -\int_i^f (F_x\,dx + F_y\,dy + F_z\,dz). \qquad (12\text{-}24)$$

To apply this equation, the path from i to f must be specified; the equation for the path of the particle gives the relationship between dx, dy, and dz. However, because the force is conservative, we get the same value of ΔU for *every* path from i to f. We can then apply the conservation of total mechanical energy in three dimensions with $E = K + U$ if we take $U = U(x, y, z)$ and $K = \frac{1}{2}mv^2 = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}mv_z^2$.

Written more compactly in vector notation, Eq. 12-24 becomes

$$\Delta U = -\int_i^f \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}}, \qquad (12\text{-}25)$$

where $d\vec{\mathbf{s}}$ is a displacement vector tangent to the path ($d\vec{\mathbf{s}} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}$). Here $\vec{\mathbf{F}} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}$, where F_x, F_y, and F_z may be functions of x, y, and z. Equation 12-25 also follows directly from Eq. 11-19 ($W = \int \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}}$) and the definition of potential energy (Eq. 12-4, $\Delta U = -W$).

We can also write Eq. 12-7 in three-dimensional form as*

$$\vec{\mathbf{F}}(x,y,z) = -\frac{\partial U}{\partial x}\hat{\mathbf{i}} - \frac{\partial U}{\partial y}\hat{\mathbf{j}} - \frac{\partial U}{\partial z}\hat{\mathbf{k}}. \qquad (12\text{-}26)$$

In the language of vectors, we say that the conservative force $\vec{\mathbf{F}}$ is written as the negative *gradient* of the potential energy $U(x, y, z)$. For motion along the x axis, Eq. 12-26 reduces to Eq. 12-7.

*The *partial derivative* $\partial U/\partial x$ means we take the derivative of $U(x, y, z)$ with respect to x as if y and z were constant. Similarly, $\partial U/\partial y$ means we differentiate with respect to y as if x and z were constant.

SAMPLE PROBLEM 12-11. In a certain system of particles confined to the xy plane, the force has the form $\vec{\mathbf{F}}(x, y) = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} = -ky\hat{\mathbf{i}} - kx\hat{\mathbf{j}}$, where k is a positive constant. (A particle located at an arbitrary point (x, y) is pushed toward the diagonal line $y = -x$ by this force. You can verify this by drawing the line $y = -x$ and sketching the force components F_x and F_y at various points in the xy plane.) (*a*) Show that the work done by this force when a particle moves from the origin (0, 0) to the point (a, b) is independent of path along the three paths shown in Fig. 12-11. (*b*) Assuming this force to be conservative, find the corresponding potential energy $U(x, y)$ of this system. Take the reference point to be $x_0 = 0$, $y_0 = 0$ and assume $U(0, 0) = 0$.

Solution (*a*) The work done along path 1 can be found by breaking the path into two parts: path 1*a* from $x = 0$ to $x = a$ along the x axis, and path 1*b* vertically from point $(a, 0)$ to point (a, b). The work along path 1*a*, where $d\vec{\mathbf{s}} = dx\,\hat{\mathbf{i}}$, is

$$W_{1a} = \int_i^f \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \int_{x_i}^{x_f} F_x\,dx = \int_{x=0}^{x=a} (-ky)\,dx = 0$$

because $y = 0$ along path 1*a*. Along path 1*b*, $d\vec{\mathbf{s}} = dy\,\hat{\mathbf{j}}$ and $x = a$, so

$$W_{1b} = \int_i^f \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \int_{y=0}^{y=b} (-kx)\,dy = (-ka)\int_0^b dy = -kab.$$

The total work along path 1 is therefore

$$W_1 = W_{1a} + W_{1b} = -kab.$$

Along path 2 we proceed in similar fashion:

$$W_{2a} = \int_i^f \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \int_{y=0}^{y=b} (-kx)\,dy = 0$$

$$W_{2b} = \int_i^f \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \int_{x=0}^{x=a} (-ky)\,dx = (-kb)\int_0^a dx = -kab.$$

Along path 3, $d\vec{\mathbf{s}} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}}$, and

$$W_3 = \int_i^f \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \int_i^f (-ky\,dx - kx\,dy).$$

Let the variable r run along the straight line from (0, 0) to (a, b). With $y = r \sin\phi$, then $dy = dr \sin\phi$ (because ϕ is constant along the line). Also, $x = r\cos\phi$ and $dx = dr\cos\phi$. We treat r as our integration variable, with values in the range from 0 at the origin

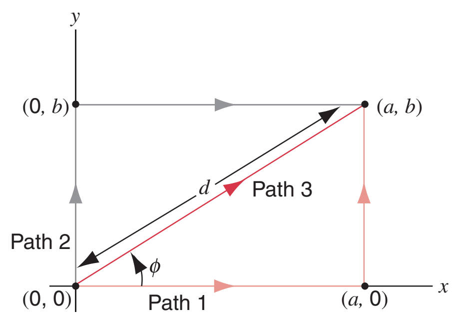


FIGURE 12-11. Sample Problem 12-11. Three different paths are used to evaluate the work done in moving a particle from the origin (0, 0) to the point (a, b).

to $d = (a^2 + b^2)^{1/2}$ at the point (a, b). The integral for W_3 then becomes

$$W_3 = \int_0^d [-k(r \sin \phi)(dr \cos \phi) - k(r \cos \phi)(dr \sin \phi)]$$

$$= -2k \sin \phi \cos \phi \int_0^d r\, dr = -kd^2 \sin \phi \cos \phi.$$

With $\sin \phi = b/d$ and $\cos \phi = a/d$, this becomes $W_3 = -kab$. Thus $W_1 = W_2 = W_3$. This does not prove that $\vec{\mathbf{F}}$ is conservative (we would need to evaluate *all* such paths to make that conclusion), but it certainly leads us to suspect the $\vec{\mathbf{F}}$ might be conservative.

(*b*) The potential energy can be found from Eq. 12-24, which we have in effect already evaluated in finding the work done along path 3. The only difference is that we must integrate to the arbitrary point (x, y) instead of to (a, b). We simply relabel point (a, b) as point (x, y) and thus

$$\Delta U = U(x, y) - U(0, 0) = -W = kxy,$$

where we have taken $U(0, 0) = 0$. You should be able to show that we can apply Eq. 12-26 to this potential energy function and obtain the force $\vec{\mathbf{F}}(x, y)$.

If we change the force slightly to $\vec{\mathbf{F}} = -k_1 y\hat{\mathbf{i}} - k_2 x\hat{\mathbf{j}}$, then the methods of part (*a*) show that this force is not conservative when $k_1 \neq k_2$. (See Exercise 33.) Even when $k_1 = -k_2$, the force is still nonconservative. Such a force has important applications to the magnetic focusing of electrically charged particles, but cannot be represented by a potential energy function, because it is not conservative.

MULTIPLE CHOICE

12-1 Conservative Forces

1. Which of the following forces is *not* conservative?
(A) $\vec{\mathbf{F}} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$ (B) $\vec{\mathbf{F}} = 3x\hat{\mathbf{i}} + 4y\hat{\mathbf{j}}$
(C) $\vec{\mathbf{F}} = 3y\hat{\mathbf{i}} + 4x\hat{\mathbf{j}}$ (D) $\vec{\mathbf{F}} = 3x^2\hat{\mathbf{i}} + 4y^2\hat{\mathbf{j}}$

2. Which of the following forces is conservative?
(A) $\vec{\mathbf{F}} = y\hat{\mathbf{i}} - x\hat{\mathbf{j}}$ (B) $\vec{\mathbf{F}} = yx\hat{\mathbf{i}} - xy\hat{\mathbf{j}}$
(C) $\vec{\mathbf{F}} = y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$ (D) $\vec{\mathbf{F}} = yx\hat{\mathbf{i}} + xy\hat{\mathbf{j}}$

3. Two conservative forces, $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$, act on an object. What is the relationship between

$$W_+ = \oint \left(\vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2\right) \cdot d\vec{\mathbf{s}}$$

and

$$W_- = \oint \left(\vec{\mathbf{F}}_1 - \vec{\mathbf{F}}_2\right) \cdot d\vec{\mathbf{s}}?$$

(The circle on the integral symbol means that the integral is to be evaluated around a closed path.)
(A) $W_+ > W_-$ (B) $W_+ = W_- \neq 0$
(C) $W_+ = W_- = 0$ (D) $W_+ < W_-$

12-2 Potential Energy

4. Which of the following can never be negative?
(A) Mass (B) Time (C) Work
(D) Potential energy (E) Kinetic energy
There may be more than one correct answer.

12-3 Conservation of Mechanical Energy

5. Two blocks are at the top of an inclined ramp. Block *A* slides down the ramp without friction; block *B* falls vertically without friction at the same instant.
(*a*) Which block reaches the bottom first?
(A) Block *A* (B) Block *B*
(C) They arrive at the same time.
(D) There is not enough information to answer the question.
(*b*) Which block reaches the bottom with the larger speed?
(A) Block *A* (B) Block *B*
(C) They arrive with the same speed.
(D) There is not enough information to answer the question.
(*c*) Which block experiences the larger acceleration?
(A) Block *A* (B) Block *B*
(C) They experience the same acceleration.
(D) There is not enough information to answer the question.

12-4 Energy Conservation in Rotational Motion

6. Three rolling objects are moving at the same speed on a level horizontal surface. The objects are a solid cylinder, a solid sphere, and a hollow sphere. All have the same mass and radius. The three objects then roll up an incline. Assuming each rolls without slipping, which will
(*a*) roll to the highest vertical point above the level surface?
(A) The solid cylinder (B) The solid sphere
(C) The hollow sphere
(D) They will all roll to the same height.
(*b*) roll the farthest distance as measured along the incline?
(A) The solid cylinder (B) The solid sphere
(C) The hollow sphere
(D) They will all roll the same distance.

7. Three rolling objects are moving at the same speed on a level surface. The objects are all solid spheres: sphere *A* has radius r and mass m, sphere *B* has radius $2r$ and mass m; sphere *C* has radius r and mass $2m$. The three objects then roll up an incline. Assuming each rolls without slipping, which will
(*a*) roll to the highest vertical point (as measured by the change in the location of the center of mass)?
(A) Sphere *A* (B) Sphere *B* (C) Sphere *C*
(D) They will all roll to the same height.
(*b*) roll the farthest distance as measured along the incline?
(A) Sphere *A* (B) Sphere *B* (C) Sphere *C*
(D) They will all roll the same distance.

8. A cylinder and a block are at the top of an inclined ramp. The cylinder rolls down the ramp without slipping; the block falls vertically without friction at the same instant.
(*a*) Which object reaches the bottom first?

(A) The cylinder (B) The block
(C) They arrive at the same time.
(D) There is not enough information to answer the question.

(*b*) Which object reaches the bottom with the larger speed?
(A) The cylinder (B) The block
(C) They arrive with the same speed.
(D) There is not enough information to answer the question.

(*c*) Which object experiences the larger acceleration?
(A) Cylinder *A* (B) Block *B*
(C) They experience the same acceleration.
(D) There is not enough information to answer the question.

9. A solid sphere of mass m and radius r is projected horizontally out of a cannon without spinning with an initial speed v_0. The sphere immediately lands on a level surface, where it skips a few times but eventually begins to roll without slipping.
(*a*) To find the final speed of the sphere, one must apply
(A) conservation of energy.
(B) conservation of linear momentum.
(C) conservation of angular momentum.
(D) at least two of the previous principles.

(*b*) The final speed for the sphere depends on
(A) the radius.
(B) the mass.
(C) both the mass and the radius.
(D) neither the mass nor the radius.

(See also Exercise 27.)

12-5 One-Dimensional Conservative Systems: The Complete Solution

10. A particle with total energy E moves in one dimension in a region where the potential energy is $U(x)$.
(*a*) The speed of the particle is zero where
(A) $U(x) = E$. (B) $U(x) = 0$.
(C) $dU(x)/dx = 0$.
(D) $d^2U(x)/dx^2 = 0$.

(*b*) The acceleration of the particle is zero where
(A) $U(x) = E$. (B) $U(x) = 0$.
(C) $dU(x)/dx = 0$.
(D) $d^2U(x)/dx^2 = 0$.

12-6 Three-Dimensional Conservative Systems

QUESTIONS

1. Consider the one-dimensional force $\vec{\mathbf{F}} = f(x)\hat{\mathbf{i}}$, where $f(x)$ is a function of x only. Is it possible to determine whether this is a conservative force without any additional information? If so, is it conservative?

2. Consider the two-dimensional force $\vec{\mathbf{F}} = f(x, y)\hat{\mathbf{i}} + g(x, y)\hat{\mathbf{j}}$, where $f(x, y)$ and $g(x, y)$ are both functions of x and y only. Is it possible to determine whether this is a conservative force without any additional information? If so, is it conservative? What if $f(x, y) = f(x)$ and $g(x, y) = g(y)$?

3. A ball is thrown up into the air; at the highest point the potential energy U is a maximum. Is the derivative of U zero at the highest point? If so, what does this say about the force on the ball at the highest point? If not, then how is U a maximum?

4. Mountain roads rarely go straight up the slope but wind up gradually. Explain why.

5. Taking into account how the potential energy of a system of two identical molecules is related to the separation of their centers, explain why a liquid that is spread out in a thin layer has more potential energy than the same mass of liquid in the shape of a sphere.

6. Pole vaulting was transformed when the wooden pole was replaced by the fiberglass pole. Explain why.

7. You drop an object and observe that it bounces to one and one-half times its original height. What conclusions can you draw?

8. An earthquake can release enough energy to devastate a city. Where does this energy reside an instant before the earthquake takes place?

9. The total mechanical energy of a certain isolated system of particles remains constant. If the individual kinetic energies of the particles are also constant, then what can be concluded about the forces that act in this system?

10. In Sample Problem 12-4 (see Fig. 12-6) we concluded that the speed of the roller coaster at the bottom does not depend at all on the shape of the track. Would this still be true if friction were present?

11. Explain, using work and energy ideas, how a child pumps a swing up to large amplitudes from a rest position. (See "How to Make a Swing Go," by R. V. Hesheth, *Physics Education,* July 1975, p. 367.)

12. Two disks are connected by a stiff spring. Can you press the upper disk down enough so that when it is released it will spring back and raise the lower disk off the table (see Fig. 12-12)? Can mechanical energy be conserved in such a case?

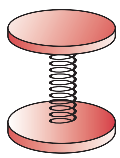


FIGURE 12-12. Question 12.

13. Discuss the words "energy conservation" as used (*a*) in this chapter and (*b*) in connection with an "energy crisis" (for example, turning off lights). How do these two usages differ?

14. Can the translational kinetic energy of a system change into rotational kinetic energy of the system in the absence of external forces? If so, give an example; if not, explain why not.

15. A bowling ball that originally is not spinning is thrown down a bowling lane; by the time the ball strikes the pins it is rolling without slipping. Is the total mechanical energy conserved?

16. Give physical examples of unstable equilibrium, neutral equilibrium, and stable equilibrium.

17. A marble can be balanced on the edge of a bowl so that with a small push it could either (1) roll into the bowl and oscillate back and forth inside it or (2) roll off the bowl, land on the floor, and break. Is this balanced position a point of stable or unstable equilibrium?

18. Is it possible to have an equilibrium point that is both unstable and stable?

EXERCISES

12-1 Conservative Forces

12-2 Potential Energy

1. In one dimension, the magnitude of the gravitational force of attraction between a particle of mass m_1 and one of mass m_2 is given by

$$F_x(x) = G\frac{m_1 m_2}{x^2},$$

where G is a constant and x is the distance between the particles. (*a*) What is the potential energy function $U(x)$? Assume that $U(x) \to 0$ as $x \to \infty$. (*b*) How much work is required to increase the separation of the particles from $x = x_1$ to $x = x_1 + d$?

2. Show that $W \propto d$ for $d \ll x_1$ in Exercise 1. Where have you seen this before?

3. A particle moves along the x axis under the influence of a conservative force that is described by

$$\vec{\mathbf{F}} = -\alpha x e^{-\beta x^2}\hat{\mathbf{i}},$$

where α and β are constants. Find the potential energy function $U(x)$.

12-3 Conservation of Mechanical Energy

4. Each minute, 73,800 m^3 of water passes over a waterfall 96.3 m high. Assuming that 58.0% of the kinetic energy gained by the water in falling is converted to electrical energy by a hydroelectric generator, calculate the power output of the generator. (The density of water is 1000 kg/m^3.)

5. To disable ballistic missiles during the early boost phase of their flight, an "electromagnetic rail gun," to be carried in low-orbit Earth satellites, has been proposed. The gun might fire a 2.38-kg maneuverable projectile at 10.0 km/s. The kinetic energy carried by the projectile is sufficient on impact to disable a missile even if it carries no explosive. (A weapon of this kind is a "kinetic energy" weapon.) The projectile is accelerated to muzzle speed by electromagnetic forces. Suppose instead that we wish to fire the projectile using a spring (a "spring" weapon). What must the force constant be in order to achieve the desired speed after compressing the spring 1.47 m?

6. A 220-lb man jumps out a window into a fire net 36 ft below. The net stretches 4.4 ft before bringing him to rest and tossing him back into the air. What is the potential energy of the stretched net?

7. A very small ice cube is released from the edge of a hemispherical frictionless bowl whose radius is 23.6 cm; see Fig. 12-13. How fast is the cube moving at the bottom of the bowl?

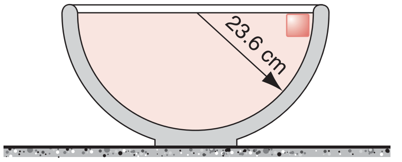


FIGURE 12-13. Exercise 7.

8. A projectile with a mass of 2.40 kg is fired from a cliff 125 m high with an initial velocity of 150 m/s, directed 41.0° above the horizontal. What are (*a*) the kinetic energy of the projectile just after firing and (*b*) its potential energy? (*c*) Find the speed of the projectile just before it strikes the ground. Which answers depend on the mass of the projectile? Ignore air drag.

9. A frictionless roller-coaster car starts at point A in Fig. 12-14 with speed v_0. What will be the speed of the car (*a*) at point B, (*b*) at point C, and (*c*) at point D? Assume that the car can be considered a particle and that it always remains on the track.

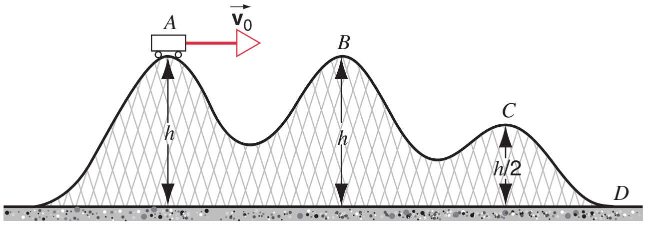


FIGURE 12-14. Exercise 9.

10. Figure 12-15 shows the force as a function of stretch or compression for the spring in a cork gun. The spring is compressed by 5.50 cm and used to propel a cork of mass 3.80 g from the gun. (*a*) What is the speed of the cork if it is released as the spring passes through its relaxed position? (*b*) Suppose now that the cork sticks to the spring, causing the spring to extend 1.50 cm beyond its unstretched length before separation occurs. What is the speed of the cork at the time of release in this case?

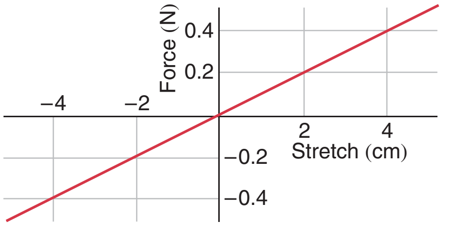


FIGURE 12-15. Exercise 10.

11. Figure 12-16 shows a 7.94-kg stone resting on a spring. The spring is compressed 10.2 cm by the stone. (*a*) Calculate the force constant of the spring. (*b*) The stone is pushed down an additional 28.6 cm and released. How much potential energy is stored in the spring just before the stone is released? (*c*) How high above this new (lowest) position will the stone rise?

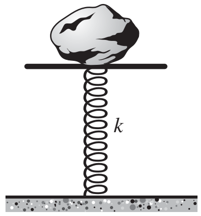


FIGURE 12-16. Exercise 11.

12. The area of the continental United States is about 8×10^6 km^2 and the average elevation of its land surface is about 500 m. The average yearly rainfall is 75 cm. Two-thirds of this rainwater returns to the atmosphere by evaporation, but the rest eventually flows into the oceans. If all this water could be used to generate electricity in hydroelectric power plants, what average power output could be produced?

13. An object falls from rest from a height h. Determine the kinetic energy and the potential energy of the object as a function (*a*) of time and (*b*) of height. Graph the expressions and show that their sum—the total mechanical energy—is constant in each case.

14. In the 1996 Olympic Games, the Bulgarian high jumper Stefka Kostadinova set a women's Olympic record for this event with a jump of 2.05 m; see Fig. 12-17. Other things being equal, how high might she have jumped on the Moon, where the surface gravity is only 1.67 m/s^2? (Hint: The height that "counts" is the vertical distance her center of gravity rose after her feet left the ground. Assume that, at the instant her feet lost contact, her center of gravity was 110 cm above ground level. Assume also that, as she clears the bar, her center of gravity is at the same height as the bar.)

FIGURE 12-17. Exercise 14.

15. A 1.93-kg block is placed against a compressed spring on a frictionless 27.0° incline (see Fig. 12-18). The spring, whose force constant is 20.8 N/cm, is compressed 18.7 cm, after which the block is released. How far up the incline will the block go before coming to rest? Measure the final position of the block with respect to its position just before being released.

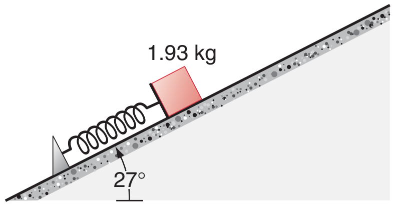


FIGURE 12-18. Exercise 15.

16. A pendulum is made by tying a 1.33-kg stone to a string 3.82 m long. The stone is projected perpendicular to the string, away from the ground, with the string at an angle of 58.0° with the vertical. It is observed to have a speed of 8.12 m/s when it passes its lowest point. (*a*) What was the speed of the stone when projected? (*b*) What is the largest angle with the vertical that the string will reach during the stone's motion? (*c*) Using the lowest point of the swing as the zero of gravitational potential energy, calculate the total mechanical energy of the system.

17. One end of a vertical spring is fastened to the ceiling. A weight is attached to the other end and slowly lowered to its equilibrium position. Show that the loss of gravitational potential energy of the weight equals one-half the gain in spring potential energy. (Why are these two quantities not equal?)

18. A 2.14-kg block is dropped from a height of 43.6 cm onto a spring of force constant $k = 18.6$ N/cm, as shown in Fig. 12-19. Find the maximum distance the spring will be compressed.

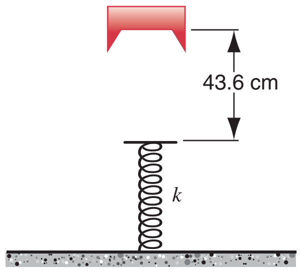


FIGURE 12-19. Exercise 18.

19. Two children are playing a game in which they try to hit a small box on the floor with a marble fired from a spring-loaded gun that is mounted on a table. The target box is 2.20 m horizontally from the edge of the table; see Fig. 12-20. Bobby compresses the spring 1.10 cm, but the marble falls 27.0 cm short. How far should Rhoda compress the spring to score a hit?

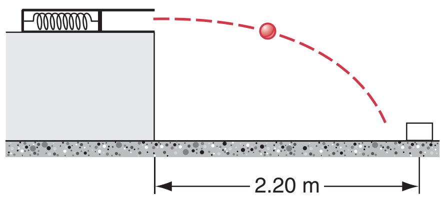


FIGURE 12-20. Exercise 19.

20. Tarzan, who weighs 180 lb, swings from a cliff at the end of a convenient 50-ft vine; see Fig. 12-21. From the top of the cliff to the bottom of the swing, Tarzan would fall by 8.5 ft. The vine has a breaking strength of 250 lb. Will the vine break?

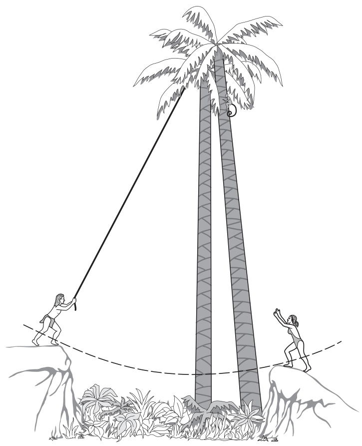


FIGURE 12-21. Exercise 20.

21. Two pendulums each of length L are initially situated as in Fig. 12-22. The first pendulum is released from height d and strikes the second. Assume that the collision is completely inelastic and neglect the mass of the strings and any frictional effects. How high does the center of mass rise after the collision?

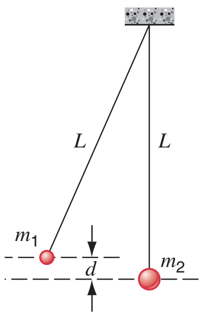


FIGURE 12-22. Exercise 21.

12-4 Energy Conservation in Rotational Motion

22. If $R = 12.3$ cm, $M = 396$ g, and $m = 48.7$ g in Sample Problem 9-10 (Fig. 9-26), find the speed of the block after it has descended 54.0 cm starting from rest. Solve the problem using energy-conservation principles.

23. A uniform spherical shell rotates about a vertical axis on frictionless bearings (Fig. 12-23). A light cord passes around the equator of the shell, over a pulley, and is attached to a small object that is otherwise free to fall under the influence of gravity. What is the speed of the object after it has fallen a distance h from rest?

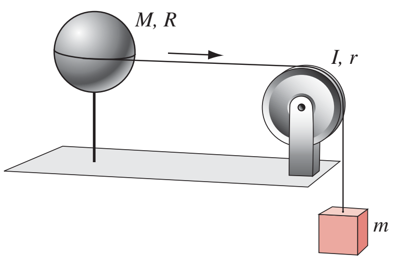


FIGURE 12-23. Exercise 23.

24. A car is fitted with an energy-conserving flywheel, which in operation is geared to the driveshaft so that it rotates at 237 rev/s when the car is traveling at 86.5 km/h. The total mass of the car is 822 kg, the flywheel weighs 194 N, and it is a uniform disk 1.08 m in diameter. The car descends a 1500-m-long, 5.00° slope, from rest, with the flywheel engaged and no power supplied from the motor. Neglecting friction and the rotational inertia of the wheels, find (*a*) the speed of the car at the bottom of the slope, (*b*) the angular acceleration of the flywheel at the bottom of the slope, and (*c*) the power being absorbed by the rotation of the flywheel at the bottom of the slope.

25. A solid sphere of radius 4.72 cm rolls up an inclined plane of inclination angle 34.0°. At the bottom of the incline the center of mass of the sphere has a translational speed of 5.18 m/s. (*a*) How far does the sphere travel up the plane? (*b*) How long does it take to return to the bottom? (*c*) How many rotations does the sphere make during the round trip?

26. A body is rolling horizontally without slipping with speed v. It then rolls up a hill to a maximum height h. If $h = 3v^2/4g$ what might the body be?

27. A solid sphere of mass m and radius r is projected horizontally out of a cannon without spinning with an initial speed v_0. The sphere immediately lands on a level surface, where it skips a few times but eventually begins to roll without slipping. Find the final speed of the sphere. (See Multiple-Choice question 9.)

12-5 One-Dimensional Conservative Systems: The Complete Solution

28. A particle moves along the x axis through a region in which its potential energy $U(x)$ varies as in Fig. 12-24. (*a*) Make a quantitative plot of the force $F_x(x)$ that acts on the particle, using the same x axis scale as in Fig. 12-24. (*b*) The particle has a (constant) mechanical energy E of 4.0 J. Sketch a plot of its kinetic energy $K(x)$ directly on Fig. 12-24.

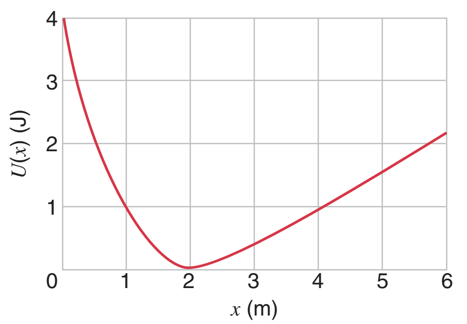


FIGURE 12-24. Exercise 28.

29. A particle of mass 2.0 kg moves along the x axis through a region in which its potential energy $U(x)$ varies as shown in Fig. 12-25. When the particle is at $x = 2.0$ m, its velocity is -2.0 m/s. (*a*) Calculate the force acting on the particle at this position. (*b*) Between what limits does the motion take place? (*c*) How fast is it moving when it is at $x = 7.0$ m?

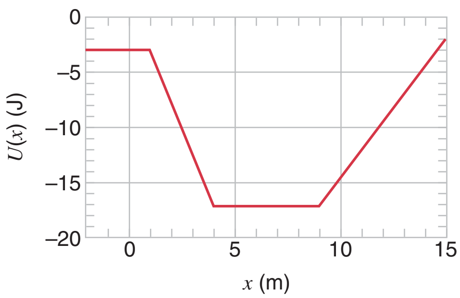


FIGURE 12-25. Exercise 29.

12-6 Three-Dimensional Conservative Systems

30. Show that for the same initial speed v_0 the speed v of a projectile will be the same at all points at the same elevation, regardless of the angle of projection. Ignore air drag.

31. The potential energy corresponding to a certain two-dimensional force is given by $U(x, y) = \frac{1}{2}k(x^2 + y^2)$. (*a*) Derive F_x and F_y and describe the vector force at each point in terms of its coordinates x and y. (*b*) Derive F_r and F_θ and describe the vector force at each point in terms of the polar coordinates r and θ of the point. (*c*) Can you think of a physical model of such a force?

32. The potential energy of a three-dimensional force is given by $U(x, y, z) = -k/\sqrt{x^2 + y^2 + z^2}$. (*a*) Derive F_x, F_y, and F_z and then describe the vector force at each point in terms of its coordinates x, y, and z. (*b*) Convert to spherical polar coordinates and find F_r.

33. By integrating along the same three paths as Sample Problem 12-11, show that the force $\vec{\mathbf{F}} = -k_1 y\hat{\mathbf{i}} - k_2 x\hat{\mathbf{j}}$ is nonconservative when $k_1 \neq k_2$.

PROBLEMS

1. The force on a particle constrained to move along the z axis is given by

$$F_z(z) = \frac{k}{(z+l)^2} - \frac{k}{(z-l)^2}$$

where k and l are fixed constants. Assume that $U(z) \to 0$ as $z \to \infty$. (*a*) Find an *exact* expression for $U(z)$ when $z > l$. (*b*) Show that $U(z) \propto 1/z^2$ for $z \gg l$.

2. A ball of mass m is attached to the end of a very light rod of length L. The other end of the rod is pivoted so that the ball can move in a vertical circle. The rod is pulled aside to the horizontal and given a downward push as shown in Fig. 12-26 so that the rod swings down and just reaches the vertically upward position. What initial speed was imparted to the ball?

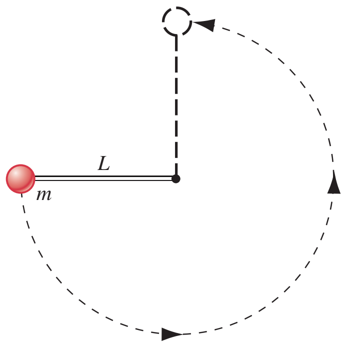


FIGURE 12-26. Problem 2.

3. An ideal massless spring can be compressed 2.33 cm by a force of 268 N. A block whose mass is $m = 3.18$ kg is released from rest at the top of the incline as shown in Fig. 12-27, the angle of the incline being 32.0°. The block comes to rest momentarily after it has compressed this spring by 5.48 cm. (*a*) How far has the block moved down the incline at this moment? (*b*) What is the speed of the block just as it touches the spring?

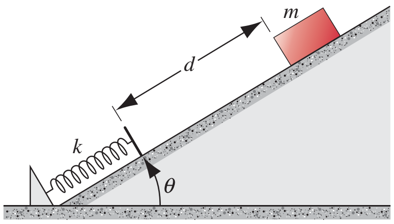


FIGURE 12-27. Problem 3.

4. A chain is held on a frictionless table with one-fourth of its length hanging over the edge, as shown in Fig. 12-28. If the chain has a length L and a mass m, how much work is required to pull the hanging part back on the table?

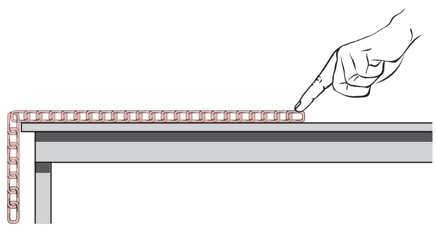


FIGURE 12-28. Problem 4.

5. A small block of mass m slides along the frictionless loop-the-loop track shown in Fig. 12-29. (*a*) The block is released from rest at point P. What is the net force acting on it at point

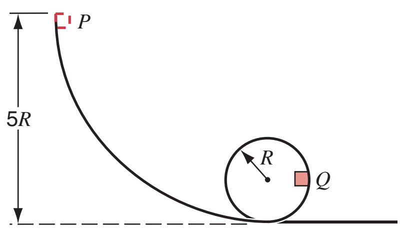


FIGURE 12-29. Problem 5.

Q? (b) At what height above the bottom of the loop should the block be released so that it is on the verge of losing contact with the track at the top of the loop?

6. A block of mass m rests on a wedge of mass M, which, in turn, rests on a horizontal table, as shown in Fig. 12-30. All surfaces are frictionless. If the system starts at rest with point P of the block a distance h above the table, find the speed of the wedge the instant point P touches the table.

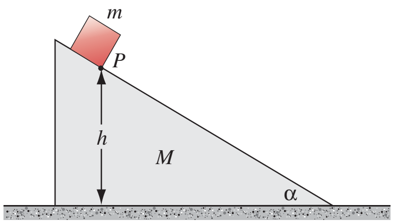


FIGURE 12-30. Problem 6.

7. A 1.18-kg object is acted on by a net conservative force given exactly by $F_x = Ax + Bx^2$ where $A = -3.00$ N/m and $B = -5.00$ N/m^2. (a) Find the potential energy of the system at $x = 2.26$ m. Assume that $U(0) = 0$. (b) The object has a speed of 4.13 m/s in the negative x direction when it is at $x = 4.91$ m. Find its speed as it passes $x = 1.77$ m.

8. The string in Fig. 12-31 has a length $L = 120$ cm, and the distance d to the fixed peg is 75.0 cm. When the ball is released from rest in the position shown, it will follow the arc shown in the figure. How fast will it be going (a) when it reaches the lowest point in its swing and (b) when it reaches its highest point, after the string catches on the peg?

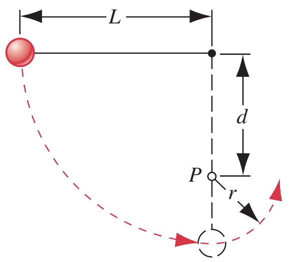


FIGURE 12-31. Problems 8 and 9.

9. In Fig. 12-31 show that, if the ball is to swing completely around the fixed peg, then $d > 3L/5$. (Hint: The ball must be moving at the top of its swing; otherwise the string will collapse.)

10. A block of mass m at the end of a string swings in a vertical circle of radius R under the influence of gravity only. Find the difference between the magnitudes of the tension in the string at the top of the loop and at the bottom of the loop assuming the block is always moving fast enough so that the string never goes slack.

11. A boy is seated on the top of a hemispherical mound of ice (Fig. 12-32). He is given a very small push and starts sliding down the ice. Show that he leaves the ice at a point whose height is $2R/3$ if the ice is frictionless. (Hint: The normal force vanishes as he leaves the ice.)

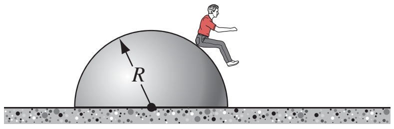


FIGURE 12-32. Problem 11.

12. The particle m in Fig. 12-33 is moving in a vertical circle of radius R inside a track. There is no friction. When m is at its lowest position, its speed is v_0. (a) What is the minimum value v_m of v_0 for which m will go completely around the circle without losing contact with the track? (b) Suppose v_0 is $0.775v_m$. The particle will move up the track to some point P at which it will lose contact with the track and travel along a path shown roughly by the dashed line. Find the angular position θ of point P.

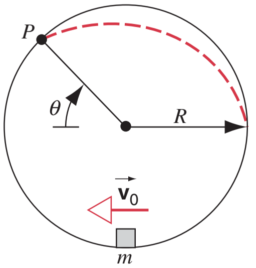


FIGURE 12-33. Problem 12.

13. A rigid body is made of three identical thin rods fastened together in the form of a letter H (Fig. 12-34). The body is free to rotate about a horizontal axis that passes through one of the legs of the H. The body is allowed to fall from rest from a position in which the plane of the H is horizontal. What is the angular speed of the body when the plane of the H is vertical?

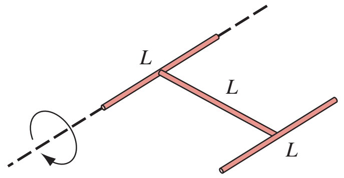


FIGURE 12-34. Problem 13.

14. A small solid marble of mass m and radius r rolls without slipping along the loop-the-loop track shown in Fig. 12-35,

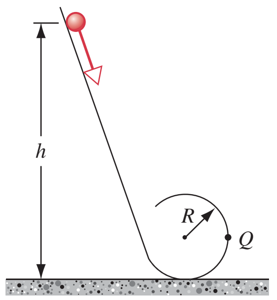


FIGURE 12-35. Problem 14.

having been released from rest somewhere on the straight section of track. (*a*) From what minimum height above the bottom of the track must the marble be released in order that it just stay on the track at the top of the loop? (The radius of the loop-the-loop is R; assume $R \gg r$.) (*b*) If the marble is released from height $6R$ above the bottom of the track, what is the horizontal component of the force acting on it at point Q?

15. A particle is projected horizontally along the interior of a frictionless hemispherical bowl of radius r, which is kept at rest (Fig. 12-36). We wish to find the initial speed v_0 required for the particle to just reach the top of the bowl. Find v_0 as a function of θ_0, the initial angular position of the particle.

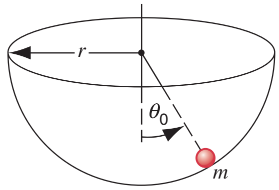


FIGURE 12-36. Problem 15.

16. Figure 12-37*a* shows an atom of mass m at a distance r from a resting atom of mass M, where $m \ll M$. Fig. 12-37*b* shows the potential energy function $U(r)$ for various positions of the lighter atom. Describe the motion of this atom if (*a*) the total mechanical energy is greater than zero, as at E_1, and (*b*) if it is less than zero, as at E_2. For $E_1 = 1.0 \times 10^{-19}$ J and $r = 0.30$ nm, find (*c*) the potential energy, (*d*) the kinetic energy, and (e) the force (magnitude and direction) acting on the moving atom.

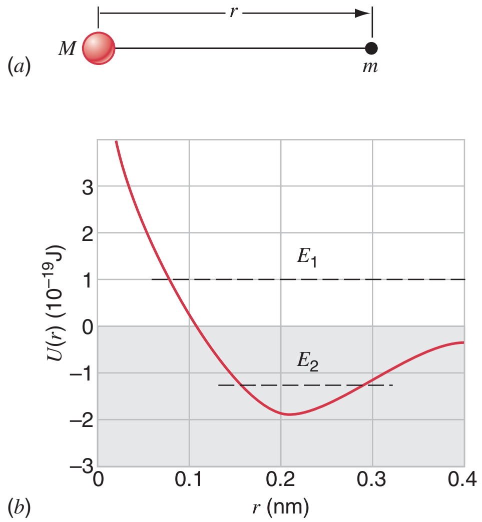


FIGURE 12-37. Problem 16.

17. An alpha particle (helium nucleus) inside a large nucleus is bound by a potential energy like that shown in Fig. 12-38. (*a*) Construct a function of x that has this general shape, with a minimum value U_0 at $x = 0$ and a maximum value U_1 at $x = x_1$ and $x = -x_1$. (*b*) Determine the force between the alpha particle and the nucleus as a function of x. (*c*) Describe the possible motions.

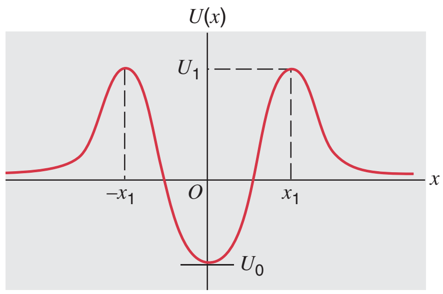


FIGURE 12-38. Problem 17.

18. The so-called Yukawa potential energy

$$U(r) = -\frac{r_0}{r} U_0 e^{-r/r_0}$$

gives a fairly accurate description of the interaction between nucleons (i.e., neutrons and protons, the constitutents of the nucleus). The constant r_0 is about 1.5×10^{-15} m and the constant U_0 is about 50 MeV. (*a*) Find the corresponding expression for the force of attraction. (*b*) To show the short range of this force, compute the ratio of the force at $r = 2r_0$, $4r_0$, and $10r_0$ to the force at $r = r_0$.

COMPUTER PROBLEMS

1. A particle moves along the x axis under the influence of a conservative force that is described by

$$\vec{\mathbf{F}} = -\text{sign}(x) F_0 (1 - e^{-\alpha x^2})\hat{\mathbf{i}}$$

where sign(x) is $+1$ for $x > 0$, -1 for $x < 0$, and 0 when $x = 0$. Here $F_0 = 1$ N and $\alpha = 1\ \text{m}^{-2}$. Numerically create a graph of the potential energy function $U(x)$.

2. A 1.0-kg particle moves in a one-dimensional potential described by $U(x) = Ax^4$, where $A = 1\ \text{J/m}^4$. (*a*) The particle is released from rest at $x = 1$ m; use an appropriate numerical method to find the time it will take before the particle returns to the starting point. (*b*) The particle is released from rest at $x = 2$ m; find the time it will take before the particle returns to the starting point. (*c*) Prepare a graph of return time vs.

starting position for various starting values between $x = 0.1$ m and $x = 10$ m. What is the functional form of this graph?

3. A 1.0-kg particle moves in a two-dimensional potential described by $U(x, y) = A(x^4 + y^4 - 2\alpha x^2 y^2)$, where $A = 1.00\ \text{J/m}^4$ and α is a dimensionless constant that can have any value between 0 and 1. The particle starts from rest at $x = 1.00$ m, $y = 2.00$ m. (*a*) Numerically compute the trajectory of the particle for $\alpha = 0$. Plot the trajectory on an *xy* graph. You might have to experiment with the length of time for which the trajectory is plotted. (*b*) Repeat the process, except use $\alpha = 1$. Plot the trajectory, and compare with your answer to (*a*). This is a classic example of chaotic motion.

CHAPTER 13

ENERGY 3: CONSERVATION OF ENERGY

The law of conservation of energy is one of the main guiding principles of physics. In the storage, conversion, or transfer of energy in mechanical systems, the total energy remains constant. So far we have studied energy conservation in mechanical systems in which no external work is done on the system and in which only conservative forces act among the constituents. In this chapter we consider systems of particles for which the energy can be changed by the work done by external forces, and we also consider nonconservative forces such as friction that might act among the objects within the system or between the system and its environment. These extensions of the law of conservation of energy lead us to introduce another form of energy, the internal energy.

Finally, we discuss a second method for changing the energy of a system—namely, the transfer of heat through the system boundary. This leads us to develop a more general form of the law of conservation of energy called the first law of thermodynamics.

13-1 WORK DONE ON A SYSTEM BY EXTERNAL FORCES

In Section 12-3 we defined the total mechanical energy E of an isolated system as the sum of its kinetic and potential energies, $K + U$. The potential energy arises from the forces that the objects within the system exert on one another, and we assumed those forces to be conservative. In such an isolated system, the total mechanical energy remains constant.

In this chapter we extend this approach in several different ways. We will consider systems in which: (1) external forces can change the total mechanical energy; (2) energy may be stored internally in the motions or interactions among the constituent atoms or molecules; (3) nonconservative forces may act, in particular frictional forces; (4) the energy can be changed through the transfer of heat. In each case we will see how the concept of energy and the law of conservation of energy can be broadened to include these effects. These discussions will provide further evidence of the importance and wide applicability of the law of conservation of energy in physics.

We begin by discussing the effect of external forces that may act on a system. In analyzing a problem, it is often convenient to divide the physical situation into a system and its environment. We imagine that we draw a boundary around the portion of the situation that we define to be the system; within that boundary there may be objects that exert conservative forces on one another, and we represent those forces by their potential energies. The objects in the environment may exert forces that can do external work W_{ext} on the system. Figure 13-1 represents this situation, in which external forces exerted by objects in the system's environment do work that can change the total mechanical energy $K + U$ of the system.

We can think of the external work as providing a means of transferring energy between the system and the environment. *Positive* external work done by the environment on the system carries energy into the system, thereby increasing its total energy; on the other hand, *negative* external

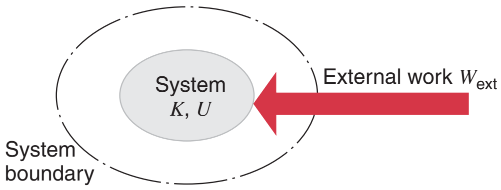


FIGURE 13-1. A system enclosed within the boundary has kinetic energy K and potential energy U (representing only the interactions among components within the system). The environment can exchange energy with the system through the performance of external work W_{ext}. The arrow indicates that energy is being transferred into the system due to the external work; energy and work are scalars and have no associated direction.

work done by the environment on the system transfers energy out of the system, and thus decreases its total energy.

Energy is not created or destroyed by the external work; the work merely represents a *transfer* of energy. For example, if $W_{ext} = +100$ J, then as a result of the external work 100 J of energy is transferred from the environment to the system. In the process, the system gains 100 J of energy and the environment loses 100 J of energy; the total energy of system + environment remains unchanged.

To analyze this case in more detail, we consider a system composed of several objects that can be treated as particles. The work done on any particle in the system may be due to forces exerted by objects inside the system as well as by objects outside of the system. We let *internal work* refer to the work on the particle due to forces exerted by other objects within the system, and we continue to assume that these forces are conservative. These internal forces might include gravitational forces, elastic spring forces, or electrical forces. The *external work* on the particle is done by forces exerted by objects that are outside the system boundary. The net work on a particular particle n is then the total of the internal and external contributions: $W_{net,\,n} = W_{int,\,n} + W_{ext,\,n}$, and the work–energy theorem (Eq. 11-24) reminds us that the net work on particle n is equal to the change in its kinetic energy: $W_{net,\,n} = \Delta K_n$.

We now consider the entire system of many particles. The total change in kinetic energy of the system is simply the sum of the changes in kinetic energy of all the n individual particles: $\Delta K = \Sigma\, \Delta K_n$, and similarly the total external work done on the system is the sum of the work done on all the n particles by the external forces: $W_{ext} = \Sigma\, W_{ext,\,n}$. If the internal forces are conservative, as we have assumed, each can be represented by a potential energy function; the *total* change in potential energy of the system can be found from Eq. 12-4 based on the total internal work done by the particles of the system on one another: $\Delta U = -W_{int} = -\Sigma\, W_{int,\,n}$. From $W_{net,\,n} = W_{int,\,n} + W_{ext,\,n}$ for particle n, we sum over all particles to obtain the work for the entire system $\Sigma\, W_{net,\,n} = \Sigma\, W_{int,\,n} + \Sigma W_{ext,\,n}$, or, making the above substitutions, $\Delta K = -\Delta U + W_{ext}$. We can then write

$$\Delta K + \Delta U = W_{ext}. \tag{13-1}$$

Equation 13-1 is the formal statement of the situation represented in Fig. 13-1: the external work can change the total mechanical energy $K + U$ of the system within the boundary. Note again that positive external work increases the energy: if $W_{ext} > 0$, then $\Delta(K + U) > 0$. Note also that Eq. 12-12 ($\Delta K + \Delta U = 0$) is a special case of Eq. 13-1 that applies to isolated systems (those for which $W_{ext} = 0$).

As an example of how we apply these results, let us consider a block of mass m attached to a vertical spring near the Earth's surface. We release the block, and as it falls the gravitational force acts downward and the spring force acts upward. We are free to choose our system boundary in any convenient way, as illustrated in Fig. 13-2.

1. *System = block (Fig. 13-2a).* Here the spring force and gravity are external forces; there are no internal forces within the system and thus no potential energy. In this case the kinetic energy K of the block changes due to the net effect of the external work done by the spring and gravity, and Eq. 13-1 becomes $\Delta K = W_{spring} + W_{grav}$.

2. *System = block + spring (Fig. 13-2b).* Now the spring is within the system, so we include its interaction with the block through their potential energy. Gravity remains an external force, so $\Delta K + \Delta U_{spring} = W_{grav}$.

3. *System = block + Earth (Fig. 13-2c).* Here gravity is an internal force, but the spring force is external, and so Eq. 13-1 becomes $\Delta K + \Delta U_{grav} = W_{spring}$.

4. *System = block + spring + Earth (Fig. 13-2d).* Now there are no external forces that do work on the system; the spring force and gravity are both internal to the system, so $\Delta K + \Delta U_{spring} + \Delta U_{grav} = 0$, because $W_{ext} = 0$.

If our goal were, for example, to calculate the change in speed of the block after falling a given distance, all of these methods would give the same result, and the choice is often just a matter of convenience.

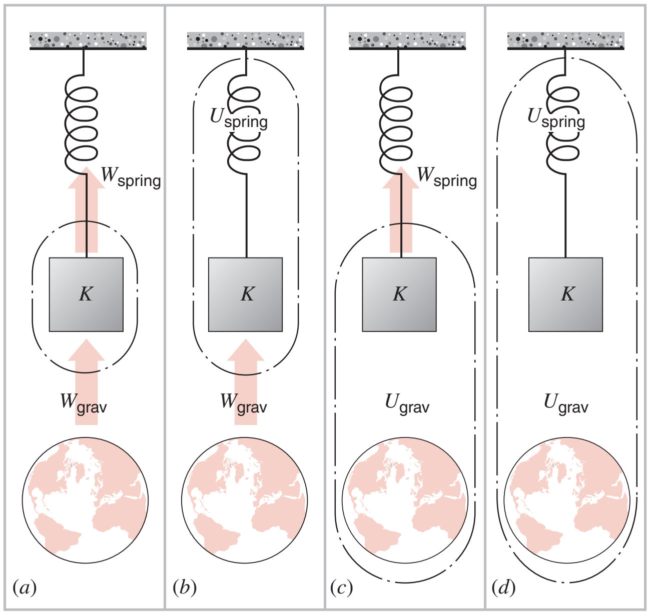


FIGURE 13-2. A block, a spring, and the Earth can be grouped in different ways to define the system and its environment.

13-2 INTERNAL ENERGY IN A SYSTEM OF PARTICLES

Consider an ice skater as she pushes herself away from a railing at the edge of a skating rink. She starts at rest against the railing, and by extending her arms to push away from the railing she begins to slide across the ice.

We will try to analyze this example by applying the law of conservation of mechanical energy in the form of Eq. 13-1 ($\Delta K + \Delta U = W_{ext}$). We define our system to include only the skater. Then clearly $\Delta U = 0$ (there are no other objects within the system to exert forces on the skater). There are three external forces exerted on the skater by bodies in the environment. Clearly gravity and the normal force do no work on the skater. The third force on the skater is that exerted on her by the railing (which is equal and opposite to the force exerted on the railing by the skater); this force likewise does no work, *because the point of application of the force does not move.* Thus for all three external forces, $W_{ext} = 0$. Applying Eq. 13-1, we would then conclude that $\Delta K = 0$, in disagreement with our observation that she accelerates away from the railing. Clearly something is missing from this calculation. Where does the skater's kinetic energy come from?

When a rule (such as Eq. 13-1) that is valid and useful in some cases appears to disagree with experiment in others, physicists usually try to broaden the rule rather than discard it. In its broadened form, the rule can often be made to apply both to the previous valid examples as well as the new examples that seem to deviate from the old rule. How can we broaden the law of conservation of mechanical energy so it can be applied to the example of the skater?

The conservation of mechanical energy was derived from Newton's laws expressed in a form that holds only for single particles. In the examples to which we applied this principle in Chapter 12, each body in the system could be treated as a particle. However, the skater is clearly *not* behaving as a particle—recall that particle behavior requires that all parts of a body move in the same way. As she extends her arm in pushing off from the railing, all parts of her body do not move in the same way, and thus she cannot be treated as a particle. The skater must be treated as a *system of particles,* which has an internal structure; within this system, something is taking place that could not take place in a single particle that, by definition, has no internal structure.

We can extend our concept of energy by postulating that a system consisting of many particles can store energy in a form that we call *internal energy* E_{int}. We then extend Eq. 13-1 to include this new form of energy:

$$\Delta K + \Delta U + \Delta E_{int} = W_{ext}. \tag{13-2}$$

What is the nature of this internal energy? Often we can represent the internal energy as the sum of the kinetic energy associated with the random motions of the atoms or molecules (which usually determines the temperature of the object) and the potential energy associated with the forces between the atoms or molecules: $E_{int} = K_{int} + U_{int}$. In most cases we shall not need to consider the forms that the internal energy may take; we simply regard it as another form of energy in the system.

For example, consider a small metal ball falling through a viscous liquid such as oil. Suppose the ball has reached its terminal speed, so that $\Delta K = 0$ as we observe it over a certain distance. We take the ball, the container of oil, and the Earth as our system. Then the only potential energy associated with forces acting between the objects in our system is that of gravity, ΔU_{grav}. No external forces act on the system, so $W_{ext} = 0$. In this case Eq. 13-2 becomes $\Delta U_{grav} + \Delta E_{int} = 0$, or $\Delta E_{int} = -\Delta U_{grav}$. As the ball falls and ΔU_{grav} decreases, the internal energy increases; that is, the loss in gravitational potential energy of the system is balanced by an increase in the internal energy, so that the total energy of the system remains constant. (The increase in the internal energy, which is associated with changes in the motion or configurations of the atoms of the ball and the oil, might be observed as a slight increase in the temperature of the oil or the ball.)

A similar explanation helps us to understand why a tennis ball released from rest does not quite bounce to the same height from which it was released. During the instant it is in contact with the ground, the flexing and deformation of the ball increase its internal energy at the expense of its kinetic energy; as a result, its speed just after the bounce is smaller than its speed just before the bounce, and so it cannot return to its original height.

You can see from this discussion how we have retained the original concept of conservation of energy. In these examples, energy was transformed from mechanical energy $K + U$ to internal energy E_{int}, but the total amount of energy remained constant.

Let us review the meanings of the terms in Eq. 13-2:

- K is the kinetic energy associated with the overall (translational or rotational) motion of the bodies in the system, measured from any convenient inertial reference frame, typically one fixed in the laboratory.
- U is the potential energy associated with conservative forces that objects within the system exert on one another.
- E_{int} is the internal energy of the system, including the microscopic kinetic and potential energies of the atoms or molecules of the system.
- W_{ext} is the work done by external forces that act on the system.

Now we can see how the inclusion of the internal energy term allows us to analyze the motion of the skater and preserve the notion of conservation of energy. From Eq. 13-2, still with $W_{ext} = 0$ and $\Delta U = 0$, we now have

$$\Delta E_{int} = -\Delta K. \tag{13-3}$$

For the skater, ΔK is positive and therefore, according to Eq. 13-3, ΔE_{int} is negative. The increase in her kinetic energy comes about at the cost of a decrease in her supply of internal energy, which her body obtains from the food she eats. Note that, even though the point of application of the force exerted on the skater by the railing does not move as she pushes away, the center of mass of the skater *does* move as she bends and then straightens her arms. This type of example will require us to examine the motion of the center of mass of a system of particles from the energy point of view; we will do so in Section 13-5.

SAMPLE PROBLEM 13-1. A Chicago Cubs fan drops a baseball (of mass $m = 0.143$ kg) from the top of the Sears Tower at a height h of 443 m (= 1450 ft). The ball reaches a terminal speed v of 42 m/s (see Section 4-4). Find the change in the internal energy of the ball and the surrounding air during the fall to the surface of the Earth.

Solution Let us regard the system as the baseball, the air through which it falls, and the Earth. No external force acts on this system; the gravitational pull of the Earth on the ball and the drag force of the air on the ball are internal forces in the system as we have defined it. The change in potential energy of the system is

$$\begin{aligned}\Delta U &= U_f - U_i = 0 - mgh \\ &= -(0.143 \text{ kg})(9.80 \text{ m/s}^2)(443 \text{ m}) = -621 \text{ J}.\end{aligned}$$

The change in kinetic energy during the fall is

$$\Delta K = K_f - K_i = \tfrac{1}{2}mv^2 - 0 = \tfrac{1}{2}(0.143 \text{ kg})(42 \text{ m/s})^2 = 126 \text{ J}.$$

(We are neglecting the motion of the Earth under the gravitational attraction of the ball.) According to Eq. 13-2, we can write conservation of energy as $\Delta U + \Delta K + \Delta E_{int} = 0$, because there is no external work done on the system. Solving for the internal energy, we obtain

$$\Delta E_{int} = -\Delta U - \Delta K = -(-621 \text{ J}) - 126 \text{ J} = 495 \text{ J}.$$

This internal energy increase might be observed as a temperature rise of the ball and the surrounding air, or perhaps as kinetic energy of the air left in the wake of the falling ball. Using Eq. 13-2 alone, we cannot allocate the energy among these forms. To do so, we must isolate the ball or the air as our system and calculate the work done by the external forces that act. This procedure, which requires knowledge of the drag force between the ball and the air as well as the details of the ball's motion, is too complex for us to solve here.

13-3 FRICTIONAL WORK

Consider a block sliding across a horizontal table and eventually coming to rest due to the frictional force exerted by the table. If we define the system to consist of the block and the tabletop, then no external force does any work on the system (the frictional force is an internal force in this system). Applying Eq. 13-2 to this system, we obtain

$$\Delta K + \Delta E_{int,\, block+table} = 0. \tag{13-4}$$

As the kinetic energy of the block decreases, there is a corresponding increase in the internal energy of the system of block + table. This increase in internal energy might be observed as a slight increase in the temperature of the surfaces of the block and the table. It is a common observation that friction between two surfaces causes an increase in the temperature, as for example in the case of holding a piece of metal against a grinding wheel or applying the brakes to an automobile or a bicycle (in which case both the brakes and the sliding tires can become warmer). You can even observe this effect by rubbing your hands together.

In Section 5-3, we showed that we could analyze mechanical systems with friction using a constant frictional force f equal in magnitude to the coefficient of friction times the normal force. We might be tempted to write the magnitude of the work done by the frictional force as the product of the frictional force times the displacement through which the object moves: $|W_f| = fs$. However, as we shall see, this gives an *incorrect* value for the frictional work. This error comes about because the basic equation for work done in one dimension by a constant force, $W = Fs$, is correct only if the object can be treated as a particle. Objects subject to sliding friction *cannot* be treated as particles from the standpoint of work and energy.

Let us consider an example in which a block is pulled across a horizontal table at constant velocity by a string that exerts a tension force of constant magnitude T (Fig. 13-3). If the velocity is constant, then the acceleration is zero and so the net force must be zero. The magnitude of the frictional force f must then equal the magnitude of the tension T. Let us try to apply Eq. 13-2 to the system consisting only of the block. We assumed that the block moves with constant velocity, and so $\Delta K = 0$. No potential energy exists within the system, and the external work on the block is due to two forces: the tension does positive work W_T and friction does negative work W_f. In this case Eq. 13-2 gives

$$\Delta E_{int,\, block} = W_T + W_f. \tag{13-5}$$

In contrast to Eq. 13-4, here the quantity ΔE_{int} refers only to the block.

Suppose the block moves through a displacement s. Then $W_T = Ts$ (a positive quantity); substituting this result into Eq. 13-5 and solving for the frictional work, we find

$$W_f = -Ts + \Delta E_{int,\, block} = -fs + \Delta E_{int,\, block}, \tag{13-6}$$

where the last result comes about because $T = f$, as we deduced because the net force on the block is zero. Equation 13-6 clearly shows that W_f is not equal to $-fs$. In fact, because $\Delta E_{int,\, block}$ is a positive quantity, we must have $|W_f| < fs$. Work represents energy that is transported across the system boundary; according to Eq. 13-6, the magnitude

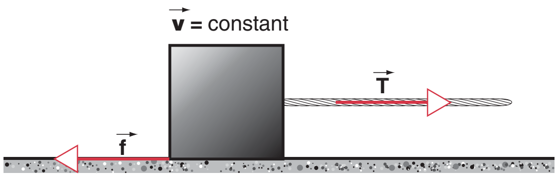


FIGURE 13-3. A block is pulled along a horizontal surface by a string that exerts a tension $\vec{T}$.

of the energy that is transported *out of* the system (the block) due to frictional work is less than fs, because some of the energy remains inside the system as internal energy. Without a more detailed model of the frictional force, we cannot carry Eq. 13-6 any further in finding the frictional work, because we don't know how much of the energy remains in the block as internal energy.

Choosing the table as our system does not improve the situation. Applying Eq. 13-2 to the table alone gives $\Delta E_{\text{int, table}} = W'_f$, where W'_f $(= -W_f)$ represents the frictional work done *on* the table *by* the block, a positive quantity. Positive work done by friction carries energy across the system boundary to increase the internal energy of the table, but again we cannot calculate the amount of this energy transfer.

Let us instead apply conservation of energy to the system consisting of block + table. Now the frictional force is an internal force, and it does not enter into the equations. The only external force is the tension, which does work W_T on the system. We can then write Eq. 13-2 as

$$\Delta E_{\text{int, block+table}} = W_T. \tag{13-7}$$

The work done by the tension force is ultimately responsible for increasing the internal energy (and thus the temperature) of the block and the table. Without a very detailed (and necessarily complicated) model of the properties of the two surfaces, we cannot separate the total internal energy increase into $\Delta E_{\text{int, block}}$ and $\Delta E_{\text{int, table}}$; Eq. 13-7 gives only their sum. Also lacking precise knowledge of the internal energy increase of the block, we cannot use Eq. 13-6 to find the frictional work.

How is it possible that a frictional force f, acting on an object that moves through a displacement s, does work that is in magnitude smaller than fs? The frictional force acting on a sliding surface is not a single force acting at a single point, but is instead due to many smaller forces acting at various surface points (see Fig. 5-14 for an indication of the microscopic character of the frictional force). This force can be regarded as the net effect of the forces at many microscopic welds, some occurring where protrusions from the table bond to the surface of the block, and others occurring where protrusions from the block encounter the surface of the table. As the block moves through a displacement s, only those welds at the moving surface contribute to the work; for the welds at the surface of the table, the displacement is zero and so their contribution to the work is zero. Thus a portion of the frictional force does not contribute to the work, and in this model it is not surprising that $|W_f| < fs$.*

This model of the frictional force is greatly oversimplified, and in fact it is hopelessly complicated to try to account for all of the microscopic welds that are responsible for the frictional force. However, consistent with the energy transfer by work represented in Fig. 13-1, we can describe the frictional process as one in which, depending on how we define the system boundary, energy can be transferred between the objects within a system or between the system and its environment, in either case changing the internal energy of the objects. Without a microscopic model we do not know how the total gain in internal energy is shared among the objects in the system, and we therefore cannot calculate the work done by the frictional force that is responsible for this transformation.

*For a more detailed explanation of this model, see "Work and Heat Transfer in the Presence of Sliding Friction," by B. A. Sherwood and W. H. Bernard, *American Journal of Physics,* November 1984, p. 1001.

SAMPLE PROBLEM 13-2. A 4.5-kg block is thrust up a 30° incline with an initial speed v of 5.0 m/s. It is found to travel a distance $d = 1.5$ m up the plane as its speed gradually deceases to zero. (a) How much internal energy does the system of block + plane + Earth gain in this process due to friction? (b) The block then slides from rest back down the plane. Assuming friction to produce the same gain in internal energy during the downward journey, what is the speed of the block as it passes through its initial location?

Solution (a) Choosing the system of block + plane + Earth, we note that the potential energy change of the block and the Earth is included in the term ΔU in Eq. 13-2. As we did in Sample Problem 13-1, we ignore the kinetic energy changes of the Earth in our calculation and consider only the change in kinetic energy of the block. The change in potential energy of the system is

$$\begin{aligned}\Delta U &= U_f - U_i = mgh - 0 = mgd \sin 30° \\ &= (4.5\text{ kg})(9.8\text{ m/s}^2)(1.5\text{ m})(\sin 30°) = 33\text{ J}.\end{aligned}$$

The change in kinetic energy of the block as it moves from the bottom to the top of the plane is

$$\Delta K = K_f - K_i = 0 - \tfrac{1}{2}mv^2 = -\tfrac{1}{2}(4.5\text{ kg})(5.0\text{ m/s})^2 = -56\text{ J}.$$

The change in mechanical energy of the system is

$$\Delta U + \Delta K = 33\text{ J} + (-56\text{ J}) = -23\text{ J}.$$

The system loses 23 J of mechanical energy. Since $W_{\text{ext}} = 0$ for this system (friction and gravity act *within* this system, as we have defined it), Eq. 13-2 gives $\Delta E_{\text{int}} = -(\Delta U + \Delta K) = +23$ J. The system gains an internal energy of 23 J, which might be revealed as a slight warming of the block and the plane.

(b) Now we consider the round-trip journey as the block moves first up the plane and then back down to its starting point. In part (a) we found the gain in internal energy for the uphill journey to be 23 J. If the downhill portion produces the same gain in internal energy, the change in internal energy for the entire trip up and back down is 46 J. Since the block returns to its starting location, $\Delta U = 0$. Thus for the round trip $\Delta K = -\Delta E_{\text{int}} = -46$ J. With $\Delta K = K_f - K_i$, we have

$$K_f = \Delta K + K_i = -46\text{ J} + 56\text{ J} = 10\text{ J}.$$

The corresponding speed is

$$v_f = \sqrt{\frac{2K_f}{m}} = \sqrt{\frac{2(10\text{ J})}{4.5\text{ kg}}} = 2.1\text{ m/s}.$$

13-4 CONSERVATION OF ENERGY IN A SYSTEM OF PARTICLES

Equation 13-2 is our first step in progressing from a law of conservation of *mechanical* energy in an isolated system (Eq. 12-15) to a more general law of the conservation of energy. The left-hand side of Eq. 13-2 represents the change in the total energy of our system, including kinetic, potential, and internal terms. As we encounter new forms that the energy can take (for example, electrostatic energy or magnetic energy) we can add corresponding terms to the left side of this equation. The right-hand side indicates one way that we can change the energy of the system: we can do external work on it. (Later in this chapter we will find that there is a second way that we can change the energy of a system—by heat transfer.)

Our statement of the law of conservation of energy in Section 12-3 was restricted to isolated systems (those on which external forces do no work). It also included only mechanical energy $K + U$. Our previous statement of the law required that the total mechanical energy remain constant, although we permitted the energy within the system to change forms (kinetic to potential or potential to kinetic).

We can broaden this statement to include the cases we have so far considered in this chapter by including other forms of energy (internal energy, for example), by relaxing the restriction that only conservative forces may act within the system (friction may act between objects in the system), and by allowing external work to be done on the system:

> *Energy may change from one form to another within a system. In an isolated system the total energy remains constant; the total energy of a system can be changed by transferring energy in the form of external work.*

The conservation of energy is, like conservation of linear and angular momentum, a law of nature that has not been contradicted by any laboratory experiment or observation.

We are free to define the system in any convenient way. Once we have drawn the system boundary, we consider all forms of energy that the objects within the system may have: kinetic, potential, or internal. Interactions between the objects within the system can change energy from one form to another, but they cannot change the total energy of the system. To determine whether the total energy changes, we look for objects in the environment of the system that can do work on it.

We illustrate these principles by considering the block–spring combination shown in Fig. 13-4. We assume that the spring is initially compressed and then released, and that a frictional force acts between the block and the table. It is instructive to define the system in several different ways, as suggested by the different system boundaries drawn in Fig. 13-4. We show energy transfers across the system boundaries as arrows that represent the work. The direction of any arrow indicates only the direction of the corresponding energy transfer (into the system or out of the system); work, being a scalar, has no direction in space.

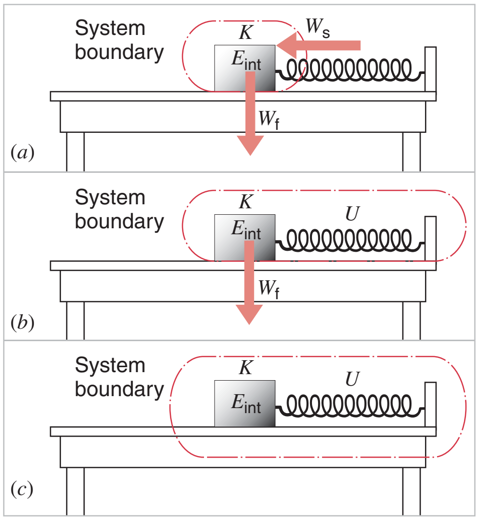


FIGURE 13-4. A block acted on by a spring slides on a table that exerts a frictional force. (*a*) The system consists only of the block; the spring force and friction do work on the system, changing its energy. (*b*) The system now consists of the block and spring, and it has both kinetic and potential energy. (*c*) The system now includes the table. The frictional force is now an internal force and contributes to the internal energy of the system.

1. *System = block.* We first define our system to be the block itself (Fig. 13-4*a*). The figure shows two transfers of energy through the system boundary: the positive work W_s done on the block by the spring and the negative work W_f done on the block by the frictional force exerted by the table. For this system, conservation of energy (Eq. 13-2) can be written as

$$\Delta K + \Delta E_{int} = W_s + W_f. \tag{13-8}$$

Here $\Delta U = 0$, because the system within the boundary experiences no change in potential energy. The spring is not part of the system, so the spring potential energy is not considered; instead, we account for the spring as a part of the environment through the work W_s it does on the system. The weight and the normal force also act on the system, but they do no work so they play no part in this energy analysis. Note the directions of the arrows indicating the energy transfers in Fig. 13-4*a*; Eq. 13-8 indicates that positive work done by the spring tends to increase the energy of the block, and negative frictional work done by the horizontal surface tends to decrease the energy of the block.

2. *System = block + spring.* Now let us consider the system to consist of the block and the spring (Fig. 13-4*b*). The system now has potential energy $\Delta U = -W_s$ (associated with the spring force). The frictional force is the only external force that does work on the system. For this definition of the system, we write conservation of energy as

$$\Delta U + \Delta K + \Delta E_{int} = W_f. \tag{13-9}$$

The energy of the system is now $U + K + E_{\text{int}}$; transfers of energy between the spring and the block do not change the energy of the system in this case. The spring force is an *internal force* that can transfer energy within the system from one form to another ($U \leftrightarrow K$), but it cannot change the *total* energy of the system. Negative (frictional) work by the horizontal surface can decrease the energy of the system.

3. *System = block + spring + table.* Finally, let us define the system to include the table (Fig. 13-4*c*). Now there is no external force responsible for energy transfers that penetrate the system boundary. With this definition of the system, the external work is zero and thus

$$\Delta U + \Delta K + \Delta E_{\text{int}} = 0. \tag{13-10}$$

The frictional force is now an internal force, along with the spring force. Energy can be transferred within the system from the mechanical energy $U + K$ of the block + spring to the internal energy of the block + table, but the total energy (mechanical + internal) remains constant. Suppose, for example, that we release the block from rest with the spring compressed. The block slides back and forth across the table and eventually comes to rest. In this case $\Delta K = 0$ (because $K_f = K_i = 0$), and so $\Delta E_{\text{int}} = -\Delta U$. The potential energy that was originally stored in the system becomes the internal energy of the system; the minus sign indicates that the internal energy increases as the potential energy decreases. From this analysis, we cannot determine the separate changes in internal energy of the block and the table, only the total change for the system as a whole.

Our analysis of Fig. 13-4 suggests that we are free to define the system to which we apply the law of conservation of energy in any way that we wish. Depending on the problem at hand, some choices will be more useful than others. Once we have made a choice we must stay with it, and we must always be clear as to whether the forces that may act and the work that may be done are internal or external to the system.

The frictional force is an example of a nonconservative, dissipative force. In a closed mechanical system such as that illustrated here, mechanical energy is transformed into internal energy by the frictional force. Mechanical energy is *not* conserved in this case, the loss in mechanical energy being compensated by an equivalent gain in the internal energy.

13-5 CENTER-OF-MASS ENERGY

Figure 13-5 shows the ice skater we discussed earlier in this chapter. The skater exerts a force on the railing, and by Newton's third law the railing exerts an equal and opposite force on the skater. This force, which is labeled $\vec{\mathbf{F}}_{\text{ext}}$ in the figure, accelerates the skater from rest to some final velocity $\vec{\mathbf{v}}_{\text{cm}}$.

Let us review what conservation of energy can teach us about this process. Taking the skater as the system, we note

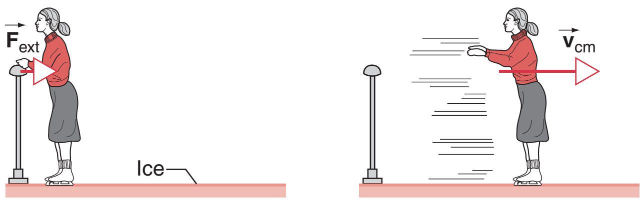


FIGURE 13-5. (*a*) A skater pushing away from a railing. The railing exerts a force $\vec{\mathbf{F}}_{\text{ext}}$ on the skater. (*b*) After pushing off, the skater is moving with velocity $\vec{\mathbf{v}}_{\text{cm}}$.

that in applying Eq. 13-2 there is no change in potential energy of our system; that is, $\Delta U = 0$. Also, there is no external work done on the system (assuming the ice to be frictionless). Even though the railing exerts a force on the skater, it does no work because *the point of application of the force does not move.* That is, in reference to Fig. 13-1, there is no transfer of energy through the system boundary. With $W_{\text{ext}} = 0$, Eq. 13-2 gives

$$\Delta K + \Delta E_{\text{int}} = 0. \tag{13-11}$$

For a skater of mass M, starting from rest, the change in kinetic energy is $\frac{1}{2}Mv_{\text{cm}}^2$ (a positive quantity), so ΔE_{int} must be negative. That is, the kinetic energy the skater gains in pushing away from the railing is derived from a decrease in her store of internal energy and not from any external source.

The conservation-of-energy equation in such a complex system provides only limited information. For instance, the external force does not appear (because it does no work), and so the equation does not permit us to determine the force.

A further complication is that the skater cannot be treated as a particle. For a body to behave like a particle, all parts of it must move in the same way. That is certainly not true of the skater—her arm and body move in different ways.

In Section 7-3, we learned how to analyze a complex system containing many particles. In particular, Eq. 7-16 ($\Sigma \vec{\mathbf{F}}_{\text{ext}} = M\vec{\mathbf{a}}_{\text{cm}}$) relates the net external force acting on a system to the motion of its center of mass. For simplicity, we assume that all forces and motions are in the x direction, and we will not explicitly write the x subscript on the x components of the force, velocity, and acceleration vectors. With only one external force acting, Eq. 7-16 becomes $F_{\text{ext}} = Ma_{\text{cm}}$, in which F_{ext} is the x component of the net external force. Suppose the center of mass moves through the small displacement dx_{cm}. Multiplying on both sides by this quantity, we obtain

$$F_{\text{ext}}dx_{\text{cm}} = Ma_{\text{cm}}dx_{\text{cm}} = M\frac{dv_{\text{cm}}}{dt}v_{\text{cm}}dt,$$

where we have replaced a_{cm} by dv_{cm}/dt and dx_{cm} by $v_{\text{cm}}dt$. This gives

$$F_{\text{ext}}dx_{\text{cm}} = Mv_{\text{cm}}dv_{\text{cm}}. \tag{13-12}$$

Let the center of mass move from x_i to x_f as the velocity changes from $v_{cm,i}$ to $v_{cm,f}$. Integrating Eq. 13-12 between these limits, we find

$$\int_{x_i}^{x_f} F_{ext}\,dx_{cm} = \int_{v_{cm,i}}^{v_{cm,f}} Mv_{cm}dv_{cm} = \tfrac{1}{2}Mv_{cm,f}^2 - \tfrac{1}{2}Mv_{cm,i}^2. \tag{13-13}$$

The terms on the right-hand side of this equation represent the kinetic energy K_{cm} of a particle of mass M moving with the velocity of the center of mass. With this identification, we obtain

$$\int_{x_i}^{x_f} F_{ext}\,dx_{cm} = K_{cm,f} - K_{cm,i} = \Delta K_{cm}. \tag{13-14}$$

In many cases of interest to us, the external force is constant and can be taken out of the integral. The remaining integral gives the net displacement s_{cm} $(= x_f - x_i)$ of the center of mass. In this case Eq. 13-14 becomes

$$F_{ext}\,s_{cm} = \Delta K_{cm}. \tag{13-15}$$

Equations 13-14 and 13-15 resemble the work–energy theorem for a particle. However, it is important to note that although the quantities on the left-hand sides of these equations look like work (and in fact have the dimension of work), they are not work in the sense we have defined it, because dx_{cm} and s_{cm} do not represent the displacement of the point of application of the external force.* (In Fig. 13-5, for example, the displacement of the point of application of the external force was zero, but s_{cm} is certainly not zero.)

Equations 13-14 and 13-15 are *not* expressions of conservation of energy. Translational kinetic energy (of the center-of-mass motion) is the only kind of energy that appears in these expressions. Other energy terms, including the real work, rotational kinetic energy, potential energy, and internal energy, do not appear.

We will refer to Eq. 13-14 or 13-15 as the *center-of-mass (COM) energy equation* and Eq. 13-2 as the *conservation-of-energy (COE) equation.* Note that the COM equation is derived directly from Newton's second law and, although it is a useful formulation, it is not a new and independent principle.

The following examples illustrate the differing and often complementary information that these two equations give.

1. *A sliding block.* A block slides across a horizontal table with initial velocity $\vec{\mathbf{v}}_{cm}$ and is brought to rest by the frictional force f exerted on it by the tabletop. The center of mass of the block moves through a displacement s_{cm}. Our two energy equations give:

COM (Eq. 13-15): $\quad -fs_{cm} = -\tfrac{1}{2}Mv_{cm}^2, \qquad$ (13-16*a*)

COE (Eq. 13-2): $\quad W_f = -\tfrac{1}{2}Mv_{cm}^2 + \Delta E_{int,\,block}. \qquad$ (13-16*b*)

The COM equation *looks like* the work–energy theorem but it is not, because, as we have seen, fs_{cm} is not the magnitude of the frictional work. In this and the following examples, we write COE (Eq. 13-2) as $W_{ext} = \Delta K + \Delta U + \Delta E_{int}$, so that the COM and COE equations look more similar.

2. *Pushing a meter stick.* Figure 13-6 shows the result of pushing on a meter stick (initially at rest) that is free to slide on a frictionless horizontal surface. A constant external force is applied at the 25-cm mark. The point of application of the force moves through the distance s as the center of mass of the stick moves through the distance s_{cm} (which is less than s), and the stick acquires a center-of-mass velocity v_{cm} and a rotational velocity ω. Our two energy equations give

COM: $\quad F_{ext}s_{cm} = \tfrac{1}{2}Mv_{cm}^2, \qquad$ (13-17*a*)

COE: $\quad F_{ext}s = \tfrac{1}{2}Mv_{cm}^2 + \tfrac{1}{2}I\omega^2. \qquad$ (13-17*b*)

The COE equation includes the actual work $(= F_{ext}s)$ done by the external force.

3. *A ball rolling down an incline.* Figure 13-7 illustrates this situation. We consider rolling without slipping (Section 9-7), so that the instantaneous point of contact between the ball and the incline (where the frictional force acts) does not move. The ball starts from rest and acquires a center-of-mass velocity $\vec{\mathbf{v}}_{cm}$ at the bottom of the incline.

COM: $\quad (Mg\sin\theta - f)s_{cm} = \tfrac{1}{2}Mv_{cm}^2, \qquad$ (13-18*a*)

COE: $\quad Mgs_{cm}\sin\theta = \tfrac{1}{2}Mv_{cm}^2 + \tfrac{1}{2}I\omega^2. \qquad$ (13-18*b*)

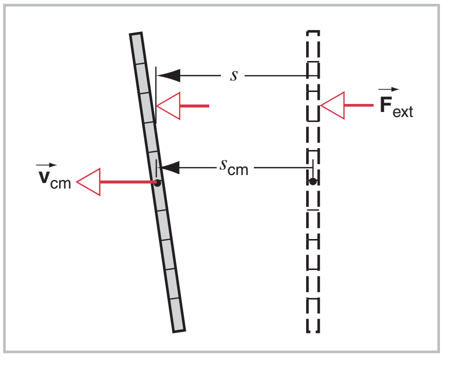


FIGURE 13-6. A meter stick is pushed across a frictionless horizontal surface by a force $\vec{\mathbf{F}}_{ext}$. The force is applied at the 25-cm mark. The stick rotates as well as translates and does not move as a particle. The force is applied through a displacement s that is greater than the displacement s_{cm} of the center of mass.

*Some authors use the terms *pseudowork* or *center-of-mass work* to describe the left side of Eq. 13-14. We prefer *not* to introduce a term closely related to work to describe a quantity that is unrelated to the accepted meaning of work. For a comprehensive summary of work and energy in a system of particles, see "Developing the Energy Concepts in Introductory Physics," by A. B. Arons, *The Physics Teacher,* October 1989, p. 506.

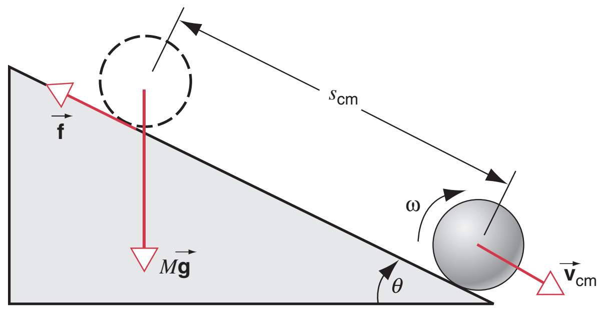


FIGURE 13-7. A ball rolling down an incline. A frictional force $\vec{f}$ acts at the instantaneous point of contact between the ball and the plane. After the ball has moved a distance s_{cm}, its velocity is $\vec{v}_{cm}$ and it is also rotating with angular speed ω.

We have applied the COE equation to the system consisting only of the ball, so gravity appears as external work. The net external force on the ball in the COM equation is $Mg \sin \theta - f$. Note that f appears in the COM equation even though it does no work (and therefore does not appear in the COE equation). Note also that if the ball were slipping as it rolled, the COM equation would be unchanged, but the COE equation would include frictional work on the left and internal energy on the right.

4. *A jumping athlete.* Figure 13-8 shows an athlete first crouching and then leaping by straightening her legs. For simplicity we assume that in straightening her legs she pushes down on the ground with a constant force F in addition to her weight, so the ground exerts a constant normal force $N = F + Mg$. At the instant her foot leaves the ground, her center of mass has risen by s_{cm} and she has a velocity v_{cm}.

COM: $$(N - Mg)\, s_{cm} = \tfrac{1}{2}Mv_{cm}^2, \tag{13-19a}$$

COE: $$-Mgs_{cm} = \tfrac{1}{2}Mv_{cm}^2 + \Delta E_{int}. \tag{13-19b}$$

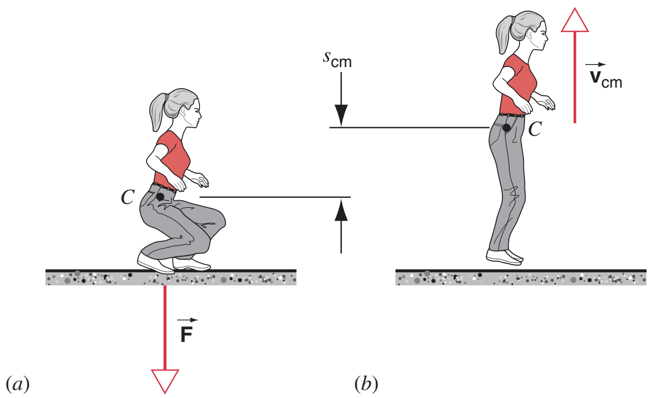


FIGURE 13-8. (*a*) A jumper in a crouching position. She pushes on the floor with force $\vec{F}$ as she straightens her legs to jump. (*b*) At the instant her feet leave the floor, she is moving upward with velocity $\vec{v}_{cm}$ and her center of mass C has risen through a distance s_{cm}.

The COE equation is applied to the system consisting only of the jumper. The normal force does no work, so it does not appear in the COE equation. The term ΔE_{int} accounts for all changes of internal energy in the jumper's body. It might include, for example, a negative term due to the stored energy in her body that she must consume to jump and a positive term from the increase in temperature in the working muscles of her legs. Subtracting the COE and COM equations, we see immediately that the net ΔE_{int} must be negative.

Sample Problem 13-3. A 50-kg ice skater pushes away from a railing as shown in Fig. 13-5, exerting a constant force $F = 55$ N as she does so. Her center of mass moves through a distance $s_{cm} = 32$ cm until she loses contact with the railing. (*a*) What is the speed of the center of mass of the skater as she breaks away from the railing? (*b*) What is the change in the stored internal energy of the skater during this process? Neglect friction between the ice and the skates.

Solution (*a*) Once again we take the skater as our system. From Newton's third law, the railing exerts on the skater a force of 55 N to the right in Fig. 13-5. This force is the only external force that we need to consider. From the COM equation (Eq. 13-15), we have

$$F_{ext}s_{cm} = \tfrac{1}{2}Mv_{cm}^2 - 0$$

or

$$v_{cm} = \sqrt{\frac{2F_{ext}s_{cm}}{M}} = \sqrt{\frac{2(55\text{ N})(0.32\text{ m})}{50\text{ kg}}} = 0.84\text{ m/s}.$$

(*b*) Now we apply the COE equation (Eq. 13-2), which, under the conditions that apply in this problem ($\Delta U = 0$ and $W_{ext} = 0$), takes the form

$$\Delta E_{int} = -\Delta K = -\tfrac{1}{2}Mv_{cm}^2 = -\tfrac{1}{2}(50\text{ kg})(0.84\text{ m/s})^2 = -17.6\text{ J}.$$

This amount of internal energy could be replenished by digesting about $\frac{1}{4}$ teaspoon of diet soda.

The analysis of this Sample Problem could be applied unchanged to the problem of a car accelerating from rest. The external force in the case—exerted by the road on the bottoms of the tires—does no work because its point of application does not move; recall that the bottom of a wheel that is rolling without slipping is instantaneously at rest. The change in the internal energy of this system is reflected in the consumption of gasoline.

Sample Problem 13-4. Skater Joan (mass 50 kg) pushes herself away from her partner Jim (mass 72 kg), who is standing with his back against a wall, as in Fig. 13-9*a*. Both have their arms bent initially. Each pushes against the other as they straighten their arms, until finally they lose contact (Fig. 13-9*b*). Jim exerts a constant force $F_{ext} = 55$ N through a distance of $s =$ 32 cm; this is the distance his hands actually move as he straightens his arms. At the instant contact is broken, Joan's center of mass has moved through a total distance of $s_{cm} = 58$ cm as a result of the extension of *both* pairs of arms. (*a*) What is Joan's speed after contact is broken? (*b*) What is the change in the stored internal energy of each skater during this process? Neglect friction between the ice and the skater.

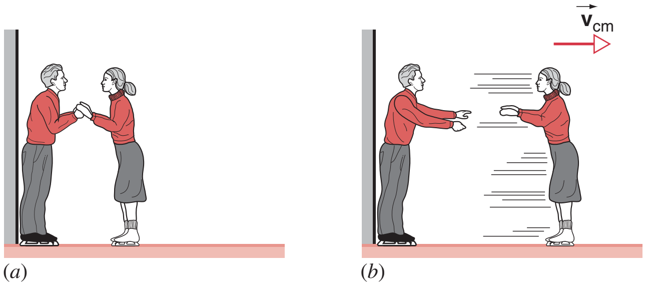


FIGURE 13-9. Sample Problem 13-4. (*a*) A skater (Joan) and her partner (Jim) are preparing to exert forces on one another by extending their arms. Jim has his back against a wall and so does not move. (*b*) After the arms have been extended, Joan is moving with speed v_{cm}.

Solution (*a*) We take Joan as our system. Note that in this case there is external work done on the system, so there is a transfer of energy through the system boundary. From the COM equation (Eq. 13-15) we have

$$\Delta K_{cm} = \tfrac{1}{2}Mv_{cm}^2 = F_{ext}\, s_{cm} = (55\text{ N})(0.58\text{ m}) = 31.9\text{ J},$$

so

$$v_{cm} = \sqrt{\frac{2\,\Delta K_{cm}}{M}} = \sqrt{\frac{2(31.9\text{ J})}{50\text{ kg}}} = 1.13\text{ m/s}.$$

(*b*) Applying the COE equation (Eq. 13-2) to Joan, we have

$$\Delta K + \Delta E_{int,\,Joan} = W_{ext},$$

where W_{ext} ($= F_{ext}s$) is the external work done on Joan by Jim. Solving for Joan's internal energy change and substituting $\Delta K = \Delta K_{cm} = F_{ext}s_{cm}$ from part (*a*), we obtain

$$\begin{aligned}\Delta E_{int,\,Joan} &= W_{ext} - \Delta K = F_{ext}s - F_{ext}s_{cm}\\ &= (55\text{ N})(0.32\text{ m}) - (55\text{ N})(0.58\text{ m})\\ &= +17.6\text{ J} - 31.9\text{ J} = -14.3\text{ J}.\end{aligned}$$

Applying the COE equation to a system consisting only of Jim, we obtain

$$\Delta E_{int,\,Jim} = W_{ext}.$$

In Jim's case, W_{ext} is negative. The external force on him is supplied by Joan as a reaction force to him pushing on her. Since the force on Jim and the displacement of his hands are in opposite directions, the external work done on Jim is negative. In this case (see Fig. 13-1), W_{ext} takes energy out of the system. For Jim,

$$\Delta E_{int,\,Jim} = W_{ext} = -(55\text{ N})(0.32\text{ m}) = -17.6\text{ J}.$$

Thus, to attain her final kinetic energy, Joan must supply 14.3 J of energy from her internal resources. Jim supplies 17.6 J by doing work on Joan, which, of course, comes from *his* internal store. If Jim were not present and Joan had attained the same kinetic energy by pushing directly on the wall, she would need to supply the full 31.9 J (= 14.3 J + 17.6 J) of her kinetic energy from her internal energy store.

SAMPLE PROBLEM 13-5. A 5.2-kg block is projected over a horizontal surface with an initial horizontal velocity of 0.65 m/s before coming to rest. The coefficient of kinetic friction between the block and the surface is 0.12. (*a*) What is the change in the internal energy of the system of block + surface? (*b*) How far does the block travel in coming to rest?

Solution (*a*) In applying energy conservation, the most useful system to consider is the block plus the portion of the horizontal surface over which it slides. In using Eq. 13-2, we have $\Delta U = 0$, because no change of potential energy occurs on the horizontal surface. Furthermore, $W_{ext} = 0$, because no external force acts on the system. (We have defined the system so that friction is an *internal* force.) Thus Eq. 13-2 becomes

$$\Delta E_{int} = -\Delta K$$

in which ΔK ($K_f - K_i$) is negative, corresponding to a loss in kinetic energy. Substituting values, we have

$$\Delta E_{int} = -(0 - \tfrac{1}{2}Mv_{cm}^2) = +\tfrac{1}{2}(5.2\text{ kg})(0.65\text{ m/s})^2 = +1.1\text{ J}.$$

This increase in internal energy of the system reveals itself as a small increase in the temperature of the block and of the horizontal surface. It is difficult to calculate how this energy is shared between the block and the surface; it is largely to avoid this difficulty that we have chosen to analyze the combined system of the block plus the surface, rather than the block alone.

(*b*) In this case we choose the block alone as our system. We cannot treat the block as a particle, because energy transfers (specifically, internal energy) other than translational kinetic energy are involved. Applying Eq. 13-15, we have

$$F_{ext}s_{cm} = \Delta K_{cm},$$

where F_{ext} is the external frictional force ($= -\mu Mg$, taking the direction of motion to be positive) that acts on the block and s_{cm} is the displacement of the center of mass of the block. Thus we have

$$(-\mu Mg)(s_{cm}) = 0 - \tfrac{1}{2}Mv_{cm}^2$$

or

$$s_{cm} = \frac{v_{cm}^2}{2\mu g} = \frac{(0.65\text{ m/s})^2}{2(0.12)(9.8\text{ m/s}^2)} = 0.18\text{ m}.$$

This analysis of this sample problem could be applied unchanged to the problem of a car braking to rest from a given initial speed. In this case, the increase in internal energy would reveal itself as a rise in temperature of the brake disks and brake pads.

13-6 REACTIONS AND DECAYS

The law of conservation of energy finds wide use in analyzing a great variety of reaction and decay processes, on a scale that ranges from atoms and molecules (chemical reactions, molecular formation) to nuclei (fusion reactions, radioactive decays) to elementary particles (high-energy collisions). In Chapter 6 we analyzed collisions using the law of conservation of linear momentum, and we classified the processes into elastic, inelastic, and explosive. In Chapter 11 we showed how we could understand those classifications in terms of the change in kinetic energy of the processes. Now we can discuss these processes from the perspective of a more general law of conservation of energy.

With this more general law, we can even analyze processes in which the identities of the objects change during the collision. For example, consider the nuclear reaction represented by $n + {}^6Li \rightarrow {}^4He + {}^3H$, in which a neutron is incident on a nucleus of lithium with a mass number (total number of protons + neutrons) of 6, containing three protons and three neutrons. After the reaction, the particles observed are a nucleus of helium with a mass number of 4 (two protons and two neutrons) and a nucleus of hydrogen with a mass number of 3 (one proton and two neutrons). Note that the total number of neutrons is unchanged in the reaction, being equal to 4 both before and after the reaction. Similarly the total number of protons remains constant at 3. However, the protons and neutrons are rearranged during the reaction. Presumably, in these rearranged groupings the neutrons and protons have different interactions with one another and thus the internal energies of the groupings may change during the reaction.

Let us analyze the reaction $A + B \rightarrow C + D$ by choosing our system boundary so that it includes objects A and B before the collision when they are far enough apart that there is no interaction between them and thus no initial potential energy. (A and B may each have an *internal* potential energy, but there is no potential energy due to any interaction of A with B.) The total initial kinetic energy of this system is $K_i = K_A + K_B$, and A and B have total internal energy $E_{int,\,i}$. During the reaction, there may be internal rearrangements so that the final particles C and D are different from A and B, but the final particles C + D remain within the system boundary and constitute the system after the reaction. The total internal energy of the system consisting of C and D after the reaction is $E_{int,f}$, and the total final kinetic energy of this system after the reaction is $K_f = K_C + K_D$; as in the initial state we assume there is no interaction between the colliding objects and therefore no final potential energy. Figure 13-10 presents a schematic view of the collision. We assume that no object in the environment does work on the objects during the collision, so $W_{ext} = 0$.

Applying our general law of conservation of energy, Eq. 13-2, to this process, and assuming $U_i = U_f = 0$, we have

$$\Delta K + \Delta E_{int} = 0 \qquad (13\text{-}20)$$

or

$$K_f - K_i = -(E_{int,\,f} - E_{int,\,i}) = E_{int,\,i} - E_{int,\,f}. \qquad (13\text{-}21)$$

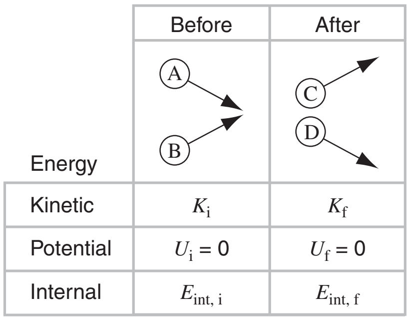


FIGURE 13-10. The energy changes in the reaction $A + B \rightarrow C + D$.

If $E_{int,\,i} > E_{int,\,f}$, the final kinetic energy is greater than the initial kinetic energy, which means that some internal energy of the colliding objects has been changed into kinetic energy. Such reactions are called *exoergic* (energy-releasing) and are analogous to the collisions we have been calling "explosive." If $E_{int,\,i} < E_{int,\,f}$, the final kinetic energy is less than the initial kinetic energy, because some of the original kinetic energy is converted into internal energy of the final particles. These reactions are called *endoergic* (energy-absorbing) and are analogous to the collisions we have been calling "inelastic." For elastic collisions, in which the kinetic energy does not change, we must have $E_{int,\,i} = E_{int,\,f}$. In practice this means that the identities of the colliding bodies do not change and that there are no internal regroupings of their constituents (that is, $A + B \rightarrow A + B$).

Decay Processes

Some nuclei and elementary particles are unstable and spontaneously decay to two or more other particles. For example, in the alpha decay ${}^{235}U \rightarrow {}^{231}Th + {}^4He$, a uranium nucleus of mass number 235 breaks apart into a nucleus of thorium of mass number 231 and a nucleus of helium of mass number 4. The ${}^4He$ nucleus is commonly known as an *alpha particle.*

We assume that the original decaying particle A is at rest ($K_i = 0$); its momentum is zero, so conservation of momentum requires that the total momentum of the product particles must be zero. If the decay occurs into only two particles B and C, their linear momenta must be equal and opposite: $m_B v_B = -m_C v_C$, so $m_B^2 v_B^2 = m_C^2 v_C^2$ or $m_B(2K_B) = m_C(2K_C)$, which gives

$$K_B/K_C = m_C/m_B. \qquad (13\text{-}22)$$

The final kinetic energy K_f, which is just the total kinetic energy of B and C, comes from the transformation of internal energy. With $U_i = 0$ and $U_f = 0$ as before, Eq. 13-21 applies but with $K_i = 0$ and $K_f = K_B + K_C$:

$$K_B + K_C = E_{int,\,i} - E_{int,\,f}. \qquad (13\text{-}23)$$

Clearly, since K_f ($= K_B + K_C$) must be positive, the decay will occur only if $E_{int,\,i} > E_{int,\,f}$. In this case, internal energy is converted into kinetic energy.

If the decay occurs into two particles, then Eq. 13-22 and 13-23 together can be solved for the final kinetic energies K_B and K_C. If the decay occurs into three or more final particles $B + C + D + \cdots$, then the equations for conservation of energy and conservation of momentum do not provide enough information to determine unique values for the kinetic energies of the product particles. In this case, the particles can have a continuous range of kinetic energies whose sum is determined by Eq. 13-23.

SAMPLE PROBLEM 13-6. The fusion reaction ${}^2H + {}^2H \rightarrow {}^1H + {}^3H$, known as the d-d reaction (d stands for *deuteron,* which is another name for ${}^2H$, the hydrogen nucleus with a mass

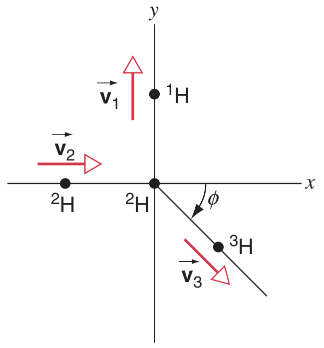


FIGURE 13-11. Sample Problem 13-6. The incident ^{2}H strikes a stationary ^{2}H target, producing the outgoing ^{1}H and ^{3}H particles.

number of 2), is important for the release of nuclear energy. The internal energy of the initial particles is greater than that of the final particles by 4.03 MeV. In one particular reaction, a beam of ^{2}H of kinetic energy 1.50 MeV is incident on a target of ^{2}H at rest. The proton (^{1}H) is observed with a kinetic energy of 3.39 MeV in a direction at an angle of 90° with respect to the original beam of ^{2}H (Fig. 13-11). Find the energy and direction of the outgoing ^{3}H. The masses are: ^{1}H – 1.01 u, ^{2}H – 2.01 u, ^{3}H – 3.02 u.

Solution From Eq. 13-21, the final kinetic energy is

$$K_f = -\Delta E_{int} + K_i = 4.03 \text{ MeV} + 1.50 \text{ MeV} = 5.53 \text{ MeV}.$$

The final kinetic energy is shared between the ^{1}H and ^{3}H nuclei. With $K_f = K_1 + K_3$, we have

$$K_3 = K_f - K_1 = 5.53 \text{ MeV} - 3.39 \text{ MeV} = 2.14 \text{ MeV}.$$

From conservation of momentum, the momentum of the original ^{2}H must equal the x component of the momentum of the ^{3}H, or $m_2 v_2 = m_3 v_3 \cos\phi$. Using $v = \sqrt{2K/m}$ we obtain

$$\cos\phi = \frac{m_2 v_2}{m_3 v_3} = \sqrt{\frac{m_2 K_2}{m_3 K_3}} = \sqrt{\frac{(2.01 \text{ u})(1.50 \text{ MeV})}{(3.02 \text{ u})(2.14 \text{ MeV})}} = 0.683$$

or $\phi = 46.9°$.

SAMPLE PROBLEM 13-7. In the alpha-decay process $^{226}\text{Ra} \rightarrow {}^{222}\text{Rn} + {}^4\text{He}$, the naturally occurring radioactive element radium decays to the gaseous element radon. The internal energy decreases by 4.87 MeV in the decay. If the radium decays from rest, find the kinetic energies of the radon and the alpha particle (^{4}He). The masses are: ^{226}Ra – 226.0 u, ^{222}Rn – 222.0 u, ^{4}He – 4.00 u.

Solution From Eq. 13-22, the ratio of the kinetic energies of the product particles is

$$\frac{K_{Rn}}{K_{He}} = \frac{m_{He}}{m_{Rn}} = \frac{4.00 \text{ u}}{222.0 \text{ u}} = 0.0180.$$

The total kinetic energy of the products is given by Eq. 13-23:

$$K_f = K_{Rn} + K_{He} = E_{int,i} - E_{int,f} = 4.87 \text{ MeV}.$$

Solving these two equations simultaneously gives $K_{Rn} = 0.086$ MeV and $K_{He} = 4.78$ MeV. Note that the lighter alpha particle takes about 98% of the energy, consistent with conservation of momentum.

13-7 ENERGY TRANSFER BY HEAT

Figure 13-1 showed that the energy of a system can be changed by work that is done on the system by its environment. Work is one of two ways that a system can exchange energy with its environment. The other way is through *heat.*

As we discussed in Section 11-1, the physics definition of "work" may differ from its common usage in the English language. The same is true for heat. Our physics definition of heat is as follows:

Heat is a means of energy transfer between a system and its environment because of the temperature difference between them.

We represent heat transfer by the symbol Q. Because heat is a form of energy, it is measured in energy units (joules, for example).

There are two important similarities between work and heat:

1. *Heat is energy in transit.* Just as we never speak of "the amount of work contained in a body," so we likewise never say "the amount of heat contained in a body." When heat is transferred from system A to system B, it is not correct to say that "system A has less heat." Instead, we should say that "system A has less energy" because some of its energy was lost due to the heat transferred to system B. Similarly, if system A does work on system B, we never say that "system A has less work," but instead that "system A has less energy" because it used some of its energy to do work on system B.

2. *The amount of heat transferred in a process depends on how the process is done.* We have seen examples of cases in which a system can be taken from a given initial state to a given final state through several different paths. If a nonconservative force (such as friction) acts on the system, then the work done by that force will in general have different values for different paths leading from the same initial state to the same final state. (This in fact was one way in which we defined nonconservative forces in Chapter 12.) In this respect, heat transfer resembles nonconservative work, in that different amounts of heat transfer may be required to take the system along different paths connecting the same initial state with the same final state.

Heat and Temperature

Often in colloquial usage we say "heat" when we mean temperature or internal energy. When we "heat" a dish in an oven to a certain temperature, we transfer energy by means of heat (with the dish surrounded by an environment at a higher temperature) until the dish reaches the desired temperature. When we take the hot dish out of the oven and place it on the table, the dish will transfer energy to its cooler environment as heat.

As with work and heat, we must give a precise definition of temperature in order for it to be a useful physical quantity. We delay a formal definition until Chapter 21, but we will give a brief summary here so that we can discuss temperature in connection with mechanical systems.

A change in the temperature of a body is accompanied by a change in the average translational kinetic energy of its atoms or molecules. If we increase the internal energy of a body, its constituent atoms or molecules might acquire this energy in several different forms—for example, increased translational kinetic energy, increased rotational kinetic energy, or an altered configuration (such as increasing their average spacing)—so that their potential energy increases. Only the portion that results in increased translational kinetic energy will produce a temperature increase.

Another way of looking at temperature is as an indicator of whether two bodies placed into contact will exchange energy as heat. If their temperatures are the same, no heat will be exchanged between them. Note that if one body is much larger than the other, it may have a much greater total internal energy, but it will transfer none of that energy to the second body if their temperatures are the same. One way of transferring heat is through collisions between the atoms or molecules of the two bodies at the surface where they are in contact. When two bodies at different temperatures are placed in contact, collisions at the contact surface between atoms or molecules of the bodies will in general transfer energy from the body whose particles have on the average more translational kinetic energy (the higher-temperature body) to the body whose particles have on the average less translational kinetic energy (the lower-temperature body).

Be careful to distinguish between the concepts of heat and temperature. Heat is always energy in transit between bodies; temperature is a measure of the internal energy of a single body. We can increase the temperature of a body with no heat transfer to it (such as by doing work on it), and we can transfer heat to a body from an environment at higher temperature with no change in the temperature of the body (for example, by melting solid ice at 0°C to liquid water at 0°C).

The First Law of Thermodynamics

In our previous general expression for conservation of energy, Eq. 13-2, we omitted one method of energy transfer—heat. Figure 13-12 shows a more complete view of the energy transfers in a system. The energy inside our system boundary can change due to either heat transferred to or from the environment or work done by or on the environment. Including the heat, we can write Eq. 13-2 as

$$\Delta E_{\text{total}} = Q + W. \qquad (13\text{-}24)$$

In Eq. 13-24, E_{total} indicates all forms of energy contained within the system boundary: kinetic, potential, internal, and perhaps other forms. For convenience we have dropped the subscript "ext" from W, but we still take it to mean the work done on the system by its external environment. Our

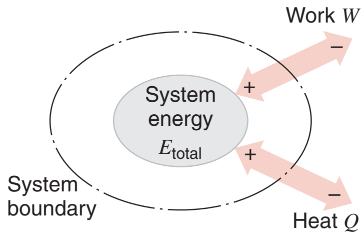


FIGURE 13-12. The energy of a system can be changed in two ways: by work done on or by the environment, and by heat transferred to or from the environment. The sign conventions for W and Q are indicated—work done *on* the system and heat transferred *to* the system both are taken to be positive and both increase the energy of the system.

sign convention for Q is similar to that for work: $Q > 0$ means that heat is transferred *to* a system and *increases* its energy, while $Q < 0$ means that heat is transferred *from* the system and *decreases* its energy.*

Equation 13-24 is the most general statement we can make about conservation of energy in a system. In this form it is commonly known as the *first law of thermodynamics.* Later in this text we will consider more detailed application of this law to a particular thermodynamic system: a gas enclosed in a container. For now we consider how this law applies to some mechanical systems.

1. *A block sliding on a horizontal surface.* A block is sliding on a flat horizontal table where a frictional force acts. The block has an initial speed v and eventually comes to rest. We first take our system to be the block. Equation 13-24 applied to the block gives

$$\Delta K + \Delta E_{\text{int, block}} = W_{\text{f}} + Q. \qquad (13\text{-}25)$$

Here $\Delta K = K_{\text{f}} - K_{\text{i}} = \frac{1}{2}Mv^2$, $\Delta E_{\text{int, block}}$ is the increase in internal energy of the block (which is measured by its rise in temperature), W_{f} is the (negative) frictional work done on the block by the table, and Q is the (negative) heat transferred from the block. We assume that the heat transferred to the air is negligible, and that the only heat transfer is from the hot block to the cooler regions of the table with which it comes into contact.

Now applying the first law of thermodynamics to the system of block + table, we find

$$\Delta K + \Delta E_{\text{int, block}} + \Delta E_{\text{int, table}} = 0. \qquad (13\text{-}26)$$

*It is important to note that W represents the external work done *on* the system. You may occasionally see Eq. 13-24 written as $\Delta E = Q - W$, in which W represents the work done *by* the system on its external environment. Since the work done by system A on system B is the negative of the work done by system B on system A, either form of the equation is correct. We have chosen to write the equation in this form so that W always represents the work done *on* a system. Otherwise it would be necessary to define thermodynamic work as the negative of mechanical work. We prefer to emphasize the connection between mechanics and thermodynamics by choosing a consistent sign convention for work.

Here the work does not appear, because it is internal to the system. Likewise, Q does not appear, because the heat transfer is also internal to this system (since we have neglected heat loss to the surrounding air). Combining Eqs. 13-25 and 13-26, we obtain

$$\Delta E_{\text{int, table}} = -W_f - Q. \qquad (13\text{-}27)$$

Both W_f and Q are negative, so both terms on the right contribute to *increase* the internal energy (temperature) of the table; $-W_f$ (a positive quantity) represents the frictional work done on the table by the block, and $-Q$ (a positive quantity) represents the heat transferred to the table by the block.

2. *Joule's experiment.* In the 19th century it was not at first realized that heat is a form of energy. As a result, heat was measured in units that differed from the standard energy units. Among the early units used for heat were the calorie (cal) and the British thermal unit (Btu), which are related to our SI energy units (joules) according to

$$1 \text{ cal} = 4.186 \text{ J} \quad \text{and} \quad 1 \text{ Btu} = 1055 \text{ J}.$$

The common use for the word calorie today is based on the energy content of food; this "calorie" is in actuality a kilocalorie (1 Cal = 1 kilocalorie = 1000 cal). The Btu is still often found today as a measure of the ability of a heater or an air conditioner to transfer energy as heat between a room and its environment.

The calorie was originally defined as the amount of heat Q that must be transferred to one gram of water to raise its temperature from 14.5°C to 15.5°C, in the process increasing its internal energy by ΔE_{int}. No external work is done in this process, and therefore we can write Eq. 13-24 as

$$\Delta E_{\text{int}} = Q. \qquad (13\text{-}28)$$

Joule's experiment was designed to raise the temperature of a quantity of water by doing work on it instead of transferring heat to it. His apparatus is shown in Fig. 13-13. The falling weights turned a set of paddle wheels that stirred the water, thus transforming the gravitational work W_g on the weights into internal energy of the water. We take Joule's entire apparatus as our system, allowing the weights to fall a fixed distance and then stop, and we wait for the paddle wheels to lose all their rotational kinetic energy to the water. Assuming that no heat is transferred through the container and no energy is dissipated in the pulleys, we can write Eq. 13-24 as

$$\Delta E_{\text{int}} = W_g. \qquad (13\text{-}29)$$

For the same change in internal energy (corresponding to the same temperature increase) as in Eq. 13-28, it was then possible for Joule to find the equivalence between a certain quantity of work (measured—using modern units—in joules) and the corresponding quantity of heat (measured in calories). This relationship is known as the *mechanical equivalent of heat:* 1 cal = 4.186 J. Today we measure heat, like other forms of energy, in joules, and so this conversion factor has lost the importance it had in Joule's time. Nevertheless, Joule's experiment, done in 1850, provided a direction in showing that heat, like work, could properly be regarded as a means of transferring energy.

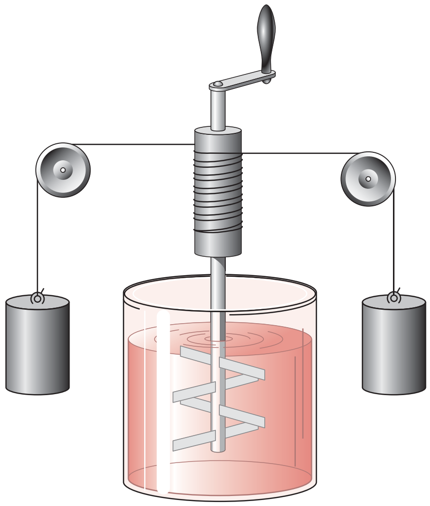


FIGURE 13-13. Joule's arrangement for measuring the mechanical equivalent of heat. The falling weights turn paddles that stir the water in the container, thus raising its temperature.

MULTIPLE CHOICE

13-1 Work Done on a System by External Forces

1. A ball is dropped from the edge of a cliff. Which of the following statements is correct? *(There may be more than one correct answer!)*
   (A) Gravity does work on the ball as it falls.
   (B) The gravitational potential energy of the ball decreases as the ball falls.
   (C) The gravitational potential energy of the Earth decreases as the ball falls.
   (D) The gravitational potential energy of the system of ball + Earth decreases as the ball falls.

2. Suppose $\Delta K = +10$ J for the block in the situation shown in Fig. 13-2. Which of the following could correctly describe the energy transfers in that situation?
   (A) $W_{\text{spring}} = +5$ J, $W_{\text{grav}} = +15$ J
   (B) $\Delta U_{\text{spring}} = +5$ J, $W_{\text{grav}} = -15$ J
   (C) $W_{\text{spring}} = -5$ J, $\Delta U_{\text{grav}} = -15$ J
   (D) $\Delta U_{\text{spring}} = -5$ J, $\Delta U_{\text{grav}} = -15$ J

3. A wooden block (mass 2.0 kg) is dropped from a high board above a swimming pool and enters the water moving at a speed of 10 m/s. The block descends to a depth of 3.0 m in the water and comes instantaneously to rest before beginning

to rise again to the surface. What is the work done on the block by the water during the 3-m descent?

(A) -159 J (B) -100 J
(C) -59 J (D) -41 J

13-2 Internal Energy in a System of Particles

4. A 2.0-kg ball is dropped from a height of 5.0 m. The ball falls, hits the ground, and bounces back up to a height of 3.0 m. What can be said about $\Delta E_{\text{int, ball}}$ between the initial and final state of the ball?

(A) $\Delta E_{\text{int, ball}} > 39.2\,\text{J}$ (B) $\Delta E_{\text{int, ball}} = 39.2\,\text{J}$
(C) $\Delta E_{\text{int, ball}} < 39.2\,\text{J}$

5. This section is concerned with "missing energy" that can be stored in an object as internal energy. Should there be a similar concern about "missing momentum" and "internal momentum"?

(A) Yes, but the effects will be much smaller because momentum is proportional to velocity whereas energy is proportional to velocity squared.
(B) Yes, but the effects can be ignored because physicists are only concerned with systems where momentum is conserved.
(C) No, because momentum is a vector whereas energy is a scalar.
(D) No, as long as "potential momentum" is not introduced.

13-3 Frictional Work

6. A 10-cm cube of metal is fastened rigidly in place. A second, identical cube of metal is pulled across the top of the first cube at a constant speed by a constant 10 N force, as shown in Fig. 13-14.

(*a*) The frictional force between the cubes

(A) is less than 10 N.
(B) is equal to 10 N.
(C) is greater than 10 N.
(D) cannot be determined without a detailed model of the two surfaces.

(*b*) How is the change in internal energy of the moving top cube, $\Delta E_{\text{int, moving}}$, related to the change in internal energy of the fixed cube $\Delta E_{\text{int, fixed}}$?

(A) $\Delta E_{\text{int, moving}} > \Delta E_{\text{int, fixed}}$
(B) $\Delta E_{\text{int, moving}} = \Delta E_{\text{int, fixed}}$
(C) $\Delta E_{\text{int, moving}} < \Delta E_{\text{int, fixed}}$
(D) There is no obvious relationship without a detailed description of the frictional force.

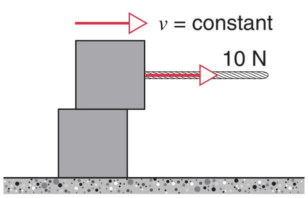


FIGURE 13-14. Multiple-choice question 6.

7. A method for determining the speed of a bullet is to shoot it into a block of wood and see how far the block slides on a surface. (See Problem 3.) Assuming (incorrectly) that the magnitude of the work done by friction is equal to the force of friction on the block times the distance the block slides, the calculated value of the bullet velocity will be

(A) lower than the actual value, because there will also be a change in the internal energy of the block and surface.
(B) higher than the actual value, because there will also be a change in the internal energy of the block and surface.
(C) correct, because errors caused by ignoring changes in internal energies are cancelled by the error in the assumption about the work done by friction.
(D) wrong, because friction invalidates conservation of momentum as well.

13-4 Conservation of Energy in a System of Particles

8. (*a*) A block slides from rest down a wedge inclined at an angle θ with the horizontal. There is friction between the block and the wedge. When the block reaches the bottom of the wedge, its kinetic energy is 3 J, and gravity has done $+10$ J of work on the block. Which of the following describes the energy transfers in this system?

(A) $\Delta E_{\text{int, block}} < +7$ J
(B) Frictional work on block by wedge $= -7$ J
(C) Frictional work on wedge by block $= +7$ J
(D) $\Delta E_{\text{int, block}} = +7$ J

(*b*) Suppose the wedge is free to slide on a frictionless horizontal table (there is still friction between the block and the wedge). The block is again released from rest and reaches the bottom of the wedge with a kinetic energy K after gravity does work W_g on the block. The masses of the block and the wedge are known. From this information, is it possible to calculate the speed of the wedge?

(A) Yes, by applying conservation of momentum in the horizontal direction
(B) No, because we do not know how much mechanical energy is lost due to friction
(C) No, because conservation of momentum does not apply when frictional forces act
(D) No, because the net external force on the system is not zero

13-5 Center-of-Mass Energy

9. Two particles collide elastically. In the laboratory frame one of the particles is originally at rest.

(*a*) In which frame is the total kinetic energy least?

(A) The laboratory frame
(B) The center-of-mass frame
(C) The kinetic energy is the same in the laboratory frame and the center-of-mass frame.
(D) The question cannot be answered without more information.

(*b*) In which frame is the magnitude of the total momentum least?

(A) The laboratory frame
(B) The center-of-mass frame
(C) The momentum is the same in the laboratory frame and the center-of-mass frame.
(D) The question cannot be answered without more information.

10. Is the rotational kinetic energy part of center-of-mass kinetic energy or part of internal energy?
(A) It is definitely part of the center-of-mass kinetic energy.
(B) It is definitely part of the internal energy.
(C) It could be either, depending on how the system is defined.
(D) It could be either, because you can always find an inertial frame where the body is not rotating.
(E) It is part of neither the center-of-mass energy nor the internal energy.

13-6 Reactions and Decays

11. Consider the decay $A \rightarrow B + C + D$ in which A is initially at rest. The masses of all the particles and energy ΔE_{int} released in the decay are known. It is desired to know the speeds and directions of all three final particles. An experiment determines the speed and direction of B. What is the minimum amount of additional experimental data needed to enable all remaining unknown variables to be calculated?
(A) No additional data are needed.
(B) Both the speed and the direction of C are needed.
(C) Either the speed or the direction of C is needed.
(D) The speeds of both C and D are needed.

12. The kinetic energy of a particle depends on the reference frame of the observer. In an exoergic reaction, the total final kinetic energy is greater than the total initial kinetic energy. Which of the following is correct?
(A) A reaction that is exoergic in one inertial reference frame is exoergic in all inertial reference frames.
(B) It is possible to find a reference frame in which an exoergic reaction could appear to be endoergic.
(C) It is possible to find a reference frame in which an exoergic reaction could appear to be elastic.
(D) The net change in kinetic energy will have the same value in all inertial reference frames.

13-7 Energy Transfer by Heat

13. How would the inclusion of energy transfer by heat affect the discussion of the block in Fig. 13-3?
(A) Energy might be transferred as heat between the block and the table, changing both $E_{\text{int, block}}$ and $E_{\text{int, table}}$ but keeping $E_{\text{int, block + table}}$ constant.
(B) Energy might be transferred as heat from the block and table to their presumably cooler surroundings, thereby decreasing both $E_{\text{int, block}}$ and $E_{\text{int, table}}$.
(C) Process of types (A) and (B) can both occur, leading to a net decrease in $E_{\text{int, block + table}}$.

QUESTIONS

1. At the highest point of its trajectory a vertically tossed ball has zero kinetic energy. Where has the energy gone? Has external work been done on the ball? Is the energy now in the form of potential energy in the ball? Potential energy in the Earth?

2. What happens to the potential energy that is lost as an elevator descends from the top of a building to a stop at the ground floor?

3. Figure 13-15 shows a circular glass tube fastened to a vertical wall. The tube is filled with water except for an air bubble that is temporarily at rest at the bottom of the tube. Discuss the subsequent motion of the bubble in terms of energy transfers. Do so both neglecting viscous and frictional forces and taking them fully into account.

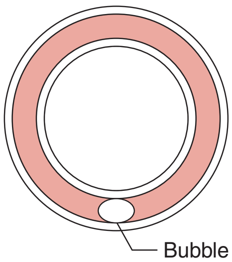


FIGURE 13-15. Question 3.

4. When the ice skater in Section 13-2 pushes herself away from the railing her internal energy E_{int} decreases. What happens to her internal energy when she skates up to the railing and then pushes herself to a stop?

5. Can internal energy be considered a special form of potential energy? Why or why not?

6. Can potential energy be considered a special case of internal energy? Why or why not?

7. An automobile is moving along a highway. The driver jams on the brakes and the car skids to a halt. In what form does the lost kinetic energy of the car appear?

8. In the above question, assume that the driver operates the brakes in such a way that there is no skidding or sliding. In this case, in what form does the lost kinetic energy of the car appear?

9. An automobile accelerates from rest to a speed v, under conditions such that no slipping of the driving wheels occurs. From where does the mechanical energy of the car come? In particular, is it true that it is provided by the (static) frictional force exerted by the road on the car?

10. W_f in Eq. 13-6 represents the transfer of energy out of the block system into the table system. Does $W_f = -\Delta E_{\text{int, table}}$? Explain. Can you conclude that $fs = \Delta E_{\text{int}}$ whenever f is an internal force of friction? If not, give a counter example.

11. In the case of work done against friction, the internal energy change is independent of the velocity (or inertial reference frame) of the observer. That is, different observers would assign the same quantity of mechanical energy transformed into internal energy due to friction. How can this be explained, considering that such observers measure different quantities of total work done and different changes in kinetic energy in general?

12. In an article "Energy and the Automobile," which appeared in the October 1980 issue of *The Physics Teacher* (p. 494), the author (Gene Waring) states: "It is interesting to note that all the fuel input energy is eventually transformed to thermal energy and strung out along the car's path." Analyze the various mechanisms by which this might come about. Consider, for example, road friction, air resistance, braking, the car radio, the headlamps, the battery, internal engine and drive train losses, the horn, and so on. Assume a straight and level roadway.

13. The electric power for a small town is provided by a hydroelectric plant at a nearby river. If you turn off a lightbulb in this closed-energy system, conservation of energy requires that an equal amount of energy, perhaps in another form, appears somewhere else in the system. Where and in what form does this energy appear?

14. Air bags greatly reduce the chance of injury in a car accident. Explain how they do so, in terms of energy transfers.

15. A ball dropped to Earth cannot rebound higher than its release point. However, spray from the bottom of a waterfall can sometimes rise higher than the top of the falls. Why is this?

16. A swinging pendulum eventually comes to rest. Is this a violation of the law of conservation of mechanical energy?

17. A scientific article ("The Energetic Cost of Moving About," by V. A. Tucker, *American Scientist,* July–August 1975, p. 413) asserts that walking and running are extremely inefficient forms of locomotion and that much greater efficiency is achieved by birds, fish, and bicyclists. Can you suggest an explanation?

18. A spring is compressed by tying its ends together tightly. It is then placed in acid and dissolves. What happens to its stored potential energy?

19. Since the left-hand sides of Eqs. 13-14 and 13-15 look so much like the definition of work in Eqs. 11-1 and 11-14, why not just call it work and move on? What is the advantage of defining work the way that physicists do? Is the same numerical answer arrived at regardless of the definition?

20. Can an external force that does no work (because the point of application is stationary) cause change in the *rotational* kinetic energy of a system?

21. Under what conditions, if any are needed, is it correct to say that the decay $A \rightarrow B + C$ is simply the reverse of the totally inelastic collision $B + C \rightarrow A$?

22. A high-school science student claims to have invented simple glass marbles that collide with perfectly elastic collisions. He demonstrates this by shooting one marble at another; you hear the snap of the collision and then see the marbles move apart. Repeated measurements always indicate that the collision is elastic within the measurement accuracy of the equipment. Is the collision elastic? Why or why not?

23. Trace back to the Sun as many of our present energy sources as you can. Can you think of any that cannot be so traced?

24. We say that a car is not accelerated by internal forces but rather by external forces exerted on it by the road. Why then do cars need engines?

25. Can the work done by internal forces decrease the kinetic energy of a body? Increase it?

26. (*a*) If you do work on a system, does the system necessarily acquire kinetic energy? (*b*) If a system acquires kinetic energy, does it necessarily mean that some external agent did work on it? Give examples. (By "kinetic energy" here we mean kinetic energy associated with the motion of the center of mass.)

27. In Sample Problem 13-3, we saw an example (a skater) in which kinetic energy appeared but no external work was done. Consider the opposite case. A screwdriver is held tightly against a rotating grinding wheel. Here external work is done but the kinetic energy of the screwdriver does not change. Explain this apparent contradiction.

28. A disgruntled hockey player throws a hockey stick along the ice. It rotates about its center of mass as it slides along and is eventually brought to rest by the action of friction. Its motion of rotation stops at the same moment that its center of mass comes to rest, not before and not after. Explain why.

EXERCISES

13-1 Work Done on a System by External Forces

1. A projectile whose mass is 9.4 kg is fired vertically upward. On its upward flight, 68 kJ of mechanical energy is dissipated because of air drag. How much higher would it have gone if the air drag had been made negligible (for example, by streamlining the projectile)?

2. While a 1700-kg automobile is moving at a constant speed of 15 m/s, the motor supplies 16 kW of power to overcome friction, wind resistance, and so on. (*a*) What power must the motor supply if the car is to move up an 8.0% grade (8.0 m vertically for each 100 m horizontally) at 15 m/s? (*b*) At what downgrade, expressed in percentage terms, would the car coast at 15 m/s?

3. In the situation of Fig. 13-2, a block of mass 1.25 kg is released from rest at a point where the spring (of force constant $k = 262$ N/m) has its relaxed length. What is the speed of the block after it has fallen a distance of 8.4 cm?

4. An automobile with passengers has a weight of 16,400 N (= 3680 lb) and is moving up a 10° slope with an initial speed of 70 mi/h (= 113 km/h) when the driver begins to apply the brakes. The car comes to a stop after traveling 225 m along the inclined road. Calculate the work done by the brakes in stopping the car, assuming that all other energy transfers in this problem (such as heat and internal energy) can be neglected.

13-2 Internal Energy in a System of Particles

5. A ball of mass 12.2 g is dropped from rest at a height of 76 cm above the surface of oil that fills a barrel to a depth of 55 cm. The ball reaches the bottom of the barrel with a speed of 1.48 m/s. (*a*) Neglecting air resistance, find the speed of the ball when it enters the oil. (*b*) What is the change in the internal energy of the system of ball + oil?

13-3 Frictional Work

6. A 25.3-kg bear slides, from rest, 12.2 m down a lodge-pole pine tree, moving with a speed of 5.56 m/s at the bottom. (*a*) What is the initial potential energy of the bear? (*b*) Find the kinetic energy of the bear at the bottom. (*c*) Assuming no other energy transfers, find the change in internal energy of the bear and tree.

7. When a space shuttle (mass 79,000 kg) returns to Earth from orbit, it enters the atmosphere at an altitude of 100 miles and a speed of 18,000 mi/h, which is gradually reduced to a touchdown speed of 190 knots (= 220 mi/h). What is its total energy (*a*) at atmospheric entry and (*b*) at touchdown? See Fig. 13-16. (*c*) What happens to the "missing" energy?

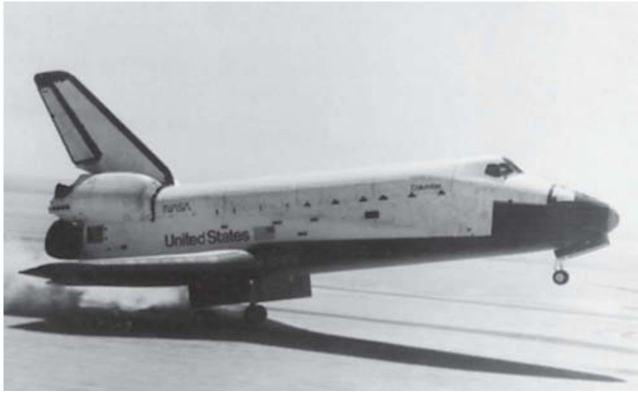


FIGURE 13-16. Exercise 7.

8. A 68-kg skydiver falls at a constant terminal speed of 59 m/s. At what rate is the internal energy of the skydiver and surrounding air increasing?

9. A river descends 15 m in passing through rapids. The speed of the water is 3.2 m/s upon entering the rapids and is 13 m/s as it leaves. What percentage of the potential energy lost by the water in traversing the rapids appears as kinetic energy of water downstream? What happens to the rest of the energy?

10. During a rockslide, a 524-kg rock slides from rest down a hill slope that is 488 m long and 292 m high. The speed of the rock as it reaches the bottom of the hill is 62.6 m/s. How much mechanical energy does the rock lose in the slide due to friction?

11. A 4.26-kg block starts up a 33.0° incline at 7.81 m/s. How far will it slide if it loses 34.6 J of mechanical energy due to friction?

12. Two snow-covered peaks are at elevations of 862 m and 741 m above the valley between them. A ski run extends from the top of the higher peak to the top of the lower one; see Fig. 13-17. (*a*) A skier starts from rest on the higher peak. At what speed would he arrive at the lower peak if he coasts without using the poles? Assume icy conditions, so that there is no friction. (*b*) After a snowfall, a 54.4-kg skier making the same run also without using the poles only just makes it to the lower peak. By how much does the internal energy of her skis and the snow over which she traveled increase?

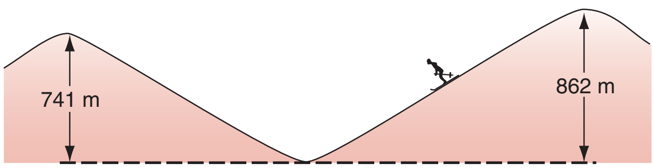


FIGURE 13-17. Exercise 12.

13-4 Conservation of Energy in a System of Particles

13. A ball loses 15.0% of its kinetic energy when it bounces back from a concrete walk. With what speed must you throw it vertically down from a height of 12.4 m to have it bounce back to that same height? Ignore air resistance.

14. A rubber ball dropped from a height of exactly 6 ft bounces (hits the floor) several times, losing 10% of its kinetic energy each bounce. After how many bounces will the ball subsequently not rise above 3 ft?

15. A steel ball of mass 0.514 kg is fastened to a cord 68.7 cm long and is released when the cord is horizontal. At the bottom of its path, the ball strikes a 2.63-kg steel block initially at rest on a frictionless surface (Fig. 13-18). On collision, one-half the mechanical kinetic energy is converted to internal energy and sound energy. Find the final speeds.

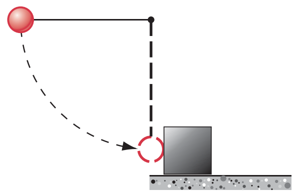


FIGURE 13-18. Exercise 15.

13-5 Center-of-Mass Energy

16. You crouch from a standing position, lowering your center of mass 18.0 cm in the process. Then you jump vertically into the air. The force that the floor exerts on you while you are jumping is three times your weight. What is your upward speed as you pass through your standing position leaving the floor?

17. A 55.0-kg woman leaps vertically into the air from a crouching position in which her center of mass is 40.0 cm above the ground. As her feet leave the floor her center of mass is 90.0 cm above the ground and rises to 120 cm at the top of her leap. (*a*) What upward force, assumed constant, does the ground exert on her? (*b*) What maximum speed does she attain?

18. A 116-kg ice hockey player skates at 3.24 m/s toward a railing at the edge of the ice and stops himself by grasping the railing with his outstretched arms. During this stopping process his center of mass moves 34.0 cm toward the rail. (*a*) Find the average force he must exert on the rail. (*b*) How much internal energy does he lose?

19. The National Transportation Safety Board is testing the crashworthiness of a new car. The 2340-kg vehicle is driven at 12.6 km/h into an abutment. During impact, the center of mass of the car moves forward 64.0 cm; the abutment is compressed by 8.30 cm. Ignore friction between the car and the road. (*a*) Find the force, assumed constant, exerted by the abutment on the car. (*b*) By how much does the internal energy of the car increase?

20. Let the total energy of a system of N particles be measured in an arbitrary frame of reference, such that $K = \Sigma \frac{1}{2} m_n v_n^2$. In the center-of-mass reference frame, the velocities are $v'_n = v_n - v_{\rm cm}$, where $v_{\rm cm}$ is the velocity of the center of mass relative to the original frame of reference. Keeping in mind that $v_n^2 = \vec{\mathbf{v}}_n \cdot \vec{\mathbf{v}}_n$, show that the kinetic energy can be written

$$K = K_{\rm int} + K_{\rm cm},$$

where $K_{int} = \Sigma \frac{1}{2} m_n v'^2_n$ and $K_{cm} = \frac{1}{2} M v^2_{cm}$. This demonstrates that the kinetic energy of a system of particles can be divided into an internal term and a center-of-mass term. The internal kinetic energy is measured in a frame of reference in which the center of mass is at rest; for example, the random motions of the molecules of gas in a container at rest are responsible for its internal translational kinetic energy.

13-6 Reactions and Decays

21. An electron, mass m, collides head-on with an atom, mass M, initially at rest. As a result of the collision, a characteristic amount of energy E is stored internally in the atom. What is the minimum initial speed v_0 that the electron must have? (Hint: Conservation principles lead to a quadratic equation for the final electron speed v and a quadratic equation for the final atom speed V. The minimum value, v_0, follows from the requirement that the radical in the solutions for v and V be real.)

13-7 Energy Transfer by Heat

PROBLEMS

1. A stone of weight w is thrown vertically upward into the air with an initial speed v_0. Suppose that the air drag force f dissipates an amount fy of mechanical energy as the stone travels a distance y. (*a*) Show that the maximum height reached by the stone is

$$h = \frac{v_0^2}{2g(1 + f/w)}.$$

(*b*) Show that the speed of the stone upon impact with the ground is

$$v = v_0 \left(\frac{w - f}{w + f} \right)^{1/2}.$$

2. A small object of mass $m = 234$ g slides along a track with elevated ends and a central flat part, as shown in Fig. 13-19. The flat part has a length $L = 2.16$ m. The curved portions of the track are frictionless; but in traversing the flat part, the object loses 688 mJ of mechanical energy, due to friction. The object is released at point A, which is a height $h = 1.05$ m above the flat part of the track. Where does the object finally come to rest?

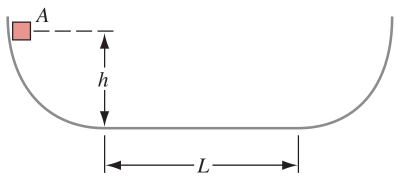


FIGURE 13-19. Problem 2.

3. A bullet of mass 4.54 g is fired horizontally into a 2.41-kg wooden block at rest on a horizontal surface. The coefficient of kinetic friction between block and surface is 0.210. The bullet comes to rest in the block, which moves 1.83 m. Assume that the work done on the block because of friction is 83% of the energy dissipated because of friction. (*a*) What is the speed of the block immediately after the bullet comes to rest within it? (*b*) What is the initial speed of the bullet?

4. A 1.34-kg block sliding on a horizontal surface collides with a spring of force constant 1.93 N/cm. The block compresses the spring 4.16 cm from the unextended position. Friction between the block and the surface dissipates 117 mJ of mechanical energy as the block is brought to rest. Find the speed of the block at the instant of collision with the spring.

5. The magnitude of the force of attraction between the positively charged proton and the negatively charged electron in the hydrogen atom is given by

$$F = k\frac{e^2}{r^2},$$

where e is the electric charge of the electron, k is a constant, and r is the separation between electron and proton. Assume that the proton is fixed. Imagine that the electron is initially moving in a circle of radius r_1 about the proton and jumps suddenly into a circular orbit of smaller radius r_2; see Fig. 13-20. (*a*) Calculate the change in kinetic energy of the electron, using Newton's second law. (*b*) Using the relation between force and potential energy, calculate the change in potential energy of the atom. (*c*) By how much has the total energy of the atom changed in this process? (This energy is often given off in the form of radiation.)

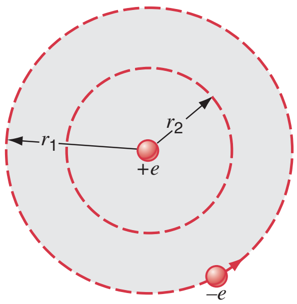


FIGURE 13-20. Problem 5.

6. The cable of a 4000-lb elevator in Fig. 13-21 snaps when the elevator is at rest at the first floor so that the bottom is a distance $d = 12.0$ ft above a cushioning spring whose force constant is $k = 10{,}000$ lb/ft. A safety device clamps the guide rails, removing 1000 ft-lb of mechanical energy for each 1.00 ft that the elevator moves. (*a*) Find the speed of the elevator just before it hits the spring. (*b*) Find the distance that the spring is compressed. (*c*) Find the distance that the elevator will bounce back up the shaft. (*d*) Calculate approximately the total distance that the elevator will move before coming to rest. Why is the answer not exact?

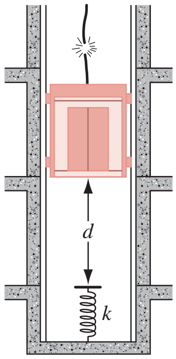


FIGURE 13-21. Problem 6.

7. A 10.0-kg block is originally at rest on a frictionless table. A 2.5-kg block is placed on the 10.0-kg block, and a 11.0-N force pulls the 2.5-kg block a distance of 30.0 cm; the blocks, however, are free to continue moving. The coefficient of friction between the two blocks is $\mu_k = 0.35$. What is the change in internal energy of the two blocks (*a*) between the start when the blocks are at rest and the moment the applied force is discontinued, and (*b*) between the moment the applied force is discontinued and the time that the blocks are at rest relative to each other?

8. Consider the reaction A + B → C + D. Show that this can be an elastic collision only if there is no change in the colliding bodies.

COMPUTER PROBLEMS

1. A small block of mass m is originally at rest on the edge of a hemispherical bowl of radius R. The block, starting at $\theta = \pi/2$, slides down to the bottom of the bowl and up the other side, but because of the energy dissipated due to friction, the block does not make it to the edge before sliding back down again. Numerically plot a graph of the angular position of the block as a function of time. (*a*) As a first approximation, solve the problem with the assumption that the amount of energy dissipated is proportional to the total *distance* traveled: $\Delta E_{\text{dissipated}} \propto \Delta\theta$. (*b*) Refine the approximation by solving the problem with the assumption that the energy dissipated also depends on the angle: $\Delta E_{\text{dissipated}} \propto \cos\theta\,\Delta\theta$.

2. Assume that 100 identical 10.0-g particles are contained in a cube 1.0 m on a side. With a spreadsheet or otherwise, use a random number generator to assign x, y, and z positions to the 100 particles, and randomly assign v_x, v_y, and v_z velocity components (between −10 and +10 m/s) to each of the 100 particles. (*a*) Calculate the location of the center of mass of the particles, the translational kinetic energy of the center of mass, the rotational kinetic energy about the center of mass, and the total kinetic energy of the system. How do the three kinetic energies compare? (*b*) Repeat the process with a new set of random numbers, and create a histogram for each of the three kinetic energies. On average, what fraction of the energy is internal for this type of system?

CHAPTER 14

GRAVITATION

So far in this book we have discussed various forces: pushes and pulls, elastic forces, friction, and other forces that act when one body is in contact with another. In this chapter we study the properties of one particularly important noncontact force, gravitation, *which is one of the fundamental and (we believe) universal forces of nature. The law that describes the gravitational force between any two bodies was discovered by Newton in 1665, and it has had spectacular success in accounting for the gravitational forces exerted on objects on Earth as well as for the motions of the planets in the solar system. A modern theory of gravitation, Einstein's general theory of relativity, is necessary to account for effects in strong gravitational fields.*

As you study this chapter, you should note that many of the basic concepts of dynamics discussed in previous chapters find application here. In particular, we shall use Newton's force laws, dynamics of circular motion, potential energy, and conservation of energy and angular momentum.

14-1 ORIGIN OF THE LAW OF GRAVITATION

From at least the time of the ancient Greeks, two problems were puzzling: (1) the falling of objects released near the Earth's surface, and (2) the motions of the planets. Although there was no reason at that time to connect these two problems, today we recognize that they result from the effect of the same force—gravitation. In fact, this force also determines the motion of the Sun in our Milky Way galaxy, as well as the motion of the galaxy in our Local Cluster of galaxies, the motion of the Local Cluster in the Local Supercluster, and so on through the universe. In short, the gravitational force, and the law that describes that force, controls the structure, the development, and the eventual fate of the universe.

The earliest serious attempt to explain the motions of the planets was due to Claudius Ptolemy (A.D. 2nd century), who developed a model of the solar system in which the planets, including the Sun and Moon, revolved about the Earth. Unfortunately, to explain the complicated orbits of the planets in this geocentric frame of reference, Ptolemy was forced to introduce epicycles, in which a planet moves around a small circle whose center moves around another larger circle centered on the Earth. Of course, today we would reject such a model because it violates the law that every accelerated motion must be accounted for by a force due to a body in its environment—there is no body at the center of the small circles that would supply the force necessary for the centripetal acceleration.

It was not until the 16th century that Nicolaus Copernicus (1473–1543) proposed a heliocentric (Sun-centered) scheme, in which the Earth and the other planets move about the Sun. Like Ptolemy's model, Copernicus' solar system was still based only on geometry because the notion of a force had not yet been introduced. Nevertheless, the Copernican model was a significant step forward because it provided the correct reference frame for our present understanding of the solar system to develop.

Based on careful analysis of observational data on planetary motions, Johannes Kepler (1571–1630) proposed three laws (which we discuss in Section 14-7) that describe those motions. However, Kepler's laws were only empirical—they simply described the motions of the planets without any basis in terms of forces. It was a great triumph for the newly developed field of mechanics later in the 17th century when Isaac Newton was able to derive Kepler's laws from his laws of mechanics and his proposed law of gravitation. With this stunning development, Newton was able to use the same concept to account for the motion of the planets and of bodies falling near the Earth's surface.

In 1665, the 23-year-old Newton left Cambridge University when the college was dismissed because of the plague. Later Newton would write: "I began to think of gravity extending to the orb of the Moon . . . and having thereby compared the force requisite to keep the Moon in her orb with the force of gravity at the surface of the Earth, and found them to answer pretty nearly." His friend William Stukeley wrote of having tea with Newton under some apple trees when Newton recalled how he got the idea for gravitation: "It was occasioned by the fall of an apple as he sat in a contemplative mood . . . and thus by degrees he began to apply this property of gravitation to the motion of the Earth and the heavenly bodies . . . "

Using modern data, let us see how Newton might have made this comparison. Figure 14-1 shows that the Moon, moving in a circular orbit about the Earth, and an apple, falling near the Earth's surface, are both accelerated toward the center of the Earth. The Moon's centripetal acceleration ($a_M = v^2/r_M$) can be found from its tangential speed $v = 2\pi r_M/T$, where T is the time for one orbit (27.3 d). Using our current value for the radius of the Moon's orbit (about 380,000 km), we obtain $a_M = 0.0027\ \text{m/s}^2$. Because the acceleration a_a of the apple is simply the free-fall acceleration g, we obtain the ratio of these two accelerations to be

$$a_M/a_a = (0.0027\ \text{m/s}^2)/(9.8\ \text{m/s}^2) = 2.8 \times 10^{-4}.$$

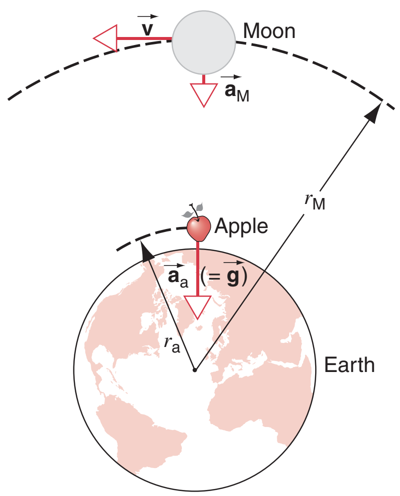


FIGURE 14-1. Both the Moon and the apple are accelerated toward the center of the Earth. The difference in their motions arises because the Moon has enough tangential speed v to maintain a circular orbit.

Guided by Kepler's laws, Newton tried to account for this difference by assuming that the gravitational force on these objects that produces the acceleration is inversely proportional to the square of their distance from the center of the Earth. Using the current value of the Earth's radius ($r_a \approx r_E = 6400$ km), Newton's prediction for the ratio of the accelerations would be

$$a_M/a_a = r_a^2/r_M^2 = (6400\ \text{km})^2/(380{,}000\ \text{km})^2 = 2.8 \times 10^{-4}.$$

In Newton's words, the two results do indeed "answer pretty nearly," suggesting that the force responsible for the fall of an apple and the force that holds the Moon in its orbit have the same origin—the Earth's gravitation.

To make this calculation, Newton had to regard the gravitational force of the Earth as if all of the Earth's mass were concentrated at its center, so that r_M and r_a are measured from the center of the Earth (see Fig. 14-1). In fact, this assumption can be rigorously proved (see Section 14-5), if we assume that the Earth is approximately spherical (a good assumption) and that the distribution of its mass may change with distance from the center but not with angular coordinate (also a good assumption, as we discuss in Section 14-4).

14-2 NEWTON'S LAW OF UNIVERSAL GRAVITATION

As we discussed in Section 5-1, gravitation, the force that acts between bodies due only to their masses, is one of the four basic forces of physics. It acts throughout the universe: between bodies on Earth, where (as we shall see below) it is weak and difficult to measure; between the Earth and bodies in its vicinity, where it is the controlling feature of our lives; and among the stars and galaxies, where it controls their evolution and structure.

Newton was the first to propose a force law for gravitation, which we can state as follows:

Every particle in the universe attracts every other particle with a force directly proportional to the product of their masses and inversely proportional to the square of the distance between them. The direction of the force is along the line joining the particles.

The magnitude of the gravitational force that two particles of masses m_1 and m_2 separated by a distance r exert on each other is

$$F = G\frac{m_1 m_2}{r^2}. \qquad (14\text{-}1)$$

Here G, called the *gravitational constant,* has the experimentally determined value

$$G = 6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2.$$

G is a universal constant, with the same value for any pair of particles at any location in the universe. It should not be confused with g, the free-fall acceleration on Earth, which is not universal and has different dimensions.

The gravitational constant is a very small number, which explains why we do not ordinarily notice the gravitational force between objects around us. For example, the force between two 1-kg particles separated by a distance of 0.1 m would be of order 10^{-8} N, about equivalent to the weight of a speck of dust! Nevertheless, using sensitive apparatus physicists can measure these small attractive forces between common objects. Normally, however, it is only when the mass of at least one of the interacting bodies is large (planet-sized) that the effects of the gravitational force become significant.

Equation 14-1 is in the form of an *inverse-square* force law, because the force depends on the inverse square of the distance. Electromagnetic forces also have the form of inverse-square laws.

None of the three fundamental quantities (force, mass, and distance) that appears in Eq. 14-1 is defined by that equation. In particular, force and mass were defined in Chapter 3. The gravitational force is just one type of force that represents the interaction of a particle with other particles in its environment. As we discuss in the next section, once G is determined by experiment for one pair of bodies, that value can then be used to find the force between any other pair of bodies.

The Vector Force

Figure 14-2*a* represents the gravitational force exerted by two particles on each other, which form an action–reaction pair according to Newton's third law. The first particle exerts an attractive force $\vec{\mathbf{F}}_{21}$ on the second particle along the line joining the two particles, and similarly the second particle exerts a force $\vec{\mathbf{F}}_{12}$ on the first. The forces are oppositely directed and always equal in magnitude, even though the two masses may not be equal.

We can express the law of universal gravitation in vector form by introducing a *unit vector* that has no units or dimensions, that has a numerical length of exactly 1, and whose sole function is to indicate a direction in space. (The unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, which respectively specify the directions of the coordinate axes x, y, and z, serve similar functions; see Appendix H.) In the case of the gravitational force, we choose the direction of the unit vector to be from one particle to the other. We denote the unit vector pointing toward m_1 from m_2 as $\hat{\mathbf{r}}_{12}$ (Fig. 14-2*b*) and that to m_2 from m_1 as $\hat{\mathbf{r}}_{21}$ (Fig. 14-2*c*). We can express these unit vectors as

$$\hat{\mathbf{r}}_{12} = \vec{\mathbf{r}}_{12}/r_{12} \quad \text{and} \quad \hat{\mathbf{r}}_{21} = \vec{\mathbf{r}}_{21}/r_{21} \qquad (14\text{-}2)$$

where, for example, $\vec{\mathbf{r}}_{12}$ is the displacement vector that locates m_1 relative to m_2 and r_{12} is its magnitude $|\vec{\mathbf{r}}_{12}|$. From Eq. 14-2 you can see that $|\hat{\mathbf{r}}_{12}| = 1$ and $|\hat{\mathbf{r}}_{21}| = 1$.

In terms of the unit vectors, we can represent the gravitational forces as

$$\vec{\mathbf{F}}_{12} = -G\frac{m_1 m_2}{r_{12}^2}\hat{\mathbf{r}}_{12} \quad \text{and} \quad \vec{\mathbf{F}}_{21} = -G\frac{m_2 m_1}{r_{21}^2}\hat{\mathbf{r}}_{21}, \qquad (14\text{-}3)$$

as shown in Figs. 14-2*b* and *c*.

The negative sign in Eq. 14-3 shows that, for example, $\vec{\mathbf{F}}_{12}$ points in a direction opposite to $\hat{\mathbf{r}}_{12}$, which indicates that the gravitational force is attractive—the direction of $\hat{\mathbf{r}}_{12}$, which is the same as that of the displacement vector $\vec{\mathbf{r}}_{12}$, points *away from* m_2, but for an attractive force the vector $\vec{\mathbf{F}}_{12}$ points *toward* m_2, as Fig. 14-2*b* shows. This notation is useful in the case in which one object (for example, the Sun) defines the origin of our coordinate system. The vector $\vec{\mathbf{r}}$ that locates the other object (the Earth, for instance) points *away from* the origin, as does the unit vector $\hat{\mathbf{r}}$, but the force on the Earth due to the Sun points *toward* the origin.

By comparing Figs. 14-2*b* and *c* we can clearly see that $\vec{\mathbf{r}}_{21} = -\vec{\mathbf{r}}_{12}$, from which Eqs. 14-3 show directly that $\vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21}$, verifying that the gravitational forces form an action–reaction pair.

Often we must consider the gravitational force when more than two bodies are interacting, for example the force on the Earth due to the Sun and the Moon. In this case the procedure is to calculate the magnitudes and directions of the forces on one body due to each of the others in turn using Eq. 14-3 and then use vector addition to find the total force on that body. Figure 14-3 shows an example in the case of one particular arrangement of the Earth, Sun, and

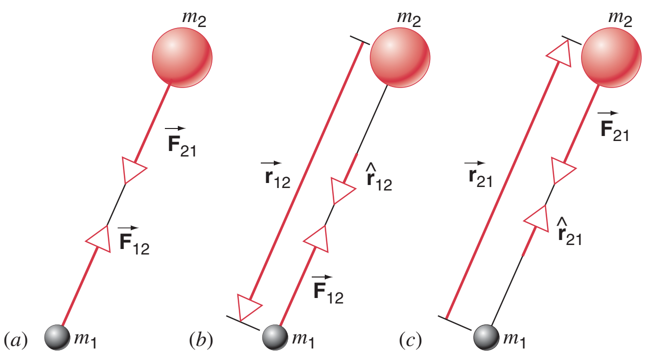


Figure 14-2. (*a*) The gravitational force between two particles, which form an action–reaction pair. (*b*) The gravitational force $\vec{\mathbf{F}}_{12}$ exerted on m_1 by m_2 and the unit vector $\hat{\mathbf{r}}_{12}$ to m_1 from m_2. (*c*) The gravitational force $\vec{\mathbf{F}}_{21}$ exerted on m_2 by m_1 and the unit vector $\hat{\mathbf{r}}_{21}$ to m_2 from m_1.

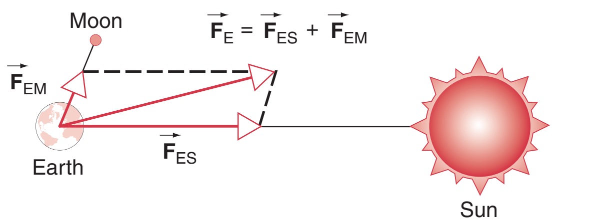


Figure 14-3. The gravitational force on the Earth due to the Sun and the Moon in one particular arrangement. The distances are not to scale, and the force vectors are also not to scale (in actuality F_{ES} is 175 times larger than F_{EM}).

Moon. We find the Earth–Sun force $\vec{\mathbf{F}}_{ES}$ as if the Moon were not present and the Earth–Moon force $\vec{\mathbf{F}}_{EM}$ as if the Sun were not present, and then we add those forces like vectors to find the resultant force on the Earth. This procedure follows the *principle of superposition,* according to which we can write the net force on body X due to N other bodies as

$$\vec{\mathbf{F}}_X = \vec{\mathbf{F}}_{X1} + \vec{\mathbf{F}}_{X2} + \cdots + \vec{\mathbf{F}}_{XN} = \sum_{n=1}^{N} \vec{\mathbf{F}}_{Xn}. \quad (14\text{-}4)$$

It is a good approximation to regard the Earth, Sun, and Moon as particles when we calculate the forces between them, because their sizes are small in comparison to the distances that separate them. However, if we wish to consider the gravitational force exerted by the Earth on a satellite in orbit 300 km above its surface, it is certainly not a good approximation to consider the Earth as a particle. To follow Eq. 14-4 and add the vector contributions to the force on the satellite due to each particle of the Earth is a task of hopeless complexity. Fortunately, this procedure is not necessary. Using calculus (which Newton developed partly for this purpose) we can show that, in the case of a spherically symmetric body, we can calculate the gravitational force as if its entire mass were concentrated in a particle at its center. We often use this important result, which we prove in Section 14-5.

SAMPLE PROBLEM 14-1. Calculate the magnitude of the gravitational force exerted on a cantaloupe of mass $m_c = 1.00$ kg on the surface of the Earth due to (*a*) the Earth, (*b*) the Moon, (*c*) the Sun.

Solution (*a*) The gravitational force on the cantaloupe due to the Earth is simply the weight of the cantaloupe:

$$F_{cE} = m_c g = (1.00\text{ kg})(9.8\text{ m/s}^2) = 9.8\text{ N}.$$

(*b*) To find the force due to the Moon, we use Eq. 14-1:

$$F_{cM} = G\frac{m_c m_M}{r_M^2}$$
$$= \frac{(6.67\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(1.00\text{ kg})(7.36\times10^{22}\text{ kg})}{(3.82\times10^8\text{ m})^2}$$
$$= 3.36\times10^{-5}\text{ N}.$$

(*c*) Again using Eq. 14-1, we have

$$F_{cS} = G\frac{m_c m_S}{r_S^2}$$
$$= \frac{(6.67\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(1.00\text{ kg})(1.99\times10^{30}\text{ kg})}{(1.50\times10^{11}\text{ m})^2}$$
$$= 5.90\times10^{-3}\text{ N}.$$

Clearly the Earth is the dominant influence on the behavior of objects on its surface. Note that the force due to the Sun on an object at the Earth's surface is much larger than the force due to the Moon. (However, the tidal effect of the Moon on the Earth's oceans is greater than that of the Sun. See Problem 5 for an explanation of this effect.)

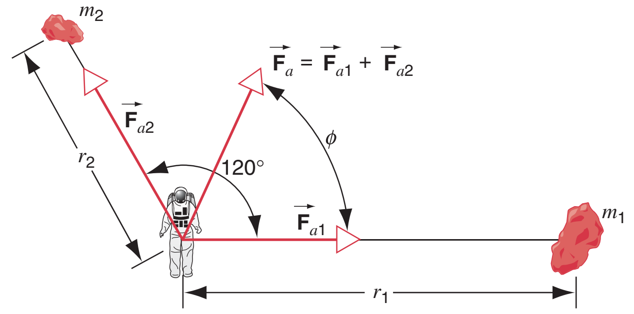


FIGURE 14-4. Sample Problem 14-2.

SAMPLE PROBLEM 14-2. A properly suited astronaut ($m_a = 105$ kg) is drifting through the asteroid belt on a mining expedition. At a particular instant he is located near two asteroids of masses $m_1 = 346$ kg ($r_1 = 215$ m) and $m_2 = 184$ kg ($r_2 = 142$ m) as shown in Fig. 14-4. The lines connecting the astronaut to the two asteroids form an angle of 120°. At that instant, what is the magnitude and direction of the gravitational force on the astronaut due to these two asteroids? Assume that the astronaut and the asteroids can be considered as particles.

Solution Equation 14-1 gives the magnitudes of the two forces:

$$F_{a1} = G\frac{m_a m_1}{r_1^2} = \frac{(6.67\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(105\text{ kg})(346\text{ kg})}{(215\text{ m})^2}$$
$$= 5.24\times10^{-11}\text{ N} = 52.4\text{ pN},$$

$$F_{a2} = G\frac{m_a m_2}{r_2^2} = \frac{(6.67\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(105\text{ kg})(184\text{ kg})}{(142\text{ m})^2}$$
$$= 6.39\times10^{-11}\text{ N} = 63.9\text{ pN}.$$

These two forces are shown in Fig. 14-4. Using either the component method or the parallelogram method, we can add these two vectors to obtain the magnitude of the total force on the astronaut to be

$$F_a = 5.80\times10^{-11}\text{ N} = 58.9\text{ pN}$$

and its direction is as shown in Fig. 14-4 ($\phi = 69.7°$).

14-3 THE GRAVITATIONAL CONSTANT *G*

Determining the value of G would seem to be a simple task. All we need to do is to measure the gravitational force F exerted by a body of known mass m_1 on a second body of known mass m_2 separated by a known distance r. We can then calculate G from Eq. 14-1.

A large-scale system such as the Earth and the Moon or the Earth and the Sun cannot serve to determine G. The distances are large enough that the objects can be regarded as approximately point masses, but the values of the masses are not determined independently. In fact, the masses of these bodies, as we shall soon discuss, are determined using the value of G.

Instead, we must turn to a small-scale measurement, in which we use two laboratory objects of known mass and measure the force between them. The force is very weak, and the masses must be placed close together to make the force as large as possible. When we do this, we can usually no longer regard the masses as point particles, and Eq. 14-1 may not be applicable. There is, however, one special case in which we can use Eq. 14-1 for large objects. As we prove in Section 14-5, for spherical mass distributions we can regard the object as a point mass concentrated at its center. This is *not* an approximation; it is an exact relationship.

The first laboratory determination of G from the force between spherical masses at close distance was done by Henry Cavendish in 1798. He used a method based on the torsion balance, illustrated in Fig. 14-5. Two small lead balls, each of mass m, are attached to the ends of a light rod. This rigid "dumbbell" is suspended, with its axis horizontal, by a fine vertical fiber. Two large lead balls each of mass M are placed near the ends of the dumbbell on opposite sides. When the large masses are in the positions A, they attract the small masses according to the law of gravitation, and a torque is exerted on the dumbbell, rotating it counterclockwise as viewed from above. The rod reaches an equilibrium position under the opposing actions of the gravitational torque exerted by the masses M and the restoring torque exerted by the twisted fiber. When the large masses are in the positions B, the dumbbell rotates clockwise to a new equilibrium position. The angle 2θ, through which the fiber is twisted when the balls are moved from one position (AA) to the other (BB), is measured by observing the deflection of a beam of light reflected from the small mirror attached to the rod. From the value of θ and the torsional constant of the fiber (determined by measuring its period of oscillation—see Section 17-5), the torque can be determined and the gravitational force can be obtained. Knowing the values of the masses m and M and the separation of their centers, we can calculate G.

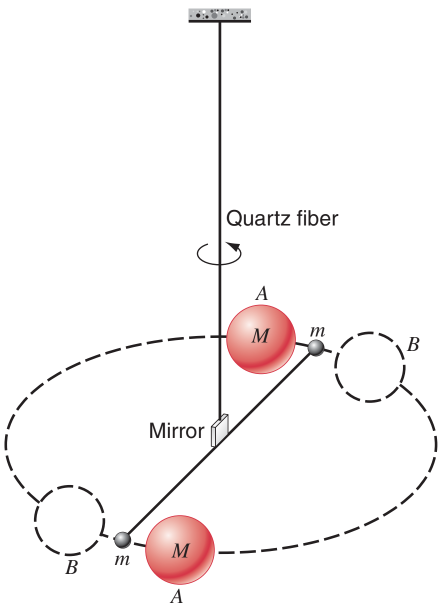


Figure 14-5. A schematic view of the apparatus used in 1798 by Henry Cavendish to measure the gravitational constant G. The large spheres of mass M, shown in location AA, can also be moved to location BB.

Cavendish's original data yielded a value for G of $6.75 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$. In the nearly 200 years since the time of Cavendish, the same basic technique using the torsion balance has been used to repeat this measurement many times, leading to the presently accepted value of G,

$$G = 6.673 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2},$$

with an uncertainty of $\pm 0.010 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$ or about $\pm 0.15\%$. Compared with the results of measuring other physical constants, this precision is not impressive; for example, the mass of the electron has been measured to a precision of about $\pm 0.000008\%$. It is difficult to improve substantially on the precision of the measured value of G because of its small magnitude and the correspondingly small value of the force between the two objects in our laboratory experiments. If we use two lead spheres of diameter 10 cm (and mass 6 kg), the maximum gravitational force between them when they are as close as possible is about 2×10^{-7} N, corresponding roughly to the weight of a piece of paper of area 1 mm^2.

This difficulty of measuring G is unfortunate, because gravitation has such an essential role in theories of the origin and structure of the universe. For example, we would like to know if G really is a constant. Does it change with time? Does it depend on the chemical or physical state of the masses? Does it depend on their temperature? Despite many experimental searches, no such variations in G have so far been unambiguously confirmed, but measurements continue to be refined and improved, and the experimental tests continue.*

The mass of the Earth can be determined from the law of universal gravitation and the value of G calculated from the Cavendish experiment. For this reason Cavendish is said to have been the first person to "weigh" the Earth. (In fact, the title of the paper written by Cavendish describing his experiments referred not to measuring G but instead to determining the density of the Earth from its weight and volume.) Consider the Earth, of mass M_E, and an object on its surface of mass m. The force of attraction is given both by

$$F = mg_0 \quad \text{and} \quad F = \frac{GmM_E}{R_E^2}.$$

Here R_E is the radius of the Earth, which is the separation of the two bodies, and g_0 is the free-fall acceleration at the Earth's surface due only to the gravitational force of the Earth (see the next section). Combining these equations we obtain

$$M_E = \frac{g_0 R_E^2}{G} = \frac{(9.83\ \mathrm{m/s^2})(6.37 \times 10^6\ \mathrm{m})^2}{6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}}$$
$$= 5.98 \times 10^{24}\ \mathrm{kg}.$$

*For a full discussion of measurements of G, see "The Newtonian Gravitational Constant: Recent Measurements and Related Studies," by George T. Gillies, *Reports on Progress in Physics,* Vol. 60, 1997, pp. 151–225.

14-4 GRAVITATION NEAR THE EARTH'S SURFACE

Let us assume, for the time being, that the Earth is spherical and that its density depends only on the radial distance from its center. The magnitude of the gravitational force acting on a particle of mass m, located at an external point a distance r from the Earth's center, can then be written, from Eq. 14-1, as

$$F = G\frac{M_E m}{r^2},$$

in which M_E is the mass of the Earth. This gravitational force can also be written, from Newton's second law, as

$$F = mg_0.$$

Here g_0 is the free-fall acceleration due only to the gravitational pull of the Earth. Combining the two equations above gives

$$g_0 = \frac{GM_E}{r^2}. \qquad (14\text{-}5)$$

Table 14-1 shows some values of g_0 at various altitudes above the surface of the Earth, calculated from this equation. Note that, contrary to the impression that gravity drops to zero in an orbiting satellite, we find $g_0 = 8.7\ \mathrm{m/s^2}$ at typical space shuttle altitudes.

The real Earth differs from our model Earth in three ways.

1. *The Earth's crust is not uniform.* There are local density variations everywhere. The precise measurement of local variations in the free-fall acceleration gives information that is useful, for example, for oil prospecting.

2. *The Earth is not a sphere.* The Earth is approximately an ellipsoid, flattened at the poles and bulging at the equator. The Earth's equatorial radius is greater than its polar radius by 21 km. Thus a point at the poles is closer to the dense core of the Earth than is a point on the equator. We would expect that the free-fall acceleration would increase as one proceeds, at sea level, from the equator toward the poles. Figure 14-6 shows that this is indeed what happens. The measured values of g in this figure include both the equatorial bulge effect and effects resulting from the rotation of the Earth.

3. *The Earth is rotating.* In Section 3-7 we defined weight as a measure of the Earth's gravitational force on a body, and we discussed how the weight can be determined from the reading of a platform scale on which the body rests. The scale reading is equal to the magnitude of the Earth's gravitational force *only* if the surface of the Earth is an inertial frame, which is only approximately true for the rotating Earth. Let us see what effect the Earth's rotation has on the scale reading.

TABLE 14-1 Variation of g_0 with Altitude

Altitude (km)	*Location*	g_0 (m/s²)
0	Earth's surface	9.83
10	Airliner cruising altitude	9.80
100	Top of atmosphere	9.53
400	Space shuttle orbit	8.70
35,700	Communication satellite orbit	0.225
380,000	Moon's orbit	0.0027

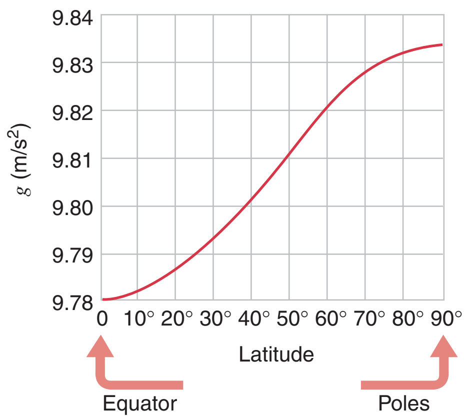


FIGURE 14-6. The variation of g with latitude at sea level. About 65% of the variation is due to the rotation of the Earth, with the remaining 35% coming from the Earth's slightly flattened shape.

Figure 14-7*a* shows the rotating Earth from an inertial frame positioned in space above the north pole. A crate of mass m rests on a platform scale at the equator. A local observer regards the scale reading to be the body's weight mg, where g is the locally measured value of the free-fall acceleration.

Because of the Earth's rotation, the crate is in uniform circular motion with radius R_E and period of rotation T (= 24 hours). As we discussed in Section 4-5, in order for a body to be in uniform circular motion at radius r and tangential speed v, the net acceleration (the centripetal acceleration) must have magnitude $a_c = v^2/r$.

Figure 14-7*b* shows the free-body diagram of the crate. There is an upward force $\vec{\mathbf{N}}$ on the crate due to the platform scale (equal in magnitude to the scale reading mg), and the

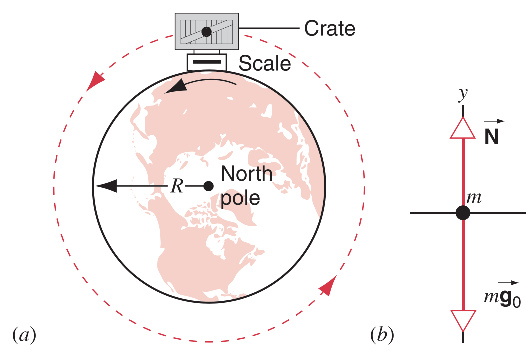


FIGURE 14-7. (*a*) A crate on the rotating Earth, resting on a platform scale at the equator. The view is along the Earth's rotational axis, looking down on the north pole. (*b*) A free-body diagram of the crate. The crate is in uniform circular motion and is thus accelerated toward the center of the Earth.

downward gravitational force is $m\vec{g}_0$. We take the y axis to be positive upward (where upward means the outward radial direction at the location of the crate), and so $\Sigma F_y = N - mg_0$. Newton's second law then gives (noting that the acceleration toward the center of the circle is in the negative y direction according to our choice of the axis)

$$N - mg_0 = -ma_c = -mv^2/R_E = -m\omega^2 R_E,$$

where $\omega = v/R_E = 2\pi/T$ is the angular speed of the Earth's rotation. We can write this expression as

$$N = m(g_0 - \omega^2 R_E). \tag{14-6}$$

Taking the magnitude of the normal force N exerted by the scale to be the weight mg of the object, we obtain

$$g_0 - g = \omega^2 R_E = 0.034 \text{ m/s}^2.$$

The free-fall acceleration g at the equator of the rotating Earth is smaller than the free-fall acceleration g_0 of a nonrotating Earth by only 0.034/9.8 or 0.35%. Equivalently, we can say that mg (the weight of the object) is less than mg_0 (the gravitational force on the object by the Earth) by an amount equal to $m\omega^2 R_E$. This effect decreases as we move north or south of the equator, and the difference $g_0 - g$ vanishes at the poles.

As we discuss later in this chapter, for a satellite in orbit at a height h above the Earth's surface and thus at a distance $r = R_E + h$ from the Earth's center, ω and r are related so that $\omega^2 r = GM_E/r^2$, which is simply g_0 as defined by Eq. 14-5. Equation 14-6 then gives $N = 0$, which accounts for our usual description of orbiting astronauts as "weightless," even though $g_0 \neq 0$ for the astronauts (the Earth's gravitational force still attracts them).

SAMPLE PROBLEM 14-3. (*a*) A neutron star is a collapsed star of extremely high density. The blinking pulsar in the Crab nebula is the best known of many examples. Consider a neutron star with a mass M equal to the mass of the Sun, 1.99×10^{30} kg, and a radius R of 12 km. What is the free-fall acceleration at its surface? Ignore rotational effects. (*b*) The asteroid Ceres has a mass of 1.2×10^{21} kg and a radius of 470 km. What is the free-fall acceleration at its surface?

Solution (*a*) From Eq. 14-5 we have

$$g_0 = \frac{GM}{R^2} = \frac{(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(1.99 \times 10^{30}\text{ kg})}{(12{,}000\text{ m})^2}$$
$$= 9.2 \times 10^{11}\text{ m/s}^2.$$

Even though pulsars rotate extremely rapidly, rotational effects have only a small influence on the value of g, because of the small size of pulsars.

(*b*) In the case of the asteroid Ceres, we have

$$g_0 = \frac{GM}{R^2} = \frac{(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(1.2 \times 10^{21}\text{ kg})}{(4.7 \times 10^5\text{ m})^2}$$
$$= 0.36\text{ m/s}^2.$$

There is quite a contrast between the gravitational forces at the surfaces of these two bodies!

14-5 THE TWO SHELL THEOREMS

In calculating the gravitational force exerted by a body such as the Earth or the Sun on another body, it would be hopelessly complicated to account for the interactions of every pair of particles in the two bodies. Fortunately, we can use two *shell theorems* to simplify the analysis of the gravitational force in certain cases.

We consider only bodies in which the mass distribution is *spherically symmetric*. That is, the density may change with radius, but the density is uniform in a thin shell at any radius. To a very good approximation, the Earth and Sun are spherically symmetric. In both bodies, the density is large near the center and decreases toward the surface. Thin shells near the center contain material of a greater density than thin shells near the surface, but within each thin shell the density has the same value at all points.

Using his law of universal gravitation and his newly developed techniques of calculus, Newton established two theorems that apply to the gravitational force exerted by a thin spherical shell of uniform density.

Shell Theorem #1:

A uniformly dense spherical shell attracts an external particle as if all the mass of the shell were concentrated at its center.

Shell Theorem #2:

A uniformly dense spherical shell exerts no gravitational force on a particle located anywhere inside it.

A spherically symmetric body such as the Earth can be regarded as composed of a series of thin spherical shells of uniform density. In calculating the force on a particle at a point beyond the Earth's radius, each of those shells can be replaced by an equivalent mass at the Earth's center, and therefore the entire Earth behaves as if it were a point mass located at its center. A corollary of shell theorem #1 is thus: *A spherically symmetric body attracts particles outside as if its mass were concentrated at its center.*

The importance of shell theorem #2 can be appreciated by imagining a tunnel drilled along a diameter of the Earth. As we descend into the tunnel, the portions of the Earth outside our radius exert no gravitational force on us. Put another way, we feel only the effect of the portion of the Earth's mass inside a sphere whose radius is our distance from the center of the Earth. Sample Problem 14-4 considers this consequence of shell theorem #2.

The shell theorems are true only for the inverse-square force. If the gravitational force depended on the separation r to some power other than -2, the shell theorems would not hold. As a consequence, it would not be possible to replace a spherically symmetric body with its equivalent point mass. In fact, the second shell theorem provides an elegant way of testing the inverse-square law—we place a small test mass at various locations inside a spherical shell and determine whether the gravitational force on the test mass due to the shell is zero everywhere in the interior. Such measurements done to the highest possible precision have shown no deviation from the Newtonian law. If we write the dependence on the separation between the masses as $1/r^{2+\delta}$, where $\delta = 0$ in Newton's theory, then experiments have set an upper limit on δ of 10^{-4}. By contrast, similar experiments designed to test the inverse-square law for electric forces give an upper limit on δ of about 10^{-16}.

Proof of the Shell Theorems (Optional)

Proving the shell theorems requires techniques of integral calculus. We wish to calculate the force exerted by a thin spherical shell of uniform density on a point mass located either outside or inside the shell. Our technique will be to imagine the shell to be sliced into thin rings. We will find the force exerted on the point mass by one arbitrary ring, and then we will add (by integration) the forces exerted by all such rings to get the total force.

Figure 14-8 shows a thin shell and the ring we will consider. The shell has total mass M, thickness t, and uniform density ρ (mass per unit volume). A point mass m is located at point P, a distance r from the center of the shell (point O). Our goal is to find the force exerted on m first by the ring and then by the entire shell.

Consider the ring shown in the figure. If the ring is very thin, all particles of the ring are a distance x from m. A particle at point A exerts a force $\vec{\mathbf{F}}_A$ on m, and a particle of equal mass at B, on the opposite side of the ring, exerts a force $\vec{\mathbf{F}}_B$. The two forces are of equal magnitude, and their resultant must lie along the line PO. This will also be true for every pair of particles located diametrically opposite one another in the ring, and so the net force exerted on m by the ring must also lie along the line PO (the symmetry axis).

Consider an element of mass dm_A at point A. The axial component (along PO) of the force that this element of mass exerts on m is

$$dF_A = G\frac{m\,dm_A}{x^2}\cos\alpha,$$

where the factor $\cos\alpha$ gives the axial component of the force. Adding the contributions for all the mass elements in the ring gives the total force dF exerted on m by the ring:

$$dF = dF_A + dF_B + \cdots = \frac{Gm}{x^2}(\cos\alpha)(dm_A + dm_B + \cdots)$$

or

$$dF = \frac{Gm\,dM}{x^2}\cos\alpha, \qquad (14\text{-}7)$$

where $dM\ (= dm_A + dm_B + \cdots)$ is the total mass of the ring.

Now we must express dM in terms of the geometrical dimensions of the ring. First we find its volume dV. Imagine the ring cut and laid out flat to form a rectangular solid of height t (the thickness of the ring), width $R\,d\theta$, and length $2\pi(R\sin\theta)$. The volume is thus $dV = t(R\,d\theta)(2\pi R\sin\theta)$. The mass of this ring is $dM = \rho\,dV$, or

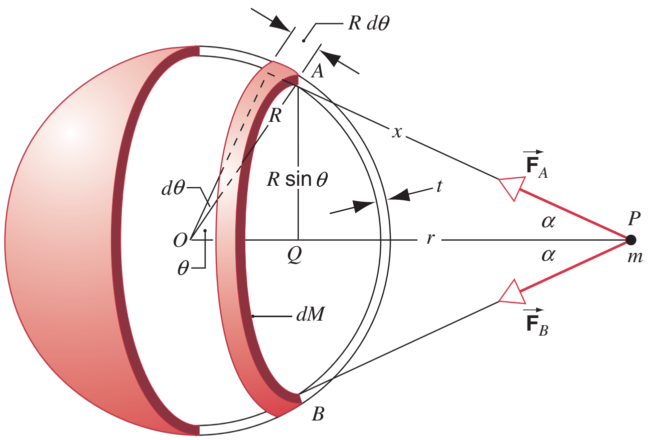


FIGURE 14-8. Gravitational attraction of a section of a spherical shell of matter on a particle of mass m at P.

$$dM = 2\pi t\rho R^2\sin\theta\,d\theta. \qquad (14\text{-}8)$$

Finally, we must choose a single variable for our integration. From the three variables in Fig. 14-8 (x, α, and θ) we choose to eliminate α and θ, leaving x as the single variable over which we plan to integrate. From the figure we see that $PQ = x\cos\alpha$ and also $PQ = r - R\cos\theta$, so

$$\cos\alpha = \frac{r - R\cos\theta}{x}. \qquad (14\text{-}9)$$

Using the law of cosines on triangle AOP we obtain $x^2 = r^2 + R^2 - 2rR\cos\theta$ or

$$R\cos\theta = \frac{r^2 + R^2 - x^2}{2r}. \qquad (14\text{-}10)$$

We now put Eq. 14-10 into Eq. 14-9 and then substitute the result for $\cos\alpha$ into Eq. 14-7. We still have not eliminated all variables except x in Eq. 14-7, because Eq. 14-8 shows that dM depends on θ. To eliminate this variable, we differentiate Eq. 14-10 to find

$$\sin\theta\,d\theta = \frac{x}{rR}\,dx. \qquad (14\text{-}11)$$

We can use Eq. 14-11 to eliminate θ from Eq. 14-8 and then substitute the result for dM into Eq. 14-7. The result is

$$dF = \frac{\pi G t\rho m R}{r^2}\left(\frac{r^2 - R^2}{x^2} + 1\right)dx. \qquad (14\text{-}12)$$

This is the force exerted by the circular ring dM on the particle m at P.

To find the total force on m due to the entire shell, we must add the effects due to all the rings into which we imagine the shell to be sliced. This involves an integral over x, which ranges from $r - R$ to $r + R$:

$$F = \int dF = \frac{\pi G t\rho m R}{r^2}\int_{r-R}^{r+R}\left(\frac{r^2 - R^2}{x^2} + 1\right)dx.$$

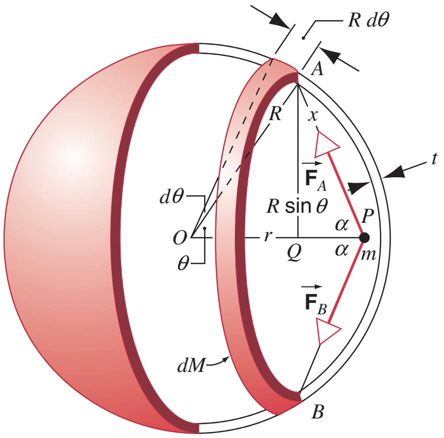


FIGURE 14-9. Gravitational attraction of a section of a spherical shell of matter on a particle of mass m at a point P inside the shell.

The integral is straightforward to evaluate and gives the value $4R$. The force then becomes

$$F = \frac{\pi G t \rho m R}{r^2}(4R) = G\frac{mM}{r^2}, \qquad (14\text{-}13)$$

where $M = 4\pi R^2 t\rho$ is the total mass of the shell. Equation 14-13 is exactly the expression for the force that a particle of mass M located at the center of the shell would exert on the particle of mass m located at P. This proves the first shell theorem.

The proof of the second shell theorem is based on the geometry shown in Fig. 14-9, with point P now inside the shell. The derivation is exactly the same up to the final step, but the lower limit of the integral is now $R - r$ rather than $r - R$. This small change causes the value of the integral to be zero, so that $F = 0$, which proves the second shell theorem. (For an alternative way of proving the second shell theorem, see Problem 17.)

These proofs apply only in the spherical geometry and only when the density of the shell is uniform. The theorems can be applied to a solid sphere even if the density changes from one shell to the next, as long as it remains uniform for each shell. ■

SAMPLE PROBLEM 14-4. Suppose a tunnel could be dug through the Earth from one side to the other along a diameter, as shown in Fig. 14-10. A particle of mass m is dropped into the tunnel from rest at the surface. (*a*) What is the force on the particle when it is a distance r from the center? (*b*) What is the speed of the particle when it is a distance r from the center? Evaluate the speed at $r = 0$. Neglect all frictional forces and assume that the Earth has a uniform density.

Solution (*a*) From shell theorem #2 we conclude that the gravitational force on the particle is due only to that portion of the Earth that lies inside the sphere of radius r, and from shell theorem #1 we conclude that we can consider that mass to be concentrated at the center of the Earth. Let the mass inside the sphere of radius r be M, and let the total mass of the Earth (of radius R_E) be M_E. Then the fraction of the mass inside radius r is the same as the

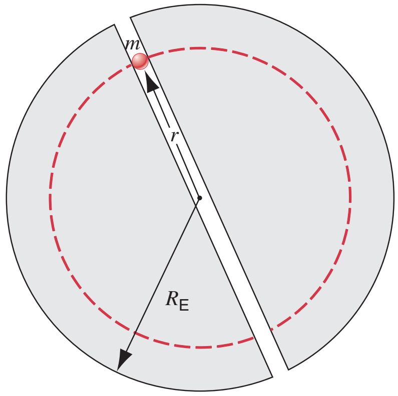


FIGURE 14-10. Sample Problem 14-4. A particle moves in a tunnel through the Earth.

fraction of the volume inside radius r (this is true only if the density is uniform, as we have assumed). Thus

$$\frac{M}{M_E} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R_E^3} \quad \text{or} \quad M = M_E\frac{r^3}{R_E^3}.$$

Regarding this mass as concentrated at the center, we find the gravitational force on mass m to be proportional to r:

$$F = G\frac{mM}{r^2} = G\frac{mM_E r^3}{r^2 R_E^3} = \frac{GmM_E}{R_E^3}r.$$

If we let $\vec{\mathbf{r}}$ be the vector from the center of the Earth to m, then from Eq. 14-3 we note that the force $\vec{\mathbf{F}}$ acting on the particle is opposite in direction to $\vec{\mathbf{r}}$, so we can write $\vec{\mathbf{F}} = -(GmM_E/R_E^3)\vec{\mathbf{r}}$ for the vector form of the force law. With the minus sign, the form of the force looks very much like that of the spring force, $F = -kx$.

(*b*) Given the similarity with the spring force, we can represent the potential energy U of the system consisting of the Earth and the falling particle as $\frac{1}{2}kr^2$, taking $U = 0$ at the center of the Earth. Here k is the constant in the force law: $k = GmM_E/R_E^3$. Applying conservation of energy at the surface and at radius r we have $K_s + U_s = K_r + U_r$ or $0 + \frac{1}{2}kR_E^2 = \frac{1}{2}mv^2 + \frac{1}{2}kr^2$. Solving for v, we have

$$v = \sqrt{\frac{k}{m}(R_E^2 - r^2)} = \sqrt{\frac{GM_E}{R_E^3}(R_E^2 - r^2)}.$$

At the center ($r = 0$) this has the value

$$v = \sqrt{\frac{GM_E}{R_E}} = \sqrt{\frac{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\ \text{kg})}{(6.37 \times 10^6\ \text{m})}}$$

$$= 7.91 \times 10^3\ \text{m/s}.$$

14-6 GRAVITATIONAL POTENTIAL ENERGY

In analyzing the motion of planets and satellites, it is often easier and more informative to use energy rather than force. In this section we shall evaluate the potential energy of a system consisting of two bodies that interact through the gravitational force. In Chapter 12 we obtained the potential energy change due to gravity for a body that moves through a height y near the Earth's surface: $\Delta U = mgy$ (Eq. 12-9).

However, this applies only near the Earth's surface, where (for changes in height that are small compared with the distance from the center of the Earth) we can regard the gravitational force as approximately constant. Our goal here is to find a general expression that applies at all locations, such as at the altitude of an orbiting satellite.

The potential energy difference can be found from Eq. 12-4: $\Delta U = U_b - U_a = -W_{ab}$, where W_{ab} is the work done by the force when the system changes from configuration a to configuration b. However, this equation applies *only* if the force is conservative. Is the gravitational force conservative?

Figure 14-11 shows a particle of mass m moving in a region where a gravitational force is exerted on it by a particle of mass M. Particle m moves from a to b along several different paths: path 1 (aAb), path 2 (aBb), and path 3 ($aCDEFGHb$). The paths consist of straight segments along a radius and curved segments along arcs of circles centered at M. Along every curved segment such as aC, $\vec{\mathbf{F}}\cdot d\vec{\mathbf{s}} = 0$ for a small displacement $d\vec{\mathbf{s}}$, because $\vec{\mathbf{F}}$ is perpendicular to $d\vec{\mathbf{s}}$. We therefore have, for the work done by the gravitational force along path 1,

$$W_1 = W_{aA} + W_{Ab} = W_{aA} = \int_{r_a}^{r_A} \vec{\mathbf{F}}\cdot d\vec{\mathbf{s}} = \int_{r_a}^{r_b} \vec{\mathbf{F}}\cdot d\vec{\mathbf{s}},$$

where the last step follows because $r_A = r_b$. Similarly,

$$W_2 = W_{aB} + W_{Bb} = W_{Bb} = \int_{r_B}^{r_b} \vec{\mathbf{F}}\cdot d\vec{\mathbf{s}} = \int_{r_a}^{r_b} \vec{\mathbf{F}}\cdot d\vec{\mathbf{s}}$$

with $r_B = r_a$. Finally, for path 3,

$$\begin{aligned} W_3 &= W_{aC} + W_{CD} + W_{DE} + W_{EF} + W_{FG} + W_{GH} + W_{Hb} \\ &= W_{CD} + W_{EF} + W_{GH} \\ &= \int_{r_C}^{r_D} \vec{\mathbf{F}}\cdot d\vec{\mathbf{s}} + \int_{r_E}^{r_F} \vec{\mathbf{F}}\cdot d\vec{\mathbf{s}} + \int_{r_G}^{r_H} \vec{\mathbf{F}}\cdot d\vec{\mathbf{s}} \\ &= \int_{r_a}^{r_D} \vec{\mathbf{F}}\cdot d\vec{\mathbf{s}} + \int_{r_D}^{r_F} \vec{\mathbf{F}}\cdot d\vec{\mathbf{s}} + \int_{r_F}^{r_b} \vec{\mathbf{F}}\cdot d\vec{\mathbf{s}} = \int_{r_a}^{r_b} \vec{\mathbf{F}}\cdot d\vec{\mathbf{s}}, \end{aligned}$$

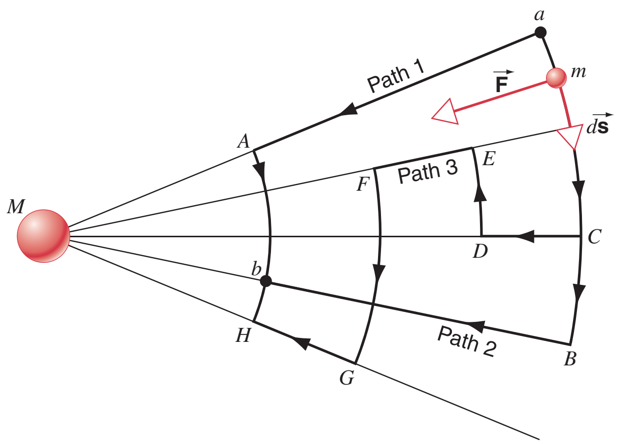


FIGURE 14-11. A particle of mass m moves from a to b along three different paths. A gravitational force $\vec{\mathbf{F}}$ is exerted on m by a particle of mass M.

where we have used the rules of calculus for combining integrals with identical upper and lower limits.

It is clear from this calculation that $W_1 = W_2 = W_3$, and you should convince yourself that *any* path from a to b can be represented as a combination of such radial and tangential segments and thus will give the same value for the work. Clearly the work is independent of the path, and the gravitational force is conservative.

Calculating the Potential Energy

Now that we have established that the gravitational force is conservative, we can calculate the potential energy. Figure 14-12 shows a particle of mass m moving from a to b along a radial path. A particle of mass M, which we assume to be at rest at the origin, exerts a gravitational force on m. The vector $\vec{\mathbf{r}}$ locates the position of m relative to M at any time. As m moves from a to b, the work done on m by the gravitational force is

$$\begin{aligned} W_{ab} &= \int_a^b \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = -\int_a^b F\,dr \\ &= -\int_{r_a}^{r_b} \frac{GMm}{r^2}\,dr = -GMm\int_{r_a}^{r_b} \frac{dr}{r^2} \\ &= -GMm\left(-\frac{1}{r}\right)\Bigg|_{r_a}^{r_b} = GMm\left(\frac{1}{r_b} - \frac{1}{r_a}\right). \end{aligned} \qquad (14\text{-}14)$$

The negative sign in the first line of this equation arises because the (attractive) force $\vec{\mathbf{F}}$ and the infinitesimal radial vector $d\vec{\mathbf{r}}$ point in opposite directions. Equation 14-14 shows that, when $r_b > r_a$ (as in Fig. 14-12), the work W_{ab} is negative, as we expect. However, we can also show that Eq. 14-14 applies when m moves inward from a point a to another point b; that is, when $r_a > r_b$, the force and displacement are in the same direction and the work is positive, consistent with Eq. 14-14.

Applying Eq. 12-4 ($\Delta U = -W_{ab}$), we can find the change in the potential energy of the system as m moves between points a and b

$$\Delta U = U_b - U_a = -W_{ab} = GMm\left(\frac{1}{r_a} - \frac{1}{r_b}\right). \qquad (14\text{-}15)$$

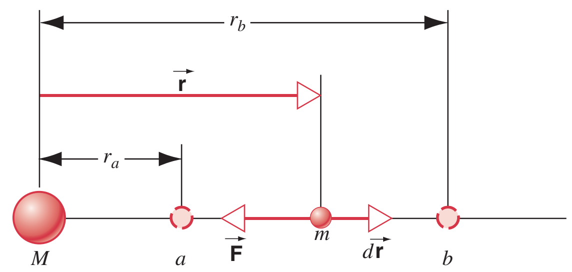


FIGURE 14-12. A particle of mass M exerts a gravitational force $\vec{\mathbf{F}}$ on a particle of mass m that moves from a to b.

If m moves outward from a to b, the change in potential energy is positive ($U_b > U_a$). That is, if the particle passes through point a with a certain kinetic energy K_a, as it travels to b its gravitational potential energy *increases* as its kinetic energy *decreases* ($K_b < K_a$). Conversely, if the particle is moving inward, its potential energy decreases as its kinetic energy increases.

Instead of *differences* in potential energy, we can consider the value of the potential energy at a single point if we define a reference point. We choose our reference configuration to be an infinite separation of the particles, and we define the potential energy to be zero in that configuration. Let us evaluate Eq. 14-15 for $r_b = \infty$ and $U_b = 0$. If a represents any arbitrary point, where the separation between the particles is r, then Eq. 14-15 becomes

$$U(\infty) - U(r) = GMm\left(\frac{1}{r} - 0\right) \qquad (14\text{-}16)$$

or

$$U(r) = -\frac{GMm}{r}. \qquad (14\text{-}17)$$

Equation 14-17 shows that, with this choice of the reference configuration, the potential energy is negative at any finite separation, and it increases toward zero as the separation increases (consistent with our discussion of the sign of ΔU following Eq. 14-15). This results from the attractive character of the gravitational force: as m moves outward from separation r to infinity, the work done on m by the gravitational force is negative, $\Delta U = U(\infty) - U(r)$ is positive, and so $U(r)$ is negative, in agreement with Eq. 14-17.

Equation 14-17 shows that the potential energy is a property of the *system* consisting of the two particles M and m, rather than of either body alone. The potential energy changes whether M or m is displaced; each is acted on by the gravitational force of the other. It also does not make any sense to assign part of the potential energy to M and part of it to m. Often, however, we do speak of the potential energy of a body m (planet or stone, say) acted on by the gravitational force of a much more massive body M (Sun or Earth, respectively). The justification for speaking as though the potential energy belongs to the planet or to the stone alone is this: When the potential energy of a system of two bodies changes into kinetic energy, the lighter body gets most of the kinetic energy. The Sun is so much more massive than a planet that the Sun receives hardly any of the kinetic energy; the same is true for the Earth in the Earth–stone system.

We can reverse the previous calculation and derive the gravitational force from the potential energy. For spherically symmetric potential energy functions, the relation $F = -dU/dr$ gives the radial component of the force; see Eq. 12-7. With the potential energy of Eq. 14-17, we obtain

$$F = -\frac{dU}{dr} = -\frac{d}{dr}\left(-\frac{GMm}{r}\right) = -\frac{GMm}{r^2}. \qquad (14\text{-}18)$$

The minus sign here shows that the force is attractive, directed inward along a radius.

We can show that the potential energy defined according to Eq. 14-15 leads to the familiar mgy for a small difference in elevation y near the surface of the Earth. Let us evaluate Eq. 14-15 for the difference in potential energy between the location at a height y above the surface (that is, $r_b = R_E + y$, where R_E is the radius of the Earth) and the surface ($r_a = R_E$):

$$\Delta U = U(R_E + y) - U(R_E) = GM_E m\left(\frac{1}{R_E} - \frac{1}{R_E + y}\right)$$

$$= \frac{GM_E m}{R_E}\left(1 - \frac{1}{1 + y/R_E}\right).$$

When $y \ll R_E$, which would be the case for small displacements of bodies near the Earth's surface, we can use the binomial expansion to approximate the last term as $(1 + x)^{-1} = 1 - x + \cdots \approx 1 - x$, which gives

$$\Delta U \approx \frac{GM_E m}{R_E}\left[1 - \left(1 - \frac{y}{R_E}\right)\right] = \frac{GM_E my}{R_E^2} = mgy,$$

using Eq. 14-5 to replace GM_E/R_E^2 with g. This shows that Eq. 14-15 for the difference in gravitational potential energy is consistent with our previous use of mgy for situations near the Earth's surface. In fact, we can use the approximation $\Delta U = mgy$ for the difference in potential energy between two elevations at any distance R from the center of the Earth, as long as $y \ll R$ and we use the value of g (see Table 14-1) appropriate for that R.

SAMPLE PROBLEM 14-5. A satellite, orbiting at an altitude of two Earth's radii above its surface, launches an equipment canister of mass m toward the Earth's center with a speed of $v_i = 525$ m/s. With what speed v_f does the canister enter the Earth's atmosphere (a distance of $h = 100$ km above its surface)?

Solution We can analyze this problem using conservation of energy. At the canister's launch it has kinetic energy $K_i = \frac{1}{2}mv_i^2$ and potential energy $U_i = -GM_E m/r_i$ (where $r_i = 3R_E$), and when it enters the atmosphere it has kinetic energy $K_f = \frac{1}{2}mv_f^2$ and potential energy $U_f = -GM_E m/r_f$ (where $r_f = R_E + h$). With $K_i + U_i = K_f + U_f$, we obtain

$$\tfrac{1}{2}mv_i^2 - \frac{GM_E m}{r_i} = \tfrac{1}{2}mv_f^2 - \frac{GM_E m}{r_f}$$

or, solving for v_f^2 and substituting $r_i = 3R_E$ and $r_f = R_E + h$,

$$v_f^2 = v_i^2 - 2GM_E\left(\frac{1}{3R_E} - \frac{1}{R_E + h}\right)$$

$$= (525\ \text{m/s})^2 - 2(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\ \text{kg})$$

$$\times \left(\frac{1}{3(6.37 \times 10^6\ \text{m})} - \frac{1}{6.47 \times 10^6\ \text{m}}\right)$$

$$= 8.18 \times 10^7\ \text{m}^2/\text{s}^2,$$

so $v_f = 9.05 \times 10^3$ m/s. Note that this value is independent of the mass of the canister and of the path it follows.

SAMPLE PROBLEM 14-6. On a straight-line path from the Earth to the Moon, the (negative) gravitational potential energy of a projectile of mass m increases as the distance from the Earth increases, reaches a maximum at point X somewhere between the two bodies, and then decreases again as the projectile approaches the surface of the Moon. (a) Find the distance of point X from the Earth's center. (b) With what minimum kinetic energy must we launch a 1-kg projectile from the Moon's surface if we want it to reach the Earth?

Solution (a) Let D represent the distance from the center of the Earth to the center of the Moon (the Moon's orbital radius). Then when the projectile is a distance x from the center of the Earth (and $D - x$ from the center of the Moon), its potential energy is

$$U(x) = -\frac{GmM_E}{x} - \frac{GmM_M}{D - x}.$$

To find where the maximum occurs, we take the derivative dU/dx and set it equal to 0:

$$\frac{dU}{dx} = \frac{GmM_E}{x^2} - \frac{GmM_M}{(D - x)^2} = 0.$$

Solving, we find $x = D\left(1 + \sqrt{M_M/M_E}\right)^{-1}$, which evaluates to 3.44×10^8 m (about 90% of the way along the line from the Earth to the Moon).

(b) As the projectile leaves the Moon, its kinetic energy decreases as it moves toward point X, and then its kinetic energy increases as it "coasts" back to Earth. The minimum launch kinetic energy corresponds to the projectile arriving at point X with zero kinetic energy. We apply conservation of energy in the form $K_i + U_i = K_X + U_X$, where i represents the Moon's surface. Taking K_X to be zero for the minimum condition, we have

$$K_i = U_X - U_i = \left(-\frac{GmM_E}{x} - \frac{GmM_M}{D - x}\right) - \left(-\frac{GmM_M}{R_M}\right).$$

Evaluating the numerical factors in this expression, we find $K_i = 1.53 \times 10^6$ J, corresponding to a speed of about 1750 m/s.

Escape Speed

A projectile fired upward from the Earth's surface will usually slow down, come momentarily to rest, and return to Earth. For a certain initial speed, however, it will move upward forever, with its speed decreasing gradually to zero just as its distance from Earth approaches infinity. The initial speed for this case is called the *escape speed.*

We can find the escape speed v for the Earth (or any other body from which a projectile might be launched) using conservation of energy. The projectile, of mass m, leaves the surface of the body, of mass M and radius R, with a kinetic energy $K_i = \frac{1}{2}mv^2$ and potential energy $U_i = -GMm/R$. When the projectile reaches infinity, it has zero potential energy and zero kinetic energy (since we are seeking the *minimum* speed for escape). Thus $U_f = 0$ and $K_f = 0$, and with $K_i + U_i = K_f + U_f$ we obtain

$$\tfrac{1}{2}mv^2 + \left(\frac{-GMm}{R}\right) = 0$$

TABLE 14-2 Some Escape Speeds

Body	*Mass* (kg)	*Radius* (m)	*Escape Speed* (km/s)
Ceres[a]	1.17×10^{21}	3.8×10^5	0.64
Moon	7.36×10^{22}	1.74×10^6	2.38
Earth	5.98×10^{24}	6.37×10^6	11.2
Jupiter	1.90×10^{27}	7.15×10^7	59.5
Sun	1.99×10^{30}	6.96×10^8	618
Sirius B[b]	2×10^{30}	1×10^7	5200
Neutron star	2×10^{30}	1×10^4	2×10^5

[a]The most massive of the asteroids.
[b]A white dwarf, the companion of the bright star Sirius.

and solving for v we find

$$v = \sqrt{\frac{2GM}{R}}. \tag{14-19}$$

Table 14-2 shows values of the escape speed for Earth and for some other bodies.

The escape speed does not depend on the direction in which the projectile is fired. The Earth's rotation—which we have not considered in this calculation—does play a role, however. Firing eastward has an advantage in that the Earth's tangential surface speed, which is 0.46 km/s at Cape Canaveral, provides part of the kinetic energy needed for escape, and thus less thrust from the rocket engines would be required to escape the Earth's gravity.

Potential Energy of Many-Particle Systems

We now consider another interpretation for $U(r)$. Consider two objects, of masses m and M, separated by an infinitely large distance and at rest. We take one of the particles (m, for example) and move it slowly and at constant velocity toward the other, until the separation of the two particles is r. To move the particle at constant velocity, the net work done on the particle must be zero: $W_{net} = W_{ext} + W_{grav} = 0$, where W_{ext} is the work done by our hand and W_{grav} is the work done by the gravitational force. From Eq. 14-14, the work done by the gravitational force as the object moves from infinite separation to a separation r is $W_{grav} = W_{\infty r} = GMm/r$. Thus the work done by our hand is $W_{ext} = -W_{grav} = -GMm/r$. Noting that this is equal to $U(r)$ as given in Eq. 14-17, we can give this alternative view of the potential energy:

> *The potential energy of a system of particles is equal to the work done by an external agent to assemble the system, starting from the standard reference configuration.*

Here "standard reference configuration" means that the particles start out at rest at infinite separation. As we have seen, we also specify that the final assembled system is at rest in the same reference frame in which the particles are at rest in their initial state.

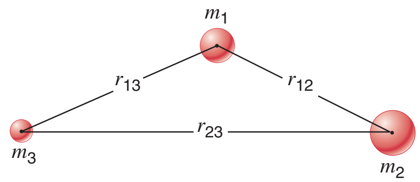


FIGURE 14-13. Three masses brought together from infinity and held in place by nongravitational forces.

These considerations also hold for systems that contain more than two particles. Consider three bodies of masses m_1, m_2, and m_3. Let them initially be at rest infinitely far from one another. The problem is to compute the work done by an external agent to bring them into the positions shown in Fig. 14-13. We first bring m_1 in from infinity to its final position and hold it in place. No work is done by gravity or the external agent because the separation between the three particles remains infinite. Let us then bring m_2 in toward m_1 from an infinite separation to the separation r_{12} and then hold *it* in place. The work done by the external agent in opposing the gravitational force exerted by m_1 on m_2 is $-Gm_1m_2/r_{12}$. Now let us bring m_3 in from infinity to the separation r_{13} from m_1 and r_{23} from m_2. The work done by the external agent opposing the gravitational force exerted by m_1 on m_3 is $-Gm_1m_3/r_{13}$, and that opposing the gravitational force exerted by m_2 on m_3 is $-Gm_2m_3/r_{23}$. The total potential energy of this system is equal to the total work done by the external agent in assembling the system, or

$$U = -\left(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}}\right). \tag{14-20}$$

Note that because work is a scalar, no vector calculations are needed in this procedure.

No matter how we assemble the system—that is, regardless of the order in which the particles are moved or the paths they take—we always find this same amount of work required to bring the bodies into the configuration of Fig. 14-13 from an initial infinite separation. The potential energy must therefore be associated with the system rather than with any one or two bodies. If we wanted to separate the system into three isolated masses once again, we would have to supply an amount of energy

$$E = +\left(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}}\right).$$

This energy is regarded as the *binding energy* holding the particles together in the configuration shown.

These concepts occur again in connection with forces of electric or magnetic origin, or, in fact, of nuclear origin. Their application is rather broad in physics. An advantage of the energy method over the dynamical method is that the energy method uses scalar quantities and scalar operations rather than vector quantities and vector operations. When the actual forces are not known, as is often the case in nuclear physics, the energy method is essential.

14-7 THE MOTIONS OF PLANETS AND SATELLITES

Using Newton's laws of motion and law of universal gravitation, we can understand and analyze the behavior of all the bodies in the solar system: the orbits of the planets and comets about the Sun and of natural and artificial satellites about their planets. We make two assumptions that simplify the analysis: (1) we consider the gravitational force only between the orbiting body (the Earth, for instance) and the central body (the Sun), ignoring the perturbing effect of the gravitational force of other bodies (such as other planets); (2) we assume that the central body is so much more massive than the orbiting body that we can ignore its motion under their mutual interaction. In reality, both objects orbit about their common center of mass, but if one object is very much more massive than the other, the center of mass is approximately at the center of the more massive body.

The empirical basis for understanding the motions of the planets is three laws deduced by Kepler (1571–1630, well before Newton) from studies of the motion of the planet Mars. We now show how Kepler's laws can be derived from Newton's laws of motion and his law of gravitation.

1. *The Law of Orbits: All planets move in elliptical orbits having the Sun at one focus.* Newton was the first to realize that there is a direct mathematical relationship between inverse-square ($1/r^2$) forces and elliptical orbits. Figure 14-14 shows a typical elliptical orbit. The origin of coordinates is at the central body, and the orbiting body is located at polar coordinates r and θ**.** The orbit is described by two parameters: the *semimajor axis a* and the *eccentricity e*. The distance from the center of the ellipse to either focus is ea. A circular orbit is a special case of an elliptical orbit with $e = 0$, in which case the two foci merge to a single point at the center of the circle. For Earth and most

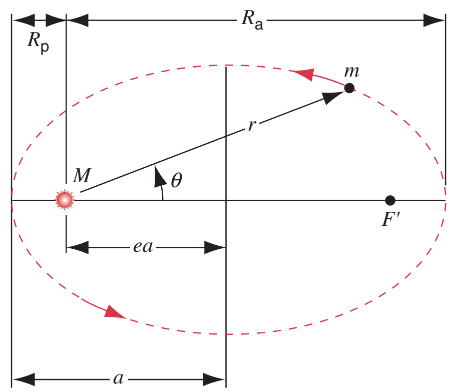


FIGURE 14-14. A planet of mass m moving in an elliptical orbit around the Sun. The Sun, of mass M, is at one focus of the ellipse. F' marks the other or "empty" focus. The semimajor axis a of the ellipse, the perihelion distance R_p, and the aphelion distance R_a are also shown. The distance ea locates the focal points, e being the eccentricity of the orbit.

other planets in the solar system, the eccentricities are small and the orbits are nearly circular, as shown in Appendix C.

The maximum distance R_a of the orbiting body from the central body is indicated by the prefix *apo-* (or sometimes *ap-*), as in *aphelion* (the maximum distance from the Sun) or *apogee* (the maximum distance from Earth). Similarly, the closest distance R_p is indicated by the prefix *peri-,* as in *perihelion* or *perigee.* As you can see from Fig. 14-14, $R_a = a(1 + e)$ and $R_p = a(1 - e)$. For circular orbits, $R_a = R_p = a$.

2. *The Law of Areas: A line joining any planet to the Sun sweeps out equal areas in equal times.* Figure 14-15 illustrates this law; in effect it says that the orbiting body moves more rapidly when it is close to the central body than it does when it is far away. We now show that the law of areas is identical with the law of conservation of angular momentum.

Consider the small area increment ΔA covered in a time interval Δt, as shown in Fig. 14-15*b*. The area of this approximately triangular wedge is one-half its base, $r\,\Delta\theta$, times its height r. The rate at which this area is swept out is $\Delta A/\Delta t = \frac{1}{2}(r\,\Delta\theta)(r)/\Delta t$. In the instantaneous limit this becomes

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t \to 0} \tfrac{1}{2} r^2 \frac{\Delta\theta}{\Delta t} = \tfrac{1}{2} r^2 \omega.$$

Assuming we can regard the more massive body M as at rest, the angular momentum of the orbiting body m relative

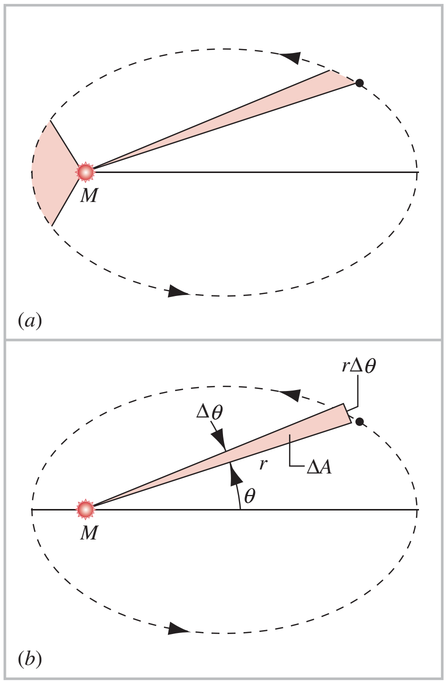


FIGURE 14-15. (*a*) The equal shaded areas are covered in equal times by a line connecting the planet to the Sun, demonstrating the law of areas. (*b*) The area ΔA is covered in a time Δt, during which the line sweeps through an angle $\Delta\theta$.

to the origin at the central body is, according to Eq. 10-12, $L_z = I\omega = mr^2\omega$ (choosing the z axis perpendicular to the plane of the orbit). Thus

$$\frac{dA}{dt} = \frac{L_z}{2m}. \qquad (14\text{-}21)$$

If the system of M and m is isolated, meaning that there is no net external torque on the system, then (see Eq. 10-9) L_z is a constant; therefore, according to Eq. 14-21, dA/dt is also constant. That is, in every interval dt in the orbit, the line connecting m and M sweeps out equal areas dA, which verifies Kepler's second law. The speeding up of a comet as it passes close to the Sun is an example of this effect and is thus a direct consequence of the law of conservation of angular momentum.

3. *The Law of Periods: The square of the period of any planet about the Sun is proportional to the cube of the planet's mean distance from the Sun.* Let us prove this result for circular orbits. The gravitational force provides the necessary centripetal acceleration for circular motion:

$$\frac{GMm}{r^2} = m\frac{v^2}{r}. \qquad (14\text{-}22)$$

Replacing the speed v with $2\pi r/T$, where T is the rotational period (the time for a full orbit), we obtain

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3. \qquad (14\text{-}23)$$

A similar result is obtained for elliptical orbits, with the radius r replaced by the semimajor axis a.*

The relationship between T^2 and a^3 should be determined by the quantity $4\pi^2/GM$. For all planets orbiting the Sun, the ratio T^2/a^3 should be a constant; Table 14-3 shows that this is indeed the case. If we can measure T and a for an orbiting body, we can determine the mass of the central body. This procedure is independent of the mass of the orbiting body, and so it gives no information about its mass.

*See, for example, *Newtonian Mechanics,* by A. P. French (Norton, 1971), pp. 585–591.

TABLE 14-3 Kepler's Law of Periods for the Solar System

Planet	*Semimajor Axis* a (10^{10} m)	*Period* T (y)	T^2/a^3 (10^{-34} y^2/m^3)
Mercury	5.79	0.241	2.99
Venus	10.8	0.615	3.00
Earth	15.0	1.00	2.96
Mars	22.8	1.88	2.98
Jupiter	77.8	11.9	3.01
Saturn	143	29.5	2.98
Uranus	287	84.0	2.98
Neptune	450	165	2.99
Pluto	590	248	2.99

Sample Problem 14-7. (*a*) Compute the mass of the Sun from the period and radius of the Earth's orbit. (*b*) Compute the mass of Jupiter from the period (1.77 d) and orbital radius (4.22×10^5 km) of its second closest moon, Io.

Solution (*a*) From Eq. 14-23, we have

$$M = \frac{4\pi^2 r^3}{GT^2} = \frac{4\pi^2(1.50 \times 10^{11}\ \text{m})^3}{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(3.15 \times 10^7\ \text{s})^2}$$
$$= 2.01 \times 10^{30}\ \text{kg}.$$

(*b*)

$$M = \frac{4\pi^2(4.22 \times 10^8\ \text{m})^3}{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(1.53 \times 10^5\ \text{s})^2}$$
$$= 1.90 \times 10^{27}\ \text{kg}.$$

Note that the mass of Jupiter cannot be obtained from the parameters of its orbit about the Sun; to determine the mass of an object from Kepler's third law, we need to know the period and semimajor axis of objects that orbit about it as the central body.

Sample Problem 14-8. It is desired to place a communications satellite into orbit so that it remains fixed above a given spot on the equator of the rotating Earth. What is the height above the Earth of such an orbit?

Solution For the satellite to remain above a given point on the Earth's surface, it must rotate with the same angular velocity as the point. The period of the satellite must therefore be 24 h or 86,400 s. The radius of the orbit must then be

$$r = \left(\frac{GT^2 M_E}{4\pi^2}\right)^{1/3}$$
$$= \left(\frac{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(86{,}400\ \text{s})^2(5.98 \times 10^{24}\ \text{kg})}{4\pi^2}\right)^{1/3}$$
$$= 4.22 \times 10^7\ \text{m},$$

and its height above the Earth's surface is

$$h = r - R_E = 4.22 \times 10^7\ \text{m} - 6.37 \times 10^6\ \text{m}$$
$$= 3.58 \times 10^7\ \text{m} = 22{,}300\ \text{mi}.$$

This orbit is called the Clarke Geosynchronous Orbit after Arthur C. Clarke, who first proposed the idea in 1948. Clarke is also well known as the author of many works of science fiction, including *2001—A Space Odyssey.*

Sample Problem 14-9. Halley's comet (Fig. 14-16) has a period of 76 years. In 1986, its closest approach to the Sun (perihelion) was 8.8×10^{10} m (between the orbits of Mercury and Venus). Find its aphelion, or farthest distance from the Sun, and the eccentricity of its orbit.

Solution From Eq. 14-23 (in which M is the mass of the Sun) we find the semimajor axis:

$$a = \left(\frac{GT^2 M}{4\pi^2}\right)^{1/3}$$
$$= \left(\frac{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(2.4 \times 10^9\ \text{s})^2(2.0 \times 10^{30}\ \text{kg})}{4\pi^2}\right)^{1/3}$$
$$= 2.7 \times 10^{12}\ \text{m}.$$

Figure 14-16. Halley's comet, photographed during its 1986 approach to the Sun.

From Fig. 14-14 we see that $R_p + R_a = 2a$, so

$$R_a = 2a - R_p = 2(2.7 \times 10^{12}\ \text{m}) - 8.8 \times 10^{10}\ \text{m}$$
$$= 5.3 \times 10^{12}\ \text{m},$$

between the orbits of Neptune and Pluto. Also from Fig. 14-14 we have $R_p = a - ea = a(1 - e)$, so

$$e = 1 - \frac{R_p}{a} = 1 - \frac{8.8 \times 10^{10}\ \text{m}}{2.7 \times 10^{12}\ \text{m}} = 0.97.$$

Such a large eccentricity (1.0 is the maximum possible) corresponds to a long, thin ellipse.

Energy Considerations in Planetary and Satellite Motion

Consider again the motion of a body of mass m (planet or satellite, say) about a massive body of mass M (Sun or Earth, say). We consider M to be at rest in an inertial reference frame, with the body m moving about it in a circular orbit with tangential speed v and angular speed ω. The potential energy of the system is

$$U(r) = -\frac{GMm}{r},$$

where r is the radius of the circular orbit. The kinetic energy of the system is

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\omega^2 r^2,$$

the Sun being at rest. From Eq. 14-22 we obtain

$$\omega^2 r^2 = \frac{GM}{r},$$

so that (with $v = \omega r$)

$$K = \frac{GMm}{2r}. \qquad (14\text{-}24)$$

The total mechanical energy is

$$E = K + U = \frac{GMm}{2r} - \frac{GMm}{r} = -\frac{GMm}{2r}. \qquad (14\text{-}25)$$

This energy is constant and negative. The kinetic energy can never be negative, but from Eq. 14-24 we see that it

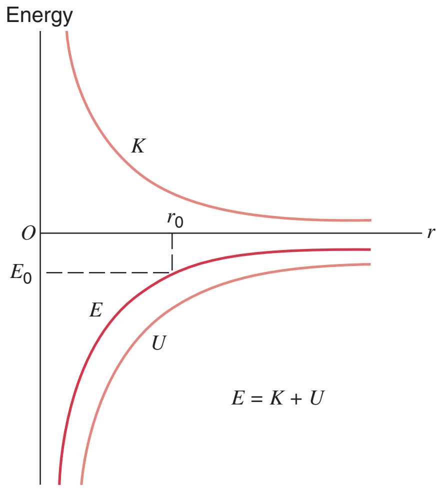


FIGURE 14-17. Kinetic energy K, potential energy U, and total energy $E = K + U$ of a body in circular planetary motion. A planet with total energy $E_0 < 0$ will remain in an orbit with radius r_0. The greater the distance from the Sun, the greater (that is, less negative) its total energy E. To escape from the center of force and still have kinetic energy at infinity, the planet would need positive total energy.

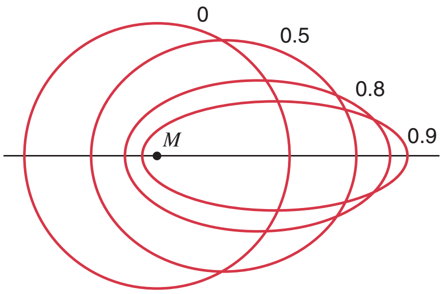


FIGURE 14-18. All four orbits have the same semimajor axis a and thus correspond to the same total energy E. Their eccentricities are marked.

must go to zero as the separation goes to infinity. The potential energy is always negative except for its zero value at infinite separation. The meaning of the total negative energy then is that the system is a closed one, the planet m always being bound to the attracting solar center M and never escaping from it (Fig. 14-17).

It can be shown* that Eq. 14-25 is also valid for elliptical orbits, if we replace r by the semimajor axis a. The total energy is still negative, and it is also constant, because gravitational forces are conservative. Hence both the total energy and the total angular momentum are constant in planetary motion. These quantities are often called *constants of the motion.*

Because the total energy does not depend on the eccentricity of the orbit, all orbits with the same semimajor axis a have the same total energy. Figure 14-18 shows several different orbits that have the same energy.

If we supply the proper amount of kinetic energy, we can arrange for the total energy to be zero or positive, in which case the orbits are no longer elliptical. The orbits are parabolic for $E = 0$ and hyperbolic for $E > 0$. This case often occurs in the scattering of particles from a nucleus, where the electrostatic force also varies as $1/r^2$. The spacecraft *Pioneer 10* was given enough initial kinetic energy to allow it to escape from the solar system; launched on March 3, 1972, it passed the orbit of Pluto, the outermost planet, on June 14, 1983, outward bound on a hyperbolic path.

Equation 14-25 shows that we cannot change the speed of an orbiting satellite without also changing the radius of its orbit. For example, suppose two satellites are following one another in the same circular orbit. If the trailing satellite tries to catch the leading one by accelerating forward, thereby increasing the kinetic energy, the total energy becomes less negative and the radius increases. Docking two spacecraft is not just a simple exercise in edging one craft forward! In fact, as the following sample problem shows, the proper procedure to use in overtaking an orbiting spacecraft often involves slowing down rather than speeding up.

SAMPLE PROBLEM 14-10. Two identical spacecraft, each with a mass of 3250 kg, are in the same circular orbit at a height of 270 km above the Earth's surface. Spacecraft A leads spacecraft B by 105 s; that is, A arrives at any fixed point 105 s before B. At a particular point P (Fig. 14-19), the pilot of B fires a short rocket burst in the *forward* direction, reducing the speed of B by 0.95%. Find the orbital parameters (energy, period, semimajor axis) of B before and after the "burn," and find the order of the two ships when they next return to point P.

Solution For $h = 270$ km, $r = R_E + h = 6370\text{ km} + 270\text{ km} = 6640$ km. Thus, before firing the rockets, $a = 6640$ km and, from Eq. 14-25,

$$E = -\frac{GmM_E}{2a}$$
$$= -\frac{(6.67\times10^{-11}\,\mathrm{N\cdot m^2/kg^2})(3250\,\mathrm{kg})(5.98\times10^{24}\,\mathrm{kg})}{2(6.64\times10^6\,\mathrm{m})}$$
$$= -9.76\times10^{10}\,\mathrm{J}.$$

The period follows from Eq. 14-23:

$$T = \left(\frac{4\pi^2a^3}{GM_E}\right)^{1/2}$$
$$= \left(\frac{4\pi^2(6.64\times10^6\,\mathrm{m})^3}{(6.67\times10^{-11}\,\mathrm{N\cdot m^2/kg^2})(5.98\times10^{24}\,\mathrm{kg})}\right)^{1/2}$$
$$= 5380\,\mathrm{s}.$$

Equations 14-24 and 14-25 show that (for a circular orbit only!) the kinetic energy is numerically equal to the negative of the total energy, so $K = +9.76\times10^{10}$ J and

$$v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{2(9.76\times10^{10}\,\mathrm{J})}{3250\,\mathrm{kg}}} = 7.75\times10^3\,\mathrm{m/s}.$$

*See reference on p. 312.

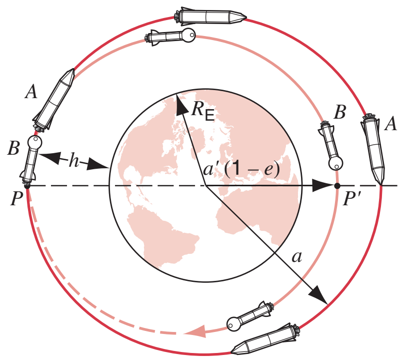


FIGURE 14-19. Sample Problem 14-10. The orbits of spacecraft A and B are shown. Note that B catches A by moving to a noncircular orbit at lower height above the Earth. The relative size of the Earth and the orbital heights is not to scale.

After the burn, the speed decreases by the given amount of 0.95% to $v' = (1 - 0.0095)v = 7.68 \times 10^3$ m/s, and the new kinetic energy of B is

$$K' = \tfrac{1}{2}(3250 \text{ kg})(7.68 \times 10^3 \text{ m/s})^2 = 9.58 \times 10^{10} \text{ J}.$$

The potential energy of B at point P immediately after the short burn is unchanged, equal to the initial value $E - K$ or $2E$, according to Eq. 14-25. The total energy E' of B after the burn must then be

$$E' = K' + U' = 9.58 \times 10^{10} \text{ J} + 2(-9.76 \times 10^{10} \text{ J})$$
$$= -9.94 \times 10^{10} \text{ J},$$

and the new semimajor axis is, from Eq. 14-25,

$$a' = -\frac{GmM_E}{2E'}$$
$$= -\frac{(6.67 \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2)(3250 \text{ kg})(5.98 \times 10^{24} \text{ kg})}{2(-9.94 \times 10^{10} \text{ J})}$$
$$= 6.52 \times 10^6 \text{ m} = 6520 \text{ km},$$

a reduction of 1.8% from the value in the original orbit. The corresponding period is

$$T' = \left(\frac{4\pi^2 a'^3}{GM_E}\right)^{1/2}$$
$$= \left(\frac{4\pi^2(6.52 \times 10^6 \text{ m})^3}{(6.67 \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24} \text{ kg})}\right)^{1/2}$$
$$= 5240 \text{ s}.$$

The difference in the periods is 140 s. That is, if A originally passes through point P at $t = 0$ and B passes through (and fires its rockets) at $t = 105$ s, then A returns to P at $t = 5380$ s (determined by the period T), and B returns to P at 5240 s after its initial passage, or at $t = 105 \text{ s} + 5240 \text{ s} = 5345$ s. Thus B is now 35 s *ahead of* A at point P. Now B can fire a second rocket burst identical in strength and duration to the first but in the reverse direction. This returns B to the original circular orbit, now 35 s ahead of A. Figure 14-19 shows the relationship between A and B during the first orbit after the burn. Note that after the burn, B moves in an elliptical orbit and so can pass A without colliding because A remains in the original circular orbit.

See Exercise 38 to help understand how B can reduce its speed at P and still get ahead of A.

14-8 THE GRAVITATIONAL FIELD (Optional)

A basic fact of gravitation is that two particles exert forces on one another. We can think of this as a direct interaction between the two particles, if we wish. This point of view is called *action-at-a-distance,* the particles interacting even though they are not in contact. Another point of view is the *field* concept, which regards a particle as modifying the space around it in some way and setting up a *gravitational field.* This field, the strength of which depends on the mass of the particle, then acts on any other particle, exerting the force of gravitational attraction on it. The field therefore plays an intermediate role in our thinking about the force that one particle exerts on another.

According to this view we have two separate parts to our problem. First, we must determine the gravitational field established by a given distribution of particles. Second, we must calculate the gravitational force that this field exerts on another particle placed in it.

We use this same approach later in the text when we study electromagnetism, in which case particles with electric charge set up an *electric* field, and the force on another charged particle is determined by the strength of the electric field at the location of the particle.

Let us consider the Earth as an isolated particle and ignore all rotational and other nongravitational effects (so that g and g_0 are equivalent). We use a small test body of mass m_0 as a probe of the gravitational field. If this body is placed in the vicinity of the Earth, it will experience a force having a definite direction and magnitude at each point in space. The direction is radially in toward the center of the Earth, and the magnitude is $m_0 g$. We can associate with each point near the Earth a vector $\vec{\mathbf{g}}$, which is the acceleration that a body would experience if it were released at this point. We define the *gravitational field strength* at a point as the *gravitational force per unit mass* at that point or, in terms of our test mass,

$$\vec{\mathbf{g}} = \frac{\vec{\mathbf{F}}}{m_0}. \tag{14-26}$$

By moving the test mass to various positions, we can make a map showing the gravitational field at any point in space. We can then find the force on a particle at any point in that field by multiplying the mass m of the particle by the value of the gravitational field $\vec{\mathbf{g}}$ at that point: $\vec{\mathbf{F}} = m\vec{\mathbf{g}}$. Figure 14-20 shows examples of gravitational fields.

The gravitational field is an example of a *vector field,* each point in this field having a vector associated with it. There are also *scalar fields,* such as the temperature field in a heat-conducting solid. The gravitational field arising from

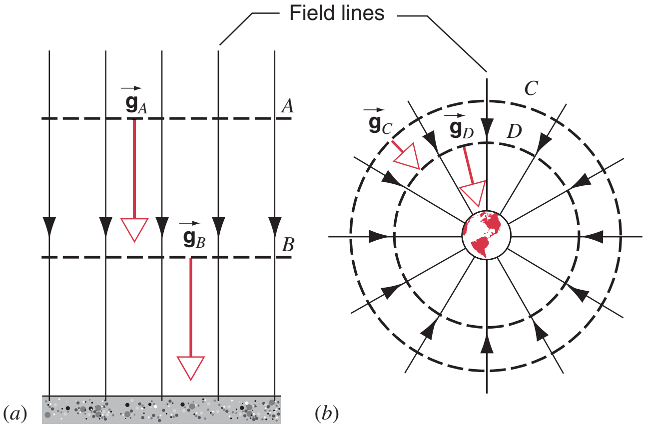


FIGURE 14-20. Examples of gravitational field line diagrams. The directions of the field lines (or that of the tangents to the lines, if they are curved) give the field direction at any point, and the density of field lines (number per unit area crossing a surface perpendicular to the lines) indicates the relative magnitudes of the fields. (*a*) The uniform field close to the Earth's surface. The field has the same magnitude and direction at all locations. The number of lines per unit area on plane surface *A* parallel to the surface is the same as that on plane surface *B*, indicating that the fields have the same magnitudes ($|\vec{g}_A| = |\vec{g}_B|$). (*b*) The field of the Earth (or any other isolated spherical body). The field points radially inward, and the density of field lines (number per unit area) on spherical surface *C* is smaller than that on surface *D* ($|\vec{g}_C| < |\vec{g}_D|$).

a fixed distribution of matter is also an example of a *static field,* because the value of the field at a given point does not change with time.

The field concept is particularly useful for understanding electromagnetic forces between moving electric charges. It has distinct advantages, both conceptually and in practice, over the action-at-a-distance concept. The field concept is particularly superior in the analysis of electromagnetic waves (for example, light or radio waves); action-at-a-distance suggests that forces can be transmitted instantly over any distance, whereas in theories based on fields the forces propagate at a finite speed (at most the speed of light). Gravitational waves, which have been predicted but not yet directly observed, would be similarly difficult to understand in the action-at-a-distance theory. The field concept, which was not used in Newton's day, was developed much later by Faraday for electromagnetism before it was applied to gravitation. Subsequently, this point of view was adopted for gravitation in the general theory of relativity. All present theories dealing with the ultimate nature of matter and the interactions between the fundamental particles are field theories of one kind or another. ■

14-9 MODERN DEVELOPMENTS IN GRAVITATION (Optional)

Newton's theory of gravitation provided the basis for understanding a wide variety of terrestrial and astronomical observations. However, discoveries in the 20th century have suggested areas in which the theory is incomplete. For example, in locations where the gravitational force is strong, such as near a neutron star (a very compact star) or a black hole, Newton's law gives incorrect results and must be replaced by a different approach, called the *general theory of relativity,* which was developed by Albert Einstein in 1916. Even in our own solar system, the planet Mercury moves sufficiently close to the Sun that it experiences a gravitational force strong enough to cause small but easily measured deviations from the Newtonian prediction for its orbit. Where the gravitational force is weaker, Einstein's theory reduces to Newton's, so we are perfectly safe in using Newton's theory to analyze the orbits of planets further from the Sun or to calculate the trajectories necessary to send space probes to the distant planets, which has been done with truly incredible precision. In this section we discuss several areas in which Newton's theory seems to be incomplete or incorrect.

Dark Matter

Figure 14-21 shows galaxies whose spiral structures are very similar to our own Milky Way galaxy. Such galaxies, which typically contain 10^{11} stars, are characterized by a bright central region and spiral arms in a flat disk. The en-

FIGURE 14-21. Typical spiral galaxies similar to our Milky Way, viewed from two different perspectives, one normal to the plane and one along the plane.

tire structure rotates about an axis perpendicular to the plane of the disk. Our Sun lies in one of the spiral arms of the Milky Way, about 2/3 of the way out from the center of the galaxy, and moves with a tangential speed of about 220 km/s, which corresponds to a full rotation every 240 million years.

The bright central region contains most of the mass of the galaxy. If we apply Kepler's third law to the rotation of a star like the Sun in one of the spiral arms a distance r from the center, we can solve Eq. 14-22 for the tangential speed of rotation v and obtain

$$v = \sqrt{\frac{GM}{r}}, \tag{14-27}$$

where M refers to the mass contained within the radius r. To the extent that we can ignore the effect of stars at larger radii, we would expect that v should decrease at increasing radii like $r^{-1/2}$. Figure 14-22 shows this expectation for stars in our galaxy and also shows that the observed data do not agree with this behavior at radii beyond the Sun. Instead, the speed seems to increase at larger r. Similar observations have been made for other galaxies.

One possible explanation for this discrepancy is the breakdown of Newton's law of gravitation at these large distances; that is, perhaps the form of the law contains an additional term that is negligible at separation distances where we have done careful measurements (in the laboratory and in the solar system) but becomes important at much larger distances. Over the years, other such small corrections to the dependence of Newton's law on separation distance have been investigated, but as yet there is no experimental evidence for anything but a $1/r^2$ behavior. A different explanation for the discrepancy of Fig. 14-22 is based on the existence of additional matter in the galaxies that is not visible to us but which exerts the gravitational force necessary to account for the data of Fig. 14-22. Several different forms for this so-called *dark matter* have been proposed: burnt-out stars, Jupiter-sized objects, and free elementary particles; however, no firm evidence for the existence of this form of matter in the quantities necessary to account for Fig. 14-22 has yet been obtained. Nevertheless, it shows our faith in Newton's law of gravitation that we are more willing to accept the existence of new forms of matter than a breakdown of the law of gravitation.

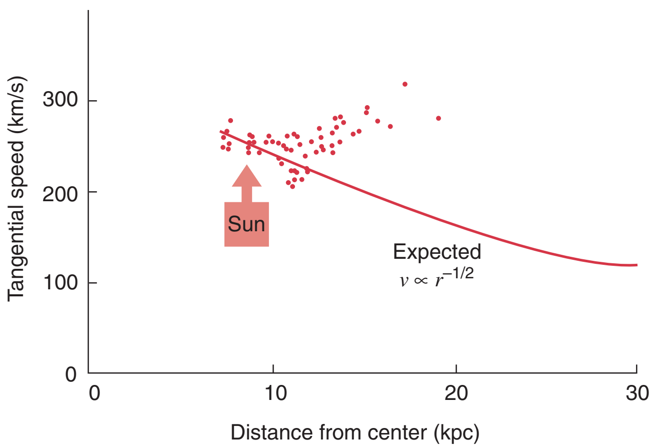


Figure 14-22. Tangential speeds of stars in our galaxy. The solid line shows the dependence of v on r given by Eq. 14-27 and calculated from Kepler's law of periods, assuming the stars to be attracted only by the large central mass of the galaxy. The discrepancy between the measured points and the curve suggests that there is unseen matter attracting the stars in the outer region of the galaxy.

Additional evidence for the existence of dark matter is found in the nature of the grouping of galaxies into clusters and superclusters. Some astronomers estimate that dark matter, made evident only by its gravitational effects, may constitute as much as 90% of all the matter in the universe. The British Astronomer Royal, Sir Martin Rees, has written, "The entities that conventional astronomers observe and call galaxies are no more than traces of sediment trapped in the centers of vast swarms of invisible objects of quite unknown structure. The gravity of this dark matter holds galaxies together and molds their structures."

Inertial Mass and Gravitational Mass

In Chapter 3 we discussed a procedure for assigning mass to an object, by comparing its response to a given force (that is, its acceleration) to that of a standard mass. This comparison is made on the basis of Newton's second law, and the mass that appears in $F = ma$ is called *inertial mass.* We can also use a procedure based on Newton's law of gravitation to measure the mass of an object. Let us measure the force on a standard kilogram in the Earth's gravitational field (that is, its weight), and let us then determine the force on our unknown mass in the same manner. According to Eq. 14-1, the ratio between those forces should be the same as the ratio of the masses, and we thus have a second method of determining mass. In this case we are measuring the *gravitational* mass.

It seems reasonable to ask whether these masses are in fact the same. Is inertial mass equal to gravitational mass? There is nothing in Newton's framework of dynamics that requires them to be equal. Their equality must be regarded in Newton's theory as an amazing coincidence, but as we shall see it arises in a natural way in Einstein's general theory of relativity.

Newton was the first to test the equality of inertial and gravitational mass, using a pendulum made in the form of an empty box. He filled the box with different quantities of material and measured the period of the resulting pendulum, which can be shown to depend on the ratio between the inertial mass and gravitational mass of the material in the box. Newton concluded that inertial and gravitational mass were the same to about one part in 10^3.

A considerable improvement in the experiment was made by Eötvös in 1909. He used a torsion balance with different materials on the two ends, and he compared for each material the gravitational mass (its weight) and the inertial mass (determined from the inertial centrifugal force owing to the Earth's rotation). Any difference in inertial and gravitational mass for the two materials would be observed as a rotation of the torsion balance. Eötvös concluded that inertial and gravitational mass were equal to within one part in 10^9. Later experiments by Dicke in 1964 and Braginsky in 1972 extended the limits to one part in 10^{11} to 10^{12} using a similar torsion balance technique but referring it to the Sun's gravitational attraction and to the inertial centrifugal force produced by the Earth's orbit about the Sun. These exceedingly precise experiments suggest that there is no difference between inertial and gravitational mass, and they force us to re-examine our laws of dynamics to account for this apparently accidental equality.*

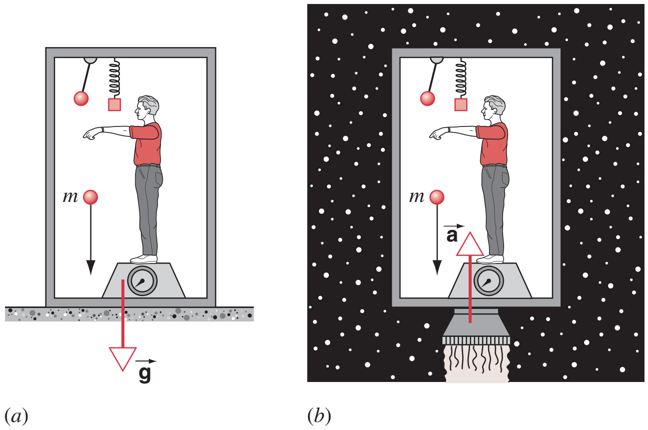


FIGURE 14-24. The effects of resting in a gravitational field of strength $\vec{g}$ (*a*) are identical to those of accelerating at $\vec{a} = -\vec{g}$ in interstellar space (*b*). No experiment done within the chamber could tell the difference. This illustrates Einstein's principle of equivalence.

The Principle of Equivalence

Here is how the idea occurred to Einstein: "I was sitting in a chair in the patent office in Bern when all of a sudden a thought occurred to me: If a person falls freely he will not feel his own weight. I was startled. This simple thought made a deep impression on me. It impelled me toward a theory of gravitation."

Figure 14-23*a* shows a person in an isolated chamber in free fall in the Earth's gravity, and Fig. 14-23*b* shows a person floating freely in interstellar space where the gravitational fields are negligibly weak. No measuring instruments that operate completely inside the chamber are able to distinguish between the two cases.

Einstein went one step further, as shown in Fig. 14-24. Consider the person in the chamber at rest on the Earth (Fig. 14-24*a*). A ball is observed to accelerate toward the floor at 9.8 m/s^2. A simple pendulum of a specified length has a certain period of oscillation. A mass hung from a spring stretches the spring by a certain amount. The floor exerts a certain normal force on bodies resting on it.

Now suppose the chamber is part of a rocket in interstellar space, and further suppose that the engines are fired to give the rocket an acceleration of exactly 9.8 m/s^2 (see Fig. 14-24*b*). Our traveler now releases a ball and observes it to move relative to the floor with that acceleration. The pendulum oscillates normally, the mass stretches the spring by the proper amount, and the floor exerts its correct normal force. In short, there is no experiment that can be done inside the chamber that will distinguish between Fig. 14-24*a*—the condition of rest in an inertial frame in a gravitational field $\vec{g}$—and Fig. 14-24*b*—acceleration $\vec{a} = -\vec{g}$ relative to an inertial frame in space of negligible gravity. This is the *principle of equivalence.*

The equality of inertial and gravitational mass follows directly from the principle of equivalence. Let an object rest on a spring scale on the floor of the chamber. When the chamber accelerates in the rocket, the floor must exert an upward force $m_i a$ to accelerate the object; here m_i is the inertial mass, and the spring balance reads the reaction force (also $m_i a$) exerted by the object. When the chamber is at rest in a gravitational field, on the other hand, the scale reads the weight $m_g g$ (which depends on the gravitational mass m_g). We have arranged our experiments so that $a = g$, and if the scale readings are to be identical (as demanded by the principle of equivalence) then the inertial and gravitational masses must be equal.

*See "Searching for the Secrets of Gravity," by John Boslough, *National Geographic,* May 1989, p. 563.

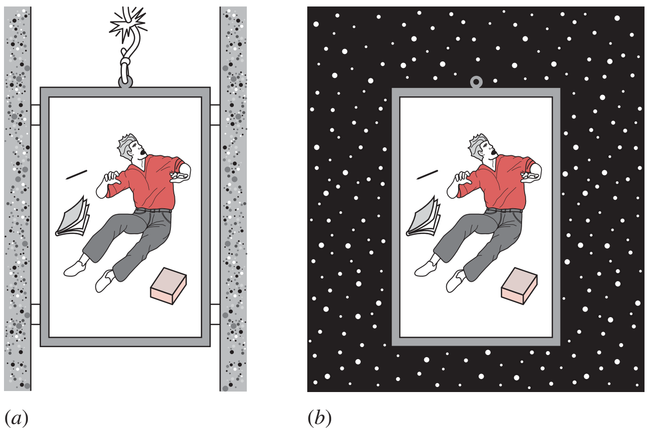


FIGURE 14-23. The effects of freely falling in the Earth's gravity (*a*) are identical to those of freely floating in interstellar space (*b*). No experiment done within the chamber could tell the difference.

The General Theory of Relativity

General relativity is essentially a theory of geometry. It provides a procedure for constructing a coordinate system whose very shape depends on the presence of matter and energy. In Einstein's theory, matter bends or curves space; our familiar rectangular coordinate system is no longer strictly valid in the presence of matter. The effect of one gravitating mass on another is then merely the movement of the second mass in the distorted geometry established by the first.

This approach is similar to the concept of fields discussed earlier in this chapter. In field theory, one mass establishes a gravitational field, and the second mass then interacts directly with the field (rather than directly with the first mass, as in the action-at-a-distance approach).

Figure 14-25 shows a two-dimensional analogy for the bending or curving of space. Imagine a rubber sheet with a coordinate grid laid out on it. All motion is confined to the coordinate system on the sheet. Now imagine a ball bearing stretching the sheet. The shortest distance between two points is no longer a straight line; in fact, in such a geometry we must redefine just what we mean by the term "straight line."

The relationship between matter and geometry in general relativity has been summarized as follows: "Geometry tells matter how to move, and matter tells geometry how to curve." The formulas of general relativity give the curvature for any given distribution of matter and energy, and the subsequent motion of beams of light or particles then follows directly.

Many experimental tests have been done to study the deviations between Newton's gravitational theory and Einstein's. These differences are apparent only in strong gravitational fields, and so measurements must be done close to the Sun or another massive body. Among the most significant experimental tests are:

1. *Precession of the perihelion of Mercury.* The orbit of the planet Mercury is not quite a closed ellipse—the axis of the ellipse rotates (or precesses) a bit upon each orbit. Most of this effect can be accounted for by Newtonian gravitation (due to the influence of the other planets, for example), but a small amount (known since 1859) cannot. This discrepancy, corresponding to a rotation of the axis of the ellipse by 43 seconds of arc per century, is nicely explained by Einstein's theory.

2. *Bending of light.* Light moving near a massive object does not follow a straight path but bends due to the curving of space as in Fig. 14-25*b*. This effect was first observed during a solar eclipse in 1919 based on the shift (of about 1.75 arc seconds) in the apparent position of stars whose light passed close to the Sun. Other observed effects of this bending include *gravitational lensing,* in which light from a distant galaxy headed toward Earth happens to pass close to a massive object (such as a black hole or another galaxy), and the bending of the light around the object causes us to observe two images of the original galaxy.

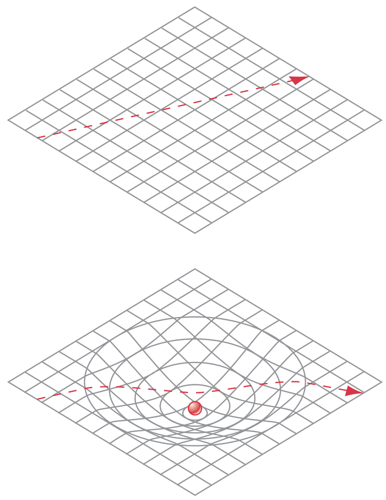


FIGURE 14-25. An analogy showing the bending or curving of space that results from the presence of gravitating mass, according to the general theory of relativity. Mass distorts the coordinate grid and changes the geometry itself.

3. *Delay of radar echoes.* Because a massive object stretches the "fabric" of space *and* time, a radar signal traveling from the Earth to another planet will be delayed slightly if it passes close to the Sun. The expected delay of a signal between Earth and Venus is only about 10^{-4} s, but it has been verified to a precision of about 0.1%.

Much effort has been spent on these experimental tests of general relativity*, and other significant tests are ongoing (including the search for "gravity waves" and the measurement of the change of the direction of the axis of a gyroscope in Earth orbit). So far the predictions of general relativity have been confirmed every time. Many of these effects are very small, but there is one consequence of general relativity that can be of great practical importance—the Global Positioning System (GPS), which uses a network of satellites to determine your position on Earth to within a few meters, must use general relativity to obtain this level of precision. ■

*For an elementary and highly readable account of these measurements, see *Was Einstein Right?,* by Clifford M. Will (Basic Books, 1986).

MULTIPLE CHOICE

14-1 Origin of the Law of Gravitation

14-2 Newton's Law of Universal Gravitation

1. The magnitude of the force of gravity between two identical objects is given by F_0. If the mass of each object is doubled but the distance between them is halved, then the new force of gravity between the objects will be
   (A) $16F_0$. (B) $4F_0$. (C) F_0. (D) $F_0/2$.
2. The magnitude of the force of gravity between two identical objects is given by F_0. If the mass of each object is doubled

and the distance between them is also doubled, then the new force of gravity between the objects will be

(A) $4F_0$. (B) $2F_0$. (C) F_0. (D) $F_0/2$.

3. Objects A and B are separated by a distance r. The magnitude of the force of gravity on A from B is given by F_{AB}, and the magnitude of the force of gravity on B from A is F_{BA}.

(*a*) If the mass of A is doubled while that of B is unchanged, then

(A) F_{AB} will double while F_{BA} will remain the same.
(B) F_{AB} will remain the same while F_{BA} will double.
(C) both F_{AB} and F_{BA} will double.
(D) both F_{AB} and F_{BA} will remain unchanged.

(*b*) If instead the mass of A is doubled while the mass of B is halved, then

(A) F_{AB} will double while F_{BA} will remain the same.
(B) F_{AB} will remain the same while F_{BA} will double.
(C) both F_{AB} and F_{BA} will double.
(D) both F_{AB} and F_{BA} will remain unchanged.

14-3 The Gravitational Constant G

4. The dimensions of G are equivalent to

(A) energy/momentum^2. (B) velocity^4/force.
(C) distance^3/force^2.
(D) velocity^3/angular momentum.

14-4 Gravitation near the Earth's Surface

5. Assuming the Earth is a uniform sphere of radius R_E, the local variation of the free-fall acceleration g_0 with respect to height h above the surface is approximately

(A) $g_0 = g_{ref}$, there is no variation,
(B) $g_0 = g_{ref}(1 - h/R_E)$,
(C) $g_0 = g_{ref}(1 - 2h/R_E)$,
(D) $g_0 = g_{ref}(1 - 3h/R_E)$,

where g_{ref} is the free-fall acceleration on the surface.

14-5 The Two Shell Theorems

6. A spherically symmetric nonrotating body has a density that varies appreciably with the radial distance from the center. At the center of the body the acceleration of free fall is

(A) definitely larger than zero.
(B) possibly larger than zero.
(C) definitely equal to zero.

7. The acceleration due to gravity in a hole dug into a nonuniform spherically symmetric body

(A) will increase as you go deeper, reaching a maximum at the center.
(B) will increase as you go deeper, but eventually reach a maximum, and then decrease until you reach the center.
(C) can increase or decrease as you go deeper.
(D) must decrease as you go deeper.

(See "Gravity in a Mine Shaft," by Peter M. Hall and David J. Hall, *The Physics Teacher,* November 1995, p. 525.)

14-6 Gravitational Potential Energy

8. Consider a spherically symmetric planet with a mass density that varies as a function of distance from the center of the planet. The magnitude of the gravitational potential energy of the system of the planet and a test mass would be

(A) zero at the center, and the maximum value would occur at the surface of the planet.
(B) nonzero at the center, and the maximum value would occur at the surface of the planet.
(C) nonzero at the center, but the maximum value would occur at some point beneath the surface but away from the center.
(D) nonzero at the center, and the maximum value would occur at the center.

14-7 The Motions of Planets and Satellites

9. Project Starshine was an inexpensive satellite launched to involve school children in orbit observations. As the satellite encountered friction from the Earth's atmosphere the radius of the near circular orbit slowly decreased over a period of many months.

(*a*) As the radius of the orbit decreased, the total energy of the satellite

(A) increased. (B) remained the same.
(C) decreased.

(*b*) As the radius of the orbit decreased, the kinetic energy of the satellite

(A) increased. (B) remained the same.
(C) decreased.

(*c*) As the radius of the orbit decreased, the average speed of the satellite

(A) increased. (B) remained the same.
(C) decreased.

10. Shown in Fig 14-26 are several possible elliptical orbits of a satellite.

(*a*) Which orbit has the largest angular momentum?
(*b*) Which orbit has the largest total energy?
(*c*) On which orbit is the largest speed acquired?

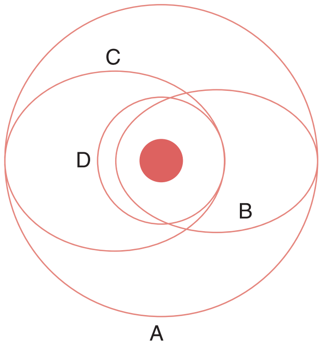


FIGURE 14-26. Multiple-choice question 10.

14-8 The Gravitational Field

14-9 Modern Developments in Gravitation

QUESTIONS

1. Modern observational astronomy and navigation procedures make use of the geocentric (or Ptolemaic) point of view (by using the rotating "celestial sphere"). Is this wrong? If not, what criterion determines the system (the Copernican or Ptolemaic) we use? When would we use the heliocentric (or Copernican) system?

2. Two planets are never seen at midnight. Which ones and why not? Can this be considered as evidence in favor of the heliocentric and against the geocentric theory?

3. If the force of gravity acts on all bodies in proportion to their masses, why does a heavy body not fall correspondingly faster than a light body?

4. How does the weight of a space probe vary en route from the Earth to the Moon? Would its mass change?

5. It is easy to calculate the ratio of the mass of the Earth to the mass of the Sun knowing only the periods of revolution and the orbital radii of the Moon around the Earth and the Earth around the Sun. Is it possible to calculate G from only astronomical observations? Explain.

6. Our analysis of the Cavendish experiment (see Fig. 14-5) considered the attraction of each large sphere only for the small sphere closest to it. Each large sphere also attracts the small sphere on the opposite end of the rod. What is the effect of this attraction on the measurement of G?

7. Is the mutual gravitational force exerted by a pair of objects affected by the nature of the intervening medium? By the temperatures of the objects? By the orientation of the objects? How could you check these effects by experiment?

8. Because the Earth bulges near the equator, the source of the Mississippi River (at about 50° N latitude), although high above sea level, is about 5 km closer to the center of the Earth than is its mouth (at about 30°N latitude). How can the river flow "uphill" as it flows south?

9. Would we have more sugar to the pound at the pole or at the equator? What about sugar to the kilogram?

10. How could you determine the mass of the Moon?

11. One clock is based on an oscillating spring, the other on a pendulum. Both are taken to Mars. Will they keep the same time there that they kept on Earth? Will they agree with each other? Explain. Mars has a mass about one-tenth that of the Earth and a radius about one-half as great.

12. At the Earth's surface, an object resting on a horizontal, frictionless surface is given a horizontal blow by a hammer. The object is then taken to the Moon, supported in the same manner, and given an equal blow by the same hammer. To the best of our knowledge, what would be the speed imparted to the object on the Moon when compared with the speed resulting from the blow on Earth (neglecting any atmospheric effects)?

13. The gravitational force exerted by the Sun on the Moon is about twice as great as the gravitational force exerted by the Earth on the Moon. Why then does the Moon not escape from the Earth?

14. Explain why the following reasoning is wrong. "The Sun attracts all bodies on the Earth. At midnight, when the Sun is directly below, it pulls on an object in the same direction as the pull of the Earth on that object; at noon, when the Sun is directly above, it pulls on an object in a direction opposite to the pull of the Earth. Hence, all objects should be heavier at midnight (or night) than they are at noon (or day)."

15. The gravitational attraction of the Sun and the Moon on the Earth produces tides. The Sun's tidal effect is about half as great as the Moon's. The direct pull of the Sun on the Earth, however, is about 175 times that of the Moon. Why is it then that the Moon causes the larger tides?

16. Particularly large tides, called spring tides, occur at full moon and at new moon, when the configurations of the Sun, Earth, and Moon are as shown in Fig. 14-27. From the figure you might conclude (incorrectly!) that the tidal effects of the Sun and of the Moon tend to add at new moon but cancel at full moon. Instead, they add at both these configurations. Explain why.

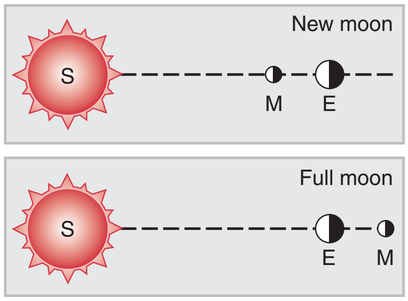


FIGURE 14-27. Question 16.

17. If lunar tides slow down the rotation of the Earth (owing to friction), the angular momentum of the Earth decreases. What happens to the motion of the Moon as a consequence of the conservation of angular momentum? Does the Sun (and solar tides) play a role here? (See "Tides and the Earth–Moon System," by Peter Goldreich, *Scientific American,* April 1972, p. 42.)

18. From Kepler's second law and observations of the Sun's motion as seen from the Earth, how can we deduce that the Earth is closer to the Sun during winter in the northern hemisphere than during summer? Why is it not colder in summer than in winter?

19. How would the results of Sample Problem 14-4 for the force and the speed at radius r differ if the density of the Earth were not uniform, but instead (a) decreased with increasing r, or (b) increased with increasing r?

20. Why can we learn more about the shape of the Earth by studying the motion of an artificial satellite than by studying the motion of the Moon?

21. A satellite in Earth orbit experiences a small drag force as it starts to enter the Earth's atmosphere. What happens to its speed? (Be careful!)

22. Would you expect the total energy of the solar system to be constant? The total angular momentum? Explain your answers.

23. Does a rocket always need the escape speed of 11.2 km/s to escape from the Earth? If not, what then does "escape speed" really mean?

24. Objects at rest on the Earth's surface move in circular paths with a period of 24 h. Are they "in orbit" in the sense that an Earth satellite is in orbit? Why not? What would the length of the "day" have to be to put such objects in true orbit?

25. Neglecting air friction and technical difficulties, can a satellite be put into an orbit by being fired from a huge cannon at the Earth's surface? Explain your answer.

26. What advantage does Florida have over California for launching (nonpolar) U.S. satellites?

27. Can a satellite coast in a stable orbit in a plane not passing through the Earth's center? Explain your answer.

28. As measured by an observer on Earth, would there be any difference in the periods of two satellites, each in a circular orbit near the Earth in an equatorial plane, but one moving eastward and the other westward?

29. Orbiting satellites occasionally burn up during their descent to Earth. However, they do not burn up during their ascent into orbit. Explain.

30. An artificial satellite is in a circular orbit about the Earth. How will its orbit change if one of its rockets is momentarily fixed (*a*) toward the Earth, (*b*) away from the Earth, (*c*) in a forward direction, (*d*) in a backward direction, and (*e*) at right angles to the plane of the orbit?

31. Inside a spaceship, what difficulties would you encounter in walking, in jumping, and in drinking?

32. We have all seen TV transmissions from orbiting shuttles and watched objects floating around in effective zero gravity. Suppose that an astronaut, braced against the shuttle frame, kicks a floating bowling ball. Will a stubbed toe result? Explain your answer.

33. If a planet of given density were made larger by accreting material from space, its force of attraction for an object on its surface would increase because of the planet's greater mass but would decrease because of the greater distance from the object to the center of the planet. Which effect dominates?

34. The orbits of satellites around the Earth are elliptical (or circular) and yet we claimed in Chapter 4 that projectiles launched from the Earth follow parabolic trajectories. Which is correct?

35. Artificial Earth satellites can locate the mean sea level with great precision. Above oil-bearing rock, however, the mean sea level can be as much as 1 m higher than that above non-oil-bearing rock (which is usually denser). Explain this.

36. (*a*) In order for two observers at any two positions on the Earth's equator to maintain radio communication by using satellites in the geosynchronous orbit, there must be at least three such satellites. Explain. (*b*) Find the maximum angular separation of any two of these satellites.

37. A stone is dropped along the center of a deep vertical mine shaft. Assume no air resistance but consider the Earth's rotation. Will the stone continue along the center of the shaft? If not, describe its motion.

38. Why is there virtually no atmosphere on the Moon?

39. Does the law of universal gravitation require the planets of the solar system to have the actual orbits observed? Would planets of another star, similar to our Sun, have the same orbits? Suggest factors that might have determined the special orbits observed.

40. Does it matter which way a rocket is pointed for it to escape from Earth? Assume, of course, that it is pointed above the horizon and neglect air resistance.

41. For a flight to Mars, a rocket is fired in the direction the Earth is moving in its orbit. For a flight to Venus, it is fired backward along that orbit. Explain why.

42. Saturn is about six times farther from the Sun than Mars. Which planet has (*a*) the greater period of revolution, (*b*) the greater orbital speed, and (*c*) the greater angular speed?

43. See Fig. 14-28. What is being plotted? Put numbers with units on each axis.

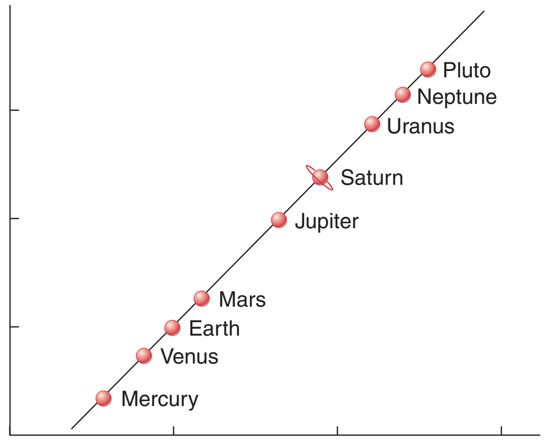


FIGURE 14-28. Question 43.

44. How can the captain of a spaceship, coasting toward a previously unknown planet, infer the value of g at the surface of the planet?

45. An iron cube is placed near an iron sphere at a location remote from the Earth's gravity. What can you say about the location of the center of gravity of the cube? Of the sphere? In general, does the location of the center of gravity of an object depend on the nature of the gravitational field in which the object is placed?

46. How could you determine whether two objects have (*a*) the same gravitational mass, (*b*) the same inertial mass, and (*c*) the same weight?

47. Consider an artificial satellite in a circular orbit about the Earth. State how the following properties of the satellite vary with the radius r of its orbit: (*a*) period, (*b*) kinetic energy, (*c*) angular momentum, and (*d*) speed.

48. You are a passenger on the *S.S. Arthur C. Clarke,* the first interstellar spaceship. The *Clarke* rotates about a central axis to simulate Earth's gravity. If you are in an enclosed cabin, how could you tell that you are not on Earth?

49. Can one regard gravity as a "fictitious" force arising from the acceleration of one's reference frame relative to an inertial reference frame, rather than a "real" force?

50. The "action-at-a-distance" view of the gravitational force implies that the action is instantaneous. Actually, present physical theory assumes that gravitation propagates with a finite speed and this is taken into account in the modification of classical physics represented by general relativity theory. What would happen to classical deductions if it were assumed that the action is not instantaneous? (See also "Infinite Speed of Propagation of Gravitation in Newtonian Physics," by I. J. Good, *American Journal of Physics,* July 1975, p. 640.)

EXERCISES

14-1 Origin of the Law of Gravitation

14-2 Newton's Law of Universal Gravitation

1. The Sun and Earth each exert a gravitational force on the Moon. Calculate the ratio F_S/F_E of these two forces. (The average Sun–Moon distance is equal to the Sun–Earth distance.)

2. How far from the Earth must a space probe be along a line toward the Sun so that the Sun's gravitational pull balances the Earth's?

3. One of the Echo satellites consisted of an inflated aluminum balloon 30 m in diameter and of mass 20 kg. A meteor having a mass of 7.0 kg passes within 3.0 m of the surface of the satellite. If the effect of all bodies other than the meteor and satellite are ignored, what gravitational force does the meteor experience at closest approach to the satellite?

14-3 The Gravitational Constant *G*

4. In the Cavendish balance (see Fig. 14-5), suppose $M = 12.7$ kg and $m = 9.85$ g. The length of the rod connecting the small spheres is 52.4 cm. When the distance between the centers of the large and small spheres is 10.8 cm, find (*a*) the gravitational force between a large sphere and the nearby small sphere, and (*b*) the torque on the rod.

14-4 Gravitation near the Earth's Surface

5. You weigh 120 lb at the sidewalk level outside the World Trade Center in New York City. Suppose that you ride from this level to the top of one of its 1350-ft towers. How much less would you weigh there because you are slightly farther away from the center of the Earth?

6. At what altitude above the Earth's surface is the free-fall acceleration equal to 7.35 m/s^2 (three-quarters of its value at the surface)?

7. A typical neutron star may have a mass equal to that of the Sun but a radius of only 10.0 km. (*a*) What is the gravitational acceleration at the surface of such a star? (*b*) How fast would an object be moving if it fell from rest through a distance of 1.20 m on such a star?

8. (*a*) Calculate g_0 on the surface of the Moon from values of the mass and radius of the Moon found in Appendix C. (*b*) What will an object weigh on the Moon's surface if it weighs 100 N on the Earth's surface? (*c*) How many Earth radii must this same object be from the surface of the Earth if it is to weigh the same as it does on the surface of the Moon?

9. If g is to be determined by dropping an object through a distance of (exactly) 10 m, how accurately must the time be measured to obtain a result good to 0.1%? Calculate a percent error and an absolute error, in milliseconds.

14-5 The Two Shell Theorems

10. Two concentric shells of uniform density having masses M_1 and M_2 are situated as shown in Fig. 14-29. Find the force on a particle of mass m when the particle is located at (*a*) $r = a$, (*b*) $r = b$, and (*c*) $r = c$. The distance r is measured from the center of the shells.

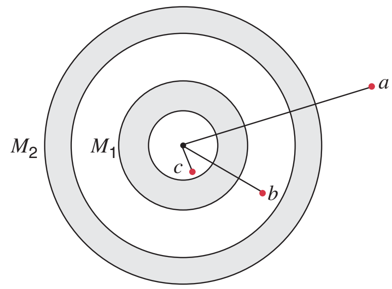


FIGURE 14-29. Exercise 10.

11. Show that, at the bottom of a vertical mine shaft dug to depth D, the measured value of g will be

$$g = g_s\left(1 - \frac{D}{R}\right),$$

g_s being the surface value. Assume that the Earth is a uniform sphere of radius R.

14-6 Gravitational Potential Energy

12. It is conjectured that a "burned-out" star could collapse to a "gravitational radius," defined as the radius for which the work needed to remove an object of mass m from the star's surface to infinity equals the rest energy mc^2 of the object. Show that the gravitational radius of the Sun is GM_S/c^2 and determine its value in terms of the Sun's present radius. (For a review of this phenomenon see "Black Holes: New Horizons in Gravitational Theory," by Philip C. Peters, *American Scientist,* September–October 1974, p. 575.)

13. A spaceship is idling at the fringes of our galaxy, 80,000 light-years from the galactic center. What minimum speed must it have if it is to escape entirely from the gravitational attraction of the galaxy? The mass of the galaxy is 1.4×10^{11} times that of our Sun. Assume, for simplicity, that the matter forming the galaxy is distributed with spherical symmetry.

14. Show that the velocity of escape from the Sun at the Earth's distance from the Sun is $\sqrt{2}$ times the speed of the Earth in its orbit, assumed to be a circle. (This is a specific case of a general result for circular orbits: $v_{esc} = \sqrt{2}v_{orb}$.)

15. A rocket is accelerated to a speed of $v = 2\sqrt{gR_E}$ near the Earth's surface and then coasts upward. (*a*) Show that it will escape from the Earth. (*b*) Show that very far from the Earth its speed is $v = \sqrt{2gR_E}$.

16. The Sun, mass 2.0×10^{30} kg, is revolving about the center of the Milky Way galaxy, which is 2.2×10^{20} m away. It completes one revolution every 2.5×10^8 years. Estimate the number of stars in the Milky Way. (Hint: Assume for simplicity that the stars are distributed with spherical symmetry about the galactic center and that our Sun is essentially at the galactic edge.)

17. A projectile is fired vertically from the Earth's surface with an initial speed of 9.42 km/s. Neglecting atmospheric friction, how far above the Earth's surface will it go?

18. (*a*) Calculate the escape speed on Europa, a satellite of the planet Jupiter. The radius of Europa is 1569 km and the free-fall acceleration at its surface is 1.30 m/s^2. (*b*) How high will a particle rise if it leaves the surface of the satellite with a vertical velocity of 1.01 km/s? (*c*) With what speed will an object hit the satellite if it is dropped from a height of 1000 km? (*d*) Calculate the mass of Europa.

19. Two neutron stars are separated by a center-to-center distance of 93.4 km. They each have a mass of 1.56×10^{30} kg and a radius of 12.6 km. They are initially at rest with respect to one another. (*a*) How fast are they moving when their separation has decreased to one-half of its initial value? (*b*) How fast are they moving just before they collide? Ignore relativistic effects.

20. Two particles of mass m and M are initially at rest an infinite distance apart. Show that at any instant their relative velocity of approach attributable to gravitational attraction is $\sqrt{2G(M+m)/d}$, where d is their separation at that instant.

21. Two point-like particles, each of mass m, are originally separated by a distance d and moving in opposite directions each with a speed of v. What is the maximum value for v so that the particles will eventually move back together under the influence of the mutual gravitational attraction?

14-7 The Motion of Planets and Satellites

22. The mean distance of Mars from the Sun is 1.52 times that of the Earth from the Sun. From this, calculate the number of years required for Mars to make one revolution about the Sun; compare your answer with the value given in Appendix C.

23. The planet Mars has a satellite, Phobos, which travels in an orbit of radius 9400 km with a period of 7 h 39 min. Calculate the mass of Mars from this information. (The mass of Phobos is negligible compared with that of Mars.)

24. Determine the mass of the Earth from the period T and the radius r of the Moon's orbit about the Earth: $T = 27.3$ days and $r = 3.82 \times 10^5$ km.

25. A satellite is placed in a circular orbit with a radius equal to one-half the radius of the Moon's orbit. What is its period of revolution in lunar months? (A lunar month is the period of revolution of the Moon.)

26. Spy satellites have been placed in the geosynchronous orbit above the Earth's equator. What is the greatest latitude L from which the satellites are visible from the Earth's surface? See Fig. 14-30.

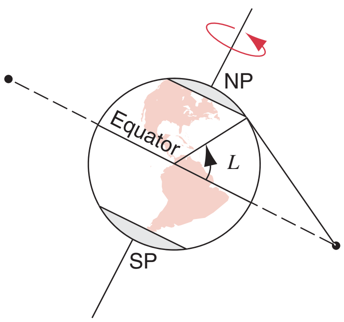


FIGURE 14-30. Exercise 26.

27. A reconnaissance spacecraft circles the Moon at very low altitude. Calculate (*a*) its speed and (*b*) its period of revolution. Take needed data for the Moon from Appendix C.

28. Use conservation of energy and Eq. 14-25 for the total energy to show that the speed v of an object in an elliptical orbit satisfies the relation

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right).$$

Here r is the distance of the orbiting body from the central body of mass M.

29. A comet moving in an orbit of eccentricity 0.880 has a speed of 3.72 km/s when it is most distant from the Sun. Find its speed when it is closest to the Sun.

30. (*a*) Express the universal gravitational constant G that appears in Newton's law of gravity in terms of the astronomical unit AU as a length unit, the solar mass M_S as a mass unit, and the year as a time unit. (1 AU $= 1.496 \times 10^{11}$ m, 1 $M_S = 1.99 \times 10^{30}$ kg, 1 y $= 3.156 \times 10^7$ s.) (*b*) What form does Kepler's third law (Eq. 14-23) take in these units?

31. Show how, guided by Kepler's third law, Newton could deduce that the force holding the Moon in its orbit, assumed circular, must vary as the inverse square of the distance from the center of the Earth.

32. As shown in Fig. 14-31, two bodies (of masses m and M) interacting through their mutual gravitational force will orbit with the same angular speed ω about their center of mass C. (*a*) Show that in this case Kepler's law of periods (Eq. 14-23) becomes

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3 \left(1 + \frac{R}{r}\right)^2.$$

(*b*) Evaluate the correction factor $(1 + R/r)^2$ for the motion of the Earth and the Sun and also for the motion of the Earth and the Moon, in each case ignoring the gravitational effect of the other bodies in the solar system.

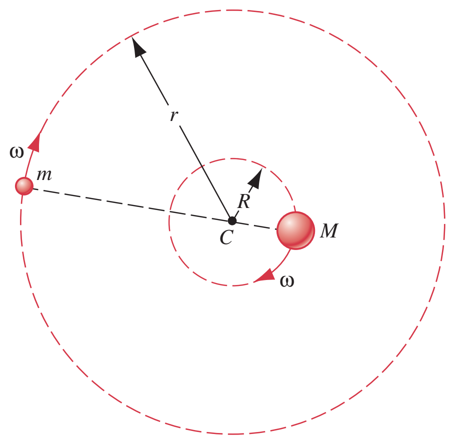


FIGURE 14-31. Exercise 32.

33. A pair of stars revolves about their common center of mass, as in Fig. 14-31. One of the stars has a mass M that is twice the mass m of the other; that is, $M = 2m$. Their centers are a distance d apart, d being large compared to the size of either star. (*a*) Derive an expression for the period of revolution of the stars about their common center of mass in terms of d, m, and G. (*b*) Compare the angular momenta of the two stars about their common center of mass by calculating the ratio L_m/L_M. (*c*) Compare the kinetic energies of the two stars by calculating the ratio K_m/K_M.

34. (*a*) Does it take more energy to get a satellite up to 1600 km above the Earth than to put it in orbit once it is there? (*b*) What about 3200 km? (*c*) What about 4800 km? Take the Earth's radius to be 6400 km.

35. The asteroid Eros, one of the many minor planets that orbit the Sun in the region between Mars and Jupiter, has a radius of 7.0 km and a mass of 5.0×10^{15} kg. (*a*) If you were standing on Eros, could you lift a 2000-kg pickup truck? (*b*) Could you run fast enough to put yourself into orbit? Ignore effects due to the rotation of the asteroid. (Note: The Olympic records for the 400-m run correspond to speeds of 9.1 m/s for men and 8.2 m/s for women.)

36. The orbit of the Earth about the Sun is almost circular. The closest and farthest distances are 1.47×10^8 km and 1.52×10^8 km, respectively. Determine the maximum variations in (*a*) potential energy, (*b*) kinetic energy, (*c*) total energy, and (*d*) orbital speed that result from the changing Earth–Sun distance in the course of 1 year. (Hint: Use conservation of energy and angular momentum.)

37. Assume that a geosynchronous communications satellite is in orbit at the longitude of Chicago. You are in Chicago and want to pick up its signals. In what direction should you point the axis of your parabolic antenna? The latitude of Chicago is 47.5° N.

38. Using the data of Sample Problem 14-10, calculate (*a*) the speed of spacecraft *B* as it passes through point *P*′, and (*b*) the average speed of spacecraft *B* in the orbit after the burn. Approximate the path of *B* as a circle. Compare these results with the corresponding quantities of spacecraft *A*.

39. Project Starshine was an inexpensive satellite (of mass 39 kg) launched to encourage worldwide participation of school children in satellite orbit measurements. The data from part of the orbit are shown in Fig. 14-32. (*a*) What was the orbital period of the satellite at the turn of the century 1999/2000? (*b*) At what rate was the satellite losing energy at the turn of the century 1999/2000?

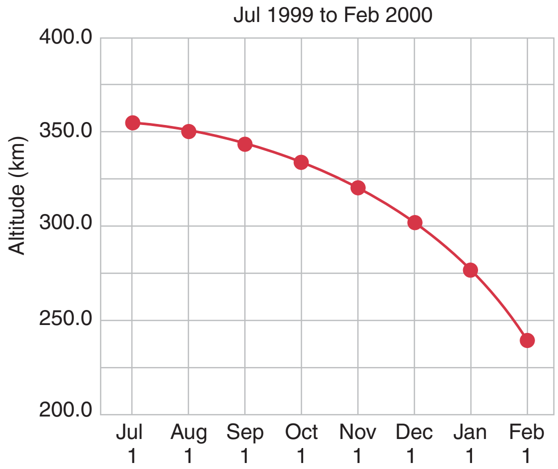


FIGURE 14-32. Exercise 39.

14-8 The Gravitational Field

14-9 Modern Developments in Gravitation

PROBLEMS

1. Two point-like objects, each with mass *m*, are connected by a massless rope of length *l*. The objects are suspended vertically near the surface of Earth, so that one object is hanging below the other. Then the objects are released. Show that the tension in the rope is

$$T = \frac{GMml}{R^3}$$

where *M* is the mass of the Earth and *R* is its radius.

2. Show that on a hypothetical planet having half the diameter of the Earth but twice its density, the acceleration of free fall is the same as on Earth.

3. Consider an inertial reference frame whose origin is fixed at the center of mass of the system Earth + falling object. (*a*) Show that the acceleration toward the center of mass of either body is independent of the mass of that body. (*b*) Show that the mutual, or relative, acceleration of the two bodies depends on the sum of the masses of the two bodies. Comment on the meaning, then, of the statement that a body falls toward the Earth with an acceleration that is independent of its mass.

4. Two objects, each of mass *m*, hang from strings of different lengths on a balance at the surface of the Earth, as shown in Fig. 14-33. If the strings have negligible mass and differ in length by *h*, (*a*) show that the error in weighing, associated with the fact that *W*′ is closer to the Earth than *W*, is $W' - W = 8\pi G\rho m h/3$ in which ρ is the mean density of the Earth (5.5 g/cm^3). (*b*) Find the difference in length that will give an error of one part in a million.

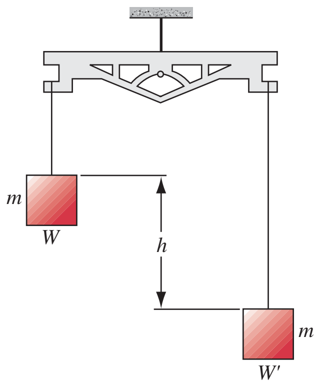


FIGURE 14-33. Problem 4.

5. (*a*) Write an expression for the force exerted by the Moon, mass *M*, on a particle of water, mass *m*, on the Earth at *A*, directly under the Moon, as shown in Fig. 14-34. The radius of the Earth is *R*, and the center-to-center Earth–Moon distance is *r*. (*b*) Suppose that the particle of water was at the center of the Earth. What force would the Moon exert on it there? (*c*) Show that the difference in these forces is given by

$$F_T = \frac{2GMmR}{r^3}$$

and represents the *tidal force,* the force on water relative to the Earth. What is the direction of the tidal force? (*d*) Repeat for a particle of water at *B*, on the far side of the Earth from the Moon. What is the direction of this tidal force? (*e*) Explain why there are two tidal bulges in the oceans (and solid Earth), one pointing toward the Moon and the other away from it.

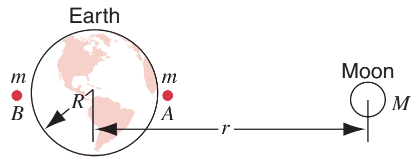


FIGURE 14-34. Problem 5.

6. An object is suspended on a spring balance in a ship sailing along the equator with a speed v. Show that the scale reading will be very close to $W_0(1 \pm 2\omega v/g)$, where ω is the angular speed of the Earth and W_0 is the scale reading when the ship is at rest. Explain the plus or minus.

7. In *3001: The Final Odyssey,* Arthur C. Clarke writes of a tower that stretches from the Earth's equator to geosynchronous orbit. (*a*) The hero, Frank Poole, finds himself in the tower and estimates the acceleration of free fall at his altitude to be $g/2$. Taking into account rotational motion, what is Poole's altitude? (*b*) Calculate the work necessary to raise a 100-kg mass from the surface of the Earth up through the tower to the geosynchronous altitude. Compare your result to the energy expenditure of a rocket that can do the same thing today. (Hint: Assume that the rotational correction is small, and solve iteratively.)

8. The fastest possible rate of rotation of a planet is that for which the gravitational force on material at the equator barely provides the centripetal force needed for the rotation. (Why?) (*a*) Show then that the corresponding shortest period of rotation is given by

$$T = \sqrt{\frac{3\pi}{G\rho}},$$

where ρ is the density of the planet, assumed to be homogeneous. (*b*) Evaluate the rotation period assuming a density of 3.0 g/cm^3, typical of many planets, satellites, and asteroids. No such object is found to be spinning with a period shorter than found by this analysis.

9. Sensitive meters that measure the local free-fall acceleration g can be used to detect the presence of deposits of near-surface rocks of density significantly greater or less than that of the surroundings. Cavities such as caverns and abandoned mine shafts can also be located. (*a*) Show that the vertical component of g a distance x from a point directly above the center of a spherical cavern (see Fig. 14-35) is less than what would be expected, assuming a uniform distribution of rock of density ρ, by the amount

$$\Delta g = \frac{4\pi}{3}R^3G\rho\frac{d}{(d^2 + x^2)^{3/2}},$$

where R is the radius of the cavern and d is the depth of its center. (*b*) These values of Δg, called anomalies, are usually very small and expressed in milligals, where 1 gal = 1 cm/s^2. Oil prospectors doing a gravity survey find Δg varying from 10.0 milligals to a maximum of 14.0 milligals over a 150-m distance. Assuming that the larger anomaly was recorded directly over the center of a spherical cavern known to be in the region, find its radius and the depth to the roof of the cavern at that point. Nearby rocks have a density of 2.80 g/cm^3. (*c*) Suppose that the cavern, instead of being empty, is completely flooded with water. What do the gravity readings in (*b*) now indicate for its radius and depth?

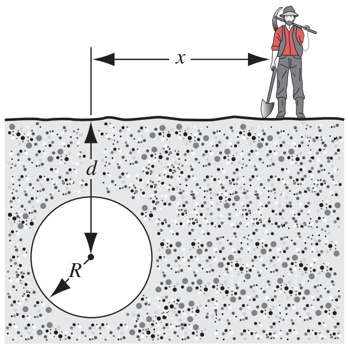


FIGURE 14-35. Problem 9.

10. Show that the acceleration of gravity in a vertical mine shaft is independent of the depth if the local density of the Earth ρ_l is 2/3 of the average density of the Earth. Assume that the Earth is a spherically symmetric, nonrotating body. (See "Gravity in a Mine Shaft," by Peter M. Hall and David J. Hall, *The Physics Teacher,* November 1995, p. 525.)

11. The following problem is from the 1946 "Olympic" examination of Moscow State University (see Fig. 14-36): A spherical hollow is made in a lead sphere of radius R, such that its surface touches the outside surface of the lead sphere and passes through its center. The mass of the sphere before hollowing was M. With what force, according to the law of universal gravitation, will the hollowed lead sphere attract a small sphere of mass m, which lies at a distance d from the center of the lead sphere on the straight line connecting the centers of the spheres and of the hollow?

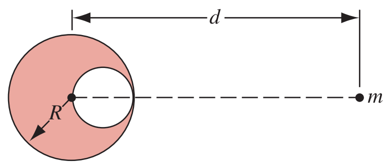


FIGURE 14-36. Problem 11.

12. (*a*) How long does it take the particle in Sample Problem 14-4 to fall from the Earth's surface to its center? (Hint: Use the expression for $v(r) = dr/dt$ given in the solution to Sample Problem 14-4 to find an expression that you can integrate to give t as a function of r. See also Section 12-5.) (*b*) After reaching the center, how long does it take for the particle to rise to the Earth's surface? What is the total time interval for the particle to make a complete round-trip and return to the starting point? (*c*) Compare the total round-trip time with the time for one orbit of a satellite close to the Earth's surface, and explain the similarity of these two numbers.

13. Figure 14-37 shows, not to scale, a cross section through the interior of the Earth. Rather than being uniform throughout, the Earth is divided into three zones: an outer crust, a mantle, and an inner core. The dimensions of these zones and the mass contained within them are shown in the figure. The Earth has total mass 5.98×10^{24} kg and radius 6370 km. Ignore rotation and assume that the Earth is spherical. (*a*) Calculate g at the surface. (*b*) Supose that a bore hole is driven to the crust–mantle interface (the Moho); what would be the value of g at the bottom of the hole? (*c*) Suppose that the Earth were a uniform sphere with the same total mass and size. What would be the value of g at a depth of 25 km? Use the result of Exercise 11. Precise measurements of g are sensitive probes of the interior structure of the Earth, although results can be clouded by local density variations and lack of a precise knowledge of the value of G.

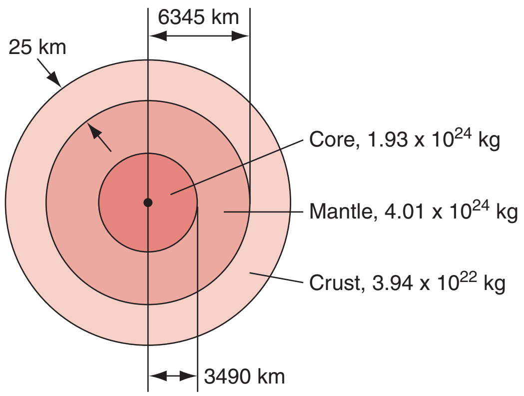


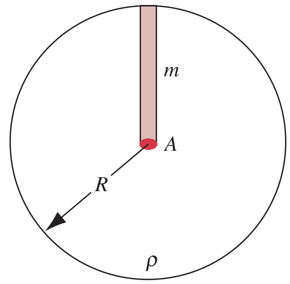
FIGURE 14-37. Problems 13 and 14.

14. Use the model of the Earth shown in Fig. 14-37 to examine the variation of g with depth in the interior of the Earth. (*a*) Find g at the core–mantle interface. How does g vary from this interface to the center of the Earth? (*b*) Show that g has a local minimum within the mantle; find the distance from the Earth's center where this occurs and the associated value of g. (*c*) Make a sketch showing the variation of g within the Earth.

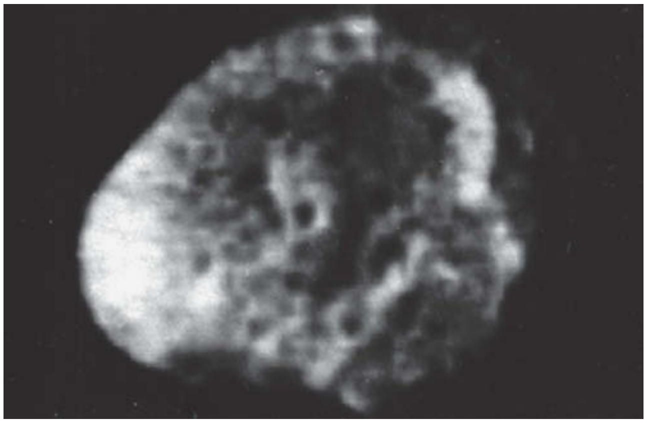
15. (*a*) Fig. 14-38*a* shows a planetary object of uniform density ρ and radius R. Show that the compressive stress S (defined as force per unit cross-sectional area) near the center is given by

$$S = \frac{2}{3}\pi G\rho^2 R^2.$$

(Hint: Construct a narrow column of cross-sectional area A extending from the center to the surface. The weight of the material in the column is mg_{av} where m is the mass of material in the column and g_{av} is the value of g midway between center and surface.) (*b*) In our solar system, objects (for example, asteroids, small satellites, comets) with "diameters" less than 600 km can be very irregular in shape (see Fig. 14-38*b*, which shows Hyperion, a small satellite of Saturn), whereas those with larger diameters are spherical. Only if the rocks have sufficient strength to resist gravity can an object maintain a nonspherical shape. Calculate the maximum compressive stress that can be sustained by the rocks making up asteroids. Assume a density of 4000 kg/m^3. (*c*) What is the largest possible size of a nonspherical self-gravitating satellite made of concrete? Assume that concrete has a maximum compressive stress of 4.0×10^7 N/m^2 and a density $\rho = 3000$ kg/m^3.



(*a*)



(*b*)

FIGURE 14-38. Problem 15.

16. A particle of mass m is located a distance y from an infinitely long, thin rod of linear mass density λ. Show that the gravitational force between the rod and the particle is $F = 2Gm\lambda/y$, directed perpendicular to the rod. (Hint: Let the perpendicular from the particle to the rod define the origin. Consider two mass increments $dm = \lambda\, dx$ located at $\pm x$ along the rod. Calculate the total force dF (magnitude and direction) exerted on the particle by these two mass increments. Then integrate over x from zero to infinity.)

17. Consider a particle at a point P anywhere inside a spherical shell of matter. Assume that the shell is of uniform thickness and density. Construct a narrow double cone with apex at P intercepting areas dA_1 and dA_2, on the shell (Fig. 14-39). (*a*) Show that the resultant gravitational force exerted on the particle at P by the intercepted mass elements is zero. (*b*) Show then that the resultant gravitational force of the entire shell on an internal particle is zero. (This method was devised by Newton.)

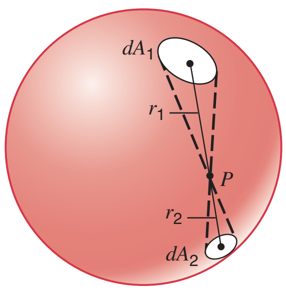


FIGURE 14-39. Problem 17.

18. A sphere of matter, of mass M and radius a, has a concentric cavity of radius b, as shown in cross section in Fig. 14-40. (a) Sketch the gravitational force F exerted by the sphere on a particle of mass m, located a distance r from the center of the sphere, as a function of r in the range $0 \leq r \leq \infty$. Consider points $r = 0$, b, a, and ∞ in particular. (b) Sketch the corresponding curve for the potential energy $U(r)$ of the system. (c) From these graphs, how would you obtain graphs of the gravitational field strength due to the sphere?

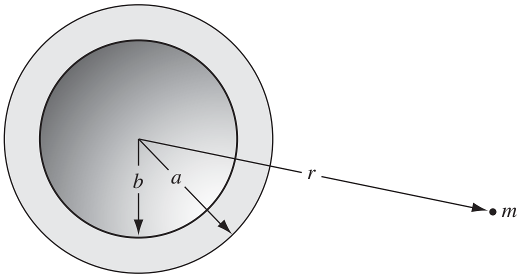


FIGURE 14-40. Problem 18.

19. Spheres of masses 2.53 kg and 7.16 kg are fixed a distance 1.56 m apart (center to center). A 212-g sphere is positioned 42.0 cm from the center of the 7.16 kg sphere, along the line of centers. How much work must be done by an external agent to move the 212-g sphere along the line of centers and place it 42.0 cm from the center of the 2.53-kg sphere?

20. A rocket burns out at an altitude h above the Earth's surface. Its speed v_0 at burnout exceeds the escape speed v_{esc} appropriate to the burnout altitude. Show that the speed v of the rocket very far from the Earth is given by

$$v = (v_0^2 - v_{esc}^2)^{1/2}.$$

21. In a particular double-star system, two stars of mass 3.22×10^{30} kg each revolve about their common center of mass, 1.12×10^{11} m away. (a) Calculate their common period of revolution, in years. (See Exercise 32.) (b) Suppose that a meteoroid (small solid particle in space) passes through this center of mass moving at right angles to the orbital plane of the stars. What must its speed be if it is to escape from the gravitational field of the double star?

22. Several planets (the gas giants Jupiter, Saturn, Uranus, and Neptune) possess nearly circular surrounding rings, perhaps composed of material that failed to form a satellite. In addition, many galaxies contain ring-like structures. Consider a homogeneous ring of mass M and radius R. (a) Find an expression for the gravitational force exerted by the ring on a particle of mass m located a distance x from the center of the ring along its axis. See Fig. 14-41. (b) Suppose that the particle falls from rest as a result of the attraction of the ring of matter. Find an expression for the speed with which it passes through the center of the ring.

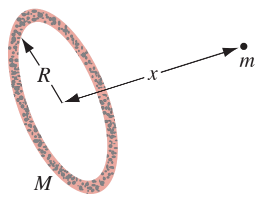


FIGURE 14-41. Problem 22.

23. Nine small particles, each with mass m, are evenly arranged around a ring of radius R. (a) Calculate the net gravitational force on one of the particles due to the other eight particles in the ring. (b) Find the rotational period of the ring necessary to prevent the ring from collapsing under the mutual gravitational attraction of the particles.

24. Two point-like particles, each of mass m, are originally at rest separated by a distance d. Show that the time for them to come together under the influence of gravity is

$$t_{meet} = \frac{\pi}{4}\sqrt{\frac{d^3}{Gm}}.$$

(See "The Period of $\vec{\mathbf{F}} = -kx^n\hat{\mathbf{x}}$ Harmonic Motion," by Chris Hirata and David Thiessen, *The Physics Teacher,* December 1995, p. 563.)

25. Consider two satellites A and B of equal mass m, moving in the same circular orbit of radius r around the Earth but in opposite senses of revolution and therefore on a collision course (see Fig. 14-42). (a) In terms of G, M_E, m, and r, find the total mechanical energy of the two-satellite-plus-Earth system before collision. (b) If the collision is completely inelastic so that wreckage remains as one piece of tangled material, find the total mechanical energy immediately after collision. (c) Describe the subsequent motion of the wreckage.

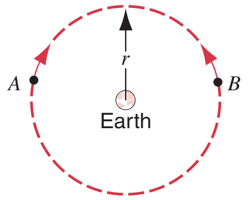


FIGURE 14-42. Problem 25.

26. The Sun's center is at one focus of the Earth's orbit. How far is it from the other focus? Express your answer in terms of the radius of the Sun $R_S = 6.96 \times 10^8$ m. The eccentricity of the Earth's orbit is 0.0167 and the semimajor axis is 1.50×10^{11} m.

27. In the year 1610, Galileo made a telescope, turned it on Jupiter, and discovered four prominent moons. Their mean orbit radii a and periods T are as follows.

Name	a (10^8 m)	T (days)
Io	4.22	1.77
Europa	6.71	3.55
Ganymede	10.7	7.16
Callisto	18.8	16.7

(a) Plot log a (y axis) against log T (x axis) and show that you get a straight line. (b) Measure its slope and compare it with the value that you expect from Kepler's law of periods. (c) Find the mass of Jupiter from the intercept of this line with the y axis. (Note: You may also use log-log graph paper.)

28. A certain triple-star system consists of two stars, each of mass m, revolving about a central star, mass M, in the same circular orbit. The two stars stay at opposite ends of a diameter of the circular orbit; see Fig. 14-43. Derive an expression for the period of revolution of the stars; the radius of the orbit is r.

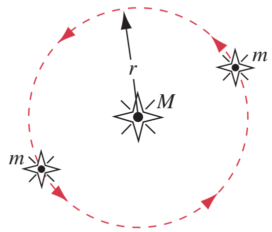


FIGURE 14-43. Problem 28.

29. A satellite travels initially in an approximately circular orbit 640 km above the surface of the Earth; its mass is 220 kg. (*a*) Determine its speed. (*b*) Determine its period of revolution. (*c*) For various reasons the satellite loses mechanical energy at the (average) rate of 1.40×10^5 J per orbital revolution. Adopting the reasonable approximation that the trajectory is a "circle of slowly diminishing radius," determine the distance from the surface of the Earth, the speed, and the period of the satellite at the end of its 1500th orbital revolution. (*d*) What is the magnitude of the average retarding force? (*e*) Is angular momentum conserved?

30. A satellite is placed at the altitude of a geosynchronous orbit, except that the plane of the orbit is inclined at an angle of 10° with respect to the equatorial plane. Describe the motion of the satellite against the background stars as seen from a point on the equator.

31. Three identical stars of mass M are located at the vertices of an equilateral triangle with side L. At what speed must they move if they all revolve under the influence of one another's gravity in a circular orbit circumscribing, while still preserving, the equilateral triangle?

32. How long will it take a comet, moving in a parabolic path, to move from its point of closest approach to the Sun at A (see Fig. 14-44) through an angle of 90°, measured at the Sun, to B? Let the distance of closest approach to the Sun be equal to the radius of the Earth's orbit, assumed circular.

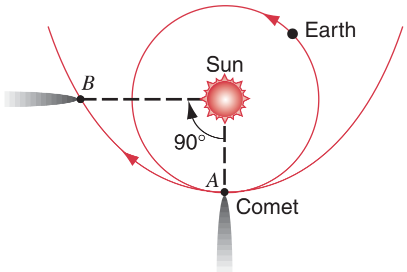


FIGURE 14-44. Problem 32.

33. Imagine a planet of mass M with a small moon of mass m and radius a orbiting it and keeping the same face toward it. If the moon now approaches the planet, there will be a critical distance from the planet's center at which loose material lying on the moon's surface will be lifted off. Show that this distance is given by $r_c = a(3M/m)^{1/3}$. This critical distance is called *Roche's limit.*

COMPUTER PROBLEMS

1. Larry Niven wrote a series of science fiction books about *Ringworld,* an inhabited, manufactured ring of metal that circled a star. Consider a uniform ring of material with total mass M and radius R. Assume that the ring is infinitesimally thin. In terms of G, M, and R, (*a*) calculate the gravitational potential energy at a point $r = R/2$ in the plane of the ring, and (*b*) calculate the magnitude and direction of the force of gravity on a 1-kg mass located at that same point. (*c*) Repeat (*a*) and (*b*) for a point $r = 3R/2$ in the plane of the ring. (See "Bound Orbits with Positive Energy," by J. West, S. Dassanayake, and A. Daniel, *American Journal of Physics,* January 1998, p. 25.)

2. Repeat Problem 23 for 19 particles, 29 particles, 39 particles, and so on up to 99 particles. Plot the results on a graph of number of particles versus rotational period. Does the result converge to a limit as the number of particles becomes infinite? If so, what is that limit? Can the problem be solved analytically?

CHAPTER 15

FLUID STATICS

Most matter can conveniently be described as being in one of three phases*—solid, liquid, or gas. Solids and liquids (also called* condensed matter*) have a certain set of properties in common; for example, they are relatively incompressible, and their densities stay relatively constant as we vary the temperature (keeping other properties, such as pressure, constant). Gases, on the other hand, are easily compressible, and their density changes substantially with temperature if we hold the pressure constant.*

From a different perspective, we can usually group gases and liquids together under the common designation fluids*. The word "fluid" comes from a Latin word meaning "to flow." Fluids will flow, for example, to take the shape of any container that holds them; solids do not share this property. In this chapter we consider the properties of fluids at rest and the laws that govern them. In the next chapter we discuss the dynamical properties of fluids in motion.*

15-1 FLUIDS AND SOLIDS

When we apply a force to the surface of a material—for example, a cube of copper—the material can exert a reaction force according to Newton's third law. If we apply the force perpendicular to the surface, the cube may compress (if our force is applied toward the surface) or stretch (if our force is applied by pulling on the surface) by a very small amount until the strong intermolecular forces, which can be considered to behave approximately like springs, contribute a reaction force that balances the applied force. The same result occurs if we apply a force parallel to the surface (called a shearing force)—the material may distort slightly as the configuration of its molecules changes to provide the reaction force to balance the applied force. Objects that we classify as *solids* can normally be in equilibrium under applied compression, tensile, or shearing forces with only minimal changes in their size or shape.

On the other hand, a liquid such as water is not able to produce reaction forces to applied forces in arbitrary directions. Most liquids are nearly incompressible, so they can provide reaction forces to compression forces with only imperceptible changes in the spacing of their molecules. (Hydraulic systems, which we discuss later in this chapter, depend on this property of liquids.) To a limited extent, liquids may support tensile forces, but substantial changes in the material often result (think of blowing up a soap bubble, which responds to the increased pressure of the air inside by stretching to become thinner and thinner until it bursts when it cannot provide enough tensile force). Liquids cannot support shearing forces, which cause molecules of the liquid to flow in the direction of the force.

A third state of matter, gases, cannot support compressional, tensile, or shearing forces. Compressional forces cause substantial changes in the state of the gas, and shearing forces also cause the molecules to flow in the direction of the force.

Together, liquids and gases are classified as *fluids.* These materials will easily flow under the action of a shearing force. We commonly observe this effect when a fluid flows to conform to the shape of its container. Even some materials that we ordinarily might classify as solids—for example, pitch ("solid" tar) and glaciers ("solid" ice)—can flow if we apply a strong enough force. Solid metals can be drawn into fine wires by forcing them through a die at high pressure, and where the Earth has been cut to build highways you can often see evidence that "solid" rock also flows under high pressure.

The differences between the properties of a fluid and a solid depend on the forces that are exerted between their molecules. We can picture a solid as a three-dimensional array in which each molecule is bound to all its close neighbors by strong, spring-like forces. As a result, a solid is able to provide a reaction force to oppose an applied force in any direction. In a liquid, the intermolecular forces are relatively weak, and liquids lack the long-range order that gives solids their stability. In gases, the intermolecular forces are very weak, and the average spacing between the molecules is larger than in liquids or solids. Both gases and liquids can be made to flow by applying relatively modest forces.

The methods of classical mechanics (which we have so far applied to particles) could be used to analyze the behavior of fluids, but these methods are of limited usefulness due to the large number of interacting particles in a fluid and the difficulty of specifying all the forces between the particles as well as the position and velocity of every particle. It is generally more convenient to analyze fluids using laws that depend on the statistical behavior of the particles or that involve average or bulk properties such as pressure, density, and temperature. Our approach to fluid mechanics takes its start from Newton's laws, but we will develop special formulations of these laws that apply to fluids at rest or in motion.

15-2 PRESSURE AND DENSITY

Pressure

The ability to flow makes a fluid unable to sustain a force parallel to its surface. Under static conditions the only force component that need be considered is one that acts *normal* or *perpendicular* to the surface of the fluid. No matter what the shape of a fluid, forces between the interior and exterior are everywhere at right angles to the fluid boundary.

The magnitude of the normal force per unit surface area is called the *pressure*. Pressure is a scalar quantity; it has no directional properties. When you swim underwater, for example, the water presses on your body from all directions. Even though the pressure is produced by a force that has directional properties and is a vector, the pressure itself is a scalar.

Microscopically, the pressure exerted by a fluid on a surface in contact with it is caused by collisions of molecules of the fluid with the surface. As a result of a collision, the component of a molecule's momentum perpendicular to the surface is reversed. The surface must exert an impulsive force on the molecule, and by Newton's third law the molecules exert an equal force perpendicular to the surface. The net result of the reaction force exerted by many molecules on the surface gives rise to the pressure on the surface. We develop this picture more quantitatively in the case of gases in Chapter 22.

A fluid under pressure exerts an outward force on any surface in contact with it. Consider a closed surface containing a fluid, as in Fig. 15-1. The fluid within the surface pushes out against the environment. A small element of surface area can be represented by the vector $\Delta\vec{\mathbf{A}}$, whose magnitude is numerically equal to the element of area and whose direction, by convention, is along the *outward* normal to the surface. The force $\Delta\vec{\mathbf{F}}$ exerted by the fluid against this surface depends on the pressure p according to

$$\Delta\vec{\mathbf{F}} = p\,\Delta\vec{\mathbf{A}}. \tag{15-1}$$

Since the vectors representing the force and the area are parallel, we can write the pressure in terms of the scalar relationship

$$p = \frac{\Delta F}{\Delta A}. \tag{15-2}$$

We take the element ΔA small enough that the pressure p defined according to Eq. 15-2 is independent of the size of the element. In general, the pressure may vary from point to point along the surface.

Pressure has dimensions of force divided by area, and a common unit for pressure is N/m^2. This unit is given the SI designation *pascal* (abbreviation Pa; 1 Pa = 1 N/m^2). A wide variety of other units can be found. Tire pressure gauges usually read in $lb/in.^2$ in the United States. The standard pressure exerted by the atmosphere of the Earth at sea level is designated as 1 atmosphere (atm; 1 atm = 14.7 $lb/in.^2$ = 1.01325×10^5 Pa, exactly). Because the pascal is a small unit (1 Pa $\approx 10^{-5}$ atm), weather forecasters often use the unit of the bar (1 bar = 10^5 Pa, or approximately 1 atm) to express atmospheric pressure. Other units for measuring pressure are discussed in Section 15-5.

Table 15-1 gives some representative pressures in pascal units. The term "overpressure" indicates a pressure value in excess of normal atmospheric pressure. Note that in the laboratory we can produce pressures that range over 22 orders of magnitude. In Appendix G you will find the conversion

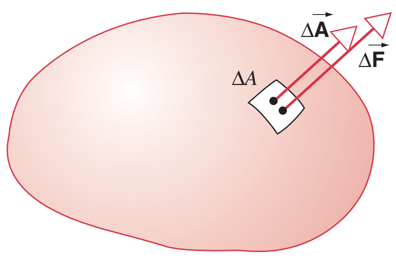


FIGURE 15-1. An element of surface ΔA can be represented by a vector $\Delta\vec{\mathbf{A}}$ of length equal to the magnitude of the area of the element and of direction perpendicular to the element. The fluid enclosed by the surface exerts a force $\Delta\vec{\mathbf{F}}$ against the element. The force is perpendicular to the element and therefore parallel to $\Delta\vec{\mathbf{A}}$.

TABLE 15-1 Some Pressures

System	*Pressure* (Pa)
Center of the Sun	2×10^{16}
Center of the Earth	4×10^{11}
Highest sustained laboratory pressure	1.5×10^{10}
Deepest ocean trench (bottom)	1.1×10^{8}
Spiked heels on a dance floor	2×10^{7}
Automobile tire (overpressure)	2×10^{5}
Atmosphere at sea level	1.0×10^{5}
Normal blood pressure[a]	1.6×10^{4}
Loudest tolerable sound[b]	30
Faintest detectable sound[b]	3×10^{-5}
Best laboratory vacuum	10^{-12}

[a] The systolic overpressure, corresponding to 120 mm Hg on the physician's pressure gauge.
[b] Overpressure at the eardrum, at 1000 Hz.

TABLE 15-2 Some Densities

Material or Object	*Density* (kg/m^3)
Interstellar space	10^{-20}
Best laboratory vacuum	10^{-17}
Air: 20°C and 1 atm	1.21
20°C and 50 atm	60.5
Styrofoam	1×10^{2}
Ice	0.917×10^{3}
Water: 20°C and 1 atm	0.998×10^{3}
20°C and 50 atm	1.000×10^{3}
Seawater: 20°C and 1 atm	1.024×10^{3}
Whole blood	1.060×10^{3}
Iron	7.8×10^{3}
Mercury	13.6×10^{3}
The Earth: average	5.5×10^{3}
core	9.5×10^{3}
crust	2.8×10^{3}
The Sun: average	1.4×10^{3}
core	1.6×10^{5}
White dwarf star (core)	10^{10}
Uranium nucleus	3×10^{17}
Neutron star (core)	10^{18}
Black hole (1 solar mass)	10^{19}

factors necessary to convert pressure measurements from one set of units to another.

Density

The density ρ of a small element of any material is the mass Δm of the element divided by its volume ΔV:

$$\rho = \frac{\Delta m}{\Delta V}. \qquad (15\text{-}3)$$

The density at a point is the limiting value of this ratio as the volume element becomes infinitesimally small. Density has no directional properties and is a scalar.

If the density of an object has the same value at all points, the density of the object is equal to the mass of the entire object divided by its volume:

$$\rho = \frac{m}{V}. \qquad (15\text{-}4)$$

Table 15-2 gives some representative densities, which vary by about 21 orders of magnitude in the laboratory and by nearly 40 orders of magnitude from the densest objects in the universe (a hypothetical black hole) to the near vacuum of space itself.

The density of a material in general depends on environmental factors, including the pressure and temperature. For liquids and solids, the variation in density is very small over wide ranges of variation of pressure and temperature.

When we increase the pressure on a material by an amount Δp, its density will correspondingly increase. The fractional change in its volume is $\Delta V/V$, which is negative if the volume decreases. The ratio between these quantities is called the *bulk modulus B*:

$$B = -\frac{\Delta p}{\Delta V/V}. \qquad (15\text{-}5)$$

The minus sign is inserted in this definition to make B a positive quantity, because Δp and ΔV have opposite signs. That is, an *increase* in pressure ($\Delta p > 0$) causes a *decrease* in volume ($\Delta V < 0$). Note that B has the same dimension as pressure, because $\Delta V/V$ is a dimensionless quantity.

If the bulk modulus of a material is large, then (according to Eq. 15-5) a large pressure change Δp produces only a small change in its volume. In this case, we can regard the material as being nearly incompressible. The bulk modulus of water, for example, is 2.2×10^9 N/m^2. At the pressure at the bottom of the Pacific Ocean (4.0×10^7 N/m^2, about 400 atm), the relative change in volume caused by pressure alone is only 1.8%. Solids usually have higher bulk moduli than liquids, because of the tighter coupling of the atoms in solids. A given pressure thus produces a smaller change in volume of a solid than a liquid. Under ordinary circumstances, we can therefore regard both solids and liquids as incompressible; that is, their densities do not change as the applied pressure changes.

If B is small, the volume can be changed by a modest change in pressure, and the material is said to be compressible. Typical gases have bulk moduli of about 10^5 N/m^2. A small pressure change of 0.1 atm can change the volume of a gas by 10%. Gases are thus easily compressible.

15-3 VARIATION OF PRESSURE IN A FLUID AT REST

If a fluid is in equilibrium, every portion of the fluid is in equilibrium. That is, both the net force and the net torque on every element of the fluid must be zero. Consider a small element of fluid volume submerged within the body

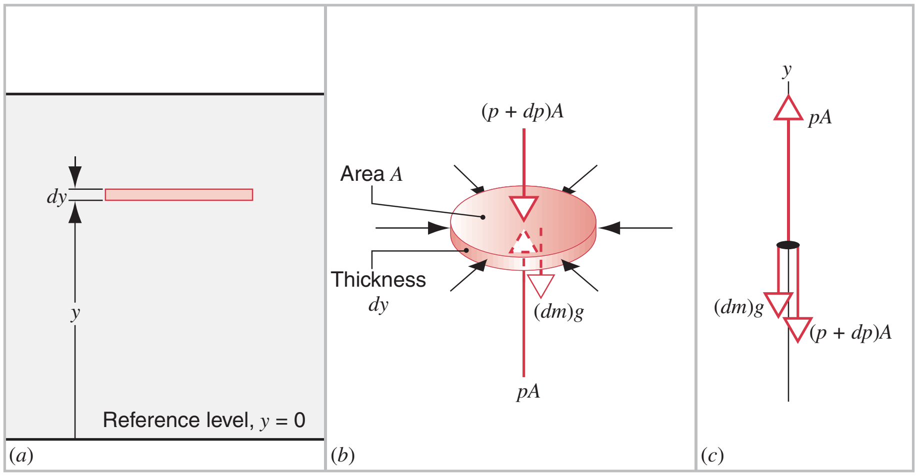


FIGURE 15-2. (*a*) A small volume element of the fluid at rest. (*b*) The forces on the element. (*c*) A free-body diagram of the element.

of the fluid. Let this element have the shape of a thin disk and be a distance y above some reference level, as shown in Fig. 15-2*a*. The thickness of the disk is dy and each face has area A. The mass of this element is $dm = \rho\, dV = \rho A\, dy$, and its weight is $(dm)g = \rho g A\, dy$. The forces exerted on the element by the surrounding fluid are perpendicular to its surface at each point (Fig. 15-2*b*).

The resultant horizontal force is zero, because the element has no horizontal acceleration. The horizontal forces are due only to the pressure of the fluid, and by symmetry the pressure must be the same at all points within a horizontal plane at y.

The fluid element is also unaccelerated in the vertical direction, so the resultant vertical force on it must be zero. A free-body diagram of the fluid element is shown in Fig. 15-2*c*. The vertical forces are due not only to the pressure of the surrounding fluid on its faces but also to the weight of the element. If we let p be the pressure on the lower face and $p + dp$ the pressure on its upper face, the upward force on the lower face is pA, and the downward forces are $(p + dp)A$ on the upper face and the weight of the element $(dm)g = \rho g A\, dy$. Hence, for vertical equilibrium,

$$\sum F_y = pA - (p + dp)A - \rho g A\, dy = 0,$$

from which we obtain

$$\frac{dp}{dy} = -\rho g. \tag{15-6}$$

This equation tells us how the pressure varies with elevation above some reference level in a fluid in static equilibrium. As the elevation increases (dy positive), the pressure decreases (dp negative). The cause of this pressure variation is the weight per unit cross-sectional area of the layers of fluid lying between the points whose pressure difference is being measured.

The quantity ρg is often called the *weight density* of the fluid; it is the weight per unit volume of the fluid. For water, for example, the weight density is 9800 N/m^3 = 62.4 lb/ft^3.

If p_1 is the pressure at elevation y_1, and p_2 the pressure at elevation y_2 above some reference level, integration of Eq. 15-6 gives

$$\int_{p_1}^{p_2} dp = -\int_{y_1}^{y_2} \rho g\, dy$$

or

$$p_2 - p_1 = -\int_{y_1}^{y_2} \rho g\, dy. \tag{15-7}$$

For liquids, which are nearly incompressible, ρ is practically constant, and differences in level are rarely so great that any change in g need be considered. Hence, taking ρ and g as constants, we obtain

$$p_2 - p_1 = -\rho g(y_2 - y_1) \tag{15-8}$$

for a homogeneous liquid.

If a liquid has a free surface, this is the natural level from which to measure distances (Fig. 15-3). Let y_2 be the elevation of the surface, at which point the pressure p_2 acting on the fluid is usually that exerted by the Earth's atmo-

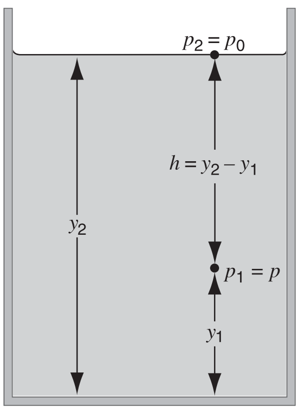


FIGURE 15-3. A container holds a quantity of a liquid whose top surface is open to the atmosphere. The pressure at any point in the liquid depends on the depth h.

sphere p_0. We take y_1 to be at any level in the fluid, and we represent the pressure there as p. Then

$$p_0 - p = -\rho g(y_2 - y_1).$$

However, $y_2 - y_1$ is the depth h below the surface at which the pressure is p (see Fig. 15-3), so that

$$p = p_0 + \rho gh. \tag{15-9}$$

This shows clearly that the pressure in a homogeneous, incompressible liquid increases with depth but is the same at all points at the same depth. The second term on the right of Eq. 15-9 gives the contribution to the pressure at a point in the liquid due to the weight of the fluid of height h above that point.

Equation 15-8 gives the relation between the pressures at any two points in a fluid, regardless of the shape of the containing vessel—for no matter what the shape of the containing vessel, two points in the fluid can be connected by a path made up of vertical and horizontal steps. For example, consider points A and B in the homogeneous liquid contained in the U-tube of Fig. 15-4*a*. Along the zigzag path from A to B there is a difference in pressure $\rho g\ \Delta y$ for each vertical segment of length Δy, whereas along each horizontal segment there is no change in pressure. Hence the difference in pressure $p_B - p_A$ is ρg times the algebraic sum of the vertical segments from A to B, or $\rho g(y_2 - y_1)$.

If the U-tube contains different immiscible liquids—say, a dense liquid in the right tube and a less dense one in the left tube, as shown in Fig. 15-4*b*—the pressure can be different at the same level (points A and B) on different sides. The liquid below the line CC is in equilibrium; thus the force exerted by the left column above C must equal the force exerted by the right column above C. The pressure at C is the same on both sides, but the pressure falls less from C to A than from C to B, because the liquid on the left is less dense than the liquid on the right. Thus the pressure at A is greater than at B.

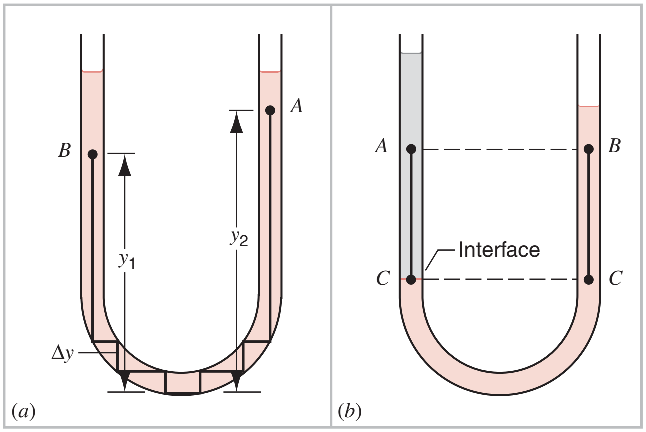


FIGURE 15-4. (*a*) The difference in pressure between two points A and B in a homogeneous liquid depends only on their difference in elevation $y_2 - y_1$. (*b*) Two points A and B at the same elevation can be at different pressures if the densities there differ.

Variation of Pressure in the Atmosphere

For gases ρ is comparatively small, and the difference in pressure at two nearby points is usually negligible (see Eq. 15-8). Thus in a reasonably small vessel containing a gas, the pressure can be taken as the same everywhere. However, this is not the case if $y_2 - y_1$ is very great. The pressure of the air varies greatly as we ascend to great heights in the atmosphere. Moreover, since gases are compressible, the variation in pressure causes a variation in the density ρ with altitude, and ρ must be known as a function of y before we can integrate Eq. 15-7.

We can get a reasonable idea of the variation of pressure with altitude in the Earth's atmosphere if we assume that the density ρ is proportional to the pressure. This would be very nearly true (according to the ideal gas law, which we discuss in Chapter 22) if the temperature of the air remained the same at all altitudes. Using this assumption, and also assuming that the variation of g with altitude is negligible, we can find the pressure p at any altitude y above sea level.

From Eq. 15-6 we have

$$\frac{dp}{dy} = -\rho g.$$

Since ρ is proportional to p, we have

$$\frac{\rho}{\rho_0} = \frac{p}{p_0}, \tag{15-10}$$

where ρ_0 and p_0 are the values of density and pressure at sea level. Then

$$\frac{dp}{dy} = -g\rho_0 \frac{p}{p_0},$$

so that

$$\frac{dp}{p} = -\frac{g\rho_0}{p_0}\, dy. \tag{15-11}$$

Integrating Eq. 15-11 from the pressure p_0 at the altitude $y = 0$ (sea level) to the pressure p at the altitude h, we obtain

$$\int_{p_0}^{p} \frac{dp}{p} = -\int_0^h \frac{g\rho_0}{p_0}\, dy,$$

which gives

$$\ln \frac{p}{p_0} = -\frac{g\rho_0}{p_0} h$$

or

$$p = p_0 e^{-(g\rho_0/p_0)h}, \tag{15-12}$$

which we can write as

$$p = p_0 e^{-h/a}, \tag{15-13}$$

where

$$a = \frac{p_0}{g\rho_0}.$$

Using the values $g = 9.80\ \text{m/s}^2$, $\rho_0 = 1.21\ \text{kg/m}^3$ (at 20°C), and $p_0 = 1.01 \times 10^5$ Pa, we obtain $a = 8.55$ km. The

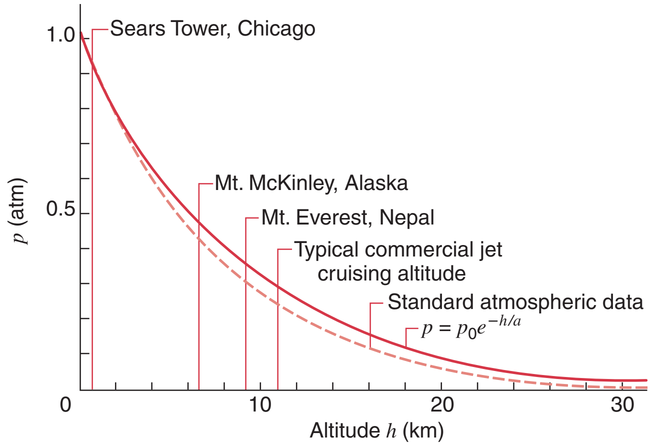


FIGURE 15-5. Comparison of standard atmospheric pressure data (dashed line) with predictions of Eq. 15-13 (solid line). The two curves differ because our calculation neglected the variation of the density with temperature as the altitude increases.

constant a gives the difference in altitude over which the pressure drops by a factor of e. Put another way, the atmospheric pressure drops by a factor of 10 when the altitude changes by $a \ln 10 = 2.30a = 20$ km. At an altitude of $h = 20$ km above sea level, the atmospheric pressure would thus be 0.1 atm; at $h = 40$ km above sea level, it would be 0.01 atm. Figure 15-5 shows a comparison between the pressure variation with altitude predicted by Eq. 15-13 and that measured for the atmosphere.

For gases at uniform temperature the density ρ of any layer is proportional to the pressure p at that layer. Liquids, however, are almost incompressible, so the lower layers are not noticeably compressed by the weight of the upper layers superimposed on them, and the density ρ is practically constant at all levels. Thus the variation of pressure with distance above the bottom of the fluid for a gas is different from that for a liquid, as indicated by Eq. 15-9 for a liquid and Eq. 15-13 for a gas.

SAMPLE PROBLEM 15-1. A U-tube, in which both ends are open to the atmosphere, is partly filled with water. Oil, which does not mix with water, is poured into one side until it stands a distance $d = 12.3$ mm above the water level on the other side, which has meanwhile risen a distance $a = 67.5$ mm from its original level (Fig. 15-6). Find the density of the oil.

Solution In Fig. 15-6 points C are at the same pressure. (If this were not true, then the U-shaped fluid element below the CC level would experience a net unbalanced force and would accelerate, violating the static assumption we make in this problem.) The pressure drop from C to the surface on the water side is $\rho_w g 2a$, where $2a$ is the height of the water column above C. The pressure drop on the other side from C to the surface is $\rho g(2a + d)$, where ρ is the unknown density of the oil. Equating the pressures at point C on each side, we obtain

$$p_0 + \rho_w g 2a = p_0 + \rho g(2a + d),$$

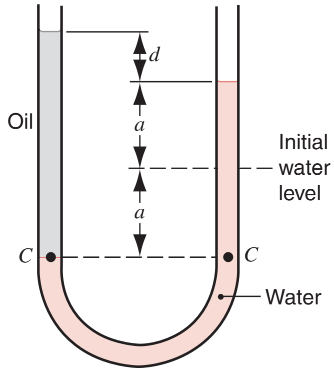


FIGURE 15-6. Sample Problem 15-1. A U-tube is filled partly with water and partly with oil of unknown density.

and so

$$\rho = \rho_w \frac{2a}{(2a + d)}$$
$$= (1.000 \times 10^3 \text{ kg/m}^3) \frac{2(67.5 \text{ mm})}{2(67.5 \text{ mm}) + 12.3 \text{ mm}}$$
$$= 916 \text{ kg/m}^3.$$

The ratio of the density of a substance to the density of water is called the *relative density* (or the *specific gravity*) of that substance. In this case the specific gravity of the oil is 0.916.

Note that in solving this problem, we have assumed that the pressure is continuous across the interface between the oil and the water at point C on the left side of the tube. If this were not so and the pressures were different, then the force exerted by the fluid on one side of the interface would differ from that of the fluid on the other side, and the interface would accelerate under the influence of the unbalanced force. Since we are assuming a static situation, there can be no motion and the pressures must therefore be the same. When we first pour the oil into the tube, however, there may be a difference in pressure and an unbalanced force that would cause the system to move until it reached the static situation shown in Fig. 15-6.

15-4 PASCAL'S PRINCIPLE AND ARCHIMEDES' PRINCIPLE

When you squeeze a tube of toothpaste, the toothpaste flows out of the open top of the tube. This demonstrates the action of *Pascal's principle*. When pressure is applied anywhere on the tube, it is felt everywhere in the tube and forces the toothpaste out of the top. Here is the statement of Pascal's principle, which was first stated by Blaise Pascal in 1652:

> *Pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid and to the walls of the containing vessel.*

That is, if you increase the external pressure on a fluid at one location by an amount Δp, the same increase in pressure is experienced everywhere in the fluid.

Pascal's principle is the basis for the operation of all hydraulic force-transmitting mechanisms, such as might be found in earth-moving machinery or the brake system of your car. It enables us to amplify a relatively small applied force to raise a much greater weight (as in the automobile lift or the dentist's chair) and to transmit forces over long distances to relatively inaccessible locations (as in the control mechanisms for the wing flaps used in aircraft).

We shall prove Pascal's principle for an incompressible liquid. Figure 15-7 shows the liquid in a cylinder that is fitted with a piston. An external force is applied to the piston, for instance, by the weight of some objects stacked on it. The external force results in an external pressure p_{ext} being applied to the liquid immediately beneath the piston. If the liquid has a density ρ, then from Eq. 15-9 we can write the pressure at an arbitrary point P a distance h below the surface:

$$p = p_{ext} + \rho g h. \tag{15-14}$$

Suppose now the external pressure is increased by an amount Δp_{ext}, perhaps by adding some more weight to the piston. How does the pressure p in the fluid change as a result of this change in the external pressure? We assume the liquid to be incompressible, so that the density ρ remains constant. The change in external pressure results in a change in pressure in the fluid that follows from Eq. 15-14:

$$\Delta p = \Delta p_{ext} + \Delta(\rho g h). \tag{15-15}$$

Since the liquid is incompressible, the density is constant, and the second term on the right of Eq. 15-15 equals zero. In this case, we obtain

$$\Delta p = \Delta p_{ext}. \tag{15-16}$$

The change in pressure at any point in the fluid is simply equal to the change in the externally applied pressure. This result confirms Pascal's principle and shows that it follows directly from our previous consideration of the static pressure in a fluid. It is therefore not an independent principle but a direct consequence of our formulation of fluid statics.

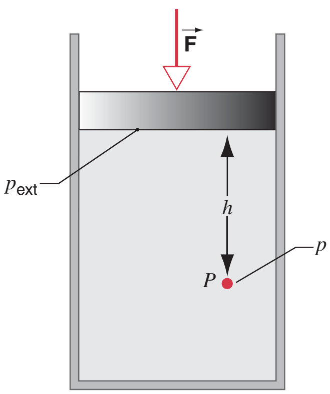


Figure 15-7. A fluid in a cylinder fitted with a movable piston. The pressure at any point P is due not only to the weight of the fluid above the level of P but also to the force exerted by the piston.

Although we derived the above result for incompressible liquids, Pascal's principle is true for all real (compressible) fluids, gases as well as liquids. The change in external pressure causes a change in density that quickly spreads throughout the fluid, but once the disturbance has died out and equilibrium has been established, Pascal's principle is found to remain valid.

The Hydraulic Lever

Figure 15-8 shows an arrangement that is often used to lift a heavy object such as an automobile. An external force F_i is exerted on a piston of area A_i. The object to be lifted exerts a force Mg on the larger piston of area A_o. In equilibrium, the magnitude of the upward force F_o exerted by the fluid on the larger piston must equal that of the downward force Mg of the weight of the object (neglecting the weight of the piston itself). We wish to find the relationship between the applied force F_i and the "output force" F_o that the system can exert on the larger piston.

The pressure on the fluid at the smaller piston, due to our externally applied force, is $p_i = F_i/A_i$. According to Pascal's principle, this "input" pressure must be equal to the "output" pressure $p_o = F_o/A_o$, which is exerted *by* the fluid *on* the larger piston. Thus $p_i = p_o$, and so

$$\frac{F_i}{A_i} = \frac{F_o}{A_o},$$

or

$$F_i = F_o \frac{A_i}{A_o} = Mg \frac{A_i}{A_o}. \tag{15-17}$$

The ratio A_i/A_o is generally much smaller than 1, and thus the applied force can be much smaller than the weight Mg that is lifted.

The downward movement of the smaller piston through a distance d_i displaces a volume of fluid $V = d_i A_i$. If the fluid is incompressible, then this volume must be equal to the volume displaced by the upward motion of the larger piston:

$$V = d_i A_i = d_o A_o,$$

or

$$d_o = d_i \frac{A_i}{A_o}. \tag{15-18}$$

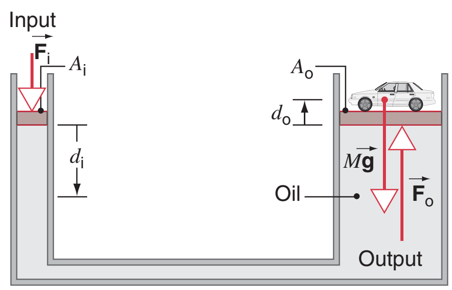


Figure 15-8. The hydraulic lever. A Force $\vec{\mathbf{F}}_i$ applied to the smaller piston can give a much larger force $\vec{\mathbf{F}}_o$ on the larger piston, which can lift a weight Mg.

If A_i/A_o is a small number, then the distance moved by the larger piston is much smaller than the distance the applied force moves the smaller piston. The price we pay for gaining the ability to lift a large load is losing the ability to move it very far.

By combining Eqs. 15-17 and 15-18, we see that $F_i d_i = F_o d_o$, which shows that the work done by the external force on the smaller piston is equal to the work done by the fluid on the larger piston. Thus (ignoring friction and other dissipative forces), there is no net gain (or loss) in energy in using this hydraulic system.

SAMPLE PROBLEM 15-2. Figure 15-9 shows a schematic view of a hydraulic jack used to lift an automobile. The hydraulic fluid is oil (density = 812 kg/m³). A hand pump is used, in which a force of magnitude F_i is applied to the smaller piston (of diameter 2.2 cm) when the hand applies a force of magnitude F_h to the end of the pump handle. The combined mass of the car to be lifted and the lifting platform is M = 1980 kg, and the large piston has a diameter of 16.4 cm. The length L of the pump handle is 36 cm, and the distance x from the pivot to the piston is 9.4 cm. (*a*) What is the applied force F_h needed to lift the car? (*b*) For each downward stroke of the pump, in which the hand moves a vertical distance of 28 cm, how far is the car raised?

Solution (*a*) From Eq. 15-17,

$$F_i = Mg\frac{A_i}{A_o} = (1980\text{ kg})(9.8\text{ m/s}^2)\frac{\pi(1.1\text{ cm})^2}{\pi(8.2\text{ cm})^2} = 349\text{ N}.$$

Taking torques on the pump handle about an axis perpendicular to the page through the pivot point O, neglecting the masses of the pump handle and the small piston, and assuming the pump handle moves with a negligibly small angular acceleration, we obtain

$$\sum \tau = F_h L - F_i x = 0,$$

where we have used Newton's third law to relate the force $\vec{\mathbf{F}}_i$ exerted *by* the pump handle *on* the piston to the force $-\vec{\mathbf{F}}_i$ exerted *by* the piston *on* the pump handle. Solving for F_h, we find that

$$F_h = F_i\frac{x}{L} = (349\text{ N})\frac{9.4\text{ cm}}{36\text{ cm}} = 91\text{ N}.$$

Such a force, about 20 lb, can easily be applied by hand.

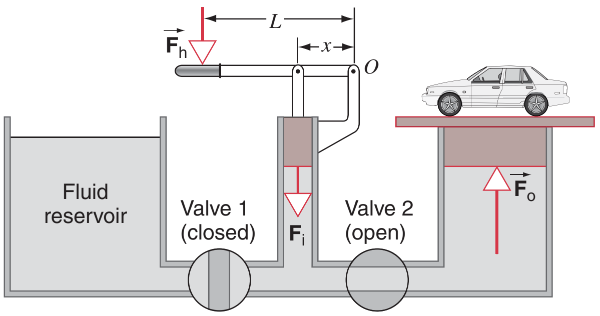


FIGURE 15-9. Sample Problem 15-2. A hydraulic pump is used to lift a car. For the downstroke, valve 1 is closed and valve 2 is open. During the upstroke, valve 1 is opened and valve 2 is closed, permitting additional fluid to be drawn into the hydraulic chamber.

(*b*) When the hand moves through a vertical distance h, the smaller piston will move through the distance

$$d_i = h\frac{x}{L} = (28\text{ cm})\frac{9.4\text{ cm}}{36\text{ cm}} = 7.3\text{ cm}.$$

Equation 15-18 then gives the distance moved by the larger piston:

$$d_o = d_i\frac{A_i}{A_o} = (7.3\text{ cm})\frac{\pi(1.1\text{ cm})^2}{\pi(8.2\text{ cm})^2} = 0.13\text{ cm} = 1.3\text{ mm}.$$

Raising the car by only such a tiny distance is the price we pay for exerting such a small force to lift the car. Of course, to make a useful device we must be able to lift the car by a larger distance, which is accomplished through many strokes of the pump. To keep the car from moving downward during the upward stroke of the pump, the valve arrangement shown in Fig. 15-9 is used. During the downstroke, the valves are in the positions shown in Fig. 15-9, and the car is raised by the distance d_o. During the return stroke valve 2 is closed, trapping fluid in the right side of the chamber and keeping the car at a fixed height, and valve 1 is opened, so the return stroke draws additional fluid from the reservoir into the left side of the chamber. For the next downstroke, the valves return to the positions shown in the figure, and the car is raised by another increment d_o. In effect, the volume of hydraulic fluid drawn into the left side of the chamber during the upstroke is pumped into the right side of the chamber during the downstroke. When the process is completed, the car can be lowered by opening both valves and allowing fluid to drain directly into the reservoir.

How does the operation of the hydraulic jack change as the car is raised and the height of the fluid in the right-hand column increases? Make a numerical estimate.

Archimedes' Principle

Figure 15-10*a* shows a volume of water contained in a thin plastic sack placed underwater. The water in the sack is in static equilibrium. Therefore its weight must be balanced by an upward force of equal magnitude. This upward force is the vector sum of all the inward forces exerted by the fluid that surrounds the sack. The arrows in Fig. 15-10*a* represent the forces exerted on the volume of liquid as a result of the pressure of the surrounding fluid. Note that the upward forces on the bottom of the sack are greater than the downward forces on the top, because the pressure increases with depth. The net upward force $\vec{\mathbf{F}}_b$ resulting from this pressure difference is called the *buoyant force* or *buoyancy.*

The pressure exerted on a submerged object by the surrounding fluid certainly cannot depend on the material of which the object is made. We could therefore replace the sack of water by a piece of wood of exactly the same size and shape, and the buoyant force would be unchanged. The upward force is still equal to the weight of the original volume of water. This leads us to Archimedes' principle:

> *A body wholly or partially immersed in a fluid is buoyed up by a force equal in magnitude to the weight of the fluid displaced by the body.*

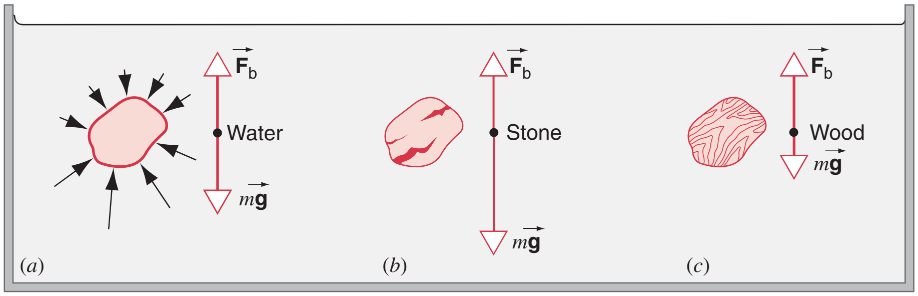


Figure 15-10. (*a*) A thin plastic sack full of water in equilibrium underwater. The water surrounding the sack exerts pressure forces on its surface, the resultant being an upward buoyant force $\vec{F}_b$ acting on the sack. (*b*) For a stone of the same volume, the buoyant force is the same, but the weight exceeds the buoyant force so the stone is not in equilibrium. (*c*) For a piece of wood of the same volume, the weight is less than the buoyant force.

An object of density greater than water (Fig. 15-10*b*) displaces a volume of water whose weight is less than the weight of the object. The object therefore sinks in the water, because the magnitude of the buoyant force is less than the weight of the object. If the submerged object were resting on a spring scale at the bottom of the water, the scale would read the upward force on the object, which is equal in magnitude to $mg - F_b$; thus submerged objects appear to weigh less than they normally do. Astronauts prepare for their voyages by practicing tasks under water in huge tanks, to simulate somewhat the weightless condition of space.

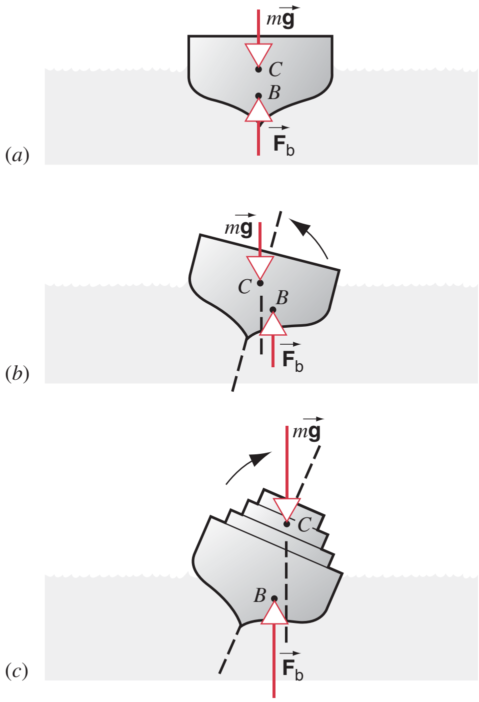


Figure 15-11. (*a*) A cross section of a ship floating upright. The buoyant force $\vec{F}_b$ acts at the center of buoyancy *B*, and the weight acts at the center of gravity *C*. The ship is in equilibrium under the action of these forces. (*b*) When the ship tips, the center of buoyancy may no longer lie on the same vertical line as the center of gravity, and a net torque may act on the ship. Here the torque about *C* acts to restore the ship to the upright position. (*c*) Here the center of gravity lies higher, so that the torque about *C* due to the buoyant force tends to tip the ship even further.

An object of density less than water (Fig. 15-10*c*) experiences a net upward force when completely submerged, because the weight of the water displaced is greater than the weight of the object. The object therefore rises until it breaks the surface, and it continues to rise until the only part of it still submerged is that volume necessary to displace water whose weight is equal to the total weight of the object. In that situation the object floats in equilibrium.

The buoyant force can be regarded as acting at the center of gravity of the fluid displaced by the submerged part of a floating object. This point is known as the *center of buoyancy.* The weight acts at the center of gravity of the entire object. These two points are in general not the same (Fig. 15-11*a*). If the two points lie on the same vertical line, then the object can float in equilibrium: both the net force and the net torque are zero. If the floating object is tipped slightly from its equilibrium position, then in general the shape of the displaced fluid changes, and the center of buoyancy shifts its position with respect to the center of gravity of the floating object. Thus a torque acts on the floating object that might tip the object back to its equilibrium position (Fig. 15-11*b*), or it might act in the other direction to tip it completely over (Fig. 15-11*c*).

Sample Problem 15-3. What fraction of the total volume of an iceberg is exposed?

Solution The weight of the iceberg is

$$W_i = \rho_i V_i g,$$

where V_i is the volume of the iceberg. The weight of the volume V_w of seawater displaced (or, equivalently, the volume of the submerged part of the iceberg) is the buoyant force

$$F_b = \rho_w V_w g.$$

However F_b equals W_i, because the iceberg is in equilibrium, so that

$$\rho_w V_w g = \rho_i V_i g,$$

and, using densities from Table 15-2,

$$\frac{V_w}{V_i} = \frac{\rho_i}{\rho_w} = \frac{917\ \text{kg/m}^3}{1024\ \text{kg/m}^3} = 0.896 = 89.6\%.$$

The volume of water displaced V_w is the volume of the submerged portion of the iceberg, so that 10.4% of the iceberg is exposed.

15-5 MEASUREMENT OF PRESSURE

The pressure exerted by a fluid can be measured using either static or dynamic techniques. The dynamic methods are based on the speed of flow of a moving fluid and are discussed in Chapter 16. In this section, we discuss static methods for measuring pressure.

Most pressure gauges use atmospheric pressure as a reference level and measure the difference between the actual pressure and atmospheric pressure, called the *gauge pressure.* The actual pressure at a point in a fluid is called the *absolute pressure,* which is then the atmospheric pressure plus the gauge pressure. Gauge pressure is given either above or below atmospheric pressure and may thus be positive or negative; absolute pressure is always positive.

The mercury barometer is a long glass tube that has been filled with mercury and then inverted into a dish of mercury, as in Fig. 15-12. The space above the mercury column is in effect a vacuum containing only mercury vapor, whose pressure p_2 is so small at ordinary temperatures that it can be neglected. The pressure p_1 on the surface of the dish of mercury is the unknown pressure p we wish to measure. From Eq. 15-8, we obtain

$$p_2 - p_1 = 0 - p = -\rho g(y_2 - y_1) = -\rho gh,$$

or

$$p = \rho gh. \tag{15-19}$$

Measuring the height of the column above the surface of the dish then gives the pressure.

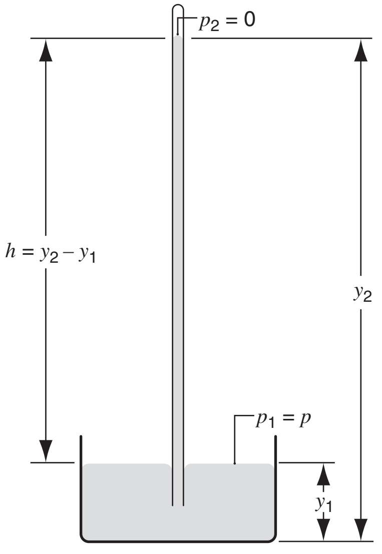


FIGURE 15-12. The mercury barometer. The mercury in the dish is in equilibrium under the influence of atmospheric pressure and the weight of the mercury in the vertical column.

The mercury barometer is often used for measuring atmospheric pressure p_0. The height of a column of mercury at normal atmospheric pressure (1 atm = 1.01325×10^5 N/m^2) is, according to Eq. 15-19:

$$h = \frac{p_0}{\rho g} = \frac{1.01325 \times 10^5 \text{ Pa}}{(13.5955 \times 10^3 \text{ kg/m}^3)(9.80665 \text{ m/s}^2)}$$
$$= 0.7600 \text{ m} = 760.0 \text{ mm},$$

where we have used a standard value for g and the density of mercury at 0°C. It is thus often said that 1 atm = 760 mm of Hg; equivalently, 1 mm of Hg = 1/760 atm. The pressure exerted by a column of mercury 1 mm high (again at 0°C and with g at its standard value) is called 1 torr, and so

$$1 \text{ torr} = 1 \text{ mm of Hg} = 133.322 \text{ Pa}.$$

We can also express 1 atm as 29.9 inches of Hg; ordinary barometers (and TV weather forecasters) often give atmospheric pressure in inches of mercury. These calculations may suggest why mercury, with its large density, is chosen to measure atmospheric pressure; a liquid of smaller density would require a proportionately higher column. To measure atmospheric pressure using a "water" barometer would require a column more than 10 m high!

Normal atmospheric pressure can also be expressed as 14.7 lb/in.^{2}, which means that the weight of the vertical column of air that extends from each square inch of the Earth's surface to the top of the atmosphere has a weight of 14.7 pounds. You should be able to show that a column of mercury 760 mm high and one square inch in cross-sectional area also weighs 14.7 pounds.

The mercury barometer was invented by the Italian Evangelista Torricelli (1608–1647), after whom the unit torr was named. Pascal, also working in the 17th century, was the first to use the barometer to show that atmospheric pressure varied with altitude. These experiments had significant impact because they demonstrated for the first time that it was possible to create a vacuum (in this case in the small volume at the top of the vertical tube). This demonstration led to the development of the vacuum pump in the latter part of the 17th century.

The open-tube manometer (Fig. 15-13) measures gauge pressure. It consists of a U-shaped tube containing a liquid, one end of the tube being open to the atmosphere and the other end being connected to the system (tank) whose pressure p we want to measure. From Eq. 15-9 we obtain

$$p - p_0 = \rho gh.$$

Thus the gauge pressure, $p - p_0$, is proportional to the difference in height of the liquid columns in the U-tube. If the vessel contains gas under high pressure, a dense liquid such as mercury is used in the tube; water or other low-density liquids can be used when low gas pressures are involved.

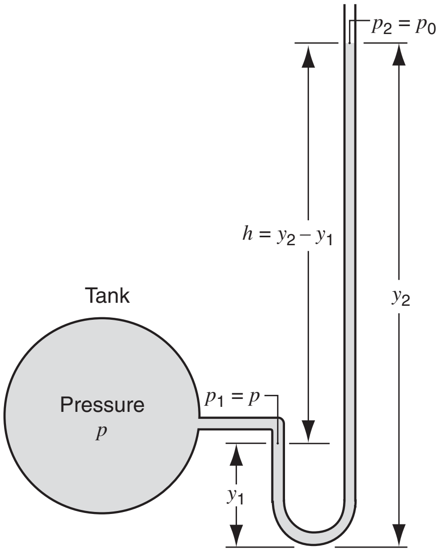


FIGURE 15-13. An open-tube manometer, which might be used to measure the pressure of a fluid in a tank.

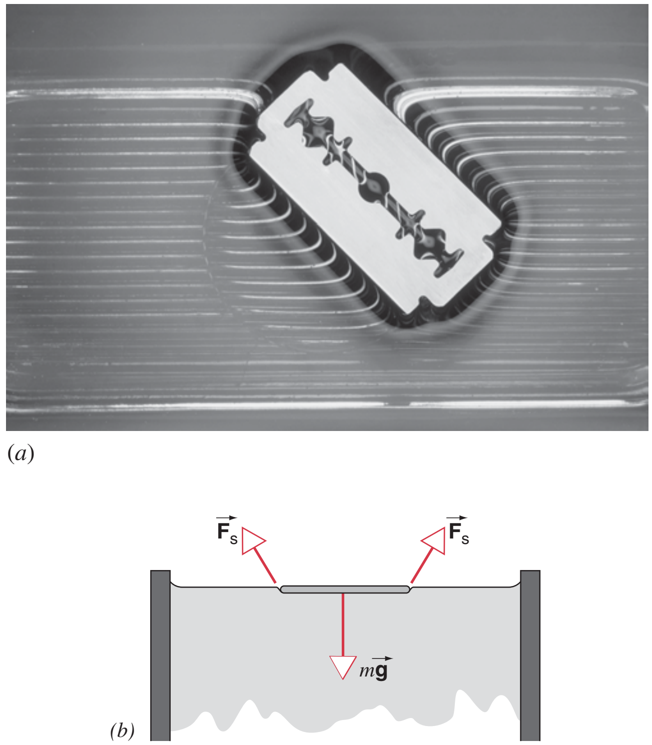


FIGURE 15-14. (*a*) A razor blade floating on the surface of water, supported only by surface tension. (*b*) The surface is distorted by the floating object, which is kept afloat by the vertical components of the surface force $\vec{F}_s$.

SAMPLE PROBLEM 15-4. The mercury column in a barometer has a measured height h of 740.35 mm. The temperature is $-5.0°C$, at which temperature the density of mercury is $1.3608 \times 10^4\ kg/m^3$. The free-fall acceleration g at the site of the barometer is $9.7835\ m/s^2$. What is the atmospheric pressure?

Solution From Eq. 15-19 we have

$$\begin{aligned} p_0 &= \rho g h \\ &= (1.3608 \times 10^4\ kg/m^3)(9.7835\ m/s^2)(0.74035\ m) \\ &= 9.8566 \times 10^4\ Pa = 739.31\ torr. \end{aligned}$$

Note that the value of the pressure in torr (739.31 torr) is numerically close to the value of the height h of the mercury column expressed in mm (740.35 mm). These two quantities will be numerically equal only if the barometer is located at a place where g has its standard value and where the mercury temperature is 0°C.

Another way to express the result of this sample problem would be as 0.98566 bar or 985.66 millibar, where 1 bar $= 10^5$ Pa.

15-6 SURFACE TENSION (Optional)

Leaves and insects can be observed to float on the *surface* of a body of water. They are *not* partially submerged and thus *not* buoyed up because of Archimedes' principle. In this case the object is completely on the surface and none of it is submerged.

The object is kept afloat by the *surface tension* of the liquid. You can demonstrate the surface tension of water by carefully floating a steel needle or a razor blade (Fig. 15-14*a*). There is of course no way for steel to float by Archimedes' principle, since its density is greater than that of water. If you submerge the needle or the razor blade, it will sink as Archimedes' principle predicts. Only when it is entirely on the surface can it float. You can add to the water a chemical, called a surface-active agent or surfactant, which reduces the surface tension (by reducing the cohesive force between molecules) and makes it more difficult to float the object. Detergents are common surfactants. If you carefully introduce detergent into the water on which a razor blade is floating, the surface tension suddenly decreases and the razor blade sinks to the bottom.

A floating object, such as that shown in Fig. 15-14*a*, depresses the surface layer of the fluid slightly (Fig. 15-14*b*), which stretches the surface layer and thus tends to increase its potential energy. Somewhat like a trampoline, the stretched surface layer exerts a restoring force, the vertical component of which can maintain equilibrium with the weight of the object. We shall soon see, however, that this analogy of the behavior of the surface layer is not strictly correct.

Figure 15-15 shows a way of measuring the surface tension of a liquid. A thin wire is bent into the shape of three sides of a rectangle and fitted with a sliding wire as the fourth side. If a film of the liquid covers the vertical loop (established perhaps by dipping the loop into a container of the liquid), the surface tension will tend to draw the sliding

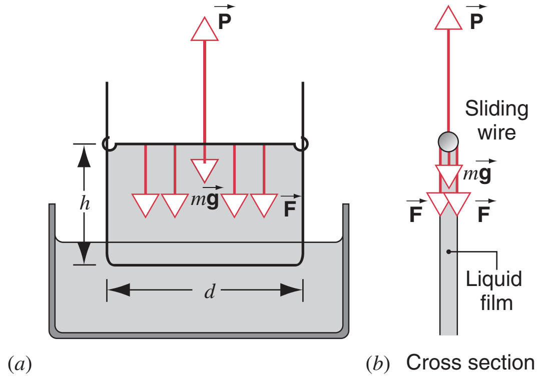


FIGURE 15-15. (*a*) Schematic diagram of an experiment to measure the surface tension of a liquid. A film of the liquid is supported in the vertical rectangular area, the top border of which is a sliding wire. An external force balances the weight of the sliding wire plus the total downward force $\vec{F}$ of the surface tension. (*b*) A cross-sectional sketch of the film, showing that the surface tension acts on two surfaces.

wire downward. We apply an external upward force P necessary to maintain the sliding wire in equilibrium. This upward force must balance the total downward force on the sliding wire, equal to its weight plus the force F due to the surface tension.

By experiment we find that the force F depends on the length d of the slide wire but does not depend at all on the height h of the rectangle. Although it is tempting to regard the surface layer as a sort of elastic sheet stretched over the liquid, this observation shows that such a picture is incorrect. Imagine the film of Fig. 15-15 to be cut into a large number N of narrow vertical strips of length h and width $\Delta d = d/N$. If the film behaved like an elastic sheet, each strip would behave like a spring, and so the total force would depend both on the number of spring-like strips (and hence on d) and on the length h of each strip. Because the surface tension depends only on d and not on h, the analogy of the elastic sheet is not correct.

The surface tension γ is defined as *the surface force F per unit length L over which it acts,* or

$$\gamma = \frac{F}{L}. \tag{15-20}$$

Note that the surface tension γ is not a force but a force per unit length. Our previous use of the term *tension* has always indicated a force, but here the usage is somewhat different.

For the film of Fig. 15-15, the force acts over a length L of $2d$, because there are two surface layers each of length d. For this arrangement the surface tension would be $\gamma = F/2d$.

We can also analyze the surface tension in terms of energy rather than force. Suppose we move the sliding wire of Fig. 15-15 through an upward displacement Δx. The work W done by the downward-acting surface force is $-F\,\Delta x$, and we can associate the work done by this conservative force with a change in potential energy $\Delta U = -W = F\,\Delta x$. Moving the wire upward increases the area of the surface by $\Delta A = L\,\Delta x$. Equation 15-20 then becomes

$$\gamma = \frac{F}{L} = \frac{F\,\Delta x}{L\,\Delta x} = \frac{\Delta U}{\Delta A}. \tag{15-21}$$

According to Eq. 15-21 we can also regard the surface tension γ as *the surface potential energy per unit area of surface.*

For water at room temperature, the value of the surface tension is $\gamma = 0.073$ N/m. Adding soap reduces the surface tension to 0.025 N/m. Organic liquids and aqueous solutions typically have surface tensions in this range. The surface tension of liquid metals is typically an order of magnitude larger than that of water. Liquid mercury at room temperature, for example, has a surface tension of 0.487 N/m. (This higher surface tension in metals occurs because the forces between molecules are typically an order of magnitude larger in metals than in water. For this same reason, the boiling points of metals are typically much higher than that of water.)

Surface tension causes suspended droplets of a liquid to acquire a spherical shape (Fig. 15-16). For a drop of a given mass or volume, the surface energy (equal to γ times the surface area) is least when the area is smallest, and a sphere has the smallest surface-to-volume ratio of any geometric shape. If no other forces act on the drop, it will natu-

FIGURE 15-16. Freely floating droplets of liquid naturally assume a spherical shape. Here astronaut Dr. Joseph P. Allen, in Earth orbit on space shuttle *Columbia,* watches a ball of orange juice he created using a beverage dispenser.

rally assume a spherical shape. In equilibrium, the surface tension gives a net inward force on an element of surface, which is balanced by an equal outward force due the pressure of the liquid within the drop. In a soap bubble (which has two surfaces and therefore twice the surface tension of a liquid drop of equal size), the gauge pressure of the gas confined in the bubble provides the outward force needed for equilibrium.

Because the protons and neutrons in the nucleus experience short-range forces somewhat like the molecules in a liquid, a nucleus experiences a surface tension similar to that of a liquid drop. For many nuclei, the shape is determined by the balance between the outward force due to the electrical repulsion of the protons and the inward force due to surface tension. For such nuclei, the preferred shape is usually spherical, like the liquid drop. Analyzing the nucleus as a charged liquid drop has been especially successful in helping us to understand nuclear fission, in which the nucleus splits into two parts of comparable size.

SAMPLE PROBLEM 15-5. In the experiment shown in Fig. 15-15*a*, it is found that the movable wire is in equilibrium when the upward force P is 3.45×10^{-3} N. The wire has a length d of 4.85 cm and a linear mass density μ of 1.75×10^{-3} kg/m. Find the surface tension of the liquid.

Solution From the equilibrium condition of Fig. 15-15*b*, we have

$$\sum F_y = P - F - mg = 0,$$

or

$$F = P - mg.$$

With $F = 2d\gamma$ (because there are two surface layers each of length d) and $m = \mu d$, we obtain

$$2d\gamma = P - \mu dg$$

or

$$\gamma = \frac{P - \mu dg}{2d}$$

$$= \frac{3.45 \times 10^{-3}\,\text{N} - (1.75 \times 10^{-3}\,\text{kg/m})(0.0485\,\text{m})(9.80\,\text{m/s}^2)}{2(0.0485\,\text{m})}$$

$$= 0.027\,\text{N/m}.$$

MULTIPLE CHOICE

15-1 Fluids and Solids

1. Consider the following types of forces: (A) compressional, (B) tensile, or (C) shearing. Which of these forces can be supported by
(*a*) a solid? (*b*) a liquid?

15-2 Pressure and Density

2. Object B has twice the density and half of the mass of object A. The ratio of the volume of A to the volume of B is
(A) 4. (B) 2. (C) 1.
(D) 1/2. (E) 1/4.

3. A suction cup is attached a smooth metal ceiling. The maximum weight that can be supported by the suction cup is dependent on
(A) its area of contact with the ceiling.
(B) the air pressure outside the cup.
(C) both (A) and (B).
(D) neither (A) nor (B).

15-3 Variation of Pressure in a Fluid at Rest

4. The top surface of an incompressible liquid is open to the atmosphere. The pressure at a depth h_1 below the surface is p_1. How does the pressure p_2 at depth $h_2 = 2h_1$ compare with p_1?
(A) $p_2 > 2p_1$ (B) $p_2 = 2p_1$ (C) $p_2 < 2p_1$

15-4 Pascal's Principle and Archimedes' Principle

5. A large rock is tied to a balloon filled with air. Both are placed in a lake. As the balloon sinks:
(*a*) The air pressure inside the balloon
(A) increases. (B) remains the same. (C) decreases.
(D) varies in an unpredictable manner.
(*b*) The average density of the balloon + air + rock
(A) increases. (B) remains the same. (C) decreases.
(D) varies in an unpredictable manner.
(*c*) The magnitude of the net force on the balloon + air + rock
(A) increases. (B) remains the same. (C) decreases.
(D) varies in an unpredictable manner.

6. The (average) human body floats in water. SCUBA divers wear weights and a flotation vest that can fill with a varying amount of air to establish neutral buoyancy. Assume that a diver originally establishes neutral buoyancy at one depth. To establish neutral buoyancy at a lower depth, the diver should
(A) let some air out of the vest.
(B) add some air to the vest.
(C) do nothing, because neutral buoyancy already exists.

7. An automobile tire is filled completely with water. The tire is mounted on an axle so that the tire is in a vertical plane. How does the pressure vary inside the tire with (*a*) no additional force applied; (*b*) a strong force pushing up on the bottom of the tire; (*c*) a strong force pushing down on the top of the tire?
(A) The pressure is significantly greater at the top.
(B) The pressure is approximately the same everywhere.
(C) The pressure is significantly greater at the bottom.
(D) The pressure variation cannot be determined without more information.

8. A wooden block floats in water in a sealed container. When the container is at rest, 25% of the block is above the water.

Consider the following five situations: (*a*) The container is lifted up at constant speed. (*b*) The container is lowered at constant speed. (*c*) The container is lifted up at an increasing speed. (*d*) The container is lowered at a decreasing speed. (*e*) The air pressure above the water in the container is increased. What happens in each situation?

(A) The block floats higher in the water.
(B) The block floats at the same level in the water.
(C) The block floats lower in the water.
(D) The fraction of the block above the water cannot be determined from the information given.

9. Bucket *A* contains only water; an identical bucket *B* contains water, but also contains a solid object in the water. Consider the following four situations: (*a*) The object floats in bucket *B*, and the buckets have the same water level. (*b*) The object floats in bucket *B*, and the buckets have the same volume of water. (*c*) The object sinks completely in bucket *B*, and the buckets have the same water level. (*d*) The object sinks completely in bucket *B*, and the buckets have the same volume of water.

In each of the above situations, which bucket has the greater total weight?

(A) Bucket *A*
(B) Bucket *B*
(C) Both buckets have the same weight.
(D) The answer cannot be determined from the information given.

15-5 Measurement of Pressure

15-6 Surface Tension

10. A spherical soap bubble has a radius r and a surface tension γ and contains air at a pressure p. More air is blown into the bubble, causing the radius to increase to $2r$.

(*a*) The surface tension in this now inflated soap bubble is

(A) slightly less than γ. (B) equal to γ.
(C) slightly more than γ. (D) 2γ.

(*b*) The pressure of the air inside this now inflated soap bubble is

(A) slightly less than p. (B) equal to p.
(C) slightly more than p. (D) $2p$.

QUESTIONS

1. Explain how it can be that pressure is a scalar quantity when forces, which are vectors, can be produced by the action of pressures.

2. Make an estimate of the average density of your body. Explain a way in which you could get an accurate value using ideas in this chapter.

3. In Chapter 19 we shall learn that an overpressure of only 20 Pa corresponds to the threshold of pain for intense sound. Yet a diver 2 m below the surface of water experiences a much greater pressure than this (how much?) and feels no pain. Why this difference?

4. Persons confined to bed are less likely to develop sores on their bodies if they use a water bed rather than an ordinary mattress. Explain.

5. Explain why one could lie on a bed of nails without pain.

6. Explain the statement "water seeks its own level."

7. Water is poured to the same level in each of the vessels shown, all having the same base area (Fig. 15-17). If the pressure is the same at the bottom of each vessel, the force experienced by the base of each vessel is the same. Why then do the three vessels have different weights when put on a scale? This apparently contradictory result is commonly known as the *hydrostatic paradox.*

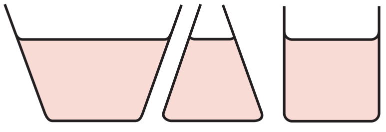


FIGURE 15-17. Question 7.

8. Does Archimedes' principle hold in a vessel in free fall? In a satellite moving in a circular orbit?

9. A spherical bob made of cork floats half submerged in a pot of tea at rest on the Earth. Will the cork float or sink aboard a spaceship (*a*) coasting in free space and (*b*) on the surface of Mars?

10. How does a suction cup work?

11. Is the buoyant force acting on a submerged submarine the same at all depths?

12. Explain how a submarine rises, falls, and maintains a fixed depth. Do fish use the same principles? (See "The Buoyancy of Marine Animals," by Eric Denton, *Scientific American,* July 1960, p. 118, and "Submarine Physics," by G. P. Harnwell, *American Journal of Physics,* March 1948, p. 127.)

13. A block of wood floats in a pail of water in an elevator. When the elevator starts from rest and accelerates down, does the block float higher above the water surface?

14. Two identical buckets are filled to the rim with water, but one has a block of wood floating in the water. Which bucket, if either, is heavier?

15. Estimate with some care the buoyant force exerted by the atmosphere on you.

16. According to Sample Problem 15-3, 89.6% of an iceberg is submerged. Yet occasionally icebergs turn over, with possibly disastrous results to nearby shipping. How can this happen considering that so much of their mass is below sea level?

17. Can you sink an iron ship by siphoning seawater into it?

18. SCUBA divers are warned not to hold their breath when swimming upward. Why?

19. A beaker is exactly full of liquid water at its freezing point and has an ice cube floating in it, also at its freezing point. As the cube melts, what happens to the water level in these three cases: (*a*) the cube is solid ice; (*b*) the cube contains some grains of sand; and (*c*) the cube contains some bubbles?

20. Although parachutes are supposed to slow your fall, they are often designed with a hole at the top. Explain why.
21. A ball floats on the surface of water in a container exposed to the atmosphere. Will the ball remain immersed at its former depth or will it sink or rise somewhat if (*a*) the container is covered and the air is removed or (*b*) the container is covered and the air is compressed?
22. Explain why an inflated balloon will rise only to a definite height once it starts to rise, whereas a submarine will always sink to the very bottom of the ocean once it starts to sink, if no changes are made.
23. Why does a balloon weigh the same when empty as when filled with air at atmospheric pressure? Would the weights be the same if measured in a vacuum?
24. Liquid containers tend to leak when taken aloft in an airplane. Why? Does it matter whether or not they are right-side up? Does it matter whether or not they are initially completely full?
25. During World War II, a damaged freighter that was barely able to float in the North Sea steamed up the Thames estuary toward the London docks. It sank before it could arrive. Why?
26. Is it true that a floating object will be in stable equilibrium only if its center of buoyancy lies above its center of gravity? Illustrate with examples.
27. Logs dropped upright into a pond do not remain upright, but float "flat" in the water. Explain.
28. Why will a sinking ship often turn over as it becomes immersed in water?
29. A barge filled with scrap iron is in a canal lock. If the iron is thrown overboard, what happens to the water level in the lock? What if it is thrown onto the land beside the canal?
30. A bucket of water is suspended from a spring balance. Does the balance reading change when a piece of iron suspended from a string is immersed in the water? When a piece of cork is put in the water?
31. If enough iron is added to one end of a uniform wooden stick or log, it will float vertically, rather than horizontally (see Question 27). Explain.
32. Although there are practical difficulties, it is possible in principle to float an ocean liner in a few barrels of water. How would you go about doing this?
33. An open bucket of water is on a frictionless plane inclined at an angle α to the horizontal. Find the equilibrium inclination to the horizontal of the free surface of the water when (*a*) the bucket is held at rest; (*b*) the bucket is allowed to slide down at constant speed ($a = 0$, $v =$ constant); and (*c*) the bucket slides down without restraint ($a =$ constant). If the plane is curved so that $a \neq$ constant, what will happen?
34. In a barometer, how important is it that the inner diameter of the barometer be uniform? That the barometer tube be absolutely vertical?
35. An open-tube manometer has one tube twice the diameter of the other. Explain how this would affect the operation of the manometer. Does it matter which end is connected to the chamber whose pressure is to be measured?
36. We have considered liquids under compression. Can liquids be put under tension? If so, will they tear under sufficient tension as do solids? (See "The Tensile Strength of Liquids," by Robert E. Apfel, *Scientific American,* December 1972, p. 58.)
37. Explain why two glass plates with a thin film of water between them are difficult to separate by a direct pull but can easily be separated by sliding.
38. Give a molecular explanation of why surface tension decreases with increasing temperature.
39. Soap films are much more stable than films of water. Why? (Consider how surface tension reacts to stretching.)
40. Explain why a soap film collapses if a small hole appears in it.
41. Explain these observations: (*a*) water forms globules on a greasy plate but not on a clean one; (*b*) small bubbles on the surface of water cluster together.
42. If soap reduces the surface tension of water, why do we blow soap bubbles instead of water bubbles?
43. Some water beetles can walk on water. Estimate the maximum weight such an insect can have and still be supported in this way.
44. What is the source of the energy that allows a fluid in a capillary (e.g., a thin, hollow, glass tube) to rise?
45. What does it mean to say that certain liquids can exert a small negative pressure?

EXERCISES

15-1 Fluids and Solids

15-2 Pressure and Density

1. Find the pressure increase in the fluid in a syringe when a nurse applies a force of 42.3 N to the syringe's piston of diameter 1.12 cm.
2. Three liquids that will not mix are poured into a cylindrical container. The amounts and densities of the liquids are 0.50 L, 2.6 g/cm^3; 0.25 L, 1.0 g/cm^3; and 0.40 L, 0.80 g/cm^3 (L = liter). Find the total force on the bottom of the container. (Ignore the contribution due to the atmosphere.) Does it matter whether the fluids mix?
3. An office window is 3.43 m by 2.08 m. As a result of the passage of a storm, the outside air pressure drops to 0.962 atm, but inside the pressure is held at 1.00 atm. What net force pushes out on the window?
4. A solid copper cube has an edge length of 85.5 cm. How much pressure must be applied to the cube to reduce the edge length to 85.0 cm? The bulk modulus of the copper is 140 GPa.
5. An airtight box having a lid with an area of 12 in.^{2} is partially evacuated. If a force of 108 lb is required to pull the lid off the box, and the outside atmospheric pressure is 15 lb/in.^{2}, what was the pressure in the box?

15-3 Variation of Pressure in a Fluid at Rest

6. The human lungs can operate against a pressure differential of less than 0.050 atm. How far below the water level can a diver, breathing through a snorkel (long tube), swim?

7. Calculate the hydrostatic difference in blood pressure in a person of height 1.83 m between the brain and the foot.

8. Find the total pressure, in pascal, 118 m below the surface of the ocean. The density of seawater is 1.024 g/cm^3 and the atmospheric pressure at sea level is 1.013×10^5 Pa.

9. The sewer outlets of a house constructed on a slope are 8.16 m below street level. If the sewer is 2.08 m below street level, find the minimum pressure differential that must be created by the sewage pump to transfer waste of average density 926 kg/m^3.

10. According to the constant temperature model of the Earth's atmosphere, (*a*) what is the pressure (in atm) at an altitude of 5.00 km, and (*b*) at what altitude is the pressure equal to 0.500 atm? Compare your answers with Fig. 15-5.

11. A simple U-tube contains mercury. When 11.2 cm of water is poured into the right arm, how high does the mercury rise in the left arm from its initial level?

12. A swimming pool has the dimensions 80 ft $\times$ 30 ft $\times$ 8.0 ft. (*a*) When it is filled with water, what is the force (due to the water alone) on the bottom? On the ends? On the sides? (*b*) If you are concerned with whether or not the concrete walls will collapse, is it appropriate to take the atmospheric pressure into account?

13. What would be the height of the atmosphere if the air density (*a*) were constant and (*b*) decreased linearly to zero with height? Assume a sea-level density of 1.21 kg/m^3.

14. Crew members attempt to escape from a damaged submarine 112 m below the surface. How much force must they apply to a pop-out hatch, which is 1.22 m by 0.590 m, to push it out?

15. The surface of contact of two fluids of different densities that are at rest and do not mix is horizontal. Prove this general result (*a*) from the fact that the potential energy of a system must be a minimum in stable equilibrium, (*b*) from the fact that at any two points in a horizontal plane in either fluid the pressures are equal.

16. Two identical cylindrical vessels with their bases at the same level each contain a liquid of density ρ. The area of either base is A, but in one vessel the liquid height is h_1 and in the other h_2. Find the work done by gravity in equalizing the levels when the two vessels are connected.

15-4 Pascal's Principle and Archimedes' Principle

17. The tension in a string holding a solid block below the surface of a liquid (of density greater than the solid) is T_0 when the containing vessel (Fig. 15-18) is at rest. Show that the tension T, when the vessel has an upward vertical acceleration a, is given by $T_0(1 + a/g)$.

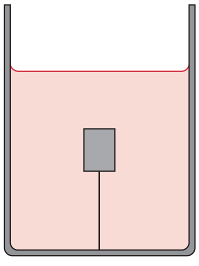


FIGURE 15-18. Exercise 17.

18. (*a*) If the small piston of a hydraulic lever has a diameter of 3.72 cm, and the large piston one of 51.3 cm, what weight on the small piston will support 18.6 kN (e.g., a car) on the large piston? (*b*) Through what distance must the small piston move to raise the car 1.65 m?

19. A boat floating in fresh water displaces 35.6 kN of water. (*a*) What weight of water would this boat displace if it were floating in salt water of density 1024 kg/m^3? (*b*) Would the volume of water displaced change? If so, by how much?

20. A block of wood floats in water with 0.646 of its volume submerged. In oil it has 0.918 of its volume submerged. Find the density of (*a*) the wood and (*b*) the oil.

21. A tin can has a total volume of 1200 cm^3 and a mass of 130 g. How many grams of lead shot could it carry without sinking in water? The density of lead is 11.4 g/cm^3.

22. About one-third of the body of a physicist swimming in the Dead Sea will be above the water line. Assuming that the human body density is 0.98 g/cm^3, find the density of the water in the Dead Sea. Why is it so much greater than 1.0 g/cm^3?

23. Assume the density of brass weights to be 8.0 g/cm^3 and that of air to be 0.0012 g/cm^3. What fractional error arises from neglecting the buoyancy of air in weighing an object of density 3.4 g/cm^3 on a beam balance?

24. An iron casting containing a number of cavities weighs 6130 N in air and 3970 N in water. What is the volume of the cavities in the casting? The density of iron is 7870 kg/m^3.

25. A cubic object of dimensions $L = 0.608$ m on a side and weight $W = 4450$ N in a vacuum is suspended by a wire in an open tank of liquid of density $\rho = 944$ kg/m^3, as in Fig. 15-19. (*a*) Find the total downward force exerted by the liquid and the atmosphere on the top of the object. (*b*) Find the total upward force on the bottom of the object. (*c*) Find the tension in the wire. (*d*) Calculate the buoyant force on the object using Archimedes' principle. What relation exists among all these quantities?

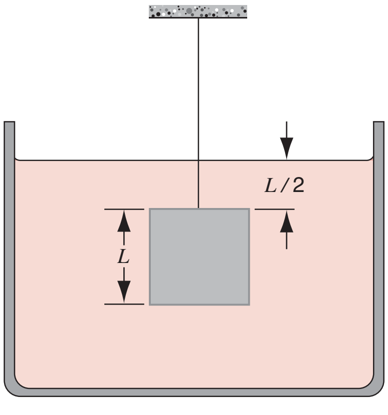


FIGURE 15-19. Exercise 25.

26. A fish maintains its depth in seawater by adjusting the air content of porous bone or air sacs to make its average density the same as that of the water. Suppose that with its air sacs collapsed a fish has a density of 1.08 g/cm^3. To what fraction of its expanded body volume must the fish inflate the air sacs to reduce its average density to that of the water? Assume that the air density is 0.00121 g/cm^3.

27. It has been proposed to move natural gas from the North Sea gas fields in huge dirigibles, using the gas itself to provide lift. Calculate the force required to tether such an airship to the ground for off-loading when it arrives fully loaded with $1.17 \times 10^6\ m^3$ of gas at a density of $0.796\ kg/m^3$. The density of the air is $1.21\ kg/m^3$. (The weight of the airship is negligible by comparison.)

28. The Goodyear blimp *Columbia* (see Fig. 15-20) is cruising slowly at low altitude, filled as usual with helium gas. Its maximum useful payload, including crew and cargo, is 1280 kg. How much more payload could the *Columbia* carry if you replaced the helium with hydrogen? Why not do it? The volume of the helium-filled interior space is $5000\ m^3$. The density of helium gas is $0.160\ kg/m^3$ and the density of hydrogen is $0.0810\ kg/m^3$.

FIGURE 15-20. Exercise 28.

29. Three children each of weight 82.4 lb make a log raft by lashing together logs of diameter 1.05 ft and length 5.80 ft. How many logs will be needed to keep them afloat? Take the density of the wood to be $47.3\ lb/ft^3$.

30. (*a*) What is the minimum area of a block of ice 0.305 m thick floating on water that will hold up an automobile of mass 1120 kg? (*b*) Does it matter where the car is placed on the block of ice? The density of ice is $917\ kg/m^3$.

15-5 Measurement of Pressure

31. A student constructs a water barometer out of a 15-m-long tube. The student attempts to measure the air pressure near sea-level when the temperature is 25°C. Estimate the relative error in pressure caused by neglecting the vapor pressure of water.

32. Estimate the density of the red wine that Pascal used in his 14-m-long barometer. Assume that the wine filled the tube.

33. The pressure at the surface of the planet Venus is 90 atm (i.e., 90 times the pressure at the surface of the Earth). How long would a mercury barometer have to be to measure this pressure? Assume that the mercury is maintained at 0°C.

15-6 Surface Tension

34. How much energy is stored in the surface of a soap bubble 2.1 cm in radius if its surface tension is $4.5 \times 10^{-2}\ N/m$?

35. A thin film of water of thickness 80.0 pm is sandwiched between two glass plates and forms a circular patch of radius 12.0 cm. Calculate the normal force needed to separate the plates if the surface tension of water is 0.072 N/m.

36. Using a soap solution for which the surface tension is 0.025 N/m, a child blows a soap bubble of radius 1.40 cm. How much energy is expended in stretching the soap surface?

PROBLEMS

1. In 1654 Otto von Guericke, Burgermeister of Magdeburg and inventor of the air pump, gave a demonstration before the Imperial Diet in which two teams of horses could not pull apart two evacuated brass hemispheres. (*a*) Show that the force F required to pull apart the hemispheres is $F = \pi R^2 \Delta p$, where R is the (outside) radius of the hemispheres and Δp is the difference in pressure outside and inside the sphere (Fig. 15-21). (*b*) Taking R equal to 0.305 m and the inside pressure as 0.100 atm, what force would the team of horses have had to exert to pull apart the hemispheres? (*c*) Why were two teams of horses used? Would not one team prove the point just as well?

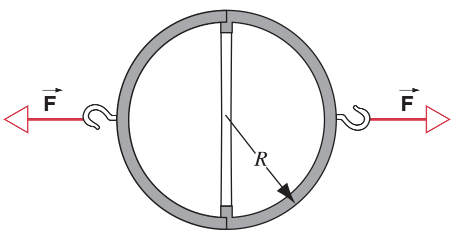


FIGURE 15-21. Problem 1.

2. Figure 15-22 displays the phase diagram of carbon, showing the ranges of temperature and pressure in which carbon will crystallize either as diamond or graphite. What is the minimum depth at which diamonds can form if the local temperature is 1000°C and the subsurface rocks have density 3.1 g/cm^3. Assume that, as in a fluid, the pressure is due to the weight of material lying above.

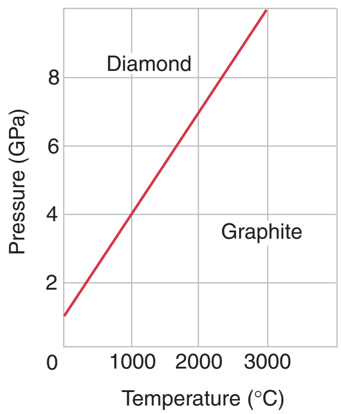


FIGURE 15-22. Problem 2.

3. Water stands at a depth D behind the vertical upstream face of a dam, as shown in Fig. 15-23. Let W be the width of the dam. (*a*) Find the resultant horizontal force exerted on the dam by the gauge pressure of the water and (*b*) the net torque due to the gauge pressure of the water exerted about a line through O parallel to the width of the dam. (*c*) Where is the line of action of the equivalent resultant force?

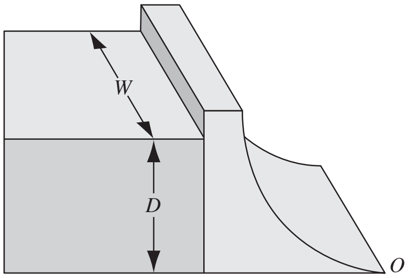


FIGURE 15-23. Problem 3.

4. A cylindrical barrel has a narrow tube fixed to the top, as shown with dimensions in Fig. 15-24. The vessel is filled with water to the top of the tube. Calculate the ratio of the hydrostatic force exerted on the bottom of the barrel to the weight of the water contained inside. Why is the ratio not equal to one? (Ignore the presence of the atmosphere.)

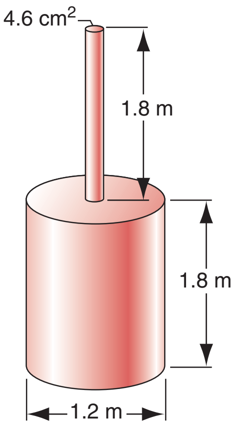


FIGURE 15-24. Problem 4.

5. In analyzing certain geological features of the Earth, it is often appropriate to assume that the pressure at some horizontal level of compensation, deep in the Earth, is the same over a large region and is equal to that exerted by the weight of the overlying material. That is, the pressure on the level of compensation is given by the hydrostatic (fluid) pressure formula. This requires, for example, that mountains have low-density roots; see Fig. 15-25. Consider a mountain 6.00 km high. The continental rocks have a density of 2.90 g/cm^3; beneath the continent is the mantle, with a density of 3.30 g/cm^3. Calculate the depth D of the root. (Hint: Set the pressure at points a and b equal; the depth y of the level of compensation will cancel out.)

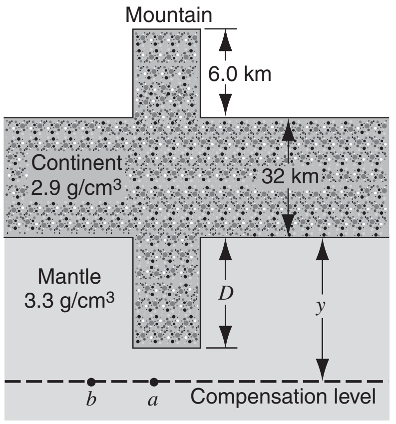


FIGURE 15-25. Problem 5.

6. (*a*) Show that the density ρ of water at a depth y in the ocean is related to the surface density ρ_s by

$$\rho \approx \rho_s[1 + (\rho_s g/B)y],$$

where $B = 2.2$ GPa is the bulk modulus of water. Ignore temperature variations. (*b*) By what fraction does the density at a depth of 4200 m exceed the surface density?

7. (*a*) Show that Eq. 15-13, the variation of pressure with altitude in the atmosphere (temperature assumed to be uniform), can be written in terms of density ρ as

$$\rho = \rho_0 e^{-h/a},$$

where ρ_0 is the density at the ground ($h = 0$). (*b*) Assume that the drag force D due to the air on an object moving at speed v is given by $D = CA\rho v^2$ where C is a constant, A is the frontal cross-sectional area of the object, and ρ is the local air density. Find the altitude at which the drag force on a rocket is a maximum if the rocket is launched vertically and moves with constant upward acceleration a_r.

8. (*a*) Consider a container of fluid subject to a vertical upward acceleration a. Show that the pressure variation with depth in the fluid is given by

$$p = \rho h(g + a),$$

where h is the depth and ρ is the density. (*b*) Show also that if the fluid as a whole undergoes a vertical downward acceleration a, the pressure at depth h is given by

$$p = \rho h(g - a).$$

(*c*) What is the state of affairs in free fall?

9. (*a*) Consider the horizontal acceleration of a mass of liquid in an open tank. Acceleration of this kind causes the liquid surface to drop at the front of the tank and to rise at the rear. Show that the liquid surface slopes at an angle θ with the horizontal, where $\tan\theta = a/g$, a being the horizontal acceleration. (*b*) How does the pressure vary with h, the vertical depth below the surface?

10. Derive the expression for the pressure as a function of the radial distance from the center of a spherical planet of radius R and uniform density ρ.

11. Show that the variation of pressure with altitude for a planetary atmosphere (assuming constant temperature) is

$$p = p_0 e^{k(1/r - 1/R)}$$

where g is taken to vary as $1/r^2$ (with r being the distance from the center of the planet), p_0 is the pressure at the surface, R is the radius of the planet, and k is a constant. Verify that this result reduces to Eq. 15-12 for locations close to the surface.

12. (*a*) A fluid is rotating at constant angular velocity ω about the central vertical axis of a cylindrical container. Show that the variation of pressure in the radial direction is given by

$$\frac{dp}{dr} = \rho\omega^2 r.$$

(*b*) Take $p = p_c$ at the axis of rotation ($r = 0$) and show that the pressure p at any point r is

$$p = p_c + \tfrac{1}{2}\rho\omega^2 r^2.$$

(*c*) Show that the liquid surface is of paraboloidal form (Fig. 15-26); that is, a vertical cross section of the surface is the curve $y = \omega^2 r^2/2g$. (*d*) Show that the variation of pressure with depth is $p = \rho g h$.

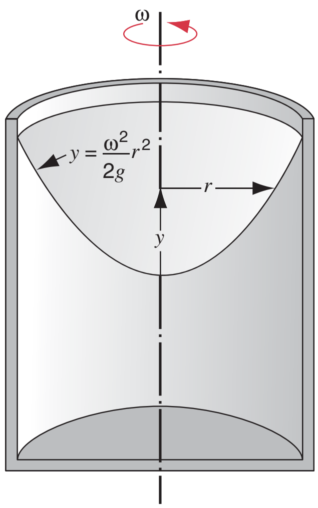


FIGURE 15-26. Problem 12.

13. A hollow spherical iron shell floats almost completely submerged in water; see Fig. 15-27. The outer diameter is 58.7 cm and the density of iron is 7.87 g/cm^3. Find the inner diameter of the shell.

FIGURE 15-27. Problem 13.

14. A block of wood has a mass of 3.67 kg and a density of 594 kg/m^3. It is to be loaded with lead so that it will float in water with 0.883 of its volume immersed. What mass of lead is needed (*a*) if the lead is on top of the wood and (*b*) if the lead is attached below the wood? The density of lead is 1.14×10^4 kg/m^3.

15. An object floating in mercury has one-fourth of its volume submerged. If enough water is added to cover the object, what fraction of its volume will remain immersed in mercury?

16. A car has a total mass of 1820 kg. The volume of air space in the passenger compartment is 4.87 m^3. The volume of the motor and front wheels is 0.750 m^3, and the volume of the rear wheels, gas tank, and luggage is 0.810 m^3. Water cannot enter these areas. The car is parked on a hill; the hand-brake cable snaps and the car rolls down the hill into a lake; see Fig. 15-28. (*a*) At first, no water enters the passenger compartment. How much of the car, in cubic meters, is below the water surface with the car floating as shown? (*b*) As water slowly enters, the car sinks. How many cubic meters of water are in the car as it disappears below the water surface? (The car remains horizontal, owing to a heavy load in the trunk.)

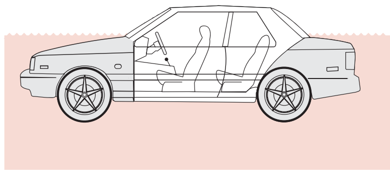


FIGURE 15-28. Problem 16.

17. You place a glass beaker, partially filled with water, in a sink (Fig. 15-29). It has a mass of 390 g and an interior volume of 500 cm^3. You now start to fill the sink with water and you find, by experiment, that if the beaker is less than half full, it will float; but if it is more than half full, it remains on the bottom of the sink as the water rises to its rim. What is the density of the material of which the beaker is made?

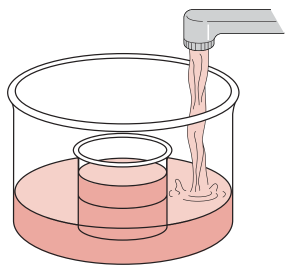


FIGURE 15-29. Problem 17.

18. The surface tension of liquid ^{4}He is 0.35 mN/m and the liquid density is 145 kg/m^3. Estimate (*a*) the number of atoms/m^2 in the surface and (*b*) the energy per bond, in eV, in the liquid at this temperature. The mass of a helium atom is 6.64×10^{-27} kg. Picture each atom as a cube and assume that each atom interacts only with its four nearest neighbors.

19. Show that the pressure difference between the inside and the outside of a bubble of radius r is $4\gamma/r$, where γ is the surface tension of the liquid from which the bubble is blown.

20. A soap bubble floating in a vacuum bell jar has a radius of 1.0 mm when the pressure inside the jar is 100 kPa. The pump is turned on for a short time and the soap bubble is seen to expand to a radius of 1.0 cm. Find the new pressure inside the bell jar. Assume that pV is a constant, where p is the pressure of the gas inside the bubble and V is the volume of the bubble.

21. A solid glass rod of radius $r = 1.3$ cm is placed inside and coaxial with a glass cylinder of internal radius $R = 1.7$ cm. Their bottom ends are aligned and placed in contact with, and perpendicular to, the surface of an open tank of water (see Fig. 15-30). To what height will the water rise in the region between the rod and the cylinder? Assume that the angle of contact is 0° and use 72.8 mN/m for the surface tension of water.

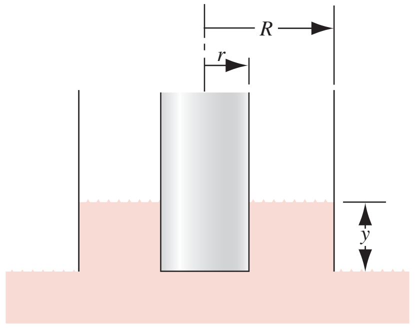


FIGURE 15-30. Problem 21.

22. A soap bubble in air has a radius of 3.20 cm. It is then blown up to a radius of 5.80 cm. Use 26.0 mN/m for the (constant) surface tension of the bubble. (*a*) What is the initial pressure difference across the bubble film? (*b*) Find the pressure difference across the film at the larger size. (*c*) How much work was done on the atmosphere in blowing up the bubble? (*d*) How much work was done in stretching the bubble surface?

COMPUTER PROBLEM

1. (*a*) Show that the equations that govern the pressure as a function of the radial distance from the center of a spherical gaseous planet, in which the density is proportional to the pressure ($\rho = kp$), are $dp/dr = -(Gm/r^2)kp$ and $dm/dr = 4\pi r^2 kp$, where m is the mass contained within the sphere of radius r. (*b*) Numerically integrate these coupled equations outward from the point r_0, where $r_0 = 10^3$ m, $p_0 = 2 \times 10^{16}$ Pa, $m_0 = 7 \times 10^{14}$ kg. Take the constant k to be 8×10^{-12} s^2/m^2. Generate a graph of pressure against radial distance. (*c*) At what distance is the pressure less than one atmosphere?

CHAPTER 16

FLUID DYNAMICS

We now turn from fluid statics to the dynamics of fluids in motion. We use familiar concepts to analyze fluid dynamics, including Newton's laws of motion and the conservation of energy. In this chapter we apply these principles to fluids, which are described using variables such as pressure and density that we introduced in Chapter 15.

We begin with a simplified model of fluid flow, in which we ignore dissipative forces. This approach is similar to our previous study of particle dynamics, in which we at first ignored dissipative (frictional) forces. An advantage of this approach is that it permits an analysis in terms of conservation of mechanical energy, as we did in Chapter 12 in the case of particles. Later in this chapter we give a brief description of the interesting and unusual results that occur in real fluids when dissipative forces, called viscous forces, *are taken into account.*

16-1 GENERAL CONCEPTS OF FLUID FLOW

One way of describing the motion of a fluid is to divide the fluid into infinitesimal volume elements, which we may call *fluid particles,* and to follow the motion of each particle. If we knew the forces acting on each fluid particle, we could then solve for the positions and velocities of each particle as functions of the time. This procedure, which is a direct generalization of particle mechanics, was first developed by Joseph Louis Lagrange (1736–1813). Because the number of fluid particles is generally very large, using this method is a formidable task.

There is a different treatment, developed by Leonhard Euler (1707–1783), that is more convenient for most purposes. In it we give up the attempt to specify the history of each fluid particle and instead specify the density and the velocity of the fluid at each point in space at each instant of time. This is the method we shall use. We describe the motion of the fluid by specifying the density $\rho(x, y, z, t)$ and the velocity $\vec{v}(x, y, z, t)$ at the point x, y, z at the time t. We thus focus our attention on what is happening at a particular point in space at a particular time, rather than on what is happening to a particular fluid particle. Any quantity used in describing the state of the fluid—for example, the pressure p—will have a definite value at each point in space and at each instant of time. Although this description of fluid motion focuses attention on a point in space rather than on a fluid particle, we cannot avoid following the fluid particles themselves, at least for short time intervals dt. After all, the laws of mechanics apply to particles and not to points in space.

We first consider some general characteristics of fluid flow.

1. *Fluid flow can be steady or nonsteady.* We describe the flow in terms of the values of such variables as pressure, density, and flow velocity at every point of the fluid. If these variables are constant in time, the flow is said to be *steady.* The values of these variables will generally change from one point to another, but they do not change with time at any particular point. This condition can often be achieved at low flow speeds; a gently flowing stream is an example. In nonsteady flow, as in a tidal bore, the velocities $\vec{v}$ are functions of the time. In the case of *turbulent* flow, such as rapids or a waterfall, the velocities vary erratically from point to point as well as from time to time.

2. *Fluid flow can be compressible or incompressible.* If the density ρ of a fluid is a constant, independent of x, y, z, and t, its flow is called *incompressible flow.* Liquids can usually be considered as flowing incompressibly. However, even for a highly compressible gas the variation in density may be insignificant, and for practical purposes we can consider its flow to be incompressible. For example, in flight at speeds much lower than the speed of sound in air (described by subsonic aerodynamics), the flow of the air over the wings is nearly incompressible.

3. *Fluid flow can be viscous or nonviscous.* Viscosity in fluid motion is the analogue of friction in the motion of solids—kinetic energy associated with the fluid flow can be transformed into internal energy by *viscous* forces. The greater the viscosity, the greater the external force or pressure that must be applied to maintain the flow; under similar conditions, honey and motor oil are more viscous than water and air. The viscosity of fluids depends on the temperature; motor oils, for example, are rated not only according to their viscosity but also by its variation with temperature. Although viscosity is present in all fluid flow, in some cases its effects (like those of friction in the mechanics of solids) may be negligible, in which case we can regard the flow as being *nonviscous.*

4. *Fluid flow can be rotational or irrotational.* Imagine a tiny bit of matter, such as a small insect, that is carried along by a flowing stream. If the particle, as it moves with the stream, does not rotate about an axis through its center of mass, the flow is *irrotational;* otherwise it is *rotational.* An element of fluid may move in a circular path and still experience irrotational flow—for example, the vortex formed when water drains from a bathtub. A mechanical analogy can be found in the motion of a Ferris wheel; even though the wheel rotates, the passengers do not rotate about their centers of mass.

We will mostly consider the motion of ideal fluids, which can be regarded as steady, incompressible, nonviscous, and irrotational. This greatly simplifies the mathematics of fluid dynamics and is often a good approximation to the behavior of real fluids. However, as in the case of friction in the dynamics of solids, in each application we must take care to examine the validity of these assumptions and the consequences if they are found not to be valid.

16-2 STREAMLINES AND THE EQUATION OF CONTINUITY

In steady flow the velocity $\vec{v}$ at a given point is constant in time. Consider the point P (Fig. 16-1) within the fluid. Since $\vec{v}$ at P does not change in time in steady flow, every fluid particle arriving at P will pass on with the same speed in the same direction. The motion of every particle passing through P thus follows the same path, called a *streamline.* Every fluid particle that passes through P will later pass through points further along the streamline, such as Q and R in Fig. 16-1. Moreover, every fluid particle that passes through R must have previously passed through P and Q.

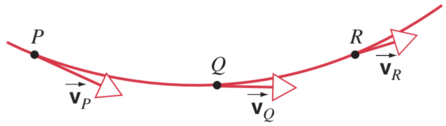


FIGURE 16-1. In steady flow, a fluid particle passing through P traces out a streamline, later passing through downstream points Q and R. Any other particle passing through P must follow this same path.

The magnitude of the velocity vector of the fluid particle will, in general, change as it moves along the streamline. The direction of the velocity vector at any point along the streamline is always tangent to the streamline.

No two streamlines can cross one another, for if they did, an oncoming fluid particle could go either one way or the other, and the flow could not be steady. In steady flow the pattern of streamlines does not change with time. Figure 16-2 shows an example of streamlines in fluid flow.

In principle we can draw a streamline through every point in the fluid. Assuming steady flow, we select a finite number of streamlines to form a bundle, like the streamline pattern of Fig. 16-3. This tubular region is called a *tube of flow.* Because the boundary of such a tube consists of streamlines, no fluid can cross the boundaries of a tube of flow, and the tube behaves somewhat like a pipe of the same shape. The fluid that enters at one end must leave at the other. The tube must be narrow enough that we can take the velocity of the fluid to be nearly constant over the cross section of the tube.

Let us consider in detail the flow of fluid through the tube of flow shown in Fig. 16-3. Fluid enters at P where the cross-sectional area is A_1 and leaves at Q where the area is A_2. Let the speed be v_1 for fluid particles at P and v_2 for fluid particles at Q. In the time interval δt a fluid element travels approximately the distance $v\ \delta t$. Then the fluid that

FIGURE 16-2. In a wind tunnel, the aerodynamics of a car can be evaluated by examining the streamlines of the air flow, here made visible by the addition of smoke to the air.

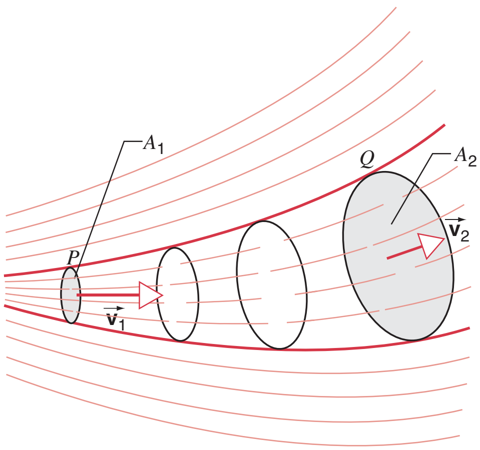


FIGURE 16-3. A bundle of streamlines forms a tube of flow, which has cross-sectional area A_1 at P and A_2 at Q.

crosses A_1 in the time interval δt has a volume δV_1 of approximately $A_1 v_1\ \delta t$. If its density at that location is ρ_1, then the mass of fluid $\delta m_1 (= \rho_1\ \delta V_1)$ crossing A_1 is approximately

$$\delta m_1 = \rho_1 A_1 v_1\ \delta t.$$

The *mass flux,* defined as the mass of fluid per unit time passing through any cross section, is thus approximately $\delta m_1/\delta t = \rho_1 A_1 v_1$ at P. We must take δt small enough so that in this time interval neither v nor A varies appreciably over the distance the fluid travels. In the limit as $\delta t \to 0$, we obtain the precise result:

$$\text{mass flux at } P = \rho_1 A_1 v_1,$$

and, from a similar analysis,

$$\text{mass flux at } Q = \rho_2 A_2 v_2,$$

where ρ_2, A_2, and v_2 represent, respectively, the density, cross-sectional area, and flow speed at Q.

We have assumed that fluid enters the tube only at P and leaves only at Q. That is, between P and Q there are no other "sources" where fluid can enter the tube or "sinks" where it can leave. Furthermore, the flow is steady, so the density of fluid between P and Q does not change with time (even though it may change from place to place). Under these conditions, fluid mass enters the tube at P at the same rate as it leaves at Q. Thus the mass flux at P must equal that at Q:

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2, \qquad (16\text{-}1)$$

or, in more general terms referring to any location in the tube of flow,

$$\rho A v = \text{constant}. \qquad (16\text{-}2)$$

This result expresses the *law of conservation of mass* in fluid dynamics.

If the fluid is incompressible, as we shall assume from now on, then $\rho_1 = \rho_2$, and Eq. 16-1 takes on the simpler form

$$A_1 v_1 = A_2 v_2, \qquad (16\text{-}3)$$

or, defining R to be the *volume flow rate* (or *volume flux*) Av,

$$R = Av = \text{constant}. \qquad (16\text{-}4)$$

The SI units of R are m^3/s. Note that Eq. 16-3 predicts that in steady incompressible flow the speed of flow varies inversely with the cross-sectional area, being larger in narrower parts of the tube.

Equations 16-2 and 16-4 are examples of mathematical relationships known as *equations of continuity,* which are in effect conservation laws for mass. The equation of continuity states that if within any volume element of space (*not* volume of fluid) there are no *sources* (where additional matter is introduced into the flow) or *sinks* (where matter is removed from the flow), then the total mass within that volume element must remain constant. In more general cases, if sources or sinks are present, the equation of continuity gives the mathematical representation of the very reasonable assertion that the rate of outflow or inflow of matter is equal to the rate at which the mass contained within the volume element is changing. Equations of continuity are common in physics and appear in any subject in which a flow is involved. For example, there is an equation of continuity for electric charge that is a conservation law for charge rather than mass.

The constancy of the volume flux along a tube of flow gives an important graphical interpretation to the streamlines, as shown in Fig. 16-4. In a narrow part of the tube, the streamlines must crowd closer together than in a wide part. Hence, as the distance between streamlines decreases, the fluid speed must increase. Therefore we conclude that widely spaced streamlines indicate regions of relatively low speed, and closely spaced streamlines indicate regions of relatively high speed.

We can obtain another interesting result by applying Newton's second law of motion to the flow of fluid between P and Q (Fig. 16-4). A fluid particle at P with speed v_1 must be accelerated in the forward direction in acquiring the higher forward speed v_2 at Q. This acceleration can come about only from a force exerted in the direction PQ, and (if there is no other external force, for instance, gravity) the force must arise from a change in pressure within the fluid. To provide this force, the pressure must be greater at P than at Q. Therefore, in the absence of other sources of acceleration, regions of higher fluid velocity must be associated with lower fluid pressure. We make this preliminary conclusion about fluid dynamics more rigorous in the next section.

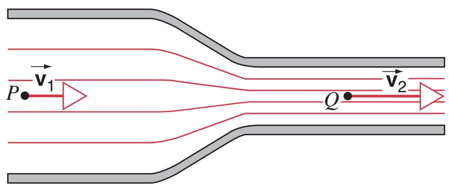


FIGURE 16-4. As the area of a horizontal tube narrows, the flow velocity must increase. If no other force acts on the fluid, the pressure at P must be greater than the pressure at Q, so that a force acts in the direction PQ to provide the necessary acceleration.

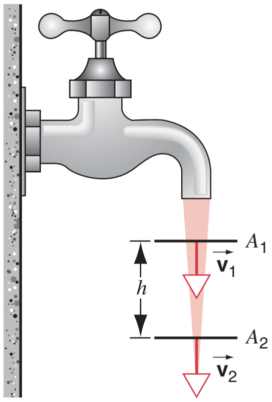


FIGURE 16-5. Sample Problem 16-1. As water falls from a tap, its speed increases. Because the flow rate must be the same at all cross sections, the stream must become narrower as it falls. (Effects associated with surface tension are neglected.)

SAMPLE PROBLEM 16-1. Figure 16-5 shows how the stream of water emerging from a faucet "necks down" as it falls. The cross-sectional area A_1 is 1.2 cm^2 and that of A_2 is 0.35 cm^2. The two levels are separated by a vertical distance h (= 45 mm). At what rate does water flow from the tap?

Solution From the equality of the volume flux (Eq. 16-3) we have

$$A_1v_1 = A_2v_2,$$

where v_1 and v_2 are the water speeds at the corresponding levels. Applying conservation of energy in the form of Eq. 12-15 to an element of fluid of mass m, we have $K_2 + U_2 = K_1 + U_1$, or $\frac{1}{2}mv_2^2 + 0 = \frac{1}{2}mv_1^2 + mgh$. Thus

$$v_2^2 = v_1^2 + 2gh.$$

Eliminating v_2 between these two equations and solving for v_1, we obtain

$$v_1 = \sqrt{\frac{2ghA_2^2}{A_1^2 - A_2^2}} = \sqrt{\frac{(2)(9.8\text{ m/s}^2)(0.045\text{ m})(0.35\text{ cm}^2)^2}{(1.2\text{ cm}^2)^2 - (0.35\text{ cm}^2)^2}}$$

$$= 0.286\text{ m/s} = 28.6\text{ cm/s}.$$

The volume flow rate R is then

$$R = A_1v_1 = (1.2\text{ cm}^2)(28.6\text{ cm/s}) = 34\text{ cm}^3/\text{s}.$$

At this rate, it would take about 3 s to fill a 100-mL beaker.

16-3 BERNOULLI'S EQUATION*

As an ideal fluid flows through a pipe or a tube of flow, its condition may change in several ways: (1) the cross-sectional area of the pipe may change; (2) the inlet and outlet of the pipe may be at different elevations; (3) the inlet and outlet pressures may be different. We have already used the equation of continuity (Eq. 16-4) to relate changes in area to changes in velocity. Because both a pressure difference and a difference in elevation can accelerate an element of fluid as it travels through a tube, we also expect changes in velocity to be related to pressure and elevation. Thus changes of types 1, 2, and 3 are not independent of one another. In this section we will consider the connections between those changes.

Our analysis is based on applying conservation of energy, $\Delta K + \Delta U = W_{ext}$ (Eq. 13-1), to the fluid flow, which we assume to be ideal, as described in Section 16-1 (steady, incompressible, nonviscous, and irrotational). Figure 16-6 represents a pipe or tube of flow for this fluid. At the inlet (left-hand end), the pipe has a uniform cross-sectional area A_1 and is at an elevation y_1 above some reference level. The pipe gradually widens and rises; at the outlet (right-hand end), the pipe has a uniform cross-sectional area A_2 and an elevation y_2. As the area changes, the speed of the fluid changes from v_1 at the inlet to v_2 at the outlet.

We will apply conservation of energy to the system consisting of the entire shaded fluid between the inlet and outlet of the pipe at a particular instant of time. A pressure p_1 (exerted, perhaps, by additional fluid in the pipe to the left of our system) acts on this fluid at the inlet end and results in a force $F_1 = p_1A_1$ that pushes the system to the right. At the outlet end, there is a pressure p_2 (due, perhaps, to additional fluid in the pipe to the right of our system) that results in a force $F_2 = p_2A_2$ that acts on our system to the left.

Under the net influence of the two pressure forces and gravity, the system moves to the right. Figure 16-6a shows

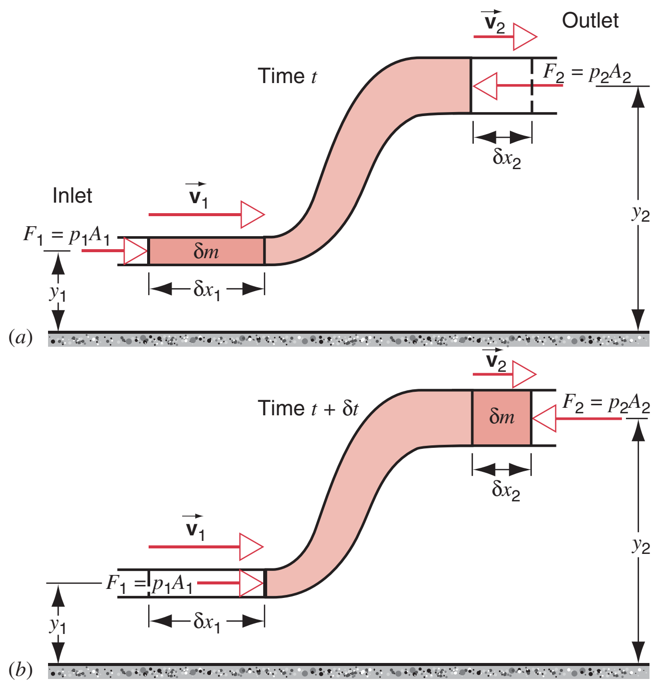


FIGURE 16-6. Fluid flows through a pipe at a steady rate. During the interval from (a) to (b), the net effect of the flow is the transfer of the element of fluid indicated by the dark shading from the inlet end of the tube to the outlet end.

*Daniel Bernoulli (1700–1782) was a Swiss mathematician, physician, and physicist who made significant discoveries not only in fluid dynamics but also in astronomy, physiology, and geology. His father and uncle were also famous for their contributions to mathematics.

the system at time t, and Fig. 16-6b shows the same system an instant of time δt later. In this short time interval, the left-hand end of the system has moved a distance δx_1 to the right, while the right-hand end has moved a distance δx_2. These distances are different, because the area of the pipe has changed and the fluid is incompressible.

The overall effect of the motion of the system is the same as if we simply moved the dark shaded element of fluid, of mass δm, from the inlet end of the pipe to the outlet end. The remainder of the light-shaded fluid is unaffected by the flow.

There are three contributions to the net external work on our system: (1) At the inlet end, the pressure force does work $W_1 = F_1 \delta x_1 = p_1 A_1 \delta x_1$ (a positive quantity, since the force and displacement are in the same direction). (2) At the outlet end, the pressure force does work $W_2 = -F_2 \delta x_2 = -p_2 A_2 \delta x_2$ (a negative quantity, because the force and displacement are in opposite directions). (3) The work done by gravity, as the dark-shaded fluid element δm moves through the vertical displacement $y_2 - y_1$, is $W_g = -\delta m\, g(y_2 - y_1)$, which is a negative quantity because the force and displacement are in opposite directions. In Eq. 13-1 for conservation of energy, ΔU represents the potential energy due to conservative forces that act among objects within the system. Here we assume that no such forces act within the fluid, so $\Delta U = 0$.

The net external work done on the system is then

$$\begin{aligned} W_{\text{ext}} &= W_1 + W_2 + W_g \\ &= p_1 A_1 \delta x_1 + (-p_2 A_2 \delta x_2) + [-\delta m\, g(y_2 - y_2)]. \end{aligned} \qquad (16\text{-}5)$$

The volume δV of the small dark-shaded fluid element can be written as $\delta V = A_1 \delta x_1$ and also as $\delta V = A_2 \delta x_2$, since we have assumed the fluid to be incompressible. In terms of the (uniform and constant) fluid density ρ, the volume element is $\delta V = \delta m/\rho$. Making these substitutions in Eq. 16-5, we obtain

$$W_{\text{ext}} = (p_1 - p_2)(\delta m/\rho) - \delta m\, g(y_2 - y_1). \qquad (16\text{-}6)$$

The change in kinetic energy of the dark-shaded fluid element is

$$\Delta K = \tfrac{1}{2}\delta m\, v_2^2 - \tfrac{1}{2}\delta m\, v_1^2. \qquad (16\text{-}7)$$

Finally, applying conservation of energy, in the form $\Delta K + \Delta U = W_{\text{ext}}$ with $\Delta U = 0$, we obtain

$$\tfrac{1}{2}\delta m\, v_2^2 - \tfrac{1}{2}\delta m\, v_1^2 = (p_1 - p_2)(\delta m/\rho) - (\delta m)\, g(y_2 - y_1),$$

which, after rearranging the terms and canceling the common factor of δm, becomes

$$p_1 + \tfrac{1}{2}\rho v_1^2 + \rho g y_1 = p_2 + \tfrac{1}{2}\rho v_2^2 + \rho g y_2. \qquad (16\text{-}8)$$

Since the subscripts 1 and 2 refer to two arbitrary locations in the pipe, we can drop the subscripts and write

$$p + \tfrac{1}{2}\rho v^2 + \rho g y = \text{constant}. \qquad (16\text{-}9)$$

Equation 16-9 is *Bernoulli's equation* for steady, incompressible, nonviscous, and irrotational flow. Strictly speaking, the points to which we apply this equation should be along the same streamline. However, if the flow is irrotational, the value of the constant is the same for *all* streamlines in the tube of flow, so Bernoulli's equation can be applied to any two points in the flow.

We have obtained two very powerful tools for analyzing the flow of fluids: the equation of continuity (Eq. 16-4), which is in effect a statement of conservation of mass, and Bernoulli's equation (Eq. 16-9), which is a statement of conservation of energy. In the next section, we apply these two equations to the analysis of several practical problems.

For now, we consider several features of Bernoulli's equation:

1. *Static pressure.* Just as statics for particles is a special case of particle dynamics, so fluid statics is also a special case of fluid dynamics. To illustrate this, consider Eq. 16-8 if the fluid is not flowing ($v_1 = v_2 = 0$):

$$p_1 + \rho g y_1 = p_2 + \rho g y_2$$

or

$$p_2 - p_1 = -\rho g(y_2 - y_1),$$

which is identical with Eq. 15-8. The pressure $p + \rho g y$, which would be present in the fluid even if $v = 0$, is called the *static pressure.*

2. *Dynamic pressure.* Suppose the fluid flows horizontally, so that gravity need not be considered. In this case Eq. 16-8 becomes, with $y_1 = y_2$,

$$p_1 + \tfrac{1}{2}\rho v_1^2 = p_2 + \tfrac{1}{2}\rho v_2^2. \qquad (16\text{-}10)$$

Equation 16-10 suggests that where the speed is large, the pressure must be small, and the converse. This verifies the discussion at the end of Section 16-2 concerning Fig. 16-4. The quantity $\frac{1}{2}\rho v^2$ (which you should check to have the dimension of pressure) is called the *dynamic pressure.*

3. *Compressible, viscous flow.* If the fluid is compressible, its internal potential energy ΔU_{int} can change as the molecules become closer together or further apart. If the flow is viscous, the internal kinetic energy ΔK_{int} of the molecules in the fluid can change in the same way that frictional forces between objects can increase their internal kinetic energies. The complete analysis of fluids using conservation of energy should therefore include an internal energy term $\Delta E_{\text{int}} = \Delta U_{\text{int}} + \Delta K_{\text{int}}$ that might account for both of these effects: $\Delta K + \Delta E_{\text{int}} = W_{\text{ext}}$, which is our generalized statement of conservation of energy (Eq. 13-2, with $\Delta U = 0$). If necessary, Bernoulli's equation could be modified to account for these other energy transformations. However, if the flow is approximately incompressible and nonviscous, these corrections are negligibly small.

Sample Problem 16-2. A storage tower of height $h = 32$ m and diameter $D = 3.0$ m supplies water to a house (Fig. 16-7). A horizontal pipe at the base of the tower has a diameter $d = 2.54$ cm (= 1 in., typical of the supply pipes for many homes in the United States). To satisfy the needs of the home, the supply pipe must be able to deliver water at a rate $R = 0.0025\ \text{m}^3/\text{s}$ (about $\frac{2}{3}$ of a gallon per second). (*a*) If water were flowing at the maximum rate, what would be the pressure in the horizontal pipe? (*b*) A smaller pipe, of diameter $d' = 1.27$ cm (= 0.5 in.), supplies the third floor of the house, a distance of 7.2 m above the ground level. What are the flow speed and water pressure in this pipe? Neglect the viscosity of the water.

Solution (*a*) We apply Bernoulli's equation along the streamline *ABC* shown in Fig. 16-7. At points *A* and *B* we have

$$p_A + \tfrac{1}{2}\rho v_A^2 + \rho g y_A = p_B + \tfrac{1}{2}\rho v_B^2 + \rho g y_B.$$

At *A*, the pressure is that of the atmosphere, p_0. With $y_A = h$ and $y_B = 0$, we obtain, for the unknown pressure,

$$p_B = p_0 + \rho g h + \tfrac{1}{2}\rho(v_A^2 - v_B^2).$$

We can find v_A and v_B from the equation of continuity (Eq. 16-4), which gives

$$v_A A_A = v_B A_B = R,$$

where *R* is the constant volume flow rate. Thus

$$v_A = \frac{R}{A_A} = \frac{0.0025\ \text{m}^3/\text{s}}{\pi(1.5\ \text{m})^2} = 3.5 \times 10^{-4}\ \text{m/s},$$

$$v_B = \frac{R}{A_B} = \frac{0.0025\ \text{m}^3/\text{s}}{\pi(0.0127\ \text{m})^2} = 4.9\ \text{m/s}.$$

Note that the term $\frac{1}{2}\rho v_A^2$ in the expression for p_B is negligible compared with the term $\frac{1}{2}\rho v_B^2$. That is, the flow speed at the top of the tank is quite small, owing to its large cross-sectional area.

We can now solve for the pressure in the pipe:

$$\begin{aligned} p_B &= p_0 + \rho g h - \tfrac{1}{2}\rho v_B^2 \\ &= 1.01 \times 10^5\ \text{Pa} + (1.0 \times 10^3\ \text{kg/m}^3)(9.8\ \text{m/s}^2)(32\ \text{m}) \\ &\quad - \tfrac{1}{2}(1.0 \times 10^3\ \text{kg/m}^3)(4.9\ \text{m/s})^2 \\ &= 1.01 \times 10^5\ \text{Pa} + 3.14 \times 10^5\ \text{Pa} - 0.12 \times 10^5\ \text{Pa} \\ &= 4.03 \times 10^5\ \text{Pa} \approx 4\ \text{atm}. \end{aligned}$$

If the water in the horizontal pipe were not flowing (that is, if the valve were closed), the *static* pressure at *B* would include only the first two terms above, which give 4.15×10^5 Pa. The pressure when the water is flowing is reduced from this static value by the amount of the dynamic pressure.

(*b*) If the narrower pipe to the third floor is to have the same flow rate *R*, the velocity at *C* must be

$$v_C = \frac{R}{A_C} = \frac{0.0025\ \text{m}^3/\text{s}}{\pi(0.00635\ \text{m})^2} = 19.7\ \text{m/s},$$

or four times the value at *B*. Bernoulli's equation gives

$$p_A + \tfrac{1}{2}\rho v_A^2 + \rho g y_A = p_C + \tfrac{1}{2}\rho v_C^2 + \rho g y_C$$

or

$$\begin{aligned} p_C &= p_0 + \tfrac{1}{2}\rho(v_A^2 - v_C^2) + \rho g(y_A - y_C) \\ &= 1.01 \times 10^5\ \text{Pa} - \tfrac{1}{2}(1.0 \times 10^3\ \text{kg/m}^3)(19.7\ \text{m/s})^2 \\ &\quad + (1.0 \times 10^3\ \text{kg/m}^3)(9.8\ \text{m/s}^2)(32\ \text{m} - 7.2\ \text{m}) \\ &= 1.01 \times 10^5\ \text{Pa} - 1.95 \times 10^5\ \text{Pa} + 2.43 \times 10^5\ \text{Pa} \\ &= 1.49 \times 10^5\ \text{Pa} \approx 1.5\ \text{atm}. \end{aligned}$$

Because of the larger flow velocity through the smaller pipe, the dynamic contribution to the pressure is much larger at *C* than it is at *B*. Both the static and dynamic effects tend to reduce the pressure at this location relative to *B*.

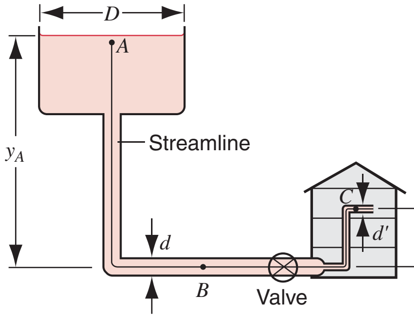


FIGURE 16-7. Sample Problem 16-2.

16-4 APPLICATIONS OF BERNOULLI'S EQUATION AND THE EQUATION OF CONTINUITY

In this section, we consider a number of applications of Bernoulli's equation, which illustrate its use and demonstrate the range of its applicability.

The Venturi Meter

This device (Fig. 16-8) is a gauge to measure the flow speed of a fluid in a pipe. A fluid of density ρ flows through a pipe of cross-sectional area A_1. At the throat the area is reduced to A_2, and a manometer tube is attached, as shown. Let the manometer liquid, such as mercury, have a density ρ'. By applying Bernoulli's equation and the equality of the volume flux at points 1 and 2, you can show (see Problem 8) that the speed of flow at point 1 is

$$v = A_2\sqrt{\frac{2(\rho' - \rho)gh}{\rho(A_1^2 - A_2^2)}}. \tag{16-11}$$

The Pitot Tube

This device (Fig. 16-9) is used to measure the flow speed of a gas. Consider the gas—say, air—flowing with density ρ and velocity $\vec{v}_1$ parallel to the planes of a number of small openings that we collectively label as point 1. The pressure

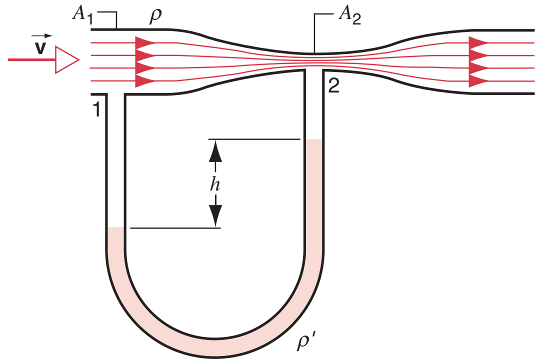


FIGURE 16-8. A Venturi meter, used to measure the speed of flow of a fluid in a pipe.

in the left arm of the manometer, which is connected to these openings, is then the static pressure in the gas stream, p_1. The opening of the right arm of the manometer is at right angles to the stream. The velocity is reduced to zero at 2, and the gas is stagnant at that point. Applying Bernoulli's equation to points 1 and 2, we obtain

$$p_1 + \tfrac{1}{2}\rho v_1^2 = p_2.$$

Substituting the manometer reading $\rho' gh$ for the pressure difference $p_2 - p_1$, we can solve for v_1 to obtain

$$v_1 = \sqrt{\frac{2gh\rho'}{\rho}}. \qquad (16\text{-}12)$$

This device can be calibrated to read v_1 directly. Pitot tubes can be commonly observed protruding from airplane wings; their readings appear as air-speed indicators on the airplane's control panel.

Dynamic Lift

Dynamic lift is the force that acts on a body, such as an airplane wing, a hydrofoil, or a helicopter rotor, by virtue of its motion through a fluid. It is not the same as static lift, which is the buoyant force that acts on a balloon or an iceberg in accord with Archimedes' principle (Section 15-4).

Familiar examples of dynamic lift occur in the flight of a baseball, tennis ball, or golf ball. The dynamic lift, which originates with the rotation of the ball in flight, can cause the ball to curve or to rise or fall relative to the parabolic trajectory that it would follow if no air were present. Because the fluid (in this case air) is somewhat viscous, there is friction as the ball travels, and the ball tends to carry with it a thin layer of fluid called the *boundary layer.* Viewed from the rest frame of a nonrotating ball, the fluid speed drops from its value beyond the boundary layer (equal to the flight speed of the ball) to zero at the surface of the ball.

Figure 16-10*a* shows, in the rest frame of the ball, streamlines for the steady flow of air rushing past a nonrotating ball, at speeds low enough so that turbulence does not occur. Figure 16-10*b* shows streamlines for the air carried around by a rapidly rotating ball. Without viscosity and the boundary layer, the spinning ball could not carry air around in this way and this circulation (as it is called) would not exist. Golf balls are systematically roughened by means of dimples to increase this circulation and the dynamic lift that results from it. Baseballs are sometimes artificially (and illegally) roughened by pitchers for the same reason.

Figure 16-10*c* shows the effect of combining the circulation (resulting from the rotation of the ball) and the steady flow (resulting from the translation of the ball through the air). For the case shown, the two velocities add above the ball and subtract below. From the spacing of the resultant streamlines, we see that the velocity of air below the ball is less than that above the ball. From Bernoulli's equation, the pressure of air below the ball must then be greater than that above, so the ball experiences a dynamic lift force.

A pitched baseball curves for essentially the same reason. For example, if Fig. 16-10 represents a top view of the spinning ball as it travels toward the batter, the "lift" acts in a sidewise direction to move the ball horizontally toward or away from the batter, as in the case of a curveball. If Fig. 16-10 represents a side view, the ball is thrown with backspin, as in the case of a fastball. The lift acts upward, causing the ball to rise relative to its parabolic trajectory.

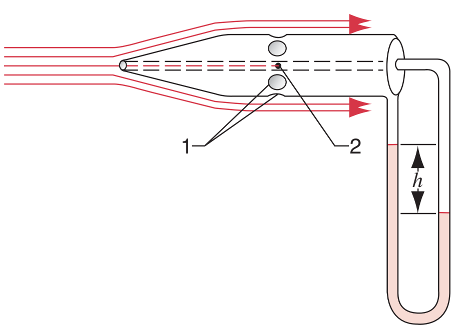


Figure 16-9. A Pitot tube, which is used to measure the flow speed of a gas.

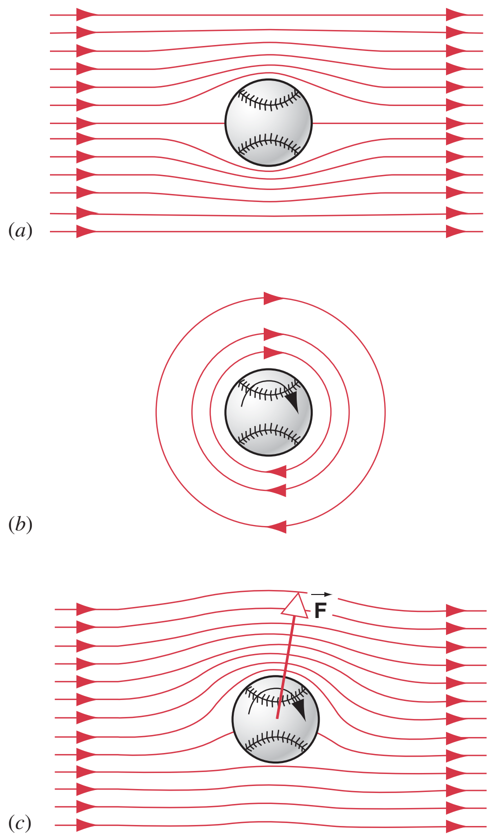


Figure 16-10. (*a*) Streamline flow around a nonrotating ball. (*b*) The circulation of air around a rotating ball, which results from the boundary layer. (*c*) The combined effects of both motions. From Bernoulli's equation, we see that a dynamic lift acts upward on the ball. The fluid exerts on the ball a net force $\vec{F}$ having a component transverse to the fluid flow (lift) and a component parallel to the fluid flow (drag).

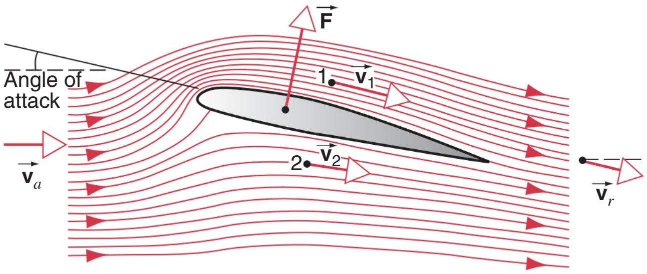


FIGURE 16-11. Streamlines around an airfoil or airplane wing. The velocity $\vec{\mathbf{v}}_a$ of the approaching air is horizontal, while the air receding from the airfoil has a velocity $\vec{\mathbf{v}}_r$ with a downward component. The airfoil has thus exerted a downward force on the air, and by Newton's third law the air must therefore have exerted an upward force on the airfoil. This upward force is represented by the lift $\vec{\mathbf{F}}$.

The lift acting on an airplane wing has a similar explanation. Figure 16-11 shows the streamlines about an airfoil (or wing cross section) attached to an aircraft. Let us choose the aircraft as our frame of reference, as in a wind tunnel experiment, and let us assume that the air is moving past the wing from left to right. Note the similarities between Figs. 16-11 and 16-10*c*. (In fact, the explanation of the lift on an airplane wing involves a circulation similar to Fig. 16-10*b*.)

Figure 16-11 shows that the streamlines are closer together above the wing than they are below it; thus the airflow speed is greater above the wing, and the pressure is smaller. This pressure difference between the upper and lower surfaces of the wing contributes to the lift. However, as shown in Fig. 16-11, there is another way to account for the lift: the wing is designed so that the air flowing past it is deflected downward, and the lift is in effect the Newton's third law reaction force to the downward force exerted by the wing on the air. Either of these explanations may be used for the lift on an airplane.*

Thrust on a Rocket

As our final example, let us compute the thrust on a rocket produced by the escape of its exhaust gases. Consider a chamber (Fig. 16-12) of cross-sectional area A filled with a gas of density ρ at a pressure p. Let there be a small orifice of cross-sectional area A_0 at the bottom of the chamber. We wish to find the speed v_0 with which the gas escapes through the orifice.

Let us write Bernoulli's equation (Eq. 16-8) as

$$p - p_0 = \rho g(y_0 - y) + \tfrac{1}{2}\rho(v_0^2 - v^2),$$

where p_0 represents atmospheric pressure just outside the orifice. For a gas the density is so small that we can neglect the variation in pressure with height in a chamber, which gives

$$p - p_0 = \tfrac{1}{2}\rho(v_0^2 - v^2)$$

or

$$v_0^2 = \frac{2(p - p_0)}{\rho} + v^2, \qquad (16\text{-}13)$$

where v is the speed of the flowing gas inside the chamber and v_0 is the speed of the gas through the orifice. Although a gas is compressible and the flow may become turbulent, we can treat the flow as steady and incompressible for pressure and exhaust speeds that are not too high.

Now let us assume continuity of mass flow (in a rocket engine this is achieved when the mass of escaping gas equals the mass of gas created by burning the fuel), so that (for an assumed constant density)

$$Av = A_0 v_0.$$

If the orifice is very small so that $A_0 \ll A$; then $v_0 \gg v$, and we can neglect v^2 compared to v_0^2 in Eq. 16-13. Hence the exhaust speed is

$$v_0 = \sqrt{\frac{2(p - p_0)}{\rho}}. \qquad (16\text{-}14)$$

If our chamber is the exhaust chamber of a rocket, the thrust on the rocket (Section 7-6) is $v_0\, dM/dt$. The mass of gas flowing out in time dt is $dM = \rho A_0 v_0\, dt$, so that

$$v_0 \frac{dM}{dt} = v_0(\rho A_0 v_0) = \rho A_0 v_0^2,$$

and using Eq. 16-14 the thrust is

$$v_0 \frac{dM}{dt} = 2A_0(p - p_0). \qquad (16\text{-}15)$$

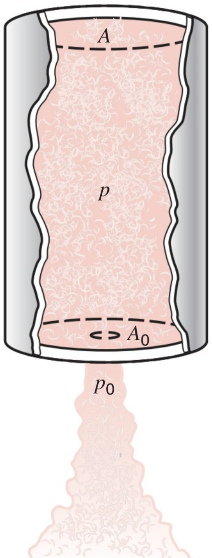


FIGURE 16-12. Fluid streaming out of a chamber, which might represent the exhaust chamber of a rocket.

*For more information on how airplanes fly, see "The Science of Flight," by Peter P. Wegener, *American Scientist,* May–June 1986, p. 268. Also see "Bernoulli's Law and Aerodynamic Lifting Force," by Klaus Weltner, *The Physics Teacher,* February 1990, p. 84. Several articles about the effect of dynamic lift in various sports are collected in *The Physics of Sports,* edited by Angelo Armenti, Jr. (American Institute of Physics, 1992). Dynamic lift can also be used to provide a horizontal force that propels a ship; see "The Flettner Ship," by Albert Einstein, in *Essays in Science* (Philosophical Library, 1955), p. 92.

16-5 FIELDS OF FLOW (Optional)

In Section 14-8 we showed how to represent the space near masses by a gravitational field. Associated with each point in the field is a vector $\vec{g}$, the gravitational force per unit mass at that point. We can make a graphical representation of the field by drawing lines in the direction of the field whose spacing suggests the strength of the field (large spacing where the field is small and small spacing where the field is large).

In fluid dynamics, we can make a similar graphical representation of the moving fluid in terms of a vector field, but in this case the field lines indicate the flow velocity $\vec{v}$ at a point. For a steady flow, the velocity at each point in space has a constant magnitude and direction, so the pattern of velocity field lines does not change with time.

We can represent the velocity field by drawing streamlines, which represent the direction of the fluid velocity at each point. The magnitude of the velocity is represented by the spacing of the streamlines: through each unit area perpendicular to the flow, we draw a number of streamlines that is proportional to the velocity at that point. That is, where the streamlines are close together the velocity is large (many lines per unit area), and where they are far apart the velocity is small (few lines per unit area).

We will illustrate the use of these field diagrams to represent the velocity field by several two-dimensional examples. In each of these, the flow velocity is the same everywhere on a line perpendicular to the plane of the drawing at each point.

Figure 16-13 shows a uniform field of flow, such as might occur in the steady, nonviscous flow of a liquid through a pipe with smooth interior walls. Here the streamlines are parallel lines, and the equal spacing suggests that the flow velocity has the same magnitude everywhere.

In Fig. 16-14 we show the field for *uniform rotational flow,* such as might be produced by rotating a bucket of water on a turntable (see Problem 12, Chapter 15). Here v is proportional to r, because the angular velocity ω is constant. In Fig. 16-15 we draw the field of flow of a *vortex,* such as might be obtained by pulling the plug in a bathtub full of water. In this case v is proportional to $1/r$, because the angular momentum $L = mvr$ is constant, and the flow is irrotational (see Problem 11). Note that both uniform rotation and vortex motion are represented by circular streamlines but are entirely different kinds of flow. Obviously, the shapes of the streamlines give only limited information; their spacing is needed too.

FIGURE 16-13. Streamlines (horizontal lines) for a homogeneous nonviscous field of flow.

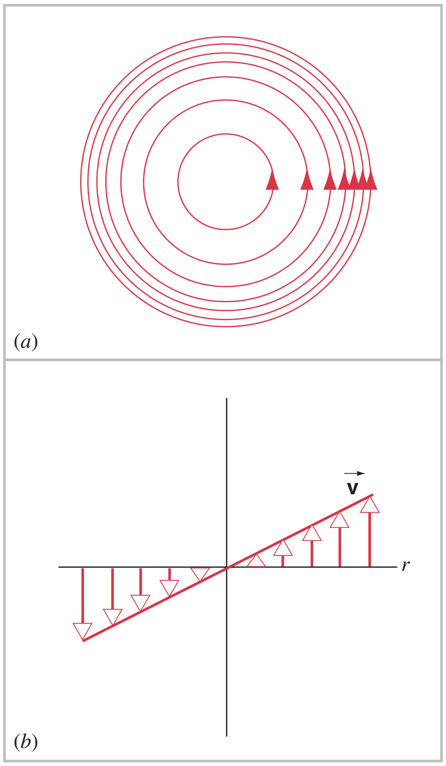


FIGURE 16-14. (*a*) A uniform rotational field of flow. (*b*) Increase of fluid velocity from the center, indicated in part (*a*) by the decreasing spacing of the field lines.

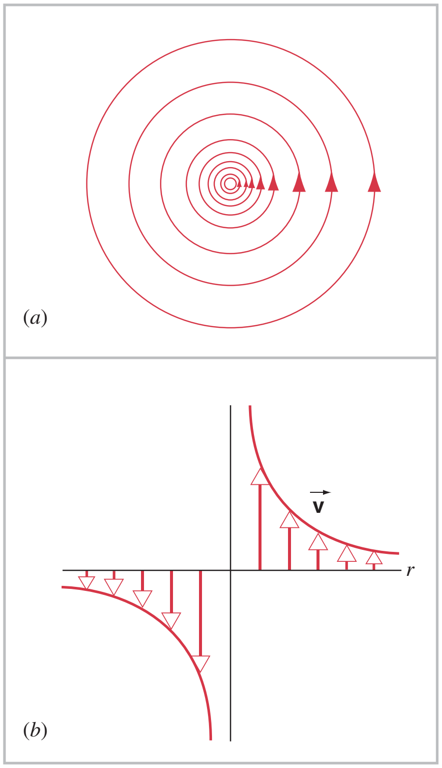


FIGURE 16-15. (*a*) Field of flow of a vortex. (*b*) Variation of fluid velocity from the center.

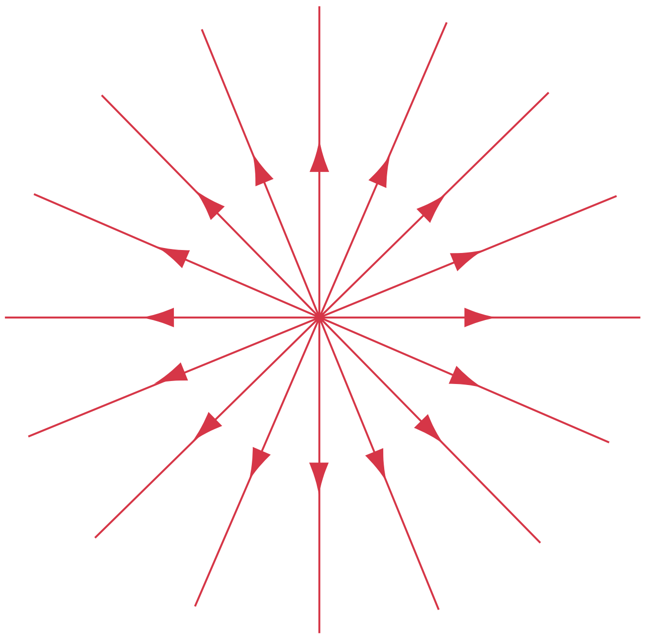


FIGURE 16-16. Flow from a linear source.

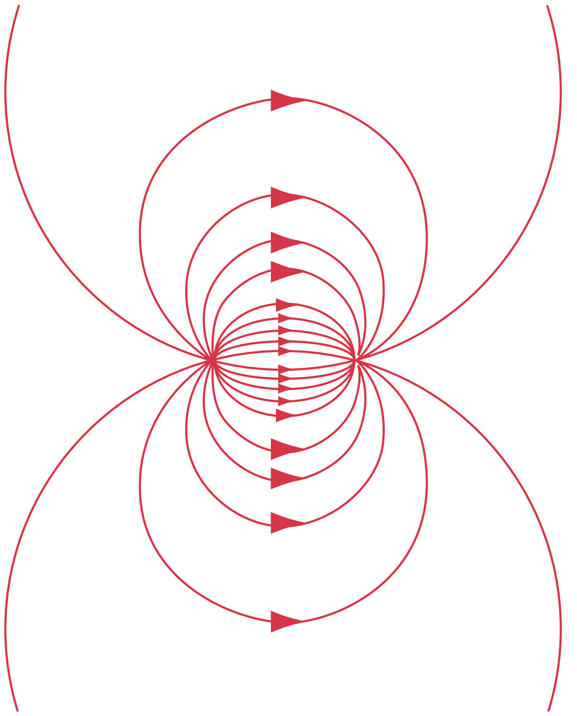


FIGURE 16-17. Linear dipole flow. The source is on the left and the sink is on the right.

Figure 16-16 represents the field of flow for a source. All streamlines are directed radially outward. The source is a line through the center perpendicular to the paper. The field of flow around a linear sink is the same as that of a source except for the sign of the flow, which is directed radially inward.

For a linear source and linear sink that have the same flow rate and are slightly separated, we obtain the combined field called linear dipole flow, shown in Fig. 16-17.

As we shall see later the electrostatic field, the magnetic field, and the field of flow for an electric current are also vector fields. In this connection, the homogeneous field (Fig. 16-13) corresponds to the electric field of a plane capacitor, the source field or sink field (Fig. 16-16) corresponds to the electric field of a cylindrical capacitor or straight wire of positive or negative charge, respectively, and the linear dipole field (Fig. 16-17) corresponds to the electric field of two oppositely charged wires.

The homogeneous field of Fig. 16-13 also represents the magnetic field inside a solenoid. The vortex field of Fig. 16-15 represents the magnetic field around a straight current-carrying wire. This last is an example of a field that is rotational (about the vortex axis).

Because of these analogies between fluid and electromagnetic fields, we can often determine a field of flow, which is difficult to calculate by present mathematical methods, by experimental measurements on appropriate electrical devices.

As we have seen throughout this chapter, the basic field ideas and conservation principles find application in many areas of physics. We shall encounter them many times again. ■

16-6 VISCOSITY, TURBULENCE, AND CHAOTIC FLOW (Optional)

Viscosity in fluid flow is similar to friction in the motion of solid bodies. When we slide one solid body over another, we must supply an external force $\vec{\mathbf{F}}$ to oppose the frictional force $\vec{\mathbf{f}}$ if we wish to keep the body in motion at constant velocity. In the case of fluid motion, we can consider a fluid between two parallel plates, illustrated in Fig. 16-18. A force $\vec{\mathbf{F}}$ is applied to the upper plate, so that it is in motion at constant velocity $\vec{\mathbf{v}}$ relative to the lower plate, which we assume to be at rest. The force $\vec{\mathbf{F}}$ opposes the viscous drag on the upper plate to keep its velocity constant.

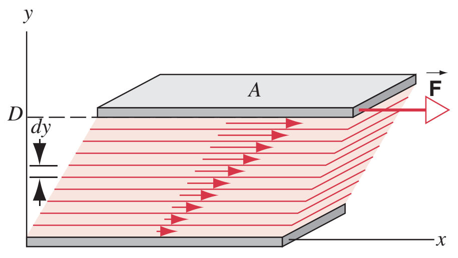


FIGURE 16-18. A viscous fluid fills the space between two flat plates separated by a distance D. The lower plate is at rest and the upper plate is pulled to the right with a constant force $\vec{\mathbf{F}}$. The velocity of each layer of fluid decreases uniformly from the upper plate to the lower plate.

The fluid can be imagined to be divided into layers parallel to the plates. Viscosity acts not only between the fluid and the upper plate, but between each layer of fluid and the adjacent layers. The speed of each layer differs by an amount dv from the one below it. Fluid flow in which the speed varies layer-by-layer is called *laminar* flow. For this discussion, we assume that the top fluid layer has the same speed v as the top plate and the bottom fluid layer has the same speed as the bottom plate—namely, zero.

The external force F that must be exerted to set up a laminar flow in the fluid is found to be directly proportional to the area A of the plate—the larger the plate, the more viscous drag and the greater the force that must be exerted. The force is also directly proportional to the change in velocity dv that occurs across each layer of thickness dy; that is, if the plates are very close together it takes a large force to maintain a particular velocity at the top plate. (Imagine a laminar flow with only two layers, one at rest in contact with the bottom plate and one moving at velocity v with the top plate. The energy dissipated depends on the relative velocity between the layers, which is large. If there are more layers, the relative velocities between the layers are smaller, which means less energy dissipation and less force necessary to maintain the motion.) We therefore have $F \propto A\, dv/dy$ or, introducing a constant of proportionality η,

$$F = \eta A \frac{dv}{dy}. \qquad (16\text{-}16)$$

The constant of proportionality η (Greek letter eta) is called the *coefficient of viscosity* (or simply the *viscosity*) of the fluid. The SI unit of viscosity is the $\mathrm{N \cdot s/m^2}$. The equivalent cgs unit is the $\mathrm{dyne \cdot s/cm^2}$, which is called the *poise.** Comparing the units shows that 1 poise $= 0.1\ \mathrm{N \cdot s/m^2}$.

The viscosity η is large for fluids that offer a large resistance to flow and small for fluids that flow easily. Table 16-1 shows some viscosities for various fluids. Note that η depends on the temperature of the fluid.

In the case of the rectangular plates shown in Fig. 16-18, the *velocity gradient dv/dy* is a constant for all layers, because the velocity increases by the same amount dv across each layer of thickness dy. With $dv/dy = v/D$, where D is the spacing between the plates, Eq. 16-16 becomes

$$F = \eta A \frac{v}{D}. \qquad (16\text{-}17)$$

*The unit is named for the French physician Jean-Louis-Marie Poiseuille (1799–1869), who first investigated the flow of viscous fluids through tubes as an aid in understanding the circulation of blood.

Table 16-1 Viscosities of Selected Fluids

Fluid	$\eta\ (\mathrm{N \cdot s/m^2})$
Glycerine (20°C)	1.5
Motor oil[a] (0°C)	0.11
Motor oil[a] (20°C)	0.03
Blood (37°C)	4.0×10^{-3}
Water (20°C)	1.0×10^{-3}
Water (90°C)	0.32×10^{-3}
Gasoline (20°C)	2.9×10^{-4}
Air (20°C)	1.8×10^{-5}
CO_2 (20°C)	1.5×10^{-5}

[a] Medium weight (S.A.E. 30).

A practical application of viscosity occurs in the fluid flow through cylindrical pipes. The flow is again laminar, but in this case the layers of fluid are thin-walled cylinders of varying radii. The flow velocity varies with the radius; its maximum value occurs on the axis and its minimum value, which we assume to be zero, at the walls (Fig. 16-19). The variation of the velocity with location across the pipe is not linear. Assuming once again that the layer next to the walls is at rest, the speed in the cylindrical shell of radius r can be shown to be (see Problem 14)

$$v = \frac{\Delta p}{4\eta L}(R^2 - r^2), \qquad (16\text{-}18)$$

which depends on the pressure difference Δp across the length L of the pipe. The speed at the center of the pipe is

$$v_0 = \frac{\Delta p\, R^2}{4\eta L}. \qquad (16\text{-}19)$$

By considering the flow through each thin cylindrical shell, we can show (see Problem 15) that the total mass flux dm/dt (fluid mass flowing through the pipe per unit time) is

$$\frac{dm}{dt} = \frac{\rho\pi R^4\, \Delta p}{8\eta L}. \qquad (16\text{-}20)$$

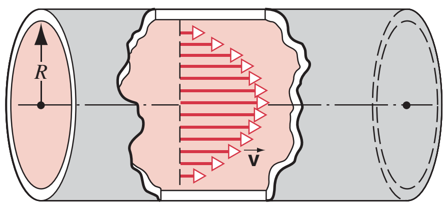


Figure 16-19. Fluid flows through a cylindrical pipe of radius R. The variation in the velocity from the wall to the center is shown.

This result is known as *Poiseuille's law.* Knowing the coefficient of viscosity of the fluid, we can then determine the pressure difference that must be provided by an external agent (a pump, perhaps) to sustain a given mass flux through the pipe. Equivalently, if we force fluid through a pipe with a known pressure difference, measuring the mass flux permits us to determine the coefficient of viscosity of the fluid.

Viscosity in liquids originates with the intermolecular cohesive forces. As the temperature increases, the coefficient of viscosity of a liquid decreases, because the increasing kinetic energy of the molecules weakens the effect of the intermolecular forces. In gases, on the other hand, the viscosity increases with increasing temperature, because the molecules themselves can migrate between the layers. At higher temperatures, there is more molecular motion and therefore more mixing. However, note that in a pipe there are always more slow molecules near the walls than there are fast molecules near the central axis, so more mixing always means more slow molecules moving toward the axis and impeding the motion of the faster-moving molecules. (The effect is similar to that of slow-moving traffic merging into the fast lane of a highway.)

FIGURE 16-20. Rising smoke is at first in laminar flow, but the flow soon becomes turbulent.

SAMPLE PROBLEM 16-3. Castor oil, which has a density of 0.96×10^3 kg/m^3 at room temperature, is forced through a pipe of circular cross section by a pump that maintains a gauge pressure of 950 Pa. The pipe has a diameter of 2.6 cm and a length of 65 cm. The castor oil emerging from the free end of the pipe at atmospheric pressure is collected. After 90 s, a total of 1.23 kg has been collected. What is the coefficient of viscosity of the castor oil at this temperature?

Solution The mass flux is

$$\frac{dm}{dt} = \frac{1.23\text{ kg}}{90\text{ s}} = 0.0137\text{ kg/s}.$$

The coefficient of viscosity can now be found directly from Eq. 16-20 if we first solve for η, which gives

$$\eta = \frac{\rho\pi R^4\,\Delta p}{8(dm/dt)L} = \frac{(0.96 \times 10^3\text{ kg/m}^3)\pi(0.013\text{ m})^4(950\text{ Pa})}{8(0.0137\text{ kg/s})(0.65\text{ m})}$$
$$= 1.15\text{ N}\cdot\text{s/m}^2.$$

Heavy oils typically have viscosities in this range.

Turbulence

After rising a short distance, the smooth column of smoke from a cigarette breaks up into an irregular and seemingly random pattern (Fig. 16-20). In a similar fashion, a stream of fluid flowing past an obstacle breaks up into eddies and vortices (Fig. 16-21), which give the flow irregular velocity components transverse to the flow direction. An example of this case is the flapping of a flag in a breeze—if the flow of air were laminar, the flag would occupy a fixed position along streamlines, but the flagpole breaks the flow into an irregular pattern similar to Fig. 16-21, which causes the transverse flapping motion of the flag. These are examples of *turbulent* fluid flow. Other examples include the wakes left in water by moving ships and in air by

FIGURE 16-21. Fluid flowing left to right past a cylindrical obstacle clearly goes from laminar to turbulent. Note the eddies and vortices that form downstream from the obstacle.

moving cars and airplanes. The sounds produced by whistling and by wind instruments result from the turbulent flow of air.

In a viscous fluid, the flow at low speed can be described as laminar, which suggests layers sliding smoothly over one another. When the flow speed is sufficiently large, the motion becomes disordered and irregular; this is turbulent flow. An analogy from mechanics is a block that is pushed across a rough surface. If the frictional force is small, the block will slide across the surface if the applied force F is at least as great as the frictional force f. If the frictional force were greater, the applied force F must also be greater, eventually becoming great enough that it tips the block over. The tipping of the block is analogous to the transition from laminar to turbulent flow.

We can determine the critical speed at which the flow becomes turbulent through a dimensional analysis. We let v_c represent the critical speed, which we take to be an average over the pipe, because, as Fig. 16-19 suggests, the speed varies over the cross section of the pipe. We expect this critical speed to depend on the viscosity η and density ρ of the fluid and the diameter D of the pipe. Using our standard technique of dimensional analysis (Section 1-7), we proceed as follows:

$$v_c \propto \eta^a \rho^b D^c$$

$$[v_c] = [\eta^a][\rho^b][D^c]$$

$$\mathrm{LT}^{-1} = (\mathrm{ML}^{-1}\mathrm{T}^{-1})^a(\mathrm{ML}^{-3})^b(\mathrm{L})^c,$$

where the dimensions of viscosity have been obtained from its units of $\mathrm{N \cdot s/m^2}$. Solving, we obtain

$$a = 1, \qquad b = -1, \qquad c = -1.$$

Thus the critical speed can be written

$$v_c \propto \frac{\eta}{\rho D},$$

or, introducing a constant of proportionality R,

$$v_c = R\frac{\eta}{\rho D}. \tag{16-21}$$

The dimensionless constant R is called the *Reynolds number*. Solving Eq. 16-21 for R, we can write the Reynolds number for *any* flow speed v as

$$R = \frac{\rho D v}{\eta}. \tag{16-22}$$

In this interpretation, the Reynolds number can be used to characterize *any* flow, and we can determine by experiment the value of the Reynolds number at which the flow becomes turbulent.

For cylindrical pipes, the Reynolds number corresponding to the critical speed is about 2000. Thus for water flowing through a pipe of diameter 2 cm (a typical household garden hose, for example), the critical speed is, using Eq. 16-21,

$$v_c = 2000\frac{1 \times 10^{-3}\ \mathrm{N \cdot s/m^2}}{(10^3\ \mathrm{kg/m^3})(0.02\ \mathrm{m})} = 0.1\ \mathrm{m/s} = 10\ \mathrm{cm/s}.$$

This is quite a low speed, which suggests that the flow of water is turbulent in ordinary household plumbing. (The flow speed from a typical household tap is about 1 m/s.)

Note from Eq. 16-21 that the critical flow speed increases with the viscosity. That is, the greater the viscous friction exerted by the surrounding fluid, the more likely the flow will be steady.

Chaotic Flow

The geometry of Fig. 16-18 is not particularly convenient for measuring viscosity. Figure 16-22 shows a more convenient arrangement. The space between the coaxial cylinders is filled with the fluid whose viscosity is to be determined. The inner cylinder is made to rotate, while the outer cylinder is held fixed. From the torque necessary to keep the inner cylinder rotating at constant angular speed, the viscosity of the fluid can be determined.

For small rotational speeds, the flow in Fig. 16-22 will be steady and laminar. As the rotational speed of the inner cylinder is increased, the flow eventually becomes turbulent. We can observe that the transition from laminar to turbulent flow takes place in an orderly fashion. Figure 16-23 shows two intermediate stages. The fluid first forms toroidal vortices (somewhat like a stack of doughnuts) and then shows a pattern of waves of definite frequency that becomes superimposed on the vortices. As the rotational speed continues to increase, waves appear with new frequencies. We can imagine the turbulent flow to be the

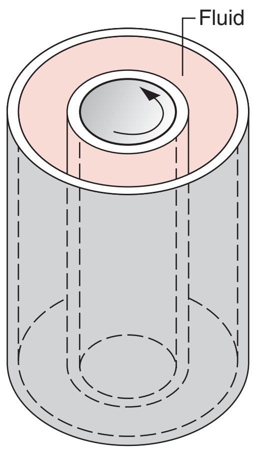
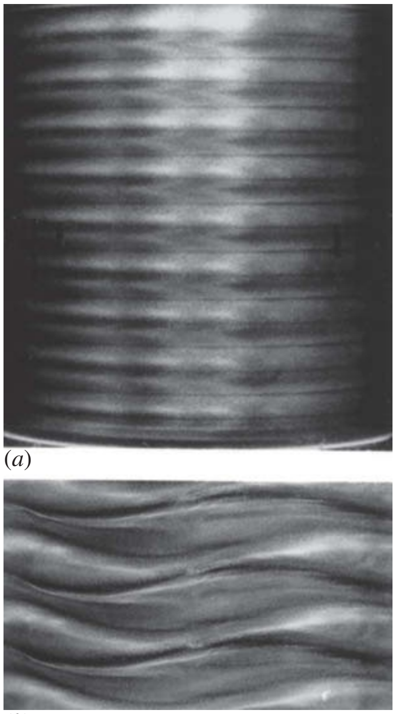


Figure 16-22. Experimental apparatus to measure fluid viscosities. The fluid is placed between the two cylinders, the outer cylinder being fixed and the inner cylinder rotating with angular velocity ω. The torque needed to turn the inner cylinder at this angular velocity is determined by the viscosity of the fluid.



(a)

(b)

FIGURE 16-23. When the fluid speed in the apparatus of Fig. 16-22 exceeds the critical velocity, the flow becomes unstable and breaks up into (*a*) toroidal vortices and then (*b*) waves superimposed on the vortices.

extension of this motion to include so many frequency components that the motion appears to become completely disordered and confused (somewhat like electronic noise). There may be an underlying periodic structure, but it is too complex to follow.

Chaos theory (see Section 5-7) takes a different approach in explaining the onset of turbulence. The turbulent motion resulting from chaos theory is truly *nonperiodic,* not simply the combination of a large number of periodic motions. There is a critical distinction between these two cases. If the transition from laminar to turbulent flow takes place through a succession of increasingly complex, but always periodic motions, then two particles of fluid that in the laminar flow are moving similarly will remain in closely related states of motion throughout the transition into turbulent flow. However, if the intermediate condition reaches a point where the motion becomes chaotic, then the motion loses its predictability, and the two particles can be found in the turbulent flow in very different states of motion. Chaos theory, which is applicable to a wide variety of physical systems, provides an alternative theoretical basis for understanding complex systems such as the turbulent motion of fluids.

MULTIPLE CHOICE

16-1 General Concepts of Fluid Flow

16-2 Streamlines and the Equation of Continuity

1. The mass flux of a fluid flowing into one side of a container is 3.0 kg/s; the mass flux flowing out the other side of the container is 2.0 kg/s. Assuming the container is completely filled with the fluid and that there are no other ways for the fluid to get in or out, one can conclude that

(A) the entrance point has a larger cross-section than the exit point.
(B) the magnitude of the entrance velocity is larger than the magnitude of the exit velocity.
(C) the density of the fluid inside the container must be increasing.
(D) the fluid is incompressible.

2. A long straight pipe of circular cross-section has a radius that varies along the length of the pipe. There is a steady flow in the pipe, with no sources or sinks. At one point P_1 in the pipe the radius is r_1 and the mass flux through P_1 is a constant Q_1. Further along the pipe is a point P_2 where the radius is $r_2 = r_1/3$.

(*a*) The mass flux through P_2 is measured to be Q_2, where Q_2/Q_1 is

(A) 9. (B) 3. (C) 1. (D) 1/9.
(E) dependent on the fluid densities at P_1 and P_2.

(*b*) The ratio of flow speeds, v_2/v_1, is

(A) 9. (B) 3. (C) 1. (D) 1/9.
(E) dependent on the fluid densities at P_1 and P_2.

3. A steady stream of water falls straight down from a pipe. Assume the flow is incompressible. At a distance d_1 beneath the pipe the speed of the falling water is 1.0 m/s. At a distance d_2 beneath the pipe the speed of the falling water is 2.0 m/s. What is the ratio of the cross section of the flow at height d_1 to the cross section at height d_2?

(A) 4 : 1 (B) 2 : 1 (C) 1 : 2 (D) 1 : 4

16-3 Bernoulli's Equation

4. A square box of wine has a small spout located in one of the bottom corners. When the box is full and sitting on a level surface, completely opening the spout results in a flow of wine with a speed of v_0 (see Fig. 16-24*a*).

(*a*) The box is now half empty and still sitting on a level surface. When the spout is completely opened the wine will flow out with a velocity of

(A) v_0. (B) $v_0/2$. (C) $v_0/\sqrt{2}$. (D) $v_0/\sqrt[4]{2}$.

(*b*) The box is still half empty, but now someone tilts it at 45° so that the spout is at the lowest point (Fig. 16-24*b*). When the spout is completely opened the wine will flow out with a speed of

(A) v_0. (B) $v_0/2$. (C) $v_0/\sqrt{2}$. (D) $v_0/\sqrt[4]{2}$.

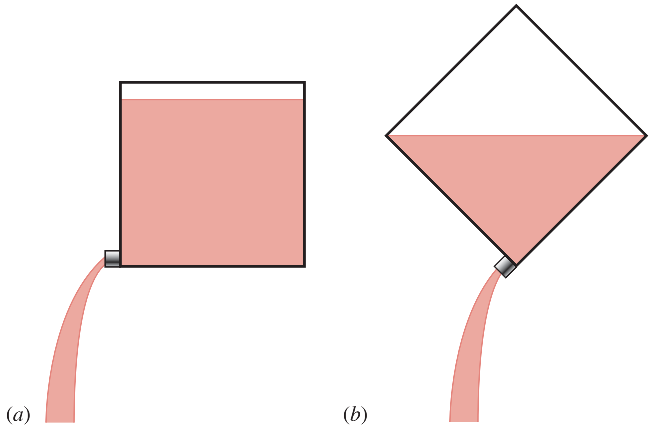


FIGURE 16-24. Multiple-choice question 4.

5. A steady stream of water falls straight down from a pipe. Assume the flow is incompressible; the flow is similar to Fig. 16-5. How does the pressure in the water vary with height in the falling stream?
   (A) The pressure in the water is higher at lower points in the stream.
   (B) The pressure in the water is lower at lower points in the stream.
   (C) The pressure in the water is the same at all points in the stream.

6. An incompressible fluid flows through a horizontal pipe. At one point in the pipe the pressure in the fluid is p_1 and the fluid speed is v_1. Further down the pipe the pressure is p_2 and the fluid speed is $2v_1$. What can be concluded about p_1 and p_2?
   (A) $p_1 = 4p_2$. (B) $p_1 = 3p_2$. (C) $p_1 = 2p_2$.
   (D) Only that $p_1 > p_2$.

7. An incompressible fluid flows through a horizontal pipe. At one point in the pipe the pressure in the fluid is p_1. Further down the pipe the pressure is $p_2 > p_1$. What can be concluded about the cross-sectional areas of the pipe A_1 at point 1 and A_2 at point 2?
   (A) $A_1 > A_2$. (B) $A_1 < A_2$.
   (C) Nothing can be concluded about the relationship between A_1 and A_2.

16-4 Applications of Bernoulli's Equation and the Equation of Continuity

16-5 Fields of Flow

16-6 Viscosity, Turbulence, and Chaotic Flow

8. A certain pump is able to maintain a pressure difference per unit length in a cylindrical pipe of radius R_1 and deliver a mass flux Q_0. It is desired to replace the single pipe with two smaller cylindrical pipes each of radius R_2. The pump will maintain the original pressure difference per unit length in each pipe, and the total mass flux through the two pipes remains equal to Q_0. What is the ratio R_1/R_2?
   (A) 2 (B) $\sqrt{2}$ (C) $\sqrt[4]{2}$ (D) 4

QUESTIONS

1. Briefly describe what is meant by each of the following and illustrate with an example: (*a*) steady fluid flow; (*b*) nonsteady fluid flow; (*c*) rotational fluid flow; (*d*) irrotational fluid flow; (*e*) compressible fluid flow; (*f*) incompressible fluid flow; (*g*) viscous fluid flow; (*h*) nonviscous fluid flow.

2. In steady flow, the velocity vector at any point is constant. Can there then be accelerated motion of the fluid particles? Explain.

3. Describe the forces acting on an element of fluid as it flows through a pipe of nonuniform cross section.

4. What effects, if any, would surface tension have on the solution to Sample Problem 16-1?

5. Explain the pressure variations in your blood as it circulates through your body.

6. Explain how a physician can measure your blood pressure.

7. In a lecture demonstration, a Ping-Pong ball is kept in midair by a vertical jet of air. Is the equilibrium stable, unstable, or neutral? Explain.

8. The height of the liquid in the standpipes of Fig. 16-25 indicates that the pressure drops along the channel, even though the channel has a uniform cross section and the flowing liquid is incompressible. Explain.

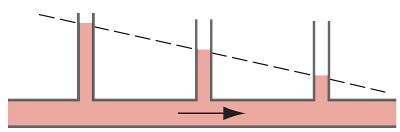


FIGURE 16-25. Question 8.

9. Explain why a taller chimney creates a better draft for taking the smoke out of a fireplace. Why doesn't the smoke pour into the room containing the fireplace?

10. (*a*) Explain how a baseball pitcher can make the baseball curve to his right or left. Can we justify applying Bernoulli's equation to such a spinning baseball? (See "Bernoulli and Newton in Fluid Mechanics," by Norman F. Smith, *The Physics Teacher,* November 1972, p. 451.) (*b*) Why is it easier to throw a curve with a tennis ball than with a baseball?

11. Not only a ball with a rough surface but also a smooth ball can be made to curve when thrown, but these balls will curve in opposite directions. Why? (See "Effect of Spin and Speed on the Curve of a Baseball and the Magnus Effect for Smooth Spheres," by Lyman J. Briggs, *American Journal of Physics,* November 1959, p. 589.)

12. Two rowboats moving parallel to one another in the same direction are pulled toward one another. Two automobiles moving parallel are also pulled together. Explain such phenomena on the basis of Bernoulli's equation.

13. In building "skyscrapers," what forces produced by the movement of air must be counteracted? How is this done? (See "The Wind Bracing of Buildings," by Carl W. Condit, *Scientific American,* February 1974, p. 92.)

14. Explain the action of a parachute in retarding free fall using Bernoulli's equation.

15. Why does a stream of water from a faucet become narrower as it falls?

16. Can you explain why water flows in a continuous stream down a vertical pipe, whereas it breaks into drops when falling freely?
17. How does the flush toilet work? Really. (See "Flushed with Pride: The Story of Thomas Crapper," by W. Reybum, Prentice-Hall, 1969.)
18. Sometimes people remove letters from envelopes by cutting a sliver from a narrow end, holding the envelope firmly, and blowing toward it. Explain, using Bernoulli's equation, why this procedure is successful.
19. On takeoff would it be better for an airplane to move into the wind or with the wind? On landing?
20. Explain how the difference in pressure between the lower and upper surfaces of an airplane wing depends on the altitude of the moving plane.
21. The accumulation of ice on an airplane wing may greatly reduce its lift. Explain. (The weight of the ice is not the issue here.)
22. How is an airplane able to fly upside down?
23. "The characteristic banana-like shape of most returning boomerangs has hardly anything to do with their ability to return. . . . The essential thing is the cross section of the arms, which should be more convex on one side than on the other, like the wing profile of an airplane." (From "The Aerodynamics of Boomerangs," by Felix Hess, *Scientific American,* November 1968, p. 124.) Explain.
24. What powers the flight of soaring birds? (See "The Soaring Flight of Birds," by C. D. Cone, Jr., *Scientific American,* April 1962, p. 130.)
25. Why does the factor "2" appear in Eq. 16-15, rather than "1"? One might naively expect that the thrust would simply be the pressure difference times the area—that is, $A_0(p - p_0)$.
26. Explain why the destructive effect of a tornado is greater near the center of the disturbance than near the edge.
27. When a stopper is pulled from a filled basin, the water drains out while circulating like a small whirlpool. The angular velocity of a fluid element about a vertical axis through the orifice appears to be greatest near the orifice. Explain.
28. Is it true that in bathtubs in the northern hemisphere the water drains out with a counterclockwise rotation and in those in the southern hemisphere with a clockwise rotation? If so, explain and predict what would happen at the equator. (See "Bath-Tub Vortex," by Ascher H. Shapiro, *Nature,* December 15, 1962, p. 1080.)
29. Explain why you cannot remove the filter paper from the funnel of Fig. 16-26 by blowing into the narrow end.

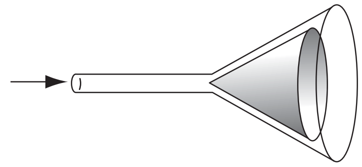


FIGURE 16-26. Question 29.

30. According to Bernoulli's equation, an increase in velocity should be associated with a decrease in pressure. Yet, when you put your hand outside the window of a moving car, increasing the speed at which the air flows by, you sense an increase in pressure. Why is this not a violation of Bernoulli's equation?
31. Why is it that the presence of the atmosphere reduces the maximum range of some objects (for example, tennis balls) but increases the maximum range of others (for example, Frisbees and golf balls)?
32. A discus can be thrown farther against a 25-mi/h wind than with it. What is the explanation? (Hint: Think about dynamic lift and drag.)
33. Explain why golf balls are dimpled.
34. The longer the board and the shallower the water, the farther will a surfboard skim across the water. Explain. (See "The Surf Skimmer," by R. D. Edge, *American Journal of Physics,* July 1968, p. 630.)
35. When poured from a teapot, water has a tendency to run along the underside of the spout. Explain. (See "The Teapot Effect . . . a Problem," by Markus Reiner, *Physics Today,* September 1956, p. 16.)
36. Prairie dogs live in large colonies in complex interconnected burrow systems. They face the problem of maintaining a sufficient air supply to their burrows to avoid suffocation. They avoid this by building conical earth mounds about some of their many burrow openings. In terms of Bernoulli's equation, how does this air conditioning scheme work? Note that because of viscous forces the wind speed over the prairie is less at close to ground level than it is even a few inches higher up. (See *New Scientist,* January 27, 1972, p. 191.)
37. Viscosity is an example of a transport phenomenon. What property is being transported? Can you think of other transport phenomena and their corresponding properties?
38. Why do auto manufacturers recommend using "multi-viscosity" engine oil in cold weather?
39. Why is it more important to take viscosity into account for a fluid flowing in a narrow channel than in a relatively unconfined channel?
40. Viscosity can delay the onset of turbulence in fluid flow; that is, it tends to stabilize the flow. Consider syrup and water, for example, and make this plausible.

EXERCISES

16-1 General Concepts of Fluid Flow

16-2 Streamlines and the Equation of Continuity

1. A pipe of diameter 34.5 cm carries water moving at 2.62 m/s. How long will it take to discharge 1600 m^3 of water?
2. A garden hose having an internal diameter of 0.75 in. is connected to a lawn sprinkler that consists merely of an enclosure with 24 holes, each 0.050 in. in diameter. If the water in the hose has a speed of 3.5 ft/s, at what speed does it leave the sprinkler holes?

3. Figure 16-27 shows the confluence of two streams to form a river. One stream has a width of 8.2 m, depth of 3.4 m, and current speed of 2.3 m/s. The other stream is 6.8 m wide, 3.2 m deep, and flows at 2.6 m/s. The width of the river is 10.7 m and the current speed is 2.9 m/s. What is its depth?

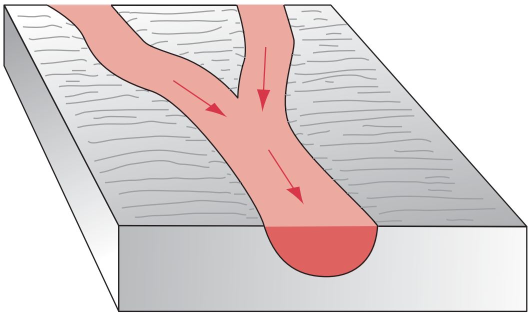


FIGURE 16-27. Exercise 3.

4. Water is pumped steadily out of a flooded basement at a speed of 5.30 m/s through a uniform hose of radius 9.70 mm. The hose passes out through a window 2.90 m above the water line. How much power is supplied by the pump?

5. A river 21 m wide and 4.3 m deep drains a 8500-km^2 land area in which the average precipitation is 48 cm/y. One-fourth of this subsequently returns to the atmosphere by evaporation, but the remainder ultimately drains into the river. What is the average speed of the river current?

16-3 Bernoulli's Equation

6. How much work is done by pressure in forcing 1.4 m^3 of water through a 13-mm internal diameter pipe if the difference in pressure at the two ends of the pipe is 1.2 atm?

7. A water intake at a storage reservoir (see Fig. 16-28) has a cross-sectional area of 7.60 ft^2. The water flows in at a speed of 1.33 ft/s. At the generator building 572 ft below the intake point, the water flows out at 31.0 ft/s. (*a*) Find the difference in pressure, in lb/in.^{2}, between inlet and outlet. (*b*) Find the area of the outlet pipe. The weight density of water is 62.4 lb/ft^3.

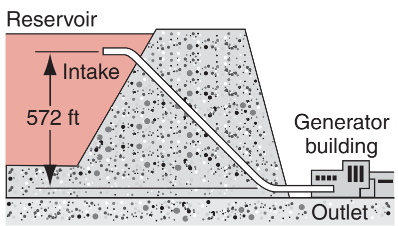


FIGURE 16-28. Exercise 7.

8. Models of torpedoes are sometimes tested in a horizontal pipe of flowing water, much as a wind tunnel is used to test model airplanes. Consider a circular pipe of internal diameter 25.5 cm and a torpedo model, aligned along the axis of the pipe, with a diameter of 4.80 cm. The torpedo is to be tested with water flowing past it at 2.76 m/s. (*a*) With what speed must the water flow in the unconstricted part of the pipe? (*b*) Find the pressure difference between the constricted and unconstricted parts of the pipe.

9. A reservoir is used to collect all of the rain water that falls over an area $A = 100$ m^2. The reservoir has a small hole of cross-sectional area a located $h = 2$ m beneath the surface of the water. (*a*) Assuming an annual rainfall of 1.6 m/year distributed evenly throughout the year, estimate the largest possible value for a that will allow the water level to remain constant in the tank. (*b*) Find, in liters/day, the amount of water this reservoir can deliver. (*c*) How many people can this reservoir support?

10. Water is moving with a speed of 5.18 m/s through a pipe with a cross-sectional area of 4.20 cm^2. The water gradually descends 9.66 m as the pipe increases in area to 7.60 cm^2. (*a*) What is the speed of flow at the lower level? (*b*) The pressure at the upper level is 152 kPa; find the pressure at the lower level.

11. In a hurricane, the air (density 1.2 kg/m^3) is blowing over the roof of a house at a speed of 110 km/h. (*a*) What is the pressure difference between inside and outside that tends to lift the roof? (*b*) What would be the lifting force on a roof of area 93 m^2?

12. The windows in an office building are 4.26 m by 5.26 m. On a stormy day, air is blowing at 28.0 m/s past a window on the 53rd floor. Calculate the net force on the window. The density of the air is 1.23 kg/m^3.

13. A liquid flows through a horizontal pipe whose inner radius is 2.52 cm. The pipe bends upward through a height of 11.5 m where it widens and joins another horizontal pipe of inner radius 6.14 cm. What must the volume flux be if the pressure in the two horizontal pipes is the same?

14. Figure 16-29 shows liquid discharging from an orifice in a large tank at a distance h below the liquid surface. The tank is open at the top. (*a*) Apply Bernoulli's equation to a streamline connecting points 1, 2, and 3, and show that the speed of efflux is

$$v = \sqrt{2gh}.$$

This is known as *Torricelli's law.* (*b*) If the orifice were curved directly upward, how high would the liquid stream rise? (*c*) How would viscosity or turbulence affect the analysis?

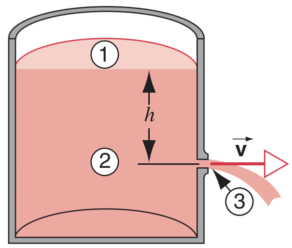


FIGURE 16-29. Exercise 14.

15. A submarine at a depth of 200 m develops a relatively small leak. At what speed does the water enter the sub? Assume that the air pressure inside the sub is the same as the air pressure at sea-level.

16. A sniper fires a rifle bullet into a gasoline tank, making a hole 53.0 m below the surface of the gasoline. The tank was sealed and is under 3.10-atm absolute pressure, as shown in Fig. 16-30. The stored gasoline has a density of 660 kg/m^3. At what speed does the gasoline begin to shoot out of the hole?

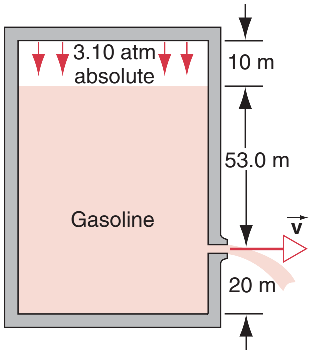


FIGURE 16-30. Exercise 16.

17. Consider a uniform U-tube with a diaphragm at the bottom and filled with a liquid to different heights in each arm (see Fig. 16-31). Now imagine that the diaphragm is punctured so that the liquid flows from left to right. (*a*) Show that the application of Bernoulli's equation to points 1 and 3 leads to a contradiction. (*b*) Explain why Bernoulli's equation is not applicable here. (Hint: Is the flow steady?)

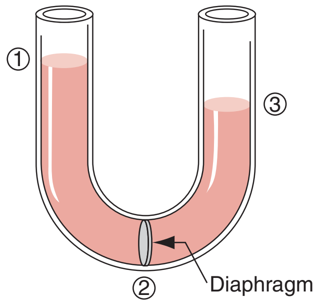


FIGURE 16-31. Exercise 17.

18. If a person blows air with a speed of 15.0 m/s across the top of one side of a U-tube containing water, what will be the difference between the water levels on the two sides? Assume that the density of air is 1.20 kg/m^3.

19. The fresh water behind a reservoir dam is 15.2 m deep. A horizontal pipe 4.30 cm in diameter passes through the dam 6.15 m below the water surface, as shown in Fig. 16-32. A plug secures the pipe opening. (*a*) Find the frictional force between plug and pipe wall. (*b*) The plug is removed. What volume of water flows out of the pipe in 3.00 h?

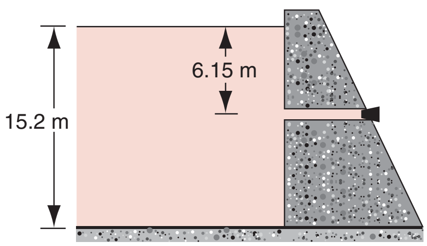


FIGURE 16-32. Exercise 19.

16-4 Applications of Bernoulli's Equation and the Equation of Continuity

20. A Pitot tube is mounted on an airplane wing to determine the speed of the plane relative to the air, which has a density of 1.03 kg/m^3. The tube contains alcohol and indicates a level difference of 26.2 cm. What is the plane's speed relative to the air? The density of alcohol is 810 kg/m^3.

21. A hollow tube has a disk DD attached to its end (Fig. 16-33). When air of density ρ is blown through the tube, the disk attracts the card CC. Let the area of the card be A and let v be the average air speed between the card and the disk. Calculate the resultant upward force on CC. Neglect the card's weight; assume that $v_0 \ll v$, where v_0 is the air speed in the hollow tube.

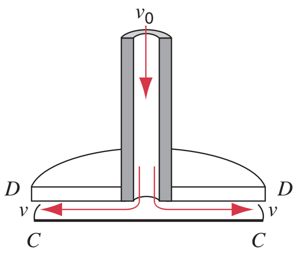


FIGURE 16-33. Exercise 21.

22. A square plate with edge length 9.10 cm and mass 488 g is hinged along one side. If air is blown over the upper surface only, what speed must the air have to hold the plate horizontal? The air has density 1.21 kg/m^3.

23. Air flows over the top of an airplane wing, area A, with speed v_t and past the underside of the wing with speed v_u. Show that Bernoulli's equation predicts that the upward lift force L on the wing will be

$$L = \tfrac{1}{2}\rho A(v_t^2 - v_u^2),$$

where ρ is the density of the air. (Hint: Apply Bernoulli's equation to a streamline passing just over the upper wing surface and to a streamline passing just beneath the lower wing surface. Can you justify setting the constants for the two streamlines equal?)

24. An airplane has a wing area (each wing) of 12.5 m^2. At a certain air speed, air flows over the upper wing surface at 49.8 m/s and over the lower wing surface at 38.2 m/s. (*a*) Find the mass of the plane. Assume that the plane travels with constant velocity and that lift effects associated with the fuselage and tail assembly are small. Discuss the lift if the airplane, flying at the same air speed, is (*b*) in level flight, (*c*) climbing at 15°, and (*d*) descending at 15°. The air density is 1.17 kg/m^3. See Exercise 23.

25. A Venturi tube has a pipe diameter of 25.4 cm and a throat diameter of 11.3 cm. The water pressure in the pipe is 57.1 kPa and in the throat is 32.6 kPa. Calculate the volume flux of water through the tube.

16-5 Fields of Flow

26. Show that the constant in Bernoulli's equation is the same for all streamlines in the case of the steady, irrotational flow of Fig. 16-13.

27. Before Newton proposed his theory of gravitation, a model of planetary motion proposed by René Descartes was widely accepted. In Descartes' model the planets were caught in and dragged along by a whirlpool of ether particles centered around the Sun. Newton showed that this vortex scheme contradicted observations because: (*a*) the speed of an ether particle in the vortex varies inversely as its distance from the Sun; (*b*) the period of revolution of such a particle varies directly as the square of its distance from the Sun; and (*c*) this result contradicts Kepler's third law. Prove (*a*), (*b*), and (*c*).

16-6 Viscosity, Turbulence, and Chaotic Flow

28. Calculate the greatest speed at which blood, at 37°C, can flow through an artery of diameter 3.8 mm if the flow is to remain laminar.

29. Liquid mercury (viscosity = 1.55×10^{-3} N·s/m^2) flows through a horizontal pipe of internal radius 1.88 cm and length 1.26 m. The volume flux is 5.35×10^{-2} L/min. (*a*) Show that the flow is laminar. (*b*) Calculate the difference in pressure between the two ends of the pipe.

PROBLEMS

1. Tidal currents in narrow channels connecting coastal bays with the ocean can be very swift; water must flow into the bay as the tide rises and back out to the sea as the tide falls. Consider the rectangular bay shown in Fig. 16-34*a*. The bay is connected to the sea by a channel 190 m wide and 6.5 m deep at mean sea level. The graph (Fig. 16-34*b*) shows the diurnal variation of the water level in the bay. Calculate the average speed of the tidal current in the channel.

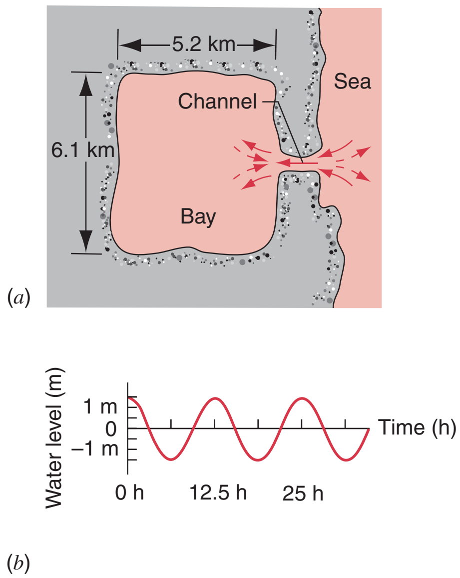


FIGURE 16-34. Problem 1.

2. Suppose that two tanks, 1 and 2, each with a large opening at the top, contain different liquids. A small hole is made in the side of each tank at the same depth h below the liquid surface, but the hole in tank 1 has half the cross-sectional area of the hole in tank 2. (*a*) What is the ratio ρ_1/ρ_2 of the densities of the fluids if it is observed that the mass flux is the same for the two holes? (*b*) What is the ratio of the flow rates (volume flux) from the two tanks? (*c*) It is desired to equalize the two flow rates by adding or draining fluid in tank 2. What should be the new height of the fluid above the hole in tank 2 to make the flow rate in tank 2 equal to that of tank 1?

3. A tank is filled with water to a height H. A hole is punched in one of the walls at a depth h below the water surface (Fig. 16-35). (*a*) Show that the distance x from the foot of the wall at which the stream strikes the floor is given by $x = 2\sqrt{h(H - h)}$. (*b*) Could a hole be punched at another depth so that this second stream would have the same range? If so, at what depth? (*c*) At what depth should the hole be placed to make the emerging stream strike the ground at the maximum distance from the base of the tank? What is this maximum distance?

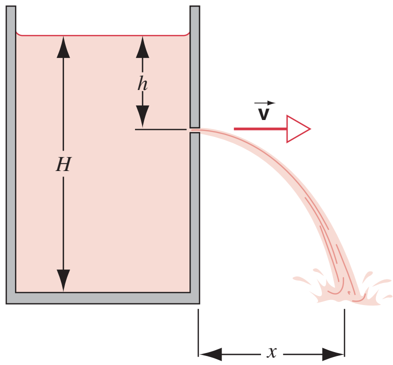


FIGURE 16-35. Problem 3.

4. A siphon is a device for removing liquid from a container that is not to be tipped. It operates as shown in Fig. 16-36. The tube must initially be filled, but once this has been done the liquid will flow until its level drops below the tube opening at

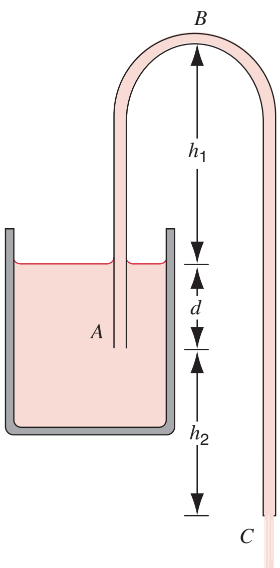


FIGURE 16-36. Problem 4.

A. The liquid has density ρ and negligible viscosity. (*a*) With what speed does the liquid emerge from the tube at *C*? (*b*) What is the pressure in the liquid at the topmost point *B*? (*c*) What is the greatest possible height *h* that a siphon may lift water?

5. (*a*) Consider a stream of fluid of density ρ with speed v_1, passing abruptly from a cylindrical pipe of cross-sectional area a_1 into a wider cylindrical pipe of cross-sectional area a_2 (see Fig. 16-37). The jet will mix with the surrounding fluid and, after the mixing, will flow on almost uniformly with an average speed v_2. Without referring to the details of the mixing, use momentum ideas to show that the increase in pressure due to the mixing is approximately

$$p_2 - p_1 = \rho v_2(v_1 - v_2).$$

(*b*) Show from Bernoulli's equation that in a gradually widening pipe we would get

$$p_2 - p_1 = \tfrac{1}{2}\rho(v_1^2 - v_2^2).$$

(*c*) Find the loss of pressure due to the abrupt enlargement of the pipe. Can you draw an analogy with elastic and inelastic collisions in particle mechanics?

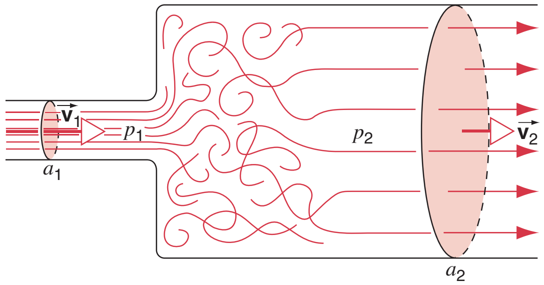


FIGURE 16-37. Problem 5.

6. A jug contains 15 glasses of orange juice. When you open the tap at the bottom it takes 12.0 s to fill a glass with juice. If you leave the tap open, how long will it take to fill the remaining 14 glasses and thus empty the jug?

7. Consider the stagnant air at the front edge of a wing and the air rushing over the wing surface at a speed *v*. Assume pressure at the leading edge to be approximately atmospheric and find the greatest value possible for *v* in streamline flow; assume that air is incompressible and use Bernoulli's equation. Take the density of air to be 1.2 kg/m^3. How does this compare with the speed of sound under these conditions (340 m/s)? Can you explain the difference? Why should there be any connection between these quantities?

8. Consider the Venturi meter of Fig. 16-8. By applying Bernoulli's equation to points 1 and 2 and the equation of continuity (Eq. 16-3), verify Eq. 16-11 for the speed of flow at point 1.

9. Consider the Venturi meter of Fig. 16-8, containing water, without the manometer. Let $A_1 = 4.75A_2$. Suppose that the pressure at point 1 is 2.12 atm. (*a*) Compute the values of v_1 at point 1 and v_2 at point 2 that would make the pressure p_2 at point 2 equal to zero. (*b*) Compute the corresponding volume flow rate if the diameter at point 1 is 5.20 cm. The phenomenon at point 2 when p_2 falls to nearly zero is known as *cavitation.* The water vaporizes into small bubbles.

10. A force field is conservative if $\oint \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = 0$. The circle on the integration sign means that the integration is to be taken along a closed curve (a round trip) in the field. A flow is a potential flow (hence irrotational) if $\oint \vec{\mathbf{v}} \cdot d\vec{\mathbf{s}} = 0$ for every closed path in the field. Using this criterion, show that the fields of (*a*) Fig. 16-13 and (*b*) Fig. 16-16 are fields of potential flow.

11. In flows that are sharply curved, centrifugal effects are appreciable. Consider an element of fluid that is moving with speed *v* along a streamline of a curved flow in a horizontal plane (Fig. 16-38). (*a*) Show that $dp/dr = \rho v^2/r$, so that the pressure increases by an amount $\rho v^2/r$ per unit distance perpendicular to the streamline as we go from the concave to the convex side of the streamline. (*b*) Then use Bernoulli's equation and this result to show that *vr* equals a constant, so that speeds increase toward the center of curvature. Hence streamlines that are uniformly spaced in a straight pipe will be crowded toward the inner wall of a curved passage and widely spaced toward the outer wall. This problem should be compared to Problem 12 of Chapter 15 in which the curved motion is produced by rotating a container. There the speed varied directly with *r*, but here it varies inversely. (*c*) Show that this flow is irrotational.

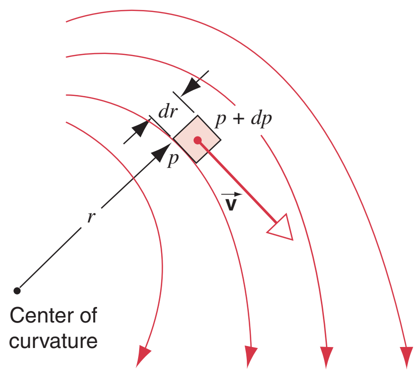


FIGURE 16-38. Problem 11.

12. Figure 16-39 shows a cross section of the upper layers of the Earth. The surface of the Earth is broken into several rigid blocks, called plates, that slide (slowly!) over a "slushy" lower layer called the asthenosphere. See the figure for typical dimensions. Suppose that the speed of the rigid plate shown is $v_0 = 48$ mm/y, and that the base of the asthenosphere does not move. Calculate the shearing stress (shearing force per unit area) on the base of the plate. The viscosity of the asthenosphere material is 4.0×10^{19} N·s/m^2. Ignore the curvature of the Earth.

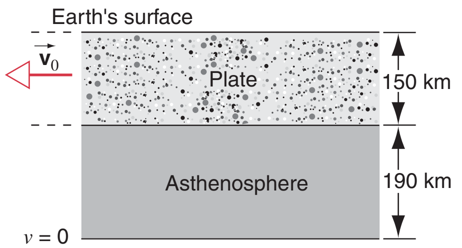


FIGURE 16-39. Problem 12.

13. The streamlines of the Poiseuille field of flow are shown in Fig. 16-40. The spacing of the streamlines indicates that although the motion is rectilinear, there is a velocity gradient in the transverse direction. Show that the Poiseuille flow is rotational.

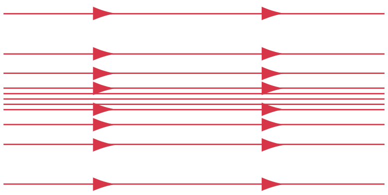


FIGURE 16-40. Problem 13.

14. A fluid of viscosity η flows steadily through a horizontal cylindrical pipe of radius R and length L, as shown in Fig. 16-41. (*a*) Consider an arbitrary cylinder of fluid of radius r. Show that the viscous force F due to the neighboring layer is $F = -\eta(2\pi rL)dv/dr$. (*b*) Show that the force F' pushing that cylinder of fluid through the pipe is $F' = (\pi r^2)\Delta p$. (*c*) Use the equilibrium condition to obtain an expression for dv in terms of dr. Integrate the expression to obtain Eq. 16-18.

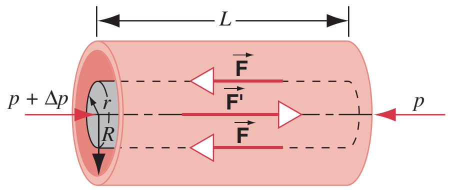


FIGURE 16-41. Problems 14 and 15.

15. Consider once again the fluid flowing through the pipe described in Problem 14 and illustrated in Fig. 16-41. Find an expression for the mass flux through the annular ring between radii r and $r + dr$; then integrate this result to find the total mass flux through the pipe, thereby verifying Eq. 16-20.

16. A soap bubble of radius 38.2 mm is blown on the end of a narrow tube of length 11.2 cm and internal diameter 1.08 mm. The other end of the tube is exposed to the atmosphere. Find the time taken for the bubble radius to fall to 21.6 mm. Assume Poiseuille flow in the tube. (For the surface tension of the soap solution use 2.50×10^{-2} N/m; the viscosity of air is 1.80×10^{-5} N·s/m^2.)

COMPUTER PROBLEM

A cylindrical water tank has a radius of 2 m and a height of 1.5 m. Originally the tank is completely filled with water, but a vertical crack appears in the tank and the water leaks out. Assuming the crack is 1 cm wide and extends from the base of the tank to the top, calculate the amount of time for the tank to completely empty. (Hint: Assume the crack is composed of 1 cm^2 holes, each one on top of the other, and solve the problem numerically.

CHAPTER 17

OSCILLATIONS

Each day we encounter many kinds of oscillatory motion. Common examples include the swinging pendulum of a clock, a person bouncing on a trampoline, and a vibrating guitar string. Examples on the microscopic scale are vibrating atoms in the quartz crystal of a wristwatch and vibrating molecules of air that transmit sound waves. In addition to these mechanical oscillations, we can also have electromagnetic oscillations, such as electrons surging back and forth in circuits that are responsible for transmitting and receiving radio or TV signals.

These oscillating systems—whether mechanical, electromagnetic, or other types—have a common mathematical formulation and are most easily expressed in terms of sine and cosine functions. In this chapter we concentrate on mechanical oscillations and their description. Later in this book we deal with various kinds of waves and with electromagnetic oscillations, which use the same mathematical description.

17-1 OSCILLATING SYSTEMS

Imagine an oscillating system, such as the pendulum of a clock or a mass on a spring. What must be the properties of the force that produces such oscillations?

If you displace a pendulum in one direction from its equilibrium position, the force (which is due to gravity) pushes it back toward equilibrium. If you displace it in the other direction, the force still acts toward the equilibrium position. *No matter what the direction of the displacement, the force always acts in a direction to restore the system to its equilibrium position.* Such a force is called a *restoring force.* (The equilibrium position is the kind we called *stable* in Chapter 12; the system tends to return to equilibrium when slightly displaced.)

Let us consider a simple example. Suppose we have a particle that is free to move only in the x direction, and let the particle experience a force of constant magnitude F_m that acts in the $+x$ direction when $x < 0$ and in the $-x$ direction when $x > 0$, as shown in Fig. 17-1*a*. The force, which is shown in Fig. 17-1*b*, is similar to the forces that give the piecewise constant accelerations we considered in Chapter 2.

A particle of mass m initially at rest at coordinate $x = +x_m$ experiences a force whose x component is $-F_m$, and the corresponding x component of the acceleration of the particle is $-a_m = -F_m/m$. The particle moves toward its equilibrium position at $x = 0$ and reaches that position with velocity $v_x = -v_m$. When it passes through the origin to negative x, the force becomes $+F_m$, and the acceleration is $+a_m$. The particle slows and comes to rest for an instant at $x = -x_m$ before reversing its motion through the origin and returning eventually to $x = +x_m$. In the absence of friction and other dissipative forces, the cycle repeats endlessly.

Figure 17-2 shows the resulting motion, plotted in the style of the examples we considered in Chapter 2. The position $x(t)$ consists of a sequence of smoothly joined segments of parabolas, as is always the case for motion at constant acceleration. The particle oscillates back and forth between $x = +x_m$ and $x = -x_m$. The magnitude of the maximum displacement from equilibrium (x_m in this case) is called the *amplitude* of the motion. The time necessary for one complete cycle (a complete repetition of the motion) is called the *period T,* as indicated in Fig. 17-2. The

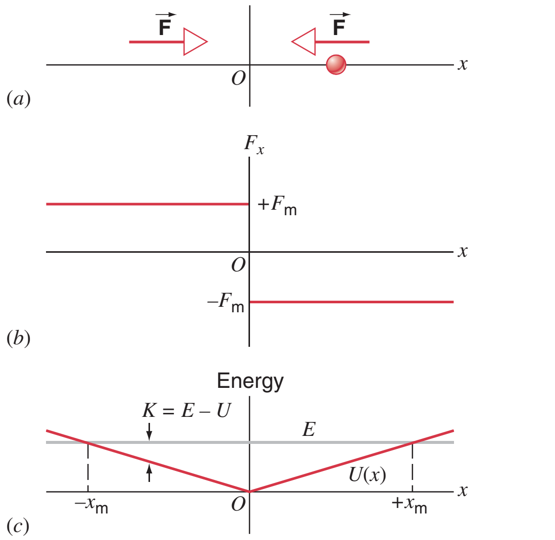


FIGURE 17-1. (*a*) A particle is acted on by a constant force $\vec{\mathbf{F}}$ that is always directed toward the origin. (*b*) A plot of this piecewise constant force, equal to $+F_m$ when $x < 0$ and to $-F_m$ when $x > 0$. Any real force of this type must be represented by a continuous function, even though it may be very steep as it goes through $x = 0$. (*c*) The potential energy corresponding to this force. If the system has total mechanical energy E, then at any location the difference $E - U$ gives the kinetic energy.

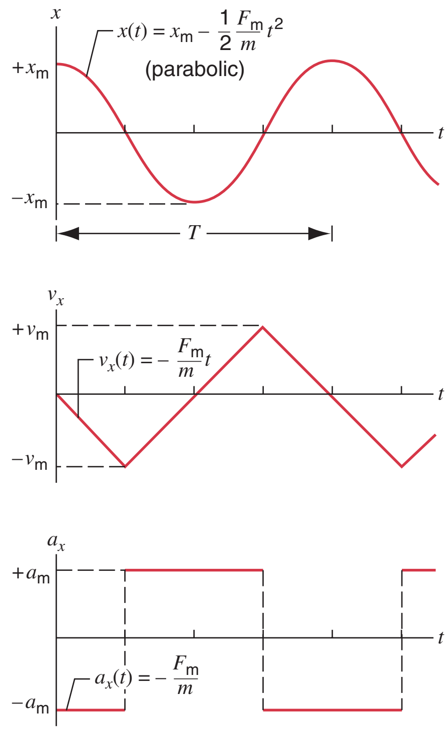


FIGURE 17-2. The position, velocity, and acceleration of the particle of Fig. 17-1 are plotted as functions of the time. The acceleration consists of alternating horizontal segments with values $+F_m/m$ and $-F_m/m$, the velocity consists of alternating linear segments with slopes $+F_m/m$ and $-F_m/m$, and the position consists of smoothly joined sections of parabolas. Because the force $F_x(x)$ is in reality a continuous function, $a_x(t)$ is also continuous, the horizontal segments having steep connections. Moreover, the sharp corners of $v_x(t)$ are rounded. The curves shown, however, are excellent approximations if the force changes from $+F_m$ to $-F_m$ over a very short interval.

number of cycles per unit time is called the *frequency f.* The frequency and the period are reciprocals of one another:

$$f = 1/T. \qquad (17\text{-}1)$$

Period is measured in time units (seconds, for instance), whereas frequency is measured in the SI unit of hertz (Hz),* where 1 Hz = 1 cycle/s. Thus, for example, an oscillation with a period of $T = 5$ s has a frequency $f = 0.2$ Hz.

So far we have used a dynamical description of the oscillation. Often a description in terms of energy is useful. Figure 17-1*c* shows the potential energy corresponding to the force of Fig. 17-1*b*. Note that, as indicated by the expression $F = -dU/dx$, the negative of the slope $U(x)$ gives the force. The mechanical energy $E = K + U$ remains constant for an isolated system. At every point, the difference $E - U$ gives the kinetic energy K at that point. If we extended the graph to sufficiently large displacements, we would eventually reach locations where $E = U$ and thus $K = 0$. At these points, as Fig. 17-2 shows, the velocity is zero and the position is $x = \pm x_m$. These points are called the *turning points* of the motion.

Figures 17-1*b* and 17-1*c* illustrate two equivalent ways of describing the conditions for oscillation: the force must always act to restore the particle to equilibrium, and the potential energy must have a minimum at the equilibrium position.

The case of constant acceleration is always pleasant to work with, because the mathematics is simple, but it is seldom an accurate description of nature. Figure 17-3*a* shows an example of a more realistic force that can produce oscillatory motion. Such a force is responsible for the binding of molecules containing two atoms. The force increases rapidly as we try to push one atom close to the other; this repulsive component keeps the molecule from collapsing. As we try to pull the atoms to larger spacings, the force tends to oppose our efforts; this force may be an electrostatic force between two opposite electric charges, but often it is more complex and involves the spatial distribution of electronic orbits in atoms.

Figure 17-3*b* shows the corresponding potential energy function $U(x)$. Note that, as was the case in Fig. 17-1, the force changes sign at the equilibrium position, and the potential energy has a minimum at that position. Note that in this case the turning points (labeled x_1 and x_2 in Fig. 17-3) are *not* symmetrically located about the equilibrium position. If we were to stretch the molecule a bit beyond its equilibrium configuration and release it (which often occurs when a molecule absorbs infrared radiation), it would execute periodic motion about equilibrium, although the mathematical description would be more complex than that of Fig. 17-2. The study of these oscillations is an important

* The frequency unit is named after Heinrich Hertz (1857–1894), whose research provided the experimental confirmation of electromagnetic waves.

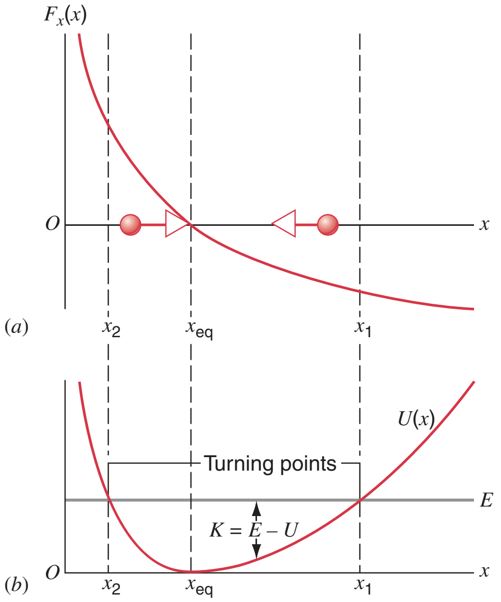


FIGURE 17-3. (*a*) The force that acts on a particle oscillating between the limits x_1 and x_2. Note that the force always tends to push the particle toward its equilibrium position, as in Fig. 17-1. Such a force might act on an atom in a molecule. (*b*) The potential energy corresponding to this force.

technique for learning about molecular structure, as we discuss in Section 17-9.

17-2 THE SIMPLE HARMONIC OSCILLATOR

The motion of a particle in a complex system, such as an atom in the vibrating molecule discussed in the previous section, is easier to analyze if we consider the motion to be a superposition of *harmonic* oscillations, which can be described in terms of sine and cosine functions.

Consider an oscillating system in one dimension, consisting of a particle subject to a force

$$F_x(x) = -kx, \tag{17-2}$$

in which k is a constant and x is the displacement of the particle from its equilibrium position. Such an oscillating system is called a *simple harmonic oscillator,* and its motion is called *simple harmonic motion.* The potential energy corresponding to this force is

$$U(x) = \tfrac{1}{2}kx^2. \tag{17-3}$$

The force and potential energy are of course related by $F_x(x) = -dU/dx$. As indicated by Eq. 17-2 and plotted in Fig. 17-4*a*, the force acting on the particle is directly proportional to the displacement but is opposite to it in direction. Equation 17-3 shows that the potential energy varies as the square of the displacement, as illustrated by the parabolic curve in Fig. 17-4*b*.

You will recognize Eqs. 17-2 and 17-3 as the expressions for the force and potential energy of an "ideal" spring

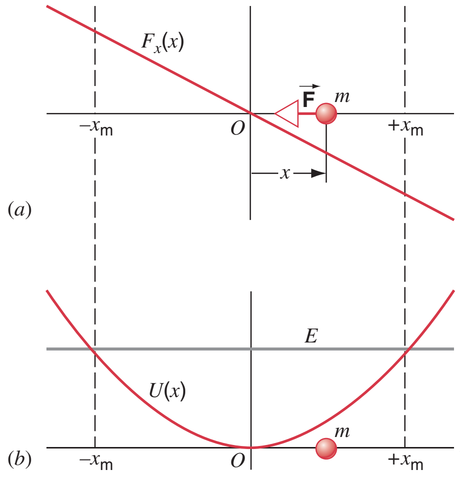


FIGURE 17-4. (*a*) The force and (*b*) the corresponding potential energy of a simple harmonic oscillator. Note the similarities and differences with Fig. 17-3.

of force constant k, compressed or extended by a distance x; see Section 11-4. Hence, *a body of mass m attached to an ideal spring of force constant k and free to move over a frictionless horizontal surface is an example of a simple harmonic oscillator* (see Fig. 17-5). Note that there is a position (the equilibrium position; see Fig. 17-5*b*) in which the spring exerts no force on the body. If the body is displaced to the right (as in Fig. 17-5*a*), the force exerted by

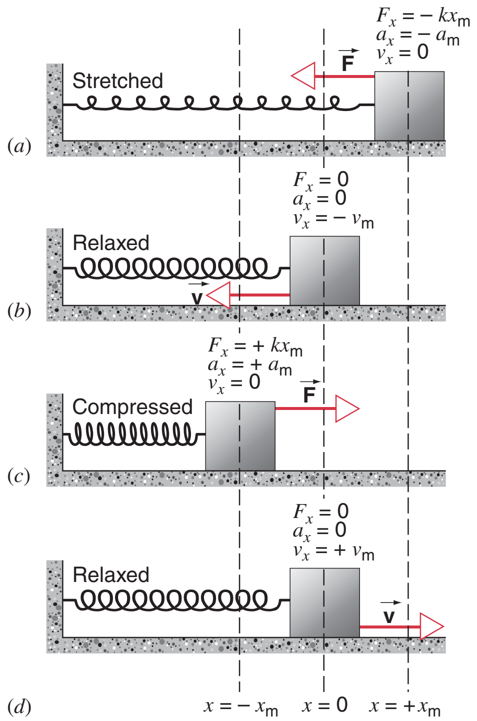


FIGURE 17-5. A simple harmonic oscillator, consisting of a spring acting on a body that slides on a frictionless horizontal surface. In (*a*), the spring is stretched so that the body has its maximum displacement from equilibrium. In (*c*) the spring is fully compressed. In (*b*) and (*d*), the body is passing through equilibrium with its maximum speed, and the spring is relaxed.

the spring on the body points to the left. If the body is displaced to the left (as in Fig. 17-5*c*), the force points to the right. In each case the force is a *restoring* force. (It is in this case a *linear* restoring force—that is, proportional to the first power of x.)

Let us apply Newton's second law, $\Sigma F_x = ma_x$, to the motion of Fig. 17-5. For ΣF_x we substitute $-kx$ and for the acceleration a_x we put in d^2x/dt^2 $(= dv_x/dt)$. This gives us

$$-kx = m\frac{d^2x}{dt^2}$$

or

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \tag{17-4}$$

Equation 17-4 is called the *equation of motion* of the simple harmonic oscillator. Its solution, which we describe in the next section, is a function $x(t)$ that describes the position of the oscillator as a function of the time, in analogy with Fig. 17-2*a*, which represents the variation of position with time of a different oscillator.

The simple harmonic oscillator problem is important for two reasons. First, many problems involving mechanical vibrations at small amplitudes reduce to that of the simple harmonic oscillator, or to a combination of such oscillators. This is equivalent to saying that if we consider a small enough portion of a restoring force curve near the equilibrium position, Fig. 17-3*a*, for instance, it becomes arbitrarily close to a straight line, which, as Fig. 17-4*a* shows, is characteristic of simple harmonic motion. Or, in other words, the potential energy curve of Fig. 17-3*b* is very nearly parabolic near the equilibrium position.

Second, as we have indicated, equations like Eq. 17-4 occur in many physical problems in acoustics, optics, mechanics, electrical circuits, and even atomic physics. The simple harmonic oscillator exhibits features common to many physical systems.

17-3 SIMPLE HARMONIC MOTION

Let us now solve the equation of motion of the simple harmonic oscillator,

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \tag{17-4}$$

We derived Eq. 17-4 for a spring force $F_x = -kx$ (where the force constant k is a measure of the stiffness of the spring) acting on a particle of mass m. We shall see later that other oscillating systems are governed by similar equations of motion, in which the constant k is related to other physical features of the system. We can use the oscillating mass–spring system as our prototype.

Equation 17-4 gives a relation between a function of the time $x(t)$ and its second time derivative d^2x/dt^2. Our goal is to find a function $x(t)$ that satisfies this relation. We begin by rewriting Eq. 17-4 as

$$\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x. \tag{17-5}$$

Equation 17-5 requires that $x(t)$ be a function whose second derivative is the negative of the function itself, except for a constant factor k/m. We know from calculus that the sine and cosine functions have this property. For example,

$$\frac{d}{dt}\cos\omega t = -\omega\sin\omega t$$

and

$$\frac{d^2}{dt^2}\cos\omega t = \frac{d}{dt}(-\omega\sin\omega t) = -\omega^2\cos\omega t.$$

The second derivative of a cosine (or of a sine) gives us back the original function multiplied by a negative factor $-\omega^2$. This property is not affected if we multiply the cosine function by any constant. We choose the constant to be x_m, so that the maximum value of x (the amplitude of the motion) will be x_m.

We write a tentative solution to Eq. 17-5 as

$$x = x_m\cos(\omega t + \phi). \tag{17-6}$$

Here, since

$$\begin{aligned} x_m\cos(\omega t + \phi) &= x_m\cos\phi\cos\omega t - x_m\sin\phi\sin\omega t \\ &= A\cos\omega t + B\sin\omega t, \end{aligned}$$

where $A = x_m\cos\phi$ and $B = -x_m\sin\phi$, the constant ϕ allows for any combination of sine and cosine solutions.

With the (as yet) unknown constants x_m, ω, and ϕ, we have written as general a solution to Eq. 17-5 as we can. To determine these constants such that Eq. 17-6 is actually the solution of Eq. 17-5, we differentiate Eq. 17-6 twice with respect to the time. We have

$$\frac{dx}{dt} = -\omega x_m\sin(\omega t + \phi)$$

and

$$\frac{d^2x}{dt^2} = -\omega^2x_m\cos(\omega t + \phi).$$

Putting this into Eq. 17-5, we obtain

$$-\omega^2x_m\cos(\omega t + \phi) = -\frac{k}{m}x_m\cos(\omega t + \phi).$$

Therefore, if we choose the constant ω such that

$$\omega^2 = \frac{k}{m}, \tag{17-7}$$

then Eq. 17-6 is in fact a solution of the equation of motion of a simple harmonic oscillator.

The constants x_m and ϕ are still undetermined and therefore still completely arbitrary. This means that *any* choice of x_m and ϕ whatsoever will satisfy Eq. 17-5, so that a large variety of motions (all of which have the same ω) is possible for the oscillator. We shall see later that x_m and ϕ are determined for a particular harmonic motion by how the motion starts.

Let us find the physical significance of the constant ω. If we increase the time t in Eq. 17-6 by $2\pi/\omega$, the function becomes

$$\begin{aligned} x &= x_m \cos[\omega(t + 2\pi/\omega) + \phi] \\ &= x_m \cos(\omega t + 2\pi + \phi) \\ &= x_m \cos(\omega t + \phi). \end{aligned}$$

That is, the function merely repeats itself after a time $2\pi/\omega$. Therefore $2\pi/\omega$ is the period of the motion T. Since $\omega^2 = k/m$, we have

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}. \quad (17\text{-}8)$$

Hence all motions given by Eq. 17-5 have the same period of oscillation, which is determined only by the mass m of the oscillating particle and the force constant k of the spring. The frequency f of the oscillator is the number of complete vibrations per unit time and is given by

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \quad (17\text{-}9)$$

Hence

$$\omega = 2\pi f = \frac{2\pi}{T}. \quad (17\text{-}10)$$

The quantity ω is called the *angular frequency;* it differs from the frequency f by a factor 2π. It has the dimension of reciprocal time (the same as angular speed), and its unit is the radian/second. In Section 17-6 we give a geometric meaning to this angular frequency.

The constant x_m has a simple physical meaning. The cosine function takes on values from -1 to $+1$. The displacement x from the central equilibrium position $x = 0$ therefore has a maximum value of x_m; see Eq. 17-6. We call x_m the *amplitude* of the motion. Because x_m is not fixed by Eq. 17-4, motions of various amplitudes are possible, but all have the same frequency and period. *The frequency of a simple harmonic motion is independent of the amplitude of the motion.*

The quantity $(\omega t + \phi)$ is called the *phase* of the motion. The constant ϕ is called the *phase constant.* Two motions may have the same amplitude and frequency but differ in phase. If $\phi = -\pi/2 = -90°$, for example,

$$\begin{aligned} x = x_m \cos(\omega t + \phi) &= x_m \cos(\omega t - 90°) \\ &= x_m \sin \omega t \end{aligned}$$

so that the displacement is zero at the time $t = 0$. If $\phi = 0$, on the other hand, the displacement $x = x_m \cos \omega t$ has its maximum value $x = x_m$ at the time $t = 0$. Other initial displacements correspond to other phase constants. See Sample Problem 17-3 for an example of the method of finding x_m and ϕ from the initial displacement and velocity.

The amplitude x_m and the phase constant ϕ of the oscillation are determined by the initial position and velocity of the particle. These two initial conditions will specify x_m and ϕ exactly (except that ϕ may be increased or decreased by any multiple of 2π without changing the motion). Once the motion has started, however, the particle will continue to oscillate with a constant amplitude and phase constant at a fixed frequency, unless other forces disturb the system.

In Fig. 17-6 we plot the displacement x versus the time t for several simple harmonic motions described by Eq. 17-6. Three comparisons are made. In Fig. 17-6*a*, the two curves have the same amplitude and frequency but differ in phase by $\phi = \pi/4$ or 45°. In Fig. 17-6*b*, the two curves have the same frequency and phase constant but differ in amplitude by a factor of 2. In Fig. 17-6*c*, the curves have the same amplitude and phase constant but differ in frequency by a factor of $\frac{1}{2}$ or in period by a factor of 2. Study these curves carefully to become familiar with the terminology used in simple harmonic motion.

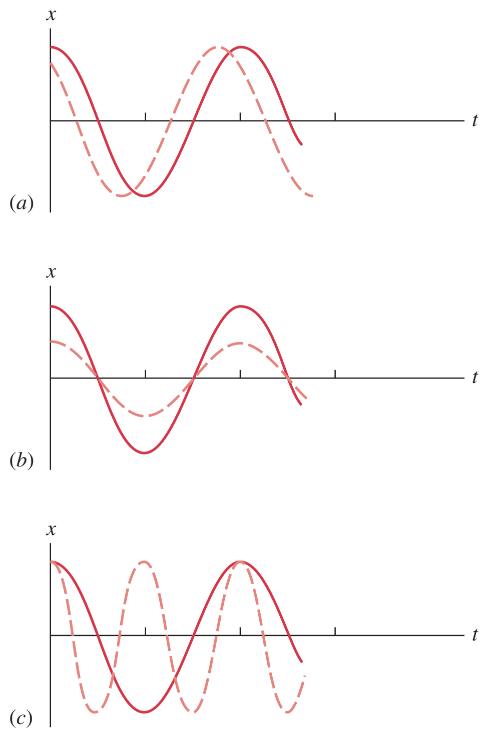


Figure 17-6. (*a*) Comparison of the motions of two simple harmonic oscillators of the same amplitude and frequency but differing in phase constant by 45°. If the motion is represented by Eq. 17-6, then the solid curve has $\phi = 0°$ and the dashed curve has $\phi = 45°$. (*b*) Two simple harmonic motions of the same phase constant and frequency but differing in amplitude by a factor of 2. (*c*) Two simple harmonic motions of the same amplitude and phase constant (0°) but differing in frequency by a factor of 2. The solid curve has twice the *period,* and therefore half the *frequency,* of the dashed curve.

Another distinctive feature of simple harmonic motion is the relation between the displacement, the velocity, and the acceleration of the oscillating particle. Let us compare these quantities. In Fig. 17-7 we plot separately the displacement x versus the time t, the velocity $v_x = dx/dt$ versus the time t, and the acceleration $a_x = dv_x/dt = d^2x/dt^2$ versus the time t. The equations of these curves are

$$x = x_m \cos(\omega t + \phi),$$
$$v_x = \frac{dx}{dt} = -\omega x_m \sin(\omega t + \phi), \qquad (17\text{-}11)$$
$$a_x = \frac{dv_x}{dt} = -\omega^2 x_m \cos(\omega t + \phi).$$

For the case plotted we have taken $\phi = 0$. The units and scale of displacement, velocity, and acceleration are omitted for simplicity of comparison. The displacement, velocity, and acceleration all oscillate harmonically. Note that the maximum displacement (amplitude) is x_m, the maximum speed (velocity amplitude) is ωx_m, and the maximum acceleration (acceleration amplitude) is $\omega^2 x_m$.

When the displacement is a maximum in either direction, the speed is zero because the velocity must now change its direction. The acceleration at this instant, like the restoring force, has a maximum magnitude but is directed opposite to the displacement. When the displacement is zero, the speed of the particle is a maximum and the acceleration is zero, corresponding to a zero restoring force. The speed increases as the particle moves toward the equilibrium position and then decreases as it moves out to the maximum displacement. Compare Fig. 17-7 with Fig. 17-2, and note their similarities and differences.

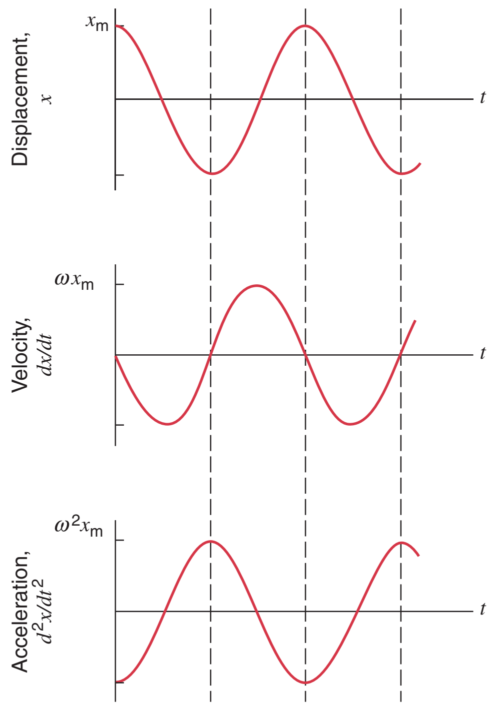


FIGURE 17-7. The displacement, velocity, and acceleration of a simple harmonic oscillator, according to Eqs. 17-11.

SAMPLE PROBLEM 17-1. A certain spring hangs vertically. When a body of mass $M = 1.65$ kg is suspended from it, its length increases by 7.33 cm. The spring is then mounted horizontally, and a block of mass $m = 2.43$ kg is attached to the spring. The block is free to slide along a frictionless horizontal surface, as in Fig. 17-5. (*a*) What is the force constant k of the spring? (*b*) What is the magnitude of the horizontal force required to stretch the spring by a distance of 11.6 cm? (*c*) When the block is displaced a distance of 11.6 cm and released, with what period will it oscillate?

Solution (*a*) The force constant k is determined from the force Mg necessary to stretch the spring by the measured vertical displacement $y = -7.33$ cm. When the suspended body is hanging at rest, $\Sigma F_y = 0$; the y component of the net force on the body is $\Sigma F_y = -ky - Mg$, so $ky = -Mg$, or

$$k = -Mg/y = -(1.65 \text{ kg})(9.80 \text{ m/s}^2)/(-0.0733 \text{ m}) = 221 \text{ N/m}.$$

(*b*) The magnitude of the horizontal force needed to stretch the spring by 11.6 cm is determined from Hooke's law (Eq. 17-2) using the force constant we found in part (*a*):

$$F = kx = (221 \text{ N/m})(0.116 \text{ m}) = 25.6 \text{ N}.$$

(*c*) The period is independent of the amplitude and depends only on the values of the mass of the block and the force constant. From Eq. 17-8,

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2.43 \text{ kg}}{221 \text{ N/m}}} = 0.6589 \text{ s} = 659 \text{ ms}.$$

(We display the value of T to four significant figures, more than are justified by the input data, because we shall need this result in the solution of Sample Problem 17-2. To avoid rounding errors in intermediate steps, it is standard practice to carry excess significant figures in this way. The final result, of course, must be properly rounded.)

17-4 ENERGY IN SIMPLE HARMONIC MOTION

In any motion in which no dissipative forces act, the total mechanical energy E $(= K + U)$ is conserved (remains constant). We can now study this in more detail for the special case of simple harmonic motion.

The potential energy U at any instant is given by

$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2}kx_m^2 \cos^2(\omega t + \phi). \qquad (17\text{-}12)$$

where we have used Eq. 17-6 for the displacement x. The potential energy thus oscillates with time and has a maximum value of $\frac{1}{2}kx_m^2$. During the motion, the potential energy varies between zero and this maximum value, as the curves in Figs. 17-8*a* and 17-8*b* show.

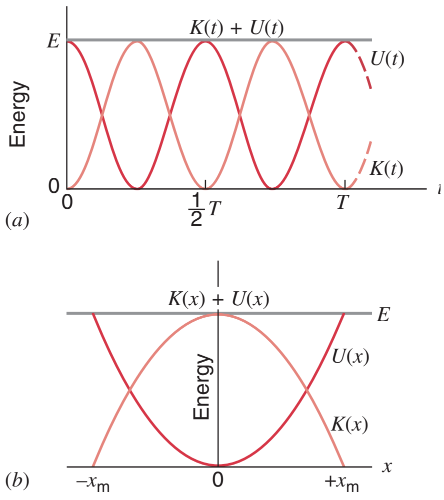


Figure 17-8. The potential energy U, kinetic energy K, and total mechanical energy E of a particle undergoing simple harmonic motion (with $\phi = 0$) are shown as functions of (*a*) the time and (*b*) the displacement. Note that in (*a*) the kinetic and potential energies each reach their maxima twice during each period of the motion. See also Fig. 12-5.

The kinetic energy K at any instant is $\frac{1}{2}mv_x^2$. Using Eq. 17-11 for $v_x(t)$ and Eq. 17-7 for ω^2, we obtain

$$\begin{aligned} K &= \tfrac{1}{2}mv_x^2 \\ &= \tfrac{1}{2}m\omega^2 x_m^2 \sin^2(\omega t + \phi) \\ &= \tfrac{1}{2}kx_m^2 \sin^2(\omega t + \phi). \end{aligned} \tag{17-13}$$

The kinetic energy, like the potential energy, oscillates with time and has a maximum value of $\frac{1}{2}kx_m^2$. During the motion, the kinetic energy varies between zero and this maximum value, as shown by the curves in Figs. 17-8*a* and 17-8*b*. Note that the kinetic and potential energies vary with twice the frequency (half the period) of the displacement and velocity. Can you explain this?

The total mechanical energy is the sum of the kinetic energy and the potential energy. Using Eqs. 17-12 and 17-13, we obtain

$$\begin{aligned} E = K + U &= \tfrac{1}{2}kx_m^2 \sin^2(\omega t + \phi) + \tfrac{1}{2}kx_m^2 \cos^2(\omega t + \phi) \\ &= \tfrac{1}{2}kx_m^2. \end{aligned} \tag{17-14}$$

We see that the total mechanical energy is constant, as we expect, and has the value $\frac{1}{2}kx_m^2$. At the maximum displacement the kinetic energy is zero, but the potential energy has the value $\frac{1}{2}kx_m^2$. At the equilibrium position the potential energy is zero, but the kinetic energy has the value $\frac{1}{2}kx_m^2$. At other positions the kinetic and potential energies each contribute terms whose sum is always $\frac{1}{2}kx_m^2$. This constant total energy E is shown in Figs. 17-8*a* and 17-8*b*. The total energy of a particle executing simple harmonic motion is proportional to the square of the amplitude of the motion. It can be shown (see Problem 14) that the average kinetic energy for the motion during one period is exactly equal to the average potential energy and that each of these average quantities is half the total energy, or $\frac{1}{4}kx_m^2$.

Equation 17-14 can be written quite generally as

$$K + U = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kx_m^2. \tag{17-15}$$

From this relation we obtain $v_x^2 = (k/m)(x_m^2 - x^2)$ or

$$v_x = \pm\sqrt{\frac{k}{m}(x_m^2 - x^2)}. \tag{17-16}$$

This relation shows clearly that the speed is a maximum at the equilibrium position ($x = 0$) and is zero at the extreme displacements ($x = \pm x_m$). In fact, we can start from the conservation of energy, Eq. 17-15 (in which $\frac{1}{2}kx_m^2 = E$), and by integration of Eq. 17-16 obtain the displacement as a function of time, as we did in Section 12-5, where we obtained a result identical to Eq. 17-6 with $\phi = 0$.

Sample Problem 17-2. The block–spring combination of Sample Problem 17-1 is stretched in the positive x direction a distance of 11.6 cm from equilibrium and released. (*a*) What is the total energy stored in the system? (*b*) What is the maximum speed of the block? (*c*) What is the magnitude of the maximum acceleration? (*d*) If the block is released at $t = 0$, what are its position, velocity, and acceleration at $t = 0.215$ s?

Solution (*a*) The amplitude of the motion is given as $x_m = 0.116$ m. The total energy is given by Eq. 17-14:

$$E = \tfrac{1}{2}kx_m^2 = \tfrac{1}{2}(221\text{ N/m})(0.116\text{ m})^2 = 1.49\text{ J}.$$

(*b*) The maximum kinetic energy is numerically equal to the total energy; when $U = 0$, $K = K_{max} = E$. The maximum speed is then

$$v_{max} = \sqrt{\frac{2K_{max}}{m}} = \sqrt{\frac{2(1.49\text{ J})}{2.43\text{ kg}}} = 1.11\text{ m/s}.$$

(*c*) The maximum acceleration occurs just at the instant of release, when the force is greatest:

$$a_{max} = \frac{F_{max}}{m} = \frac{kx_m}{m} = \frac{(221\text{ N/m})(0.116\text{ m})}{2.43\text{ kg}} = 10.6\text{ m/s}^2.$$

(*d*) From the period found in Sample Problem 17-1, we can obtain the angular frequency:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.6589\text{ s}} = 9.536\text{ rad/s}.$$

Since the block has its maximum displacement of $x_m = 0.116$ m at $t = 0$, its motion can be described by a cosine function:

$$x(t) = x_m \cos \omega t,$$

a result that follows by putting $\phi = 0$ in Eq. 17-6. At $t = 0.215$ s, we find

$$x = (0.116\text{ m}) \cos(9.536\text{ rad/s})(0.215\text{ s}) = -0.0535\text{ m}.$$

Note that the angle ωt, whose cosine we must find, is expressed in radians. The velocity is given by Eq. 17-11, which, with $\phi = 0$, becomes $v_x(t) = -\omega x_m \sin \omega t$. At 0.215 s, we obtain

$$\begin{aligned} v_x &= -(9.536\text{ rad/s})(0.116\text{ m}) \sin(9.536\text{ rad/s})(0.215\text{ s}) \\ &= -0.981\text{ m/s}. \end{aligned}$$

To find the acceleration, we again use Eq. 17-11 and note that, at all times, $a_x = -\omega^2 x$:

$$a_x = -(9.536 \text{ rad/s})^2(-0.0535 \text{ m}) = +4.87 \text{ m/s}^2.$$

Let us examine our results to see whether they are reasonable. The time $t = 0.215$ s is between $T/4 = 0.165$ s and $T/2 = 0.330$ s. If the block begins at $x = x_m = +0.116$ m, then at $T/4$ it will pass through equilibrium, and it is certainly reasonable that at $t = 0.215$ s it is at a negative x coordinate, as we found. Since it is at that time moving toward $x = -x_m$, its velocity must be negative, as we found. However, it has already passed through the point of most negative velocity, and it is slowing as it approaches $x = -x_m$; therefore the acceleration should be positive. We can check the value of the acceleration from $a_x = kx/m$. We can also check the relationship between v_x and x using Eq. 17-16.

SAMPLE PROBLEM 17-3. The block of the block–spring system of Sample Problem 17-1 is pushed from equilibrium by an external force in the positive x direction. At $t = 0$, when the displacement of the block is $x = +0.0624$ m and its velocity is $v_x = +0.847$ m/s, the external force is removed and the block begins to oscillate. Write an equation for $x(t)$ during the oscillation.

Solution Since we have the same mass (2.43 kg) and force constant (221 N/m), the angular frequency is still 9.536 rad/s, as we found in Sample Problem 17-2. The most general equation for $x(t)$ is given by Eq. 17-6,

$$x(t) = x_m \cos(\omega t + \phi),$$

and we must find x_m and ϕ to complete the solution. To find x_m, let us compute the total energy, which at $t = 0$ has both kinetic and potential terms:

$$\begin{aligned} E = K + U &= \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2 \\ &= \tfrac{1}{2}(2.43 \text{ kg})(0.847 \text{ m/s})^2 + \tfrac{1}{2}(221 \text{ N/m})(0.0624 \text{ m})^2 \\ &= 0.872 \text{ J} + 0.430 \text{ J} = 1.302 \text{ J}. \end{aligned}$$

Setting this equal to $\tfrac{1}{2}kx_m^2$, as Eq. 17-15 requires, we have

$$x_m = \sqrt{\frac{2E}{k}} = \sqrt{\frac{2(1.302 \text{ J})}{221 \text{ N/m}}} = 0.1085 \text{ m}.$$

To find the phase constant, we use the information given for $t = 0$:

$$x(0) = x_m \cos\phi$$

$$\cos\phi = \frac{x(0)}{x_m} = \frac{+0.0624 \text{ m}}{0.1085 \text{ m}} = +0.5751.$$

In the range of 0 to 2π, there are two values of ϕ whose cosine is $+0.5751$; the possible values are $\phi = 54.9°$ or $\phi = 305.1°$. Either one will satisfy the condition that $x(0)$ have the proper value, but only one will give the correct initial velocity:

$$\begin{aligned} v_x(0) = -\omega x_m \sin\phi &= -(9.536 \text{ rad/s})(0.1085 \text{ m}) \sin\phi \\ &= -(1.035 \text{ m/s}) \sin\phi \\ &= -0.847 \text{ m/s} \quad \text{for } \phi = 54.9° \\ &= +0.847 \text{ m/s} \quad \text{for } \phi = 305.1°. \end{aligned}$$

Obviously the second choice is the one we want, and we therefore take $\phi = 305.1° = 5.33$ radians. We can now write

$$x(t) = (0.109 \text{ m}) \cos[(9.54 \text{ rad/s})t + 5.33 \text{ rad}].$$

See Problem 9 for a derivation of the general relationships that permit x_m and ϕ to be calculated from $x(0)$ and $v_x(0)$.

17-5 APPLICATIONS OF SIMPLE HARMONIC MOTION

A few physical systems that move with simple harmonic motion are considered here. Others are found throughout the text.*

The Torsional Oscillator

Figure 17-9 shows a disk suspended by a wire attached to the center of mass of the disk. The wire is securely fixed to a solid support or clamp and to the disk. With the disk in equilibrium, a radial line is drawn from its center to a point P on its rim, as shown. If the disk is rotated in a horizontal (xy) plane so that the reference line OP moves to the position OQ, the wire will be twisted. The twisted wire will exert a restoring torque on the disk, tending to return the reference line to its equilibrium position. For small twists the restoring torque is found to be proportional to the angular displacement (Hooke's law), so that

$$\tau_z = -\kappa\theta. \tag{17-17}$$

Here κ (the Greek letter kappa) is a constant that depends on the properties of the wire and is called the *torsional* constant. The minus sign shows that the torque is directed op-

* See "A Repertoire of S.H.M.," by Eli Maor, *The Physics Teacher,* October 1972, p. 377, for a full discussion of 16 physical systems that exhibit simple harmonic motion.

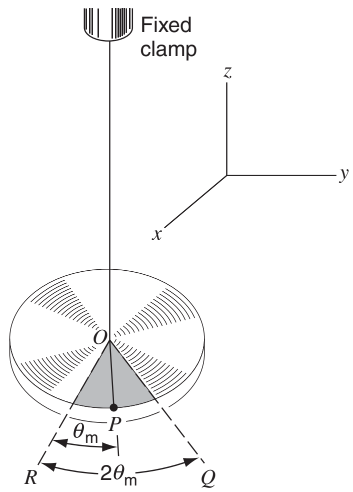


FIGURE 17-9. The torsional oscillator. The line drawn from O to P oscillates between OQ and OR, sweeping out an angle $2\theta_m$, where θ_m is the angular amplitude of the motion. The oscillation takes place in the xy plane; the z axis is along the wire.

posite to the angular displacement θ. Equation 17-17 is the condition for *angular simple harmonic motion.*

The equation of motion for such a system is based on the angular form of Newton's second law,

$$\sum \tau_z = I\alpha_z = I\frac{d^2\theta}{dt^2}, \qquad (17\text{-}18)$$

where I is the rotational inertia of the disk about the z axis. Using Eq. 17-17, we obtain

$$-\kappa\theta = I\frac{d^2\theta}{dt^2}$$

or

$$\frac{d^2\theta}{dt^2} = -\left(\frac{\kappa}{I}\right)\theta. \qquad (17\text{-}19)$$

Note the similarity between Eq. 17-19 for angular simple harmonic motion and Eq. 17-5 for linear simple harmonic motion. In fact, the equations are mathematically identical. Just as in Chapter 8, we can simply substitute angular displacement θ for linear displacement x, rotational inertia I for mass m, and torsional constant κ for force constant k. By making these substitutions, we find the solution of Eq. 17-19 to be a simple harmonic oscillation in the angle coordinate θ; namely,

$$\theta = \theta_m \cos(\omega t + \phi). \qquad (17\text{-}20)$$

Here θ_m is the maximum angular displacement, that is, the amplitude of the angular oscillation. Note that ω here means angular frequency, not angular velocity. In Eq. 17-20, $\omega \neq d\theta/dt$.

In Fig. 17-9 the disk oscillates about the equilibrium position $\theta = 0$, the total angular range being $2\theta_m$ (from OQ to OR). The period of the oscillation by analogy with Eq. 17-8 is

$$T = 2\pi\sqrt{\frac{I}{\kappa}}. \qquad (17\text{-}21)$$

If κ is known and T is measured, the rotational inertia I about the axis of rotation of any oscillating rigid body can be determined. If I is known and T is measured, the torsional constant κ of any sample of wire can be determined.

A torsional oscillator like that of Fig. 17-9 is also called a *torsional pendulum.* The Cavendish balance, used to measure the gravitational force constant G (see Chapter 14), is a torsional pendulum. Like the simple pendulum (discussed below) the torsional pendulum is often used for timekeeping, a common example being the balance wheel of a mechanical watch, in which the restoring torque is supplied by a spiral hairspring.

The Simple Pendulum

A simple pendulum is an idealized body consisting of a particle suspended by a light inextensible cord. When pulled to one side of its equilibrium position and released, the pendulum swings in a vertical plane under the influence of gravity. The motion is periodic and oscillatory. We wish to determine the period of the motion.

Figure 17-10 shows a pendulum of length L and particle mass m. At the instant shown, the cord makes an angle θ with the vertical. The forces acting on m are the weight $m\vec{g}$ and the tension $\vec{T}$ in the cord. The motion will be along an arc of the circle with radius L, and so we choose axes tangent to the circle and along the radius. The weight $m\vec{g}$ is resolved into a radial component of magnitude $mg\cos\theta$ and a tangential component of magnitude $mg\sin\theta$. The radial components of the forces supply the necessary centripetal acceleration to keep the particle moving on a circular arc. The tangential component is the restoring force acting on m tending to return it to the equilibrium position. Hence the restoring force is

$$F_x = -mg\sin\theta, \qquad (17\text{-}22)$$

the minus sign indicating that F_x is opposite to the direction of increasing x and increasing θ.

Note that the restoring force is not proportional to the angular displacement θ but to $\sin\theta$ instead. The resulting motion is therefore not simple harmonic. However, if the angle θ is small, $\sin\theta$ is very nearly equal to θ in radians. For example, if $\theta = 5°$ $(= 0.0873\text{ rad})$, then $\sin\theta = 0.0872$, which differs from θ by only about 0.1%. The displacement x is then approximately equal to the arc length $L\theta$, and for small angles this is nearly straight-line motion. Hence, assuming $\sin\theta \approx \theta$, we obtain

$$F_x = -mg\theta = -mg\frac{x}{L} = -\left(\frac{mg}{L}\right)x. \qquad (17\text{-}23)$$

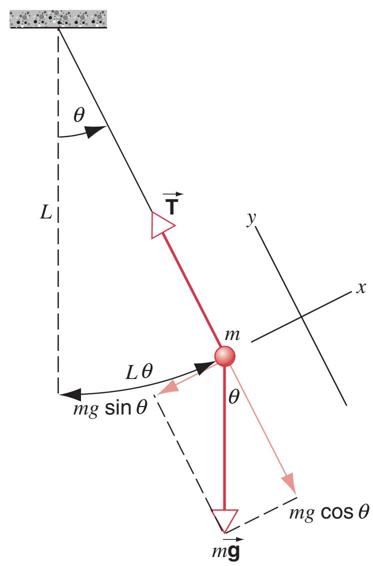


Figure 17-10. The simple pendulum. The forces acting on the pendulum are the tension $\vec{T}$ and the gravitational force $m\vec{g}$, which is resolved into its radial and tangential components. We choose the x axis to be in the tangential direction and the y axis to be in the radial direction at this particular time.

For *small displacements,* the restoring force is proportional to the displacement and is oppositely directed. This is exactly the criterion for simple harmonic motion, and in fact Eq. 17-23 has the same form as Eq. 17-2, $F_x = -kx$, with the constant mg/L representing the constant k. (Check that the dimensions of k and mg/L are the same.) The period of a simple pendulum when its amplitude is small is then found by putting $k = mg/L$ into Eq. 17-8:

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/L}}$$

or

$$T = 2\pi\sqrt{\frac{L}{g}}. \qquad (17\text{-}24)$$

Note that the period is independent of the mass of the suspended particle.

When the amplitude of the oscillation is not small, the general equation for the period can be shown* to be

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1}{2^2}\sin^2\frac{\theta_m}{2} + \frac{3^2}{2^2 4^2}\sin^4\frac{\theta_m}{2} + \cdots\right). \qquad (17\text{-}25)$$

Here θ_m is the maximum angular displacement. Note that T increases with increasing amplitude. Succeeding terms in the infinite series become smaller and smaller, and the period can be computed to any desired degree of accuracy by taking enough terms. When $\theta_m = 15°$, the true period differs from that given by Eq. 17-24 by less than 0.5%.

For the past three centuries, the pendulum has been our most reliable timekeeper, succeeded only in the last century by clocks based on atomic or electronic oscillations. For a pendulum clock to be an accurate timekeeper, the amplitude of the swing must be kept constant despite the frictional losses that affect all mechanical systems. Even so small a change in amplitude as from 5° to 4° would cause a pendulum clock to run fast by 0.25 minute per day, an unacceptable amount even for household timekeeping. To keep the amplitude constant in a pendulum clock, energy is automatically supplied in small increments from a weight or a spring by an escapement mechanism to compensate for frictional losses. The pendulum clock with escapement was invented by Christiaan Huygens (1629–1695).

The simple pendulum also provides a convenient method for measuring the value of g, the acceleration due to gravity. We can easily determine L and T using student laboratory equipment to a precision of less than 0.1%, and thus Eq. 17-24 permits us to determine g to about that precision. With better apparatus, this can be extended to about 0.0001%.

* This equation is derived in many intermediate mechanics textbooks. For example, see J.B. Marion and K.T. Thornton, *Classical Dynamics of Particles and Systems,* 3rd edition (Harcourt Brace Jovanovich, 1988), Section 3.13.

The Physical Pendulum

Any rigid body mounted so that it can swing in a vertical plane about some axis passing through it is called a *physical pendulum.* This is a generalization of the simple pendulum, in which a weightless cord holds a single particle. Actually all real pendulums are physical pendulums.

In Fig. 17-11 a body of irregular shape is pivoted about a horizontal frictionless axis through P and displaced from the equilibrium position by an angle θ. The equilibrium position is that in which the center of mass C of the body lies vertically below P. The distance from the pivot to the center of mass is d, the rotational inertia of the body about an axis through the pivot is I, and the mass of the body is M. The restoring torque for an angular displacement θ is

$$\tau_z = -Mgd\sin\theta \qquad (17\text{-}26)$$

and is due to the tangential component of the weight. Since τ_z is proportional to $\sin\theta$, rather than θ, the condition for angular simple harmonic motion does not, in general, hold here. For small angular displacements, however, the relation $\sin\theta \approx \theta$ is, as before, an excellent approximation, so that for small amplitudes,

$$\tau_z = -Mgd\,\theta. \qquad (17\text{-}27)$$

This is in the form of Eq. 17-17, and the period follows directly from Eq. 17-21 with the substitution $\kappa = Mgd$, which gives

$$T = 2\pi\sqrt{\frac{I}{Mgd}}. \qquad (17\text{-}28)$$

Equation 17-28 can be solved for the rotational inertia I, giving

$$I = \frac{T^2 Mgd}{4\pi^2}. \qquad (17\text{-}29)$$

The quantities on the right are all directly measurable. Hence the rotational inertia about an axis of rotation (other than through the center of mass) of a body of any shape can

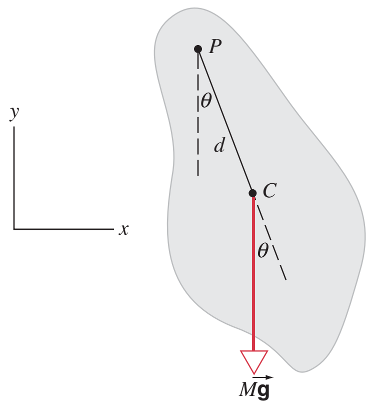


FIGURE 17-11. A physical pendulum. The center of mass is at C, and the pivot is at point P. The pendulum is displaced by an angle θ from its equilibrium position, which occurs when C hangs directly below P. The weight $M\vec{g}$ provides the restoring torque. The oscillation is in the xy plane. The z axis is out of the page.

be determined by suspending the body as a physical pendulum from that axis.

The physical pendulum includes the simple pendulum as a special case. Locating the pivot far from the object, using a weightless cord of length L, we would have $I = ML^2$ and $d = L$, so

$$T = 2\pi\sqrt{\frac{I}{Mgd}} = 2\pi\sqrt{\frac{ML^2}{MgL}} = 2\pi\sqrt{\frac{L}{g}},$$

which is the period of a simple pendulum.

If the mass of a physical pendulum were concentrated at the properly chosen distance L from the pivot, the resulting simple pendulum would have the same period as the original physical pendulum if

$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{I}{Mgd}}$$

or

$$L = \frac{I}{Md}. \qquad (17\text{-}30)$$

Hence, as far as its period of oscillation is concerned, the mass of a physical pendulum may be considered to be concentrated at a point whose distance from the pivot is $L = I/Md$. This point is called the *center of oscillation* of the physical pendulum. Note that it depends on the location of the pivot for any given body. Furthermore, if we pivot the original physical pendulum from this point, it will have the same period as it does when pivoted from point P.

The center of oscillation has another interesting property. If an impulsive force in the plane of oscillation acts at the center of oscillation, no effect of this force is felt at the pivot point. (See Problem 24 for a proof of this statement.) In this sense, the center of oscillation is often called the *center of percussion.* Baseball batters can avoid the "sting" on their hands (the pivot point of the bat) by hitting the ball at the center of percussion of the bat. The "sweet spot" on a tennis racket has a similar explanation; hitting the ball on this spot eliminates any reaction force on the hand.*

SAMPLE PROBLEM 17-4. A thin uniform rod of mass $M = 0.112$ kg and length $L = 0.096$ m is suspended by a wire that passes through its center and is perpendicular to its length. The wire is twisted and the rod set oscillating. The period is found to be 2.14 s. When a flat body in the shape of an equilateral triangle is suspended similarly through its center of mass, the period is found to be 5.83 s. Find the rotational inertia of the triangle about this axis.

Solution The rotational inertia of a rod, rotated about a central axis perpendicular to its length, is $ML^2/12$. Hence

$$I_{\text{rod}} = \frac{(0.112\ \text{kg})(0.096\ \text{m})^2}{12} = 8.60 \times 10^{-5}\ \text{kg}\cdot\text{m}^2.$$

From Eq. 17-21,

$$\frac{T_{\text{rod}}}{T_{\text{triangle}}} = \left(\frac{I_{\text{rod}}}{I_{\text{triangle}}}\right)^{1/2} \quad \text{or} \quad I_{\text{triangle}} = I_{\text{rod}}\left(\frac{T_{\text{triangle}}}{T_{\text{rod}}}\right)^2,$$

so that

$$I_{\text{triangle}} = (8.60 \times 10^{-5}\ \text{kg}\cdot\text{m}^2)\left(\frac{5.83\ \text{s}}{2.14\ \text{s}}\right)^2$$

$$= 6.38 \times 10^{-4}\ \text{kg}\cdot\text{m}^2.$$

Does the amplitude of either oscillation affect the period in these cases?

SAMPLE PROBLEM 17-5. A uniform disk is pivoted at its rim (Fig. 17-12). Find its period for small oscillations and the length of the equivalent simple pendulum.

Solution The rotational inertia of a disk about an axis through its center is $\frac{1}{2}MR^2$, where R is the radius and M is the mass of the disk. The rotational inertia about the pivot at the rim is, using the parallel axis theorem,

$$I = \tfrac{1}{2}MR^2 + MR^2 = \tfrac{3}{2}MR^2.$$

The period of this physical pendulum, found from Eq. 17-28 with $d = R$, is then

$$T = 2\pi\sqrt{\frac{I}{MgR}} = 2\pi\sqrt{\frac{3}{2}\frac{MR^2}{MgR}} = 2\pi\sqrt{\frac{3}{2}\frac{R}{g}},$$

independent of the mass of the disk.

The simple pendulum having the same period has a length (see Eq. 17-30)

$$L = \frac{I}{MR} = \tfrac{3}{2}R$$

or three-fourths the diameter of the disk. The center of oscillation of the disk pivoted at P is therefore at O, a distance $\frac{3}{2}R$ below the point of support. Is any particular mass required of the equivalent simple pendulum?

If we pivot the disk at a point midway between the rim and the center, as at O, we find that $I = \frac{1}{2}MR^2 + M(\frac{1}{2}R)^2 = \frac{3}{4}MR^2$ and $d = \frac{1}{2}R$. The period T is

$$T = 2\pi\sqrt{\frac{I}{Mgd}} = 2\pi\sqrt{\frac{\frac{3}{4}MR^2}{Mg(R/2)}} = 2\pi\sqrt{\frac{3}{2}\frac{R}{g}},$$

just as before. This illustrates the equality of the periods of the physical pendulum when pivoted about O and P.

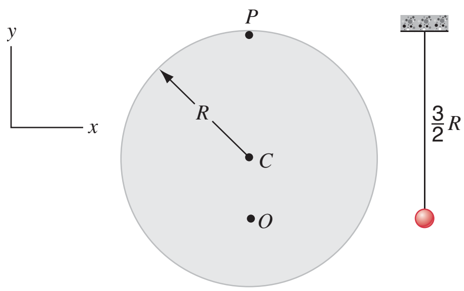


FIGURE 17-12. Sample Problem 17-5. A disk pivoted at its rim (point P) oscillates as a physical pendulum. To the right is shown a simple pendulum with the same period. Point O is the center of oscillation.

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* For an interesting collection of articles about these effects, see *The Physics of Sports,* edited by Angelo Armenti, Jr. (American Institute of Physics, 1992).

If the disk were pivoted at the center, what would be its period of oscillation?

SAMPLE PROBLEM 17-6. The period of a disk of radius 10.2 cm executing small oscillations about a pivot at its rim is measured to be 0.784 s. Find the value of g, the acceleration due to gravity at that location.

Solution From Sample Problem 17-5, we have

$$T = 2\pi\sqrt{\frac{3R}{2g}},$$

and solving for g, we obtain

$$g = \frac{6\pi^2 R}{T^2}.$$

With $T = 0.784$ s and $R = 0.102$ m, we find

$$g = \frac{6\pi^2(0.102\text{ m})}{(0.784\text{ s})^2} = 9.83\text{ m/s}^2.$$

17-6 SIMPLE HARMONIC MOTION AND UNIFORM CIRCULAR MOTION

In 1610, Galileo used his newly constructed telescope to observe the moons of Jupiter. As he watched night after night, he measured the position of each moon relative to the planet. He observed the moons to travel back and forth in motion that we would call simple harmonic. Figure 17-13 shows Galileo's original data, plotted to show the sidewise displacement of one moon (Callisto) as a function of the time. The sinusoidal dependence characteristic of simple harmonic motion is apparent.

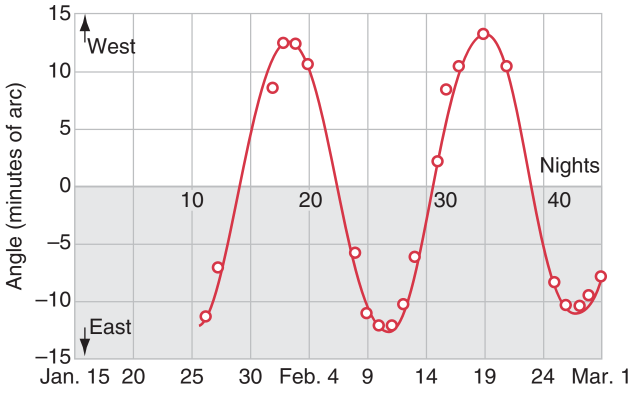


FIGURE 17-13. The angular position as a function of time of Jupiter's moon Callisto, as measured from Earth. The circles are based on Galileo's 1610 measurements. The curve is a best fit and strongly suggests simple harmonic motion. Nearly 400 years after Galileo, the motions of Jupiter's moons continue to delight the amateur astronomer. Each month the magazine *Sky and Telescope* publishes a chart showing their motions, in terms of a sinusoidally varying angular coordinate similar to this figure.

Actually, Callisto does not oscillate back and forth; it moves in a very nearly circular orbit about the planet, and what Galileo observed was uniform circular motion in a plane viewed edge on. Since this corresponds exactly with the displacement versus time relationship of simple harmonic motion, we are led to the following conclusion:

Simple harmonic motion can be described as the projection of uniform circular motion along a diameter of the circle.

Let us examine in more detail the mathematical basis for this conclusion. Figure 17-14 shows a particle P in uniform circular motion; its angular velocity is ω and the radius of the circle is r. At a time t (Fig. 17-14a), the vector $\vec{\mathbf{r}}$, which locates point P relative to the origin O, makes an angle $\omega t + \phi$ with the x axis, and the x component of $\vec{\mathbf{r}}$ is

$$x(t) = r\cos(\omega t + \phi). \tag{17-31}$$

This is of course identical to Eq. 17-6 for the displacement of the simple harmonic oscillator, with x_m corresponding to r. If we let P' represent the projection of P on the x axis, then P' executes simple harmonic motion along the x axis.

In uniform circular motion, the magnitude of the constant tangential speed is ωr. Figure 17-14b shows the vector representing the instantaneous velocity $\vec{\mathbf{v}}$ at time t. The x component of $\vec{\mathbf{v}}$, which gives the velocity of P' along the x direction, is

$$v_x(t) = -\omega r\sin(\omega t + \phi). \tag{17-32}$$

The centripetal acceleration in circular motion is $\omega^2 r$, and as shown in Fig. 17-14c, the x component of the acceleration of P is

$$a_x(t) = -\omega^2 r\cos(\omega t + \phi). \tag{17-33}$$

Equations 17-32 and 17-33 are identical with Eqs. 17-11 for simple harmonic motion, again with x_m replaced by r. Thus displacement, velocity, and acceleration are identical in simple harmonic motion and in the projection of circular motion.

Reversing the above argument, we can state that Eq. 17-31 for the displacement of a simple harmonic oscillator is sufficient to describe the x component of a vector whose tip traces a circular path at constant speed. If we can also describe the y component, then we have a complete description of the vector. Figure 17-14a shows the y projection OQ at time t, which can be described by

$$y(t) = r\sin(\omega t + \phi). \tag{17-34}$$

Note that the projection of uniform circular motion along the y direction also gives simple harmonic motion, as would projection along *any* direction. Note also that $x^2 + y^2 = r^2$ at all times, as we expect for circular motion. You should be able to find expressions for the y components of the velocity and acceleration and show that, as expected, $v_x^2 + v_y^2 = (\omega r)^2$ and $a_x^2 + a_y^2 = (\omega^2 r)^2$.

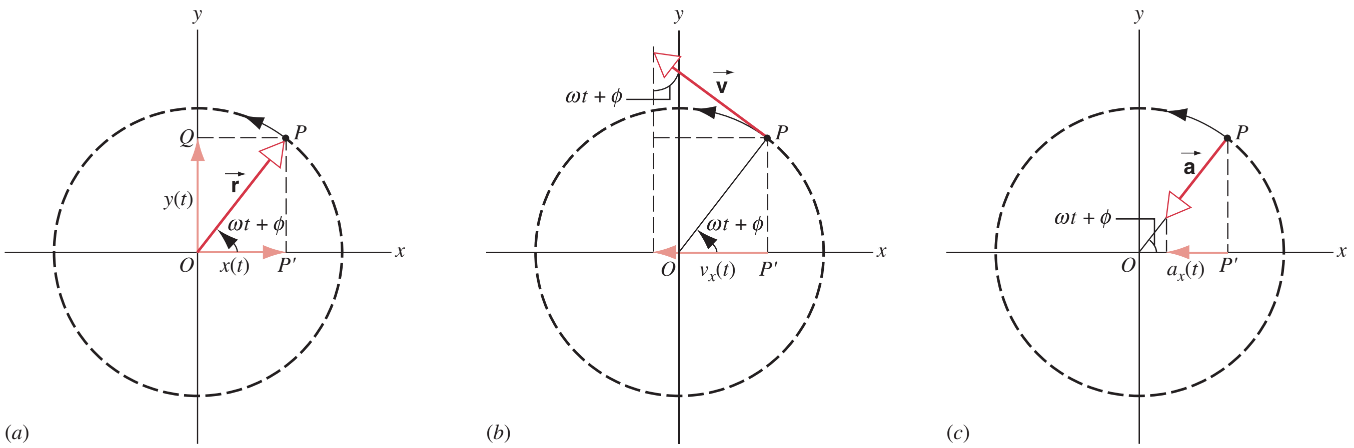


Figure 17-14. (*a*) Point *P* moves counterclockwise at constant speed around a circle of radius *r*. The vector $\vec{r}$ makes an angle $\omega t + \phi$ with the *x* axis. The projection *P′* on the *x* axis executes simple harmonic motion as *P* moves around the circle. (*b*) The velocity of *P* and its *x* component, which represents the velocity of *P′* in simple harmonic motion. (*c*) The acceleration of *P* and its *x* component.

Using the trigonometric identity $\sin\theta = \cos(\theta - \pi/2)$ we can rewrite Eq. 17-34 as

$$y(t) = r\cos(\omega t + \phi - \pi/2). \qquad (17\text{-}35)$$

Thus circular motion can be regarded as the combination of two simple harmonic motions at right angles, with identical amplitudes and frequencies but differing in phase by 90°. Other more complicated motions can be analyzed as combinations of simple harmonic motions with appropriately chosen amplitudes, frequencies, and phases. (See Exercises 41 and 42.)

Sample Problem 17-7. Consider a body executing a horizontal simple harmonic motion. The equation of that motion is

$$x = (0.35\text{ m})\cos[(8.3\text{ rad/s})t],$$

where *x* is in meters and *t* in seconds. This motion can also be represented as the projection of uniform circular motion along a horizontal diameter. (*a*) Give the properties of the corresponding uniform circular motion. (*b*) From the motion of the reference point, determine the time required for the body to come halfway in toward the center of motion from its initial position.

Solution (*a*) The *x* component of the circular motion is given by

$$x = r\cos(\omega t + \phi).$$

Therefore the reference circle must have a radius $r = 0.35$ m, the initial phase or phase constant must be $\phi = 0$, and the angular speed must be $\omega = 8.3$ rad/s, in order to obtain the given equation for the horizontal projection.

(*b*) As the body moves halfway in, the reference point moves through an angle of $\omega t = \pi/3 = 60°$ (Fig. 17-15). The angular speed is constant at 8.3 rad/s, so that the time required to move through 60° is

$$t = \frac{60°}{\omega} = \frac{\pi/3\text{ rad}}{8.3\text{ rad/s}} = 0.13\text{ s}.$$

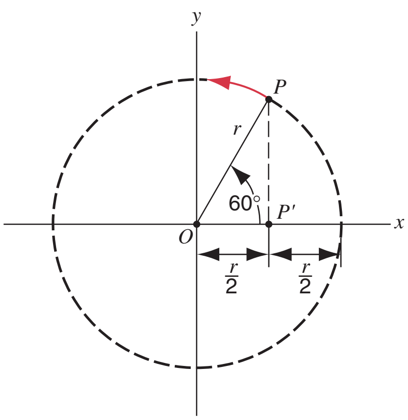


Figure 17-15. Sample Problem 17-7. The radius *OP* moves from $\phi = 0$ at $t = 0$ to $\omega t = 60°$ at time *t*. The projection *P′* moves correspondingly from $x = r$ to $x = r/2$.

17-7 DAMPED HARMONIC MOTION

Up to this point we have assumed that no frictional forces act on the oscillator. If this assumption held strictly, a pendulum or a mass on a spring would oscillate indefinitely with a constant mechanical energy (that is, with no loss in the amplitude of the oscillation). Since we observe a loss in amplitude for real oscillators, we know that this assumption is not strictly true, although it may be a good approximation for some oscillators. Fortunately, the period is nearly independent of the amplitude for small-amplitude oscillations, so the decrease in amplitude causes a negligible change in the period of the oscillator.

This loss in amplitude is called *damping* and the motion is called *damped harmonic motion.* There are many causes of damping, including friction, air resistance, and internal forces.

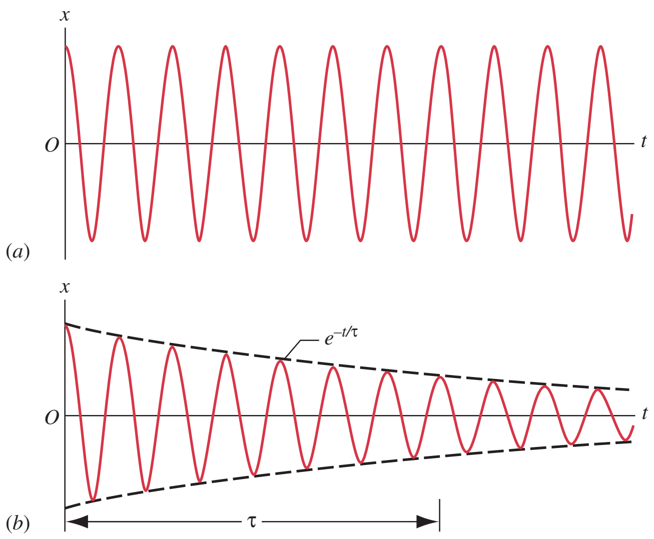


FIGURE 17-16. (*a*) Undamped oscillation, drawn for a phase constant ϕ of zero. (*b*) Damped oscillation with the same frequency as (*a*). The lifetime τ is the time necessary for the amplitude to decrease to $1/e = 0.368$ of its initial value.

Figure 17-16 compares the motion of undamped and damped oscillators. When we add a small damping force, the frequency changes by a negligible amount but the amplitude gradually decreases to zero. In many cases this decrease in amplitude can be accounted for by multiplying the equation for the undamped oscillator (Eq. 17-6) by an exponential function that describes the dashed curves in Fig. 17-16*b*:

$$x(t) = x_m e^{-t/\tau} \cos(\omega t + \phi), \tag{17-36}$$

where τ is called the *damping time constant* or the *mean lifetime* of the oscillation. Mathematically, it is the time necessary for the amplitude to drop to $1/e$ of its initial value, as shown in Fig. 17-16*b*. The solid curve in Fig. 17-16*b* is a plot of Eq. 17-36.

Each of the "peaks" in Fig. 17-16*b* represents a time when $\cos(\omega t + \phi) = 1$. When the exponential decay is slow compared with the variation in the cosine term (that is, when τ is large compared with the oscillation period $2\pi/\omega$), these points correspond to turning points of the motion, where the velocity is zero. At those instants the mechanical energy of the oscillator is all potential energy $\frac{1}{2}kx^2$ and so

$$E(t) = \tfrac{1}{2}kx_m^2 e^{-2t/\tau}. \tag{17-37}$$

Equation 17-37 shows that the mechanical energy of the oscillator decreases exponentially with time (but note that the energy decreases twice as rapidly as the amplitude—E falls to $1/e$ of its initial value in a time of $\tau/2$). The lost mechanical energy might appear in a variety of forms, depending on the nature of the damping force—for example, as increased kinetic energy (increased temperature) of the surrounding air due to air resistance, or as internal energy (also increased temperature) of the spring due to internal stretching and compression forces.

Mathematical Analysis (Optional)

If we assume a particular form for the damping force, we can use Newton's laws to solve for the equations of motion. Figure 17-17 shows a simple model of a damped oscillator. We assume the block to slide on a frictionless surface, and we represent the damping in terms of a (massless) vane that moves in a viscous fluid. We can represent the damping force due to the fluid in exactly the same way that we represented the drag force on a projectile in Section 4-4: $F_x = -bv_x$, where b is a positive constant called the *damping constant* that depends on the properties of the fluid and the size and shape of the vane that is immersed in the fluid. With $\Sigma F_x = -kx - bv_x$, Newton's second law gives

$$-kx - bv_x = ma_x$$

or, with $v_x = dx/dt$ and $a_x = d^2x/dt^2$,

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0. \tag{17-38}$$

The solution to this equation, which you can verify by direct substitution (see Exercise 45), is

$$x(t) = x_m e^{-bt/2m} \cos(\omega' t + \phi), \tag{17-39}$$

where

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}. \tag{17-40}$$

This solution assumes that the damping constant is small, so that the quantity under the square root in Eq. 17-40 cannot be negative.

Note that Eq. 17-39 has the same form as Eq. 17-36, with the lifetime $\tau = 2m/b$. The greater is the damping constant b, the more quickly the amplitude of the oscillation dies out. As b approaches zero (corresponding to no damping), then τ is infinite and the amplitude remains constant.

When damping is present, the oscillation frequency is smaller (the period is larger). That is, damping slows down the motion, as we might expect. If $b = 0$ (no damping, then $\omega' = \sqrt{k/m}$, which is simply the angular frequency ω of the undamped motion. When damping is present, ω' is slightly less than ω, but in most cases of interest the damping is sufficiently weak that $\omega' \approx \omega$. For example, in the case shown in Fig. 17-16*b*, in which the amplitude drops by half in five

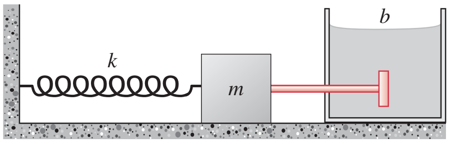


FIGURE 17-17. A representation of a damped harmonic oscillator. We consider the oscillating body (of mass m) to be attached to a (massless) vane immersed in a fluid, in which it experiences a viscous damping force $-bv_x$. We do not consider sliding friction at the horizontal surface.

cycles of oscillation, we would have $\omega' = 0.9998\omega$. It is for this reason that we used the undamped frequency ω in Eq. 17-36.

In the special case in which $b = 2\sqrt{km}$, Eq. 17-40 gives $\omega' = 0$, so the motion decays exponentially to zero with no oscillation at all. In this case the lifetime τ (see Eq. 17-36) has its smallest possible value, $1/\omega$. This condition, called *critical damping,* is often the goal of mechanical engineers in designing systems in which unwanted and often harmful oscillations can be made to disappear in the shortest possible time.

SAMPLE PROBLEM 17-8. In a damped oscillator such as that of Fig. 17-17, let $m = 250$ g, $k = 85$ N/m, and $b = 0.070$ kg/s. In how many periods of oscillation will the mechanical energy of the oscillator drop to one-half of its initial value?

Solution For small damping, $\omega' \approx \omega$ and the period is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.25\text{ kg}}{85\text{ N/m}}} = 0.34\text{ s.}$$

At $t = 0$, the initial mechanical energy is $\frac{1}{2}kx_m^2$. According to Eq. 17-37, the energy will have half this value at a time t determined from

$$\tfrac{1}{2}(\tfrac{1}{2}kx_m^2) = \tfrac{1}{2}kx_m^2 e^{-2t/\tau}.$$

Solving for t and using $\tau = 2m/b$, we obtain

$$t = \tfrac{1}{2}\tau \ln 2 = \frac{m \ln 2}{b} = \frac{(0.25\text{ kg})(\ln 2)}{0.070\text{ kg/s}} = 2.5\text{ s.}$$

The time t is about $7.4T$; thus about 7.4 cycles of the oscillation are required for the mechanical energy to drop by half.

17-8 FORCED OSCILLATIONS AND RESONANCE

Left on its own, the motion of an oscillator repeats with its *natural frequency* ω, determined, for example, according to Eqs. 17-9 and 17-10. In the presence of a small damping force, the frequency does not change very much from this value.

Another interesting class of situations occurs when we apply a sinusoidal external force to the oscillator. For example, our eardrums vibrate when exposed to the periodic force of a sound wave or a molecule vibrates when it absorbs an electromagnetic wave of a certain frequency. The resulting oscillations are called *forced oscillations* and have important applications not only in mechanics but also in acoustics, electric circuits, and atomic physics.

These forced oscillations occur at the frequency of the external force and not at the natural frequency of the vibrating system. However, the amplitude of the oscillation depends on the relationship between the natural frequency and the frequency of the applied force. A succession of small impulses applied at the proper frequency can produce an oscillation of large amplitude. For example, when you push a friend on a swing, applying your pushes precisely at the same time in each cycle causes your friend to move in an increasingly large arc.

We assume that we are dealing with a real oscillator in which a damping force is present. (Otherwise, the energy delivered to the oscillator by the external force would continue to accumulate, and the amplitude would increase without bound.) We consider the damped oscillator of Fig. 17-16*b*, which we show again in Fig. 17-18*a*. The natural frequency of the oscillator is ω, and we assume that the damping is sufficiently small so that it does not significantly change this frequency. We now apply a sinusoidal force $F_x(t) = F_m \sin \omega'' t$, which we assume to have a constant amplitude F_m.

When we first apply this force, the motion is dominated by short-lived transient terms that die away in a time characteristic of the damping lifetime τ. We examine the motion in the "steady state" after these terms have become negligible. Figure 17-18*b* shows the resulting motion when the driving frequency is half the natural frequency. Note that the motion is a simple sinusoidal oscillation, but at the driving frequency ω'' rather than at the natural frequency ω. Figure 17-18*c* shows the motion for a driving force of the same amplitude but with $\omega'' = 0.8\omega$. The amplitude of the oscillation in Fig. 17-18*c* is about twice as large as that in

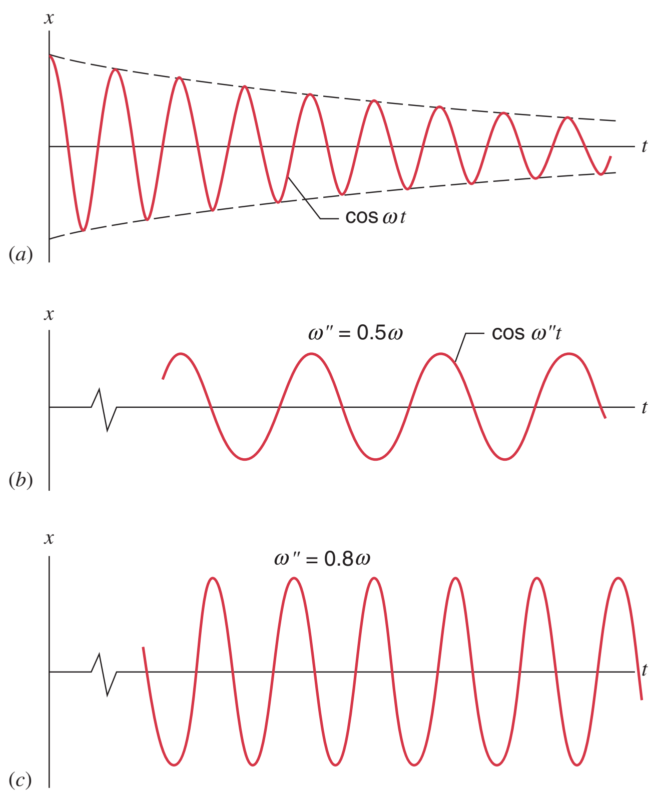


FIGURE 17-18. (*a*) A damped oscillator (identical with Fig. 16*b*). (*b*) The same oscillator subject to an applied force with $\omega'' = 0.5\omega$. (*c*) The forced oscillator with $\omega'' = 0.8\omega$. Because ω'' is closer to resonance, the amplitude of the oscillation is larger even though the applied force has the same amplitude in (*b*) and (*c*).

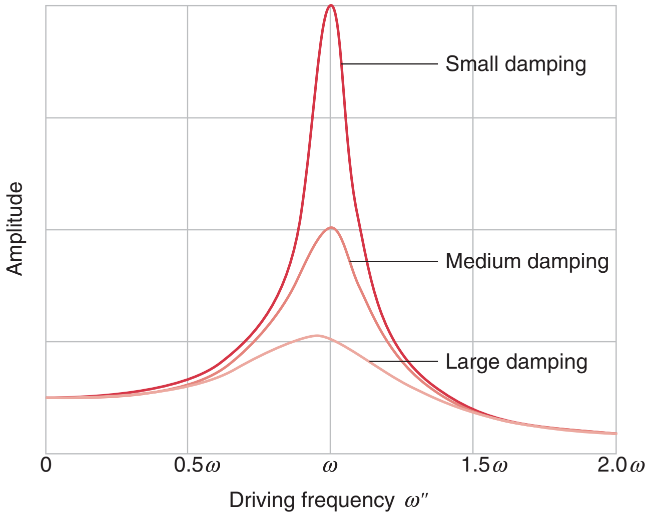


FIGURE 17-19. The amplitude of a forced oscillator as the angular frequency ω'' of the driving force is varied. The three curves correspond to different levels of damping, the smallest damping giving the sharpest resonance curve. Medium damping corresponds to twice the damping force, and large damping to four times the small damping force.

Fig. 17-18*b*. As ω'' approaches ω (with F_m held constant), the amplitude of the motion continues to increase: when $\omega'' = 0.9\omega$, the amplitude is about four times that of Fig. 17-18*b*, and it grows to about 40 times as large for $\omega'' = 0.99\omega$.

When the damping is small, the forced oscillations reach their maximum displacement amplitude when the driving frequency is equal to the natural frequency. This condition is known as *resonance* and the corresponding frequency ω'' is called the *resonant angular frequency:*

$$\omega'' = \omega \qquad \text{(resonance condition).} \qquad (17\text{-}41)$$

(Occasionally other definitions of resonance are used, for example, the frequency at which the maximum power is delivered to the oscillator or the frequency at which its velocity is a maximum. These definitions are not equivalent; for example, as we discuss in Chapter 36, resonance in oscillating electrical circuits is generally defined in terms of the current amplitude, which is analogous to a velocity resonance.)

As Fig. 17-18*b* and *c* show, at resonance the system oscillates at the frequency of the driving force with constant amplitude (if the driving force is of constant amplitude). Damping is present, which would normally cause a decrease in amplitude, but the source of the driving force provides the additional energy needed to keep the amplitude of the oscillation constant. In this steady state, the rate at which energy is provided by the driving force exactly matches the rate at which energy is dissipated by the damping force. In effect, the oscillator transfers energy from the driving source to the damping medium; there is no net energy increase to the oscillator. Note especially that at resonance the oscillation amplitude does not increase without bound, but instead remains constant.

Figure 17-19 shows the amplitude of the forced vibrations as the driving frequency is varied in the vicinity of the natural frequency ω. When the damping is small, the amplitude of the forced oscillation increases rapidly as ω'' approaches ω and reaches its maximum when $\omega'' = \omega$. For larger damping, the amplitude does not increase nearly as rapidly near resonance, and for the largest damping the resonant frequency is even displaced slightly from the natural frequency.

All mechanical structures—such as buildings, bridges, and airplanes—have one or more natural frequencies of oscillation. If the structure is subject to a driving frequency that matches one of the natural frequencies, the resulting large amplitude of oscillation can have disastrous consequences. Shattering a wine glass with a sound wave that matches one of the natural frequencies of the glass is but one demonstration of this effect; the collapse of roadways and bridges in earthquakes is a more serious outcome.

Another example of resonance occurred in the Tacoma Narrows Bridge in Washington State in 1940. The wind blowing through the Narrows broke up into vortices, in effect providing small puffs that shook the bridge at a frequency that matched one of its natural frequencies. The result was a gentle rolling motion, somewhat like a roller coaster, which earned the bridge the nickname "Galloping Gertie." About 5 months after the bridge opened, the gentle rolling oscillations became violent torsional oscillations (Fig. 17-20). These oscillations were not a result of resonance but of nonlinear effects due to particularly strong wind gusts. Such complex effects cannot be analyzed in terms of the forced oscillator we have discussed here.

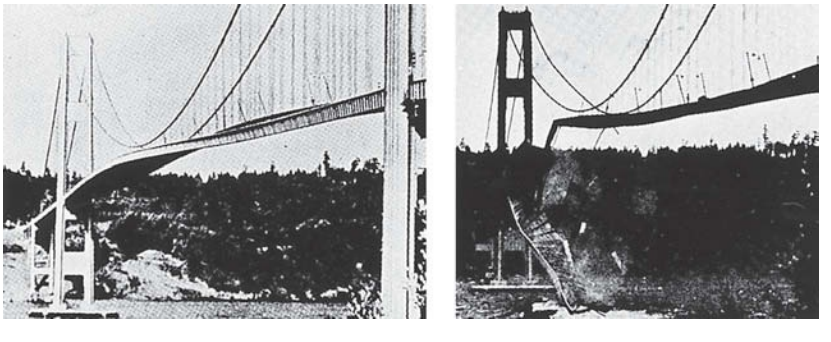


FIGURE 17-20. The Tacoma Narrows Bridge on Puget Sound, Washington. Completed and opened to traffic in July 1940, it immediately showed gentle rolling oscillations due to resonance. Later the bridge developed violent torsional oscillations shown at left. Eventually the main span broke up, sending the bridge roadway crashing into the water below, as shown at right.

Mathematical Analysis (Optional)

We again consider a damping force of the form $-bv_x$, and we take the driving force to be $F_m \cos \omega'' t$. The experimental set-up might be similar to Fig. 17-17, with the fixed wall on the left replaced by a movable support attached to the shaft of a motor that rotates at angular velocity ω''. With $\Sigma F_x = -kx - bv_x + F_m \cos \omega'' t$, Newton's second law gives

$$-kx - bv_x + F_m \cos \omega'' t = ma_x$$

or, with $v_x = dx/dt$ and $a_x = d^2x/dt^2$,

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_m \cos \omega'' t. \qquad (17\text{-}42)$$

After a sufficient time for the initial transients to die away, the solution to this equation is

$$x(t) = \frac{F_m}{G}\cos(\omega'' t - \beta), \qquad (17\text{-}43)$$

where

$$G = \sqrt{m^2(\omega''^2 - \omega^2)^2 + b^2\omega''^2} \qquad (17\text{-}44)$$

and

$$\beta = \cos^{-1}\frac{b\omega''}{G}. \qquad (17\text{-}45)$$

You can verify that Eq. 17-43 is a solution of Eq. 17-42 by calculating the first and second derivatives of $x(t)$ and substituting them into Eq. 17-42.

Note that for small values of the damping constant b, G has its smallest value for $\omega'' = \omega$, and so the amplitude of $x(t)$, which is equal to F_m/G, has its largest value there. This is responsible for the resonance peaks in Fig. 17-19, and the curves in that figure are plots of F_m/G for various values of the damping constant b.

17-9 TWO-BODY OSCILLATIONS (Optional)

In a *two-body collision,* such as is illustrated in Fig. 17-21*a*, a spring connects two objects, each of which is free to move. When the objects are displaced and released, they both oscillate. Many examples of two-body oscillations are found in nature. In diatomic molecules, two atoms are bound together by a force of the form illustrated in Fig. 17-3. Near the equilibrium position, the potential energy can be approximated by a parabolic shape, which corresponds to that of a simple harmonic oscillator. The emission and absorption of radiation by diatomic molecules can be understood based on the energy associated with this type of oscillation. Similar oscillations also occur in nuclei; in one type of motion, the protons and neutrons oscillate against each other just like the two bodies in Fig. 17-21*a*, and the nucleus can emit and absorb radiation in a manner similar to the diatomic molecule.

In general this type of motion is complicated to analyze. However, the description can be simplified if we replace the separate coordinates of the bodies (x_1 and x_2 in Fig. 17-21*a*) with two other coordinates: the relative separation $x_1 - x_2$ and the location x_{cm} of the center of mass. In the absence of external forces, the center of mass moves at constant velocity, and its motion is of no real interest in studying the oscillation of the system, so we can analyze the system in terms of the relative coordinate alone.

The relative separation $x_1 - x_2$ gives the length of the spring at any time. Suppose its unstretched length is L; then $x = (x_1 - x_2) - L$ is the change in length of the spring, and $F = kx$ is the magnitude of the force exerted on *each particle* by the spring. As shown in Fig. 17-21*a*, if the spring exerts a force $-\vec{F}$ on m_1, then it exerts a force $+\vec{F}$ on m_2.

Let us apply Newton's second law separately to the two particles, taking force components along the x axis:

$$m_1\frac{d^2x_1}{dt^2} = -kx,$$

$$m_2\frac{d^2x_2}{dt^2} = +kx.$$

We now multiply the first of these equations by m_2 and the second by m_1, and then subtract. The result is

$$m_1m_2\frac{d^2x_1}{dt^2} - m_1m_2\frac{d^2x_2}{dt^2} = -m_2kx - m_1kx,$$

which we can write as

$$\frac{m_1m_2}{m_1 + m_2}\frac{d^2}{dt^2}(x_1 - x_2) = -kx. \qquad (17\text{-}46)$$

The quantity $m_1m_2/(m_1 + m_2)$ has the dimension of mass and is known as the *reduced mass m*:

$$m = \frac{m_1m_2}{m_1 + m_2}. \qquad (17\text{-}47)$$

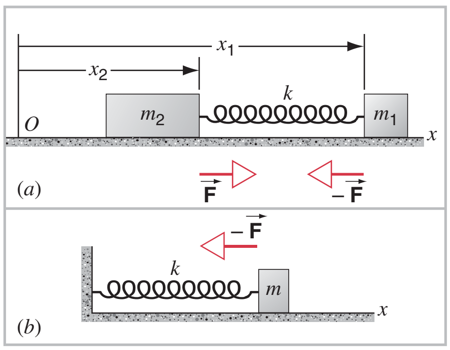


FIGURE 17-21. (*a*) Two oscillating bodies of masses m_1 and m_2 connected by a spring. (*b*) The relative motion can be represented by the oscillation of a single body having the reduced mass m.

Because the unstretched length L of the spring is a constant, the derivatives of $(x_1 - x_2)$ are the same as the derivatives of x:

$$\frac{d}{dt}(x_1 - x_2) = \frac{d}{dt}(x + L) = \frac{dx}{dt},$$

and so Eq. 17-46 becomes

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0.$$

This is identical in form to Eq. 17-4 for the single oscillating mass, thus demonstrating that, from the standpoint of oscillations, the system of Fig. 17-21*a* can be replaced by a single particle, as represented in Fig. 17-21*b*, with a mass equal to the reduced mass of the system. In particular, the frequency of oscillation of the system of Fig. 17-21 is given by Eq. 17-9, using the reduced mass.

If we wish to examine the detailed motion of the system, we can simply write down the solution for $x(t)$, $v_x(t)$, and $a_x(t)$ given by Eqs. 17-11, keeping in mind that x represents the relative coordinate of the two particles, and thus v_x and a_x represent their *relative* velocity $v_{1x} - v_{2x}$ and acceleration $a_{1x} - a_{2x}$, respectively.

Note that the reduced mass m is always smaller than either mass. If one of the masses is very much smaller than the other, then m is roughly equal to the smaller mass. If the masses are equal, then m is half as large as either mass.

SAMPLE PROBLEM 17-9. Naturally occurring chlorine consists of two isotopes: ^{35}Cl, of relative abundance 76% and atomic mass 34.968853 u, and ^{37}Cl, of relative abundance 24% and atomic mass 36.965903 u. (*a*) What is the reduced mass of a molecule of HCl when it contains ^{35}Cl and when it contains ^{37}Cl? (*b*) The vibrational frequency of a molecule of HCl is 8.5×10^{13} Hz. Assuming HCl to behave like a simple two-body oscillator, find the effective force constant k.

Solution (*a*) The reduced mass for H^{35}Cl is found from Eq. 17-47, using the H mass of 1.007825 u:

$$m = \frac{m_1 m_2}{m_1 + m_2} = \frac{(1.007825\text{ u})(34.968853\text{ u})}{1.007825\text{ u} + 34.968853\text{ u}} = 0.979593\text{ u}.$$

For H^{37}Cl we have similarly

$$m = \frac{(1.007825\text{ u})(36.965903\text{ u})}{1.007825\text{ u} + 36.965903\text{ u}} = 0.981077\text{ u}.$$

(*b*) Solving Eq. 17-9 for the force constant, we obtain

$$k = 4\pi^2 f^2 m = 4\pi^2(8.5 \times 10^{13}\text{ Hz})^2(0.98\text{ u})(1.66 \times 10^{-27}\text{ kg/u})$$
$$= 460\text{ N/m}.$$

This is of the same order of magnitude as the force constant of ordinary springs (for example, see Sample Problem 17-1). Can you explain how the force constant for one molecule can be the same as that of a spring?

Molecules can absorb or emit electromagntic radiation and change their state of vibrational motion in the process. In fact, observing the radiation that is absorbed or emitted is one of the ways we learn about the structure of molecules. Figure 17-22 shows an example of the infrared absorption spectrum of HCl. Each peak corresponds to a change in the vibrational state of the HCl when it absorbs radiation at that frequency. The two components to each peak are due to the two isotopes of Cl; their different masses result in slightly different reduced masses for molecules of H^{35}Cl and H^{37}Cl, as we found in part (*a*), and therefore in slightly different vibrational frequencies.

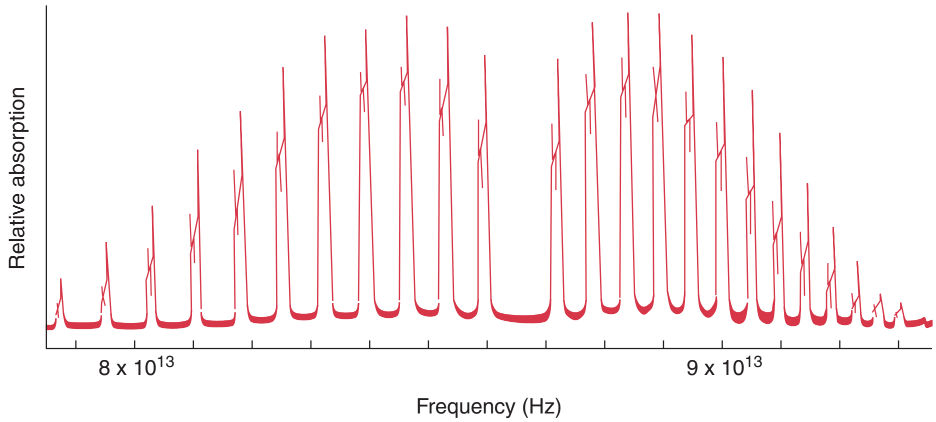


FIGURE 17-22. The absorption spectrum of infrared radiation by molecular HCl. Each peak corresponds to a change in the vibrational motion of the molecules. The closely spaced pairs of peaks are due to the two isotopes of Cl.

MULTIPLE CHOICE

17-1 Oscillating Systems

1. A particle oscillates about the equilibrium position x_0 subject to a force that has an associated potential energy $U(x)$. Which of the following statements (perhaps more than one) about $U(x)$ is true?
   (A) $U(x)$ must be symmetric about x_0.
   (B) $U(x)$ must have a minimum at x_0.
   (C) $U(x)$ may have a maximum at x_0.
   (D) $U(x)$ must be positive in the vicinity of x_0.
2. The equilibrium position of an object in an oscillating system is always the point where

(A) $x = 0$. (B) $v_x = 0$.
(C) $a_x = 0$. (D) $p_x = 0$.

17-2 The Simple Harmonic Oscillator

17-3 Simple Harmonic Motion

3. A particle on a spring executes simple harmonic motion. If the mass of the particle and the amplitude are both doubled then

(*a*) the period of oscillation will change by a factor of
(A) 4. (B) $\sqrt{8}$. (C) 2. (D) $\sqrt{2}$.
(E) 1 (it remains unchanged).

(*b*) the maximum speed of the particle will change by a factor of
(A) 4. (B) $\sqrt{8}$. (C) 2. (D) $\sqrt{2}$.
(E) 1 (it remains unchanged).

(*c*) the magnitude of the maximum acceleration of the particle will change by a factor of
(A) 4. (B) $\sqrt{8}$. (C) 2. (D) $\sqrt{2}$.
(E) 1 (it remains unchanged).

4. A particle on a spring executes simple harmonic motion; when it passes through the equilibrium position it has a speed v. The particle is stopped, and then the oscillations are restarted so that it now passes through the equilibrium position with a speed of $2v$. After this change

(*a*) the frequency of oscillation will change by a factor of
(A) 4. (B) $\sqrt{8}$. (C) 2. (D) $\sqrt{2}$.
(E) 1 (it remains unchanged).

(*b*) the maximum displacement of the particle will change by a factor of
(A) 4. (B) $\sqrt{8}$. (C) 2. (D) $\sqrt{2}$.
(E) 1 (it remains unchanged).

(*c*) the magnitude of the maximum acceleration of the particle will change by a factor of
(A) 4. (B) $\sqrt{8}$. (C) 2. (D) $\sqrt{2}$.
(E) 1 (it remains unchanged).

17-4 Energy in Simple Harmonic Motion

5. A particle on a spring executes simple harmonic motion. When the particle is found at $x = x_{max}/2$ the speed of the particle is
(A) $v_x = v_{max}$. (B) $v_x = \sqrt{3}v_{max}/2$.
(C) $v_x = \sqrt{2}v_{max}/2$. (D) $v_x = v_{max}/2$.

6. A particle on a spring executes simple harmonic motion. If the total energy of the particle is doubled then

(*a*) the period of oscillation will increase by a factor of
(A) 4. (B) $\sqrt{8}$. (C) 2. (D) $\sqrt{2}$.
(E) 1 (it remains unchanged).

(*b*) the maximum speed of the particle will increase by a factor of
(A) 4. (B) $\sqrt{8}$. (C) 2. (D) $\sqrt{2}$.
(E) 1 (it remains unchanged).

(*c*) the magnitude of the maximum acceleration of the particle will increase by a factor of
(A) 4. (B) $\sqrt{8}$. (C) 2. (D) $\sqrt{2}$.
(E) 1 (it remains unchanged).

17-5 Applications of Simple Harmonic Motion

7. A round metal hoop is suspended on the edge by a hook. The hoop can oscillate side to side in the plane of the hoop, or it can oscillate back and forth in a direction perpendicular to the plane of the hoop. For which mode will the frequency of oscillation be larger?
(A) Oscillations in the plane of the hoop
(B) Oscillations perpendicular to the plane of the hoop
(C) The frequency of oscillation will be the same in either mode.

8. What are the units for κ in Eq. 17-17?
(A) Newton/(meter·radian)
(B) Newton·meter/radian
(C) Kilogram/(radian·second^2)
(D) Kilogram·radian^2/second^2

17-6 Simple Harmonic Motion and Uniform Circular Motion

9. An object of mass m is moving in uniform circular motion in the xy plane. The circle has radius R and the object is moving around the circle with speed v. The motion is projected onto the x axis where it appears as simple harmonic motion according to $x(t) = R\cos(\omega t + \phi)$.

(*a*) In this projection ω is
(A) v/R. (B) m^2R.
(C) R/v. (D) $v/(R \sin \omega t)$.

(*b*) In this projection ϕ is
(A) 0. (B) vt/ω. (C) π.
(D) ϕ cannot be determined from the information given.

17-7 Damped Harmonic Motion

10. Let ω' be the angular frequency of a damped oscillator, and ω be the angular frequency of an identical but undamped oscillator. The damped frequency ω' will equal $\omega/2$ if:
(A) $b = m\omega$. (B) $b = \sqrt{2}m\omega$.
(C) $b = \sqrt{3}m\omega$. (D) $b = 2m\omega$.

17-8 Forced Oscillations and Resonance

11. A driven damped oscillator will, after all transient motion has died out, oscillate at
(A) the driving frequency.
(B) the frequency of the damped but free oscillator.
(C) the frequency of the undamped but free oscillator.
(D) any of the above, because the frequencies are all the same.

12. The resonant frequency of a driven damped oscillator is equal to
(A) the driving frequency.
(B) the frequency of the damped but free oscillator
(C) the frequency of the undamped but free oscillator.
(D) any of the above, because the frequencies are all the same.

17-9 Two-Body Oscillations

13. A diatomic molecule can be thought of as a dumbbell: two masses joined together by an ideal spring. The system can oscillate with a frequency ω, but it can also rotate about the center of mass. If the molecule rotates what happens to

(*a*) x_{eq}, the equilibrium separation?
(A) x_{eq} decreases.
(B) x_{eq} remains the same.
(C) x_{eq} increases.

(*b*) ω, the vibrational frequency?
(A) ω decreases.
(B) ω remains the same.
(C) ω increases.

QUESTIONS

1. Give some examples of motions that are approximately simple harmonic. Why are motions that are exactly simple harmonic rare?

2. A typical screen-door spring is tension-stressed in its normal state; that is, adjacent turns cling to each other and resist separation. Does such a spring obey Hooke's law?

3. Is Hooke's law obeyed, even approximately, by a diving board? A trampoline? A coiled spring made of lead wire?

4. What would happen to the motion of an oscillating system if the sign of the force term, $-kx$ in Eq. 17-2, were changed?

5. A spring has a force constant k, and an object of mass m is suspended from it. The spring is cut in half and the same object is suspended from one of the halves. How are the frequencies of oscillation, before and after the spring is cut, related?

6. An unstressed spring has a force constant k. It is stretched by a weight hung from it to an equilibrium length well within the elastic limit. Does the spring have the same force constant k for displacements from this new equilibrium position?

7. Suppose we have a block of unknown mass and a spring of unknown force constant. Show how we can predict the period of oscillation of this block–spring system simply by measuring the extension of the spring produced by attaching the block to it.

8. Any real spring has mass. If this mass is taken into account, explain qualitatively how this will affect the period of oscillation of a spring–block system.

9. Can one have an oscillator that even for small amplitudes is not simple harmonic? That is, can one have a nonlinear restoring force in an oscillator even at arbitrarily small amplitudes?

10. How are each of the following properties of a simple harmonic oscillator affected by doubling the amplitude: period, force constant, total mechanical energy, maximum velocity, maximum acceleration?

11. What changes could you make in a harmonic oscillator that would double the maximum speed of the oscillating object?

12. A person stands on a bathroom-type scale, which rests on a platform suspended by a large spring. The whole system executes simple harmonic motion in a vertical direction. Describe the variation in scale reading during a period of motion.

13. Could we ever construct a true simple pendulum? Explain your answer.

14. Could standards of mass, length, and time be based on properties of a pendulum? Explain.

15. Considering the elastic and the inertial aspects involved, explain the fact that whereas when an object of mass m oscillates vertically on a spring the period depends on m but is independent of g, the reverse is true for a simple pendulum.

16. Predict by qualitative arguments whether a pendulum oscillating with large amplitude will have a period longer or shorter than the period for oscillations with small amplitude. (Consider extreme cases.)

17. As the amplitude θ_m in Eq. 17-25 approaches 180°, what value do you expect the period to approach? Explain in physical terms.

18. What happens to the frequency of a swing as its oscillations die down from large amplitude to small?

19. How is the period of a pendulum affected when its point of suspension is (*a*) moved horizontally in the plane of oscillation with acceleration a; (*b*) moved vertically upward with acceleration a; (*c*) moved vertically downward with acceleration $a < g$; with acceleration $a > g$? Which case, if any, applies to a pendulum mounted on a cart rolling down an inclined plane?

20. Why was an axis through the center of mass excluded in using Eq. 17-29 to determine I? Does this equation apply to such an axis? How can you determine I for such an axis using physical pendulum methods?

21. A hollow sphere is filled with water through a small hole in it. It is hung by a long thread and, as the water flows out of the hole at the bottom, one finds that the period of oscillation first increases and then decreases. Explain.

22. (*a*) The effect of the mass, m, of the cord attached to the bob, of mass M, of a pendulum is to increase the period over that for a simple pendulum in which $m = 0$. Make this plausible. (*b*) Although the effect of the mass of the cord on the pendulum is to increase its period, a cord of length L swinging without anything on the end ($M = 0$) has a period less than that of a simple pendulum of length L. Make that plausible.

23. If taken to the Moon, will there be any change in the frequency of oscillation of a torsional pendulum? A simple pendulum? A spring–block oscillator? A physical pendulum?

24. How can a pendulum be used to trace out a sinusoidal curve?

25. Is there a connection between the F versus x relation at the molecular level and the macroscopic relation between F and x in a spring? Explain your answer.

26. (*a*) Under what circumstances would the reduced mass of a two-body system be equal to the mass of one of the bodies? Explain. (*b*) What is the reduced mass if the bodies have equal mass? (*c*) Do cases (*a*) and (*b*) give the extreme values of the reduced mass?

27. Why is the tub of a washing machine often mounted on springs?

28. Why are damping devices often used on machinery? Give an example.

29. Give some examples of common phenomena in which resonance plays an important role.

30. The lunar ocean tide is much more important than the solar ocean tide. The opposite is true for tides in the Earth's atmosphere, however. Explain this, using resonance ideas, given the fact that the atmosphere has a natural period of oscillation of nearly 12 hours.

31. In Fig. 17-19, what value does the amplitude of the forced oscillations approach as the driving frequency ω approaches (*a*) zero and (*b*) infinity?

32. Buildings of different heights sustain different amounts of damage in an earthquake. Explain why.

33. A singer, holding a note of the right frequency, can shatter a glass if the glassware is of high quality. This cannot be done if the glassware quality is low. Explain why.

EXERCISES

17-1 Oscillating Systems
17-2 The Simple Harmonic Oscillator
17-3 Simple Harmonic Motion

1. A 3.94-kg block extends a spring 15.7 cm from its unstretched position. The block is removed and a 0.520-kg object is hung from the same spring. Find the period of its oscillation.

2. An oscillator consists of a block of mass 512 g connected to a spring. When set into oscillation with amplitude 34.7 cm, it is observed to repeat its motion every 0.484 s. Find (*a*) the period, (*b*) the frquency, (*c*) the angular frequency, (*d*) the force constant, (*e*) the maximum speed, and (*f*) the maximum force exerted on the block.

3. A loudspeaker produces a musical sound by the oscillation of a diaphragm. If the amplitude of oscillation is limited to 1.20×10^{-3} mm, what frequencies will result in the acceleration of the diaphragm exceeding g?

4. A 5.22-kg object is attached to the bottom of a vertical spring and set vibrating. The maximum speed of the object is 15.3 cm/s and the period is 645 ms. Find (*a*) the force constant of the spring, (*b*) the amplitude of the motion, and (*c*) the frequency of oscillation.

5. In an electric shaver, the blade moves back and forth over a distance of 2.00 mm. The motion is simple harmonic, with frequency 120 Hz. Find (*a*) the amplitude, (*b*) the maximum blade speed, and (*c*) the maximum blade acceleration.

6. An automobile can be considered to be mounted on four springs as far as vertical oscillations are concerned. The springs of a certain car of mass 1460 kg are adjusted so that the vibrations have a frequency of 2.95 Hz. (*a*) Find the force constant of each of the four springs (assumed identical). (*b*) What will be the vibration frequency if five persons, averaging 73.2 kg each, ride in the car?

7. A body oscillates with simple harmonic motion according to the equation

$$x = (6.12\text{ m})\cos[(8.38\text{ rad/s})t + 1.92\text{ rad}].$$

Find (*a*) the displacement, (*b*) the velocity, and (*c*) the acceleration at the time $t = 1.90$ s. Find also (*d*) the frequency and (*e*) the period of the motion.

8. The scale of a spring balance reading from 0 to 50.0 lb is 4.00 in. long. A package suspended from the balance is found to oscillate vertically with a frequency of 2.00 Hz. How much does the package weigh?

9. The piston in the cylinder head of a locomotive has a stroke of 76.5 cm. What is the maximum speed of the piston if the drive wheels make 193 rev/min and the piston moves with simple harmonic motion?

10. A 2.14-kg object hangs from a spring. A 325-g body hung below the object stretches the spring 1.80 cm farther. The 325-g body is removed and the object is set into oscillation. Find the period of the motion.

11. At a certain harbor, the tides cause the ocean surface to rise and fall in simple harmonic motion, with a period of 12.5 h. How long does it take for the water to fall from its maximum height to one-half its maximum height above its average (equilibrium) level?

12. A block is on a piston that is moving vertically with simple harmonic motion. (*a*) At what amplitude of motion will the block and the piston separate if the period of the piston's motion is 1.18 s? (*b*) If the piston has an amplitude of 5.12 cm in its motion, find the maximum frequency for which the block and piston will be in contact continuously.

13. An oscillator consists of a block attached to a spring (k = 456 N/m). At some time t, the position (measured from the equilibrium location), velocity, and acceleration of the block are $x = 0.112$ m, $v_x = -13.6$ m/s, $a_x = -123$ m/s^2. Calculate (*a*) the frequency, (*b*) the mass of the block, and (*c*) the amplitude of oscillation.

14. Two particles execute simple harmonic motion of the same amplitude and frequency along the same straight line. They pass one another when going in opposite directions each time their displacement is half their amplitude. Find the phase difference between them.

15. Three 10,000-kg ore cars are held at rest on a 26.0° incline on a mine railway using a cable that is parallel to the incline (Fig. 17-23). The cable is observed to stretch 14.2 cm just before a coupling breaks, detaching one of the cars. Find (*a*) the frequency of the resulting oscillations of the remaining two cars and (*b*) the amplitude of the oscillations.

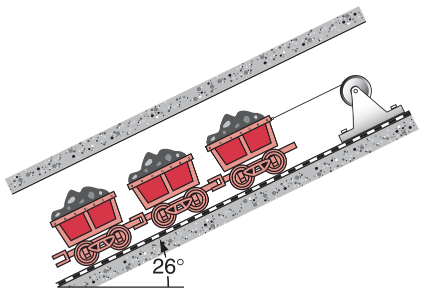


FIGURE 17-23. Exercise 15.

16. A U-tube is filled with a single homogeneous liquid. The liquid is temporarily depressed in one side by a piston. The piston is removed and the level of liquid in each side oscillates. Show that the period of oscillation is $\pi\sqrt{2L/g}$, where L is the total length of the liquid in the tube.

17. A cylindrical wooden log is loaded with lead at one end so that it floats upright in water as in Fig. 17-24. The length of the submerged portion is $L = 2.56$ m. The log is set into vertical oscillation. (*a*) Show that the oscillation is simple harmonic. (*b*) Find the period of the oscillation. Neglect the fact that the water has a damping effect on the motion.

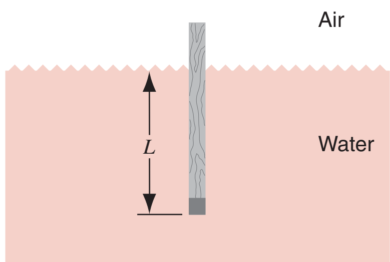


FIGURE 17-24. Exercise 17.

17-4 Energy in Simple Harmonic Motion

18. An oscillating block–spring system has a mechanical energy of 1.18 J, an amplitude of 9.84 cm, and a maximum speed of 1.22 m/s. Find (*a*) the force constant of the spring, (*b*) the mass of the block, and (*c*) the frequency of oscillation.

19. A (hypothetical) large slingshot is stretched 1.53 m to launch a 130-g projectile with speed sufficient to escape from the Earth (11.2 km/s). (*a*) What is the force constant of the device, if all the potential energy is converted to kinetic energy? (*b*) Assume that an average person can exert a force of 220 N. How many people are required to stretch the slingshot?

20. (*a*) When the displacement is one-half the amplitude x_m, what fraction of the total energy is kinetic and what fraction is potential in simple harmonic motion? (*b*) At what displacement is the energy half kinetic and half potential?

21. A 12.3-kg particle is undergoing simple harmonic motion with an amplitude of 1.86 mm. The maximum acceleration experienced by the particle is 7.93 km/s^2. (*a*) Find the period of the motion. (*b*) What is the maximum speed of the particle? (*c*) Calculate the total mechanical energy of this simple harmonic oscillator.

22. A 5.13-kg object moves on a horizontal frictionless surface under the influence of a spring with force constant 9.88 N/cm. The object is displaced 53.5 cm and given an initial velocity of 11.2 m/s back toward the equilibrium position. Find (*a*) the frequency of the motion, (*b*) the initial potential energy of the system, (*c*) the initial kinetic energy, and (*d*) the amplitude of the motion.

23. An object of mass 1.26 kg attached to a spring of force constant 5.38 N/cm is set into oscillation by extending the spring 26.3 cm and giving the object a velocity of 3.72 m/s toward the equilibrium position of the spring. Using the results obtained in Problem 9, calculate (*a*) the amplitude and (*b*) the phase angle of the resulting simple harmonic motion.

24. A 4.00-kg block is suspended from a spring with a force constant of 5.00 N/cm. A 50.0-g bullet is fired into the block from below with a speed of 150 m/s and comes to rest in the block. (*a*) Find the amplitude of the resulting simple harmonic motion. (*b*) What fraction of the original kinetic energy of the bullet appears as mechanical energy in the oscillator?

17-5 Applications of Simple Harmonic Motion

25. Find the legnth of a simple pendulum whose period is 1.00 s at a location where $g = 9.82$ m/s^2.

26. A simple pendulum of length 1.53 m makes 72.0 oscillations in 180 s at a certain location. Find the acceleration due to gravity at this point.

27. The period of a simple pendulum is given by the series in Eq. 17-25. (*a*) For what value of θ_m is the second term of the series equal to 0.02? (*b*) What is the value of the third term in the series at this amplitude?

28. If a pendulum has a period of 1.00 s at the equator, what would be its period at the south pole? See Fig. 14-6.

29. The fact that g varies from place to place over the Earth's surface drew attention when Jean Richer in 1672 took a pendulum clock from Paris to Cayenne, French Guiana, and found that it lost 2.5 min/day. If $g = 9.81$ m/s^2 in Paris, calculate g in Cayenne.

30. g is to be determined by measuring the period of a pendulum. How accurately (in seconds) would you have to measure the time for 100 oscillations of a 10-m-long pendulum to achieve a 0.1% error in the measurement of g? Calculate the percent error and an absolute error, in milliseconds. Compare your answer to Exercise 9 in Chapter 14.

31. A 2500-kg demolition ball swings from the end of a crane, as shown in Fig. 17-25. The length of the swinging segment of cable is 17.3 m. Find the period of swing, assuming that the system can be treated as a simple pendulum.

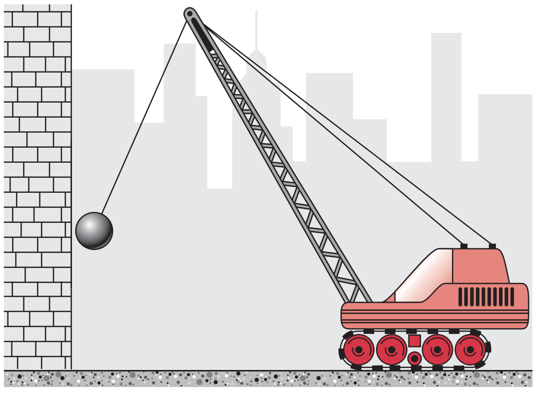


FIGURE 17-25. Exercise 31.

32. There is an interesting relation between the block–spring system and the simple pendulum. Suppose that you hang an object of mass M on the end of a spring, and when the object is in equilibrium the spring is stretched a distance h. Show that the frequency of this block–spring system is the same as that of a simple pendulum of mass m and length h, even if $m \neq M$; see Fig. 17-26.

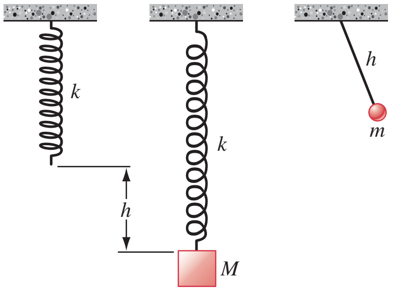


FIGURE 17-26. Exercise 32.

33. A circular hoop of radius 65.3 cm and mass 2.16 kg is suspended on a horizontal nail. (*a*) Find its frequency of oscillation for small displacements from equilibrium. (*b*) What is the legnth of the equivalent simple pendulum?

34. An engineer wants to find the rotational inertia of an odd-shaped object of mass 11.3 kg about an axis through its center of mass. The object is supported with a wire through its center of mass and along the desired axis. The wire has a torsional constant $\kappa = 0.513$ N·m. The engineer observes that this torsional pendulum oscillates through 20.0 cycles in 48.7 s. What value of the rotational inertia is calculated?

35. A 95.2-kg solid sphere with a 14.8-cm radius is suspended by a vertical wire attached to the ceiling of a room. A torque of 0.192 N·m is required to swist the sphere through an angle of

0.850 rad. Find the period of oscillation when the sphere is released from this position.

36. A physical pendulum consists of a meter stick that is pivoted at a small hole drilled through the stick a distance x from the 50.0-cm mark. The period of oscillation is observed to be 2.50 s. Find the distance x.

37. A meter stick swinging from one end oscillates with a frequency f_0. What would be the frequency, in terms of f_0, if the bottom third of the stick were cut off?

38. Figure 17-27 shows a physical pendulum constructed from equal-length sections of identical pipe. The inner radius of the pipe is 10.2 cm and the thickness is 6.40 mm. (*a*) Calculate the period of oscillation about the pivot shown. (*b*) Suppose that a new physical pendulum is constructed by rotating the bottom section 90° about a vertical axis through its center. Show that the new period of oscillation about the same pivot is about 2% less than the period of the original pendulum.

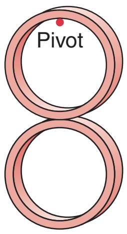


FIGURE 17-27. Exercise 38.

39. A pendulum whose upper end is attached so as to allow the pendulum to swing freely in any direction can be used to repeat an experiment first shown publicly by Foucault in Paris in 1851. If the pendulum is set oscillating, the plane of oscillation slowly rotates with respect to a line drawn on the floor, even though the tension in the wire supporting the bob and the gravitational pull of the Earth on the bob lie in a vertical plane. (*a*) Show that this is a result of the fact that the Earth is not an inertial reference frame. (*b*) Show that for a Foucault pendulum at a latitude θ, the period of rotation of the plane, in hours, is 24 sin θ. (*c*) Explain in simple terms the result at $\theta = 90°$ (the poles) and $\theta = 0°$ (the equator).

17-6 Simple Harmonic Motion and Uniform Circular Motion

40. Sketch the path of a particle that moves in the xy plane according to $x = x_m \cos(\omega t - \pi/2)$ and $y = 2x_m \cos \omega t$.

41. Electrons in an oscilloscope are deflected by two mutually perpendicular electric forces in such a way that at any time t the displacement is given by $x = A \cos \omega t$ and $y = A \cos(\omega t + \phi_y)$. Describe the path of the electrons and determine its equation when (*a*) $\phi_y = 0°$, (*b*) $\phi_y = 30°$, and (*c*) $\phi_y = 90°$.

42. A particle of mass m moves in a fixed plane along the trajectory $\vec{r} = \hat{i}A \cos \omega t + \hat{j}A \cos 3\omega t$. (*a*) Sketch the trajectory of the particle. (*b*) Find the force acting on the particle. Also find (*c*) its potential energy and (*d*) its total energy as functions of time. (*e*) Is the motion periodic? If so, find the period.

43. The orbit of the Moon around the Earth as projected along a diameter can be viewed as simple harmonic motion. Calculate the effective force constant k for this motion.

17-7 Damped Harmonic Motion

44. For the system shown in Fig. 17-17, the block has a mass of 1.52 kg and the force constant is 8.13 N/m. The frictional force is given by $-b(dx/dt)$, where $b = 227$ g/s. Suppose that the block is pulled aside a distance 12.5 cm and released. (*a*) Calculate the time interval required for the amplitude to fall to one-third of its initial value. (*b*) How many oscillations are made by the block in this time?

45. Verify by taking derivatives, that Eq. 17-39 is a solution of Eq. 17-38 for the damped oscillator, provided that the frequency ω' is given by Eq. 17-40.

46. A damped harmonic oscillator involves a block (m = 1.91 kg), a spring (k = 12.6 N/m), and a damping force $F = -bv_x$. Initially, it oscillates with an amplitude of 26.2 cm; because of the damping, the amplitude falls to three-fourths of this initial value after four complete cycles. (*a*) What is the value of b? (*b*) How much energy has been "lost" during these four cycles?

17-8 Forced Oscillations and Resonance

47. Consider the forced oscillations of a damped block–spring system. Show that at resonance (*a*) the amplitude of oscillation is $x_m = F_m/b\omega$, and (*b*) the maximum speed of the oscillating block is $v_{max} = F_m/b$.

48. Verify that Eq. 17-43 is a solution of Eq. 17-42 by direct substitution.

49. Verify that Eq. 17-43 is the most general form for the steady-state solution to the driven oscillator (Eq. 17-42). Let

$$x(t) = \frac{F_m}{G}\cos(\omega''' t - \beta)$$

and show that ω''' must equal the driving frequency ω''.

50. (*a*) Show that Eq. 17-39, the solution to the damped harmonic oscillator without a driving force, is also a solution to Eq. 17-42, the equation for a driven damped harmonic oscillator. Do so by direct substitution. (*b*) Physically, what does this solution represent?

51. A 2200-lb car carrying four 180-lb people is traveling over a rough "washboard" dirt road. The corrugations in the road are 13 ft apart. The car is observed to bounce with maximum amplitude when its speed is 10 mi/h. The car now stops and the four people get out. By how much does the car body rise on its suspension because of this decrease in weight?

52. Starting from Eq. 17-43, find the velocity v_x ($= dx/dt$) in forced oscillatory motion. Show that the velocity amplitude is

$$v_m = F_m/[(m\omega'' - k/\omega'')^2 + b^2]^{1/2}.$$

The equations of Section 17-8 are identical in form with those representing an electrical circuit containing a resistance R, and inductance L, and a capacitance C in series with an alternating emf $V = V_m \cos \omega'' t$. Hence b, m, k, and F_m, are analogous to R, L, $1/C$, and V_m, respectively, and x and v are analogous to electric charge q and current i, respectively. In the electrical case the current amplitude i_m, analogous to the velocity amplitude v_m above, is used to describe the quality of the resonance.

17-9 Two-Body Oscillations

53. Suppose that the spring in Fig. 17-21a has a force constant k = 252 N/m. Let m_1 = 1.13 kg and m_2 = 3.24 kg. Calculate the period of oscillation of the two-body system.

54. (*a*) Calculate the reduced mass of each of the following diatomic molecules: O_2, HF, and CO. Express your answers in unified atomic mass units, the mass of a hydrogen atom being

1.01 u. (*b*) An HF molecule is known to vibrate at a frequency of $f = 8.7 \times 10^{13}$ Hz. Find the effective "force constant" k for the coupling forces between the atoms. In terms of your experience with ordinary springs, would you say that this "molecular spring" is relatively stiff or not?

55. Show that the kinetic energy of the two-body oscillator of Fig. 17-21*a* is given by $K = \frac{1}{2} m v_x^2$, where m is the reduced mass and v_x $(= v_{1x} - v_{2x})$ is the relative velocity. It may help to note that linear momentum is conserved while the system oscillates.

PROBLEMS

1. The vibration frequencies of atoms in solids at normal temperatures are of the order of 10.0 THz. Imagine the atoms to be connected to one another by "springs." Suppose that a single silver atom vibrates with this frequency and that all the other atoms are at rest. Compute the effective force constant. One mole of silver has a mass of 108 g and contains 6.02×10^{23} atoms.

2. Figure 17-28 shows an astronaut on a Body Mass Measurement Device (BMMD). Designed for use on orbiting space vehicles, its purpose is to allow astronauts to measure their mass in the weightless conditions in Earth orbit. The BMMD is a spring-mounted chair; an astronaut measures his or her period of oscillation in the chair; the mass follows from the formula for the period of an oscillating block–spring system. (*a*) If M is the mass of the astronaut and m the effective mass of that part of the BMMD that also oscillates, show that

$$M = (k/4\pi^2)T^2 - m,$$

where T is the period of oscillation and k is the force constant. (*b*) The force constant is $k = 605.6$ N/m for the BMMD, and the period of oscillation of the empty chair is 0.90149 s. Calculate the effective mass of the chair. (*c*) With an astronaut in the chair, the period of oscillation becomes 2.08832 s. Calculate the mass of the astronaut.

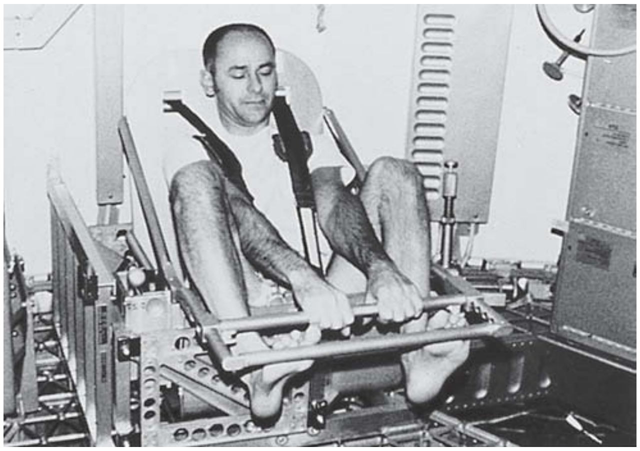


FIGURE 17-28. Problem 2.

3. Two blocks ($m = 1.22$ kg and $M = 8.73$ kg) and a spring ($k = 344$ N/m) are arranged on a horizontal, frictionless surface as shown in Fig. 17-29. The coefficient of static friction between the blocks is 0.42. Find the maximum possible amplitude of the simple harmonic motion if no slippage is to occur between the blocks.

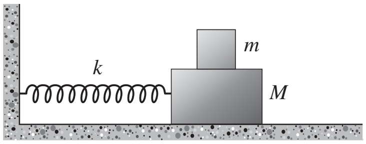


FIGURE 17-29. Problem 3.

4. The force of interaction between two atoms in certain diatomic molecules can be represented by $F = -a/r^2 + b/r^3$ in which a and b are positive constants and r is the separation distance of the atoms. Make a graph of F versus r. Then (*a*) show that the separation at equilibrium is b/a; (*b*) show that for small oscillations about this equilibrium separation the force constant is a^4/b^3; (*c*) find the period of this motion.

5. Two springs are attached to a block of mass m, free to slide on a frictionless horizontal surface, as shown in Fig. 17-30. Show that the frequency of oscillation of the block is

$$f = \frac{1}{2\pi}\sqrt{\frac{k_1 + k_2}{m}} = \sqrt{f_1^2 + f_2^2},$$

where f_1 and f_2 are the frequencies at which the block would oscillate if connected only to spring 1 or spring 2. (The electrical analog of this system is a series combination of two capacitors.)

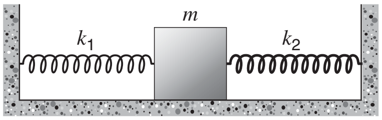


FIGURE 17-30. Problem 5.

6. Two springs are joined and connected to a block of mass m as shown in Fig. 17-31. The surfaces are frictionless. If the springs separately have force constants k_1 and k_2, show that the frequency of oscillation of the block is

$$f = \frac{1}{2\pi}\sqrt{\frac{k_1 k_2}{(k_1 + k_2)m}} = \frac{f_1 f_2}{\sqrt{f_1^2 + f_2^2}},$$

where f_1 and f_2 are the frequencies at which the block would oscillate if connected only to spring 1 or spring 2. (The electrical analog of this system is a parallel combination of two capacitors.)

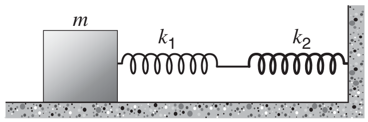


FIGURE 17-31. Problem 6.

7. A massless spring of force constant 3.60 N/cm is cut into halves. (*a*) What is the force constant of each half? (*b*) The two halves, suspended separately, support a block of mass M (see Fig. 17-32). The system vibrates at a frequency of 2.87 Hz. Find the value of the mass M.

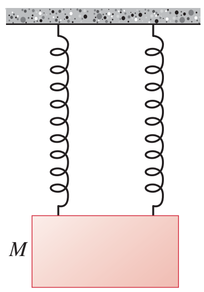


FIGURE 17-32. Problem 7.

8. If the mass of a spring m_s is not negligible but is small compared to the mass m of the object suspended from it, the period of motion is $T = 2\pi\sqrt{(m + m_s/3)/k}$. Derive this result. (Hint: The condition $m_s \ll m$ is equivalent to the assumption that the spring stretches proportionally along its length.) (See H. L. Armstrong, *American Journal of Physics,* Vol. 37, 1969, p. 447, for a complete solution of the general case.)

9. Show that the general relationships between the two initial values of position $x(0)$ and velocity $v_x(0)$, and amplitude x_m and phase angle ϕ of Eq. 17-6, are

$$x_m = \sqrt{[x(0)]^2 + [v_x(0)/\omega]^2} \quad \text{and} \quad \tan\phi = -v_x(0)/\omega x(0).$$

10. Solve Eq. 17-16, which expresses conservation of energy, for dt and integrate the result. Assume that $x = x_m$ at $t = 0$, and show that Eq. 17-6 (with $\phi = 0$), the displacement as a function of time, is obtained.

11. A block of mass M, at rest on a horizontal, frictionless table, is attached to a rigid support by a spring of force constant k. A bullet of mass m and speed v strikes the block as shown in Fig. 17-33. The bullet remains embedded in the block. Determine the amplitude of the resulting simple harmonic motion, in terms of m, M, v, and k.

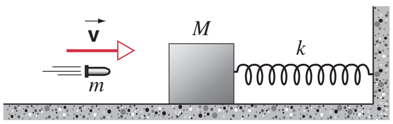


FIGURE 17-33. Problem 11.

12. Consider a massless spring of force constant k in a uniform gravitational field. Attach an object of mass m to the spring. (*a*) Show that if $x = 0$ marks the slack position of the spring, the static equilibrium position is given by $x = mg/k$ (see Fig. 17-34). (*b*) Show that the equation of motion of the mass-spring system is

$$m\frac{d^2x}{dt^2} + kx = mg$$

and that the solution for the displacement as a function of time is $x = x_m \cos(\omega t + \phi) + mg/k$, where $\omega = \sqrt{k/m}$ as before. (*c*) Show therefore that the system has the same ω, v, a, f, and T in a uniform gravitational field as in the absence of such a field, with the one change that the equilibrium position has been displaced by mg/k. (*d*) Now consider the energy of the system, $\frac{1}{2}mv^2 + \frac{1}{2}kx^2 + mg(h - x) =$ constant, and show that time differentiation leads to the equation of motion of part (*b*). (*e*) Show that when the object falls from $x = 0$ to the static equilibrium position, $x = mg/k$, the loss in gravitational potential energy goes half into a gain in elastic potential energy and half into a gain in kinetic energy. (*f*) Finally, consider the system in motion about the static equilibrium position. Compute separately the change in gravitational potential energy and in elastic potential energy when the object moves up through a displacement x_m, and when the object moves down through a displacement x_m. Show that the total change in potential energy is the same in each case—namely, $\frac{1}{2}kx_m^2$. In view of the results of (*c*) and (*f*), one can simply ignore the uniform gravitational field in the analysis merely by shifting the reference position from $x = 0$ to $x_0 = x - mg/k = 0$. The new potential energy curve $[U(x_0) = \frac{1}{2}kx_0^2 + \text{constant}]$ has the same parabolic shape as the potential energy curve in the absence of a gravitational field $[U(x) = \frac{1}{2}kx^2]$.

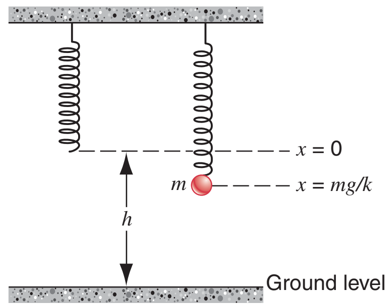


FIGURE 17-34. Problem 12.

13. A solid cylinder is attached to a horizontal massless spring so that it can roll without slipping along a horizontal surface, as in Fig. 17-35. The force constant k of the spring is 2.94 N/cm. If the system is released from rest at a position in which the spring is stretched by 23.9 cm, find (*a*) the translational kinetic energy and (*b*) the rotational kinetic energy of the cylinder as it passes through the equilibrium position. (*c*) Show that under these conditions the center of mass of the cylinder executes simple harmonic motion with a period

$$T = 2\pi\sqrt{3M/2k},$$

where M is the mass of the cylinder.

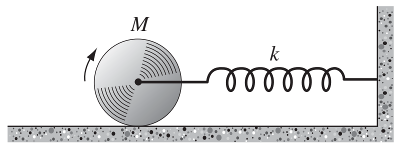


FIGURE 17-35. Problem 13.

14. (*a*) Prove that in simple harmonic motion the average potential energy equals the average kinetic energy when the average is taken with respect to time over one period of the motion, and that each average equals $\frac{1}{4}kx_m^2$. (*b*) Prove that when the average is taken with respect to position over one cycle, the average potential energy equals $\frac{1}{6}kx_m^2$ and the average kinetic energy equals $\frac{1}{3}kx_m^2$. (*c*) Explain physically why the results for (*a*) and (*b*) are different.

15. A physical pendulum consists of a uniform solid disk of mass $M = 563$ g and radius $R = 14.4$ cm supported in a vertical plane by a pivot located a distance $d = 10.2$ cm from the center of the disk, as shown in Fig. 17-36. The disk is displaced by a small angle and released. Find the period of the resulting simple harmonic motion.

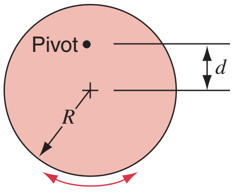


FIGURE 17-36. Problem 15.

16. A pendulum consists of a uniform disk with radius 10.3 cm and mass 488 g attached to a 52.4-cm-long uniform rod with mass 272 g; see Fig. 17-37. (*a*) Calculate the rotational inertia of the pendulum about the pivot. (*b*) What is the distance between the pivot and the center of mass of the pendulum? (*c*) Calculate the small-angle period of oscillation.

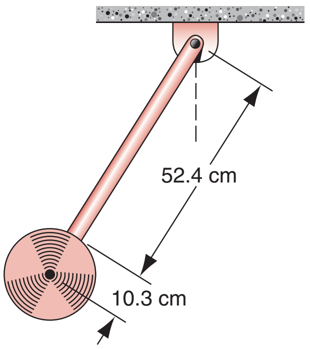


FIGURE 17-37. Problem 16.

17. A pendulum is formed by pivoting a long, thin rod of length L and mass m about a point on the rod that is a distance d above the center of the rod. (*a*) Find the small-amplitude period of this pendulum in terms of d, L, m, and g. (*b*) Show that the period has a minimum value when $d = L/\sqrt{12} = 0.289L$.

18. A wheel is free to rotate about its fixed axle. A spring is attached to one of its spokes a distance r from the axle, as shown in Fig. 17-38. Assuming that the wheel is a hoop of mass M and radius R, obtain the angular frequency of small oscillations of this system in terms of M, R, r, and the force constant k. Discuss the special cases $r = R$ and $r = 0$.

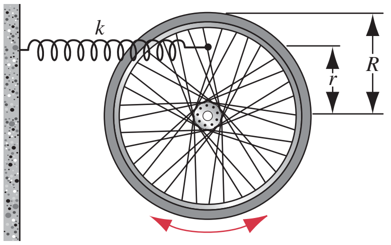


FIGURE 17-38. Problem 18.

19. A particle is released from rest at a point P inside a frictionless hemispherical bowl of radius R. (*a*) Show that when P is near the bottom of the bowl the particle undergoes simple harmonic motion. (*b*) Find the length of the equivalent simple pendulum.

20. A physical pendulum has two possible pivot points; one has a fixed position and the other is adjustable along the length of the pendulum, as shown in Fig. 17-39. The period of the pendulum when suspended from the fixed pivot is T. The pendulum is then reversed and suspended from the adjustable pivot. The position of this pivot is moved until, by trial and error, the pendulum has the same period as before—namely, T. Show that the free-fall acceleration g is given by

$$g = \frac{4\pi^2 L}{T^2},$$

in which L is the distance between the two pivot points. Note that g can be measured in this way without needing to know the rotational inertia of the pendulum or any of its other dimensions except L.

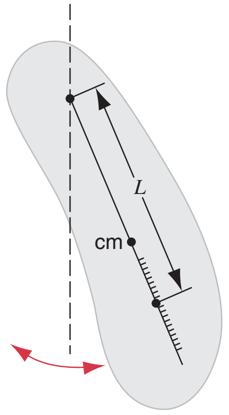


FIGURE 17-39. Problem 20.

21. A 2.50-kg disk, 42.0 cm in diameter, is supported by a light rod, 76.0 cm long, which is pivoted at its end, as shown in Fig. 17-40. (*a*) The light, torsional spring is initially not connected. What is the period of oscillation? (*b*) The torsional spring is now connected so that, in equilibrium, the rod hangs vertically. What should be the torsional constant of the spring so that the new period of oscillation is 500 ms shorter than before?

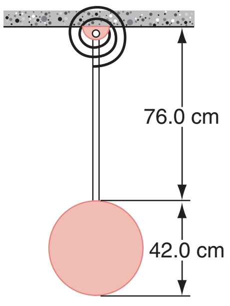


FIGURE 17-40. Problem 21.

22. A simple pendulum of length L and mass m is suspended in a car that is traveling with a constant speed v around a circle of radius R. If the pendulum undergoes small oscillations in a radial direction about its equilibrium position, what will its frequency of oscillation be?

23. Consider an unusual galaxy, in which the stars are uniformly distributed around a ring of radius R and total mass M, except for one star (mass m) that resides at the center of the ring. (*a*) Suppose the central star is displaced a distance z from the plane of the ring along its symmetry axis. Show that the gravitational force on the star due to the ring is $F_z = GMmz/(R^2 + z^2)^{3/2}$. (*b*) Assuming $z \ll R$, find the oscillation frequency f if the central star is displaced a distance z along the axis and then released. (*c*) Estimate the oscillation frequency for a galaxy of mass and radius equal to that of the Milky Way.

24. Suppose an impulsive force F acts horizontally to the right at point O (the center of oscillation) in Fig. 17-12. Assume that the pendulum is initially at rest. (*a*) By combining the effects of translation and rotation, show that the resulting acceleration of a particle at point P is zero. (*b*) What do you conclude about the force at P that results from the applied force F? Because of this property, the center of oscillation is often called the center of percussion.

25. Assume that you are examining the characteristics of a suspension system of a 2000-kg automobile. The suspension "sags" 10 cm when the weight of the entire automobile is placed on it. In addition, the amplitude of oscillation decreases by 50% during one complete oscillation. Estimate the values of k and b for the spring and shock absorber system of each wheel. Assume that each wheel supports 500 kg.

26. Driven nonlinear oscillators do not need to oscillate at the driving frequency. Consider the driven quartic oscillator

$$m\frac{d^2x}{dt^2} + kx^3 = F\cos\omega_d t,$$

where m, k, F, and ω_d are constants. Show that $x(t) = A\cos\omega t$ is a solution to this equation if (1) $\omega = \omega_d/3$ and (2) $F \propto \omega_d^3$.

27. (*a*) Show that when $m_2 \to \infty$ in Eq. 17-46, $m \to m_1$. (*b*) Show that the effect of a noninfinite wall ($m_2 < \infty$) on the oscillations of a body of mass m_1 at the end of a spring attached to the wall is to reduce the period, or increase the frequency, of oscillation compared to (*a*). (*c*) Show that when $m_2 = m_1$ the effect is as though the spring were cut in half, each body oscillating independently about the center of mass at the middle.

COMPUTER PROBLEMS

1. Consider a system composed of two objects constrained to move along the x axis. The first object is connected to a spring that is attached to the origin, and the second object is connected to a spring that is attached to the first object. Both objects have the same mass, 0.10 kg, and both springs have the same force constant, 1.0 N/m. (*a*) Numerically simulate the motion of the objects, assuming that the second object is pulled and then released a distance of 1.0 cm from the equilibrium position. Generate a graph of the motion of the objects. (*b*) Use a fast Fourier transform (available on some spread sheet programs) to show that there are two characteristic frequencies of the motion. What are these frequencies?

2. An object of mass m moves subject to a force that results in a potential energy of $U(x) = \frac{1}{4}kx^4$. This type of motion is called a *quartic oscillator*. Note that the frequency of oscillation depends on the amplitude of the oscillations here. Assuming a mass $m = 0.10$ kg and a force constant of $k = 100$ N/m^3, numerically simulate the motion for several different amplitudes. Graph the results, and find the relation between amplitude and frequency for this system.

CHAPTER 18

WAVE MOTION

Wave motion appears in almost every branch of physics. Surface waves on bodies of water are commonly observed. Sound waves and light waves are essential to our perception of the environment, because we have receptors (eyes and ears) capable of their detection. In the past century we learned how to produce and use radio waves. The similarity of the physical and mathematical descriptions of these different kinds of waves indicates that wave motion is one of the unifying themes of physics.

In this chapter and the next we develop the verbal and mathematical descriptions of waves. We use the example of mechanical waves, *in part because we have already developed the laws of mechanics in this text. Later in the text we develop the laws that govern other types of waves (light and other electromagnetic waves, for example). For simplicity, we concentrate on the study of harmonic waves (that is, those that can be represented by sine and cosine functions), but the principles that we develop apply to more complex waveforms as well.*

18-1 MECHANICAL WAVES

Waves are a common and essential part of our environment. We are surrounded by sound waves, light waves, water waves, and other kinds of waves, which we can control and use to convey information or transport energy from one location to another.

All types of waves use similar mathematical descriptions. We can therefore learn a great deal about waves in general by making a careful study of one type of wave. In this chapter we consider only *mechanical waves,* which include sound waves and water waves. In particular, we choose one special type of mechanical wave—the oscillation of a stretched string such as might be found on a guitar.

Mechanical waves travel through an elastic medium. They can originate when we cause an initial disturbance at one location in the medium. Because of the elastic properties of the medium, the disturbance travels through the medium.

On a microscopic level, the forces between atoms are responsible for the propagation of mechanical waves. Each atom exerts a force on the atoms that surround it, and through this force the motion of the atom is transmitted to its neighbors. However, the particles of the medium do not experience any net displacement in the direction of the wave—as the wave passes, the particles simply move back and forth through small distances about their equilibrium positions.

For example, a leaf floating on a lake may bob up and down as a wave passes, but after the wave has passed the leaf returns very nearly to its original position. A sound wave can travel through air, but there is no net motion of the air molecules in the direction that the wave is moving. The wave can transport energy and momentum from one location to another without any material particles making that journey. As Leonardo da Vinci observed about water waves in the 15th century: “It often happens that the wave flees the place of its creation, while the water does not; like the waves made in a field of grain by the wind, where we see the waves running across the field while the grain remains in place.”

18-2 TYPES OF WAVES

In listing water waves, light waves, and sound waves as examples of wave motion, we are classifying waves according to their broad physical properties. Waves can also be classified in other ways.

1. *Direction of particle motion.* We can classify mechanical waves by considering how the direction of motion of the particles of the medium is related to the direction of propagation of the wave. If the motion of the particles is perpendicular to the direction of propagation of the wave itself, we have a *transverse* wave. For example, when a string under tension is set oscillating back and forth at one end, a transverse wave travels along the string; the disturbance moves along the string but the string particles vibrate at right angles to the direction of propagation of the disturbance (Fig. 18-1*a*). Light waves, although they are not mechanical waves, are also transverse waves.

If, however, the motion of the particles in a mechanical wave is back and forth along the direction of propagation, we have a *longitudinal* wave. For example, when a spring under tension is set oscillating back and forth at one end, a longitudinal wave travels along the spring; the coils vibrate back and forth parallel to the direction in which the disturbance travels along the spring (Fig. 18-1*b*). Sound waves in a gas are longitudinal waves. We discuss them in greater detail in Chapter 19.

Some waves are neither purely longitudinal nor purely transverse. For example, in waves on the surface of water the particles of water move both up and down and back and forth, tracing out elliptical paths as the water waves move by.

2. *Number of dimensions.* Waves can also be classified as propagating in one, two, and three dimensions. Waves moving along the string or spring of Fig. 18-1 are one-dimensional. Surface waves or ripples on water, caused by dropping a pebble into a quiet pond, are two-dimensional (Fig. 18-2). Sound waves and light waves traveling radially outward from a small source are three-dimensional.

3. *Periodicity.* Waves may be classified further according to how the particles of the medium move in time. For example, we can produce a *pulse* traveling down a stretched string by applying a single sidewise movement at its end (Fig. 18-1*c*). Each particle remains at rest until the pulse reaches it, then it moves during a short time, and then it again remains at rest. If we continue to move the end of the string back and forth (Fig. 18-1*a*), we produce a *train of waves* traveling along the string. If our motion is periodic, we produce a *periodic train of waves* in which each particle of the string has a periodic motion. The simplest special case of a periodic wave is a *harmonic wave,* in which each particle undergoes simple harmonic motion.

4. *Shape of wavefronts.* Imagine a stone dropped in a still lake. Circular ripples spread outward from the point where the stone entered the water (Fig. 18-2). Along a given circular ripple, all points are in the same state of motion. Those points define a surface called a *wavefront.* If the medium is of uniform density, the direction of motion of the waves is at right angles to the wavefront. A line normal to the wavefronts, indicating the direction of motion of the waves, is called a *ray.*

Wavefronts can have many shapes. A point source at the surface of water produces two-dimensional waves with circular wavefronts and rays that radiate outward from the point of the disturbance (as in Fig. 18-2). On the other hand, a very long stick dropped horizontally into the water would produce (near its center) disturbances that travel as straight lines, in which the rays are parallel lines. The three-dimensional analogy, in which the disturbances travel in a single direction, is the *plane wave.* At a given instant, conditions are the same everywhere on any plane perpendicular to the direction of propagation. The wavefronts are planes, and the

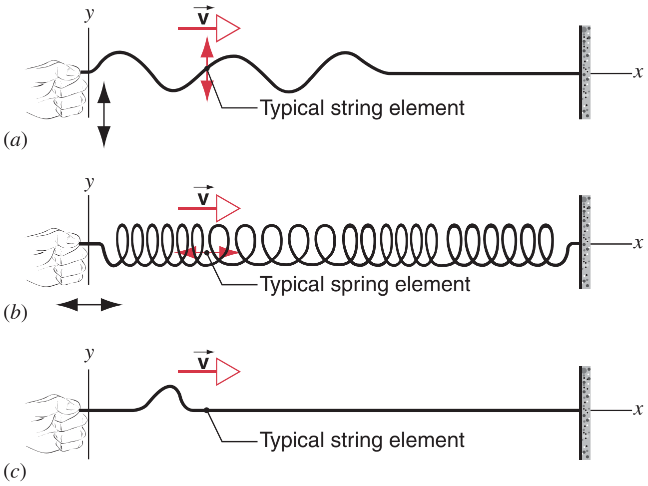


FIGURE 18-1. (*a*) Sending a transverse wave along a string. Each element of the string vibrates at right angles to the direction of propagation of the wave. (*b*) Sending a longitudinal wave along a spring. Each element of the spring vibrates parallel to the direction of propagation of the wave. (*c*) Sending a single transverse pulse along a string.

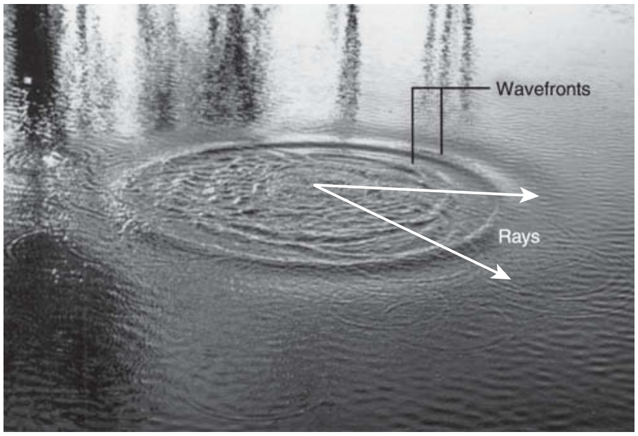


FIGURE 18-2. Waves on the surface of a lake. The circular ripples represent wavefronts. The rays, which are perpendicular to the wavefronts, indicate the direction of motion of the wave.

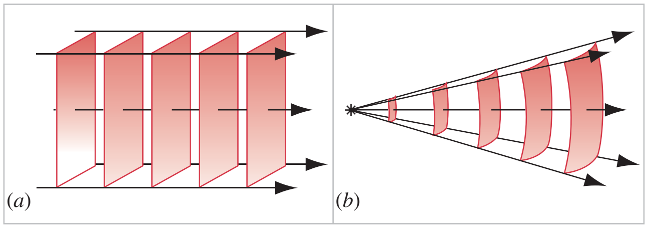


FIGURE 18-3. (*a*) A plane wave. The planes represent wavefronts spaced one wavelength apart, and the arrows represent rays. (*b*) A spherical wave. The wavefronts, spaced one wavelength apart, are spherical surfaces, and the rays are in the radial direction.

rays are parallel straight lines (Fig. 18-3*a*). The three-dimensional analogy of circular waves is spherical waves. Here the disturbance is propagated outward in all directions from a point source of waves. The wavefronts are spherical, and the rays are radial lines leaving the point source in all directions (Fig. 18-3*b*). Far from the source the spherical wavefronts have very small curvature, and over a limited region they can often be regarded as planes. Of course, there are many other possible shapes for wavefronts.

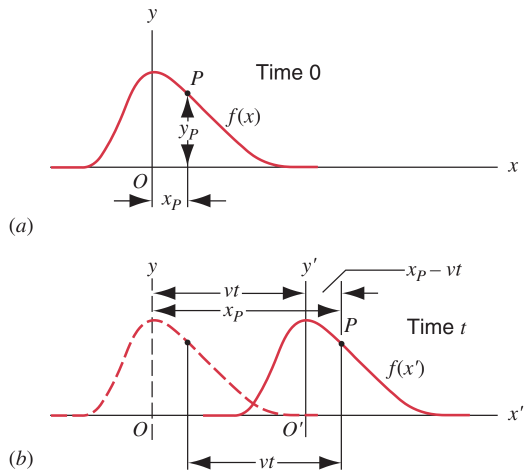


FIGURE 18-4. (*a*) A transverse pulse, shown as a snapshot at time $t = 0$. The point P represents a particular location on the phase of the pulse, *not* a particular point of the medium (the string, for instance). (*b*) At a time t later, the pulse has moved a distance vt in the positive x direction. The point P on the phase has also moved a distance vt. The peak of the pulse defines the origin of the x' coordinate.

18-3 TRAVELING WAVES

As an example of a mechanical wave, we consider a transverse waveform that travels on a long stretched string. We assume an "ideal" string, in which the disturbance, whether it is a pulse or a train of waves, keeps its form as it travels. For this to occur, frictional losses and other means of energy dissipation must be negligibly small. The disturbance lies in the xy plane and travels in the x direction.

Figure 18-4*a* shows an arbitrary waveform at $t = 0$; we can consider this to be a snapshot of the pulse traveling along the string shown in Fig. 18-1*c*. Let the pulse move in the positive x direction with speed v. At a later time t, the pulse has moved a distance vt, as shown in Fig. 18-4*b*. Note that the waveform is the same at $t = 0$ as it is at later times.

The coordinate y indicates the transverse displacement of a particular point on the string. This coordinate depends on both the position x and the time t. We indicate this dependence on two variables as $y(x, t)$.

We can represent the waveform of Fig. 18-4*a* as

$$y(x, 0) = f(x), \tag{18-1}$$

where f is a function that describes the shape of the wave. At time t, the waveform must still be described by the same function f, because we have assumed that the shape does not change as the wave travels. Relative to the origin O' of a reference frame that travels with the pulse, the shape is described by the function $f(x')$, as indicated in Fig. 18-4*b*. The relationship between the x coordinates in the two reference frames is $x' = x - vt$, as you can see from Fig. 18-4*b*. Thus, at time t, the wave is described by

$$y(x, t) = f(x') = f(x - vt). \tag{18-2}$$

That is, the function $f(x - vt)$ has the same shape relative to the point $x = vt$ at time t that the function $f(x)$ has relative to the point $x = 0$ at time $t = 0$.

To describe the wave completely, we must specify the function f. Later we shall consider harmonic waves, for which f is a sine or cosine function.

Equations 18-1 and 18-2 together indicate that we can change a function of any shape into a wave traveling in the positive x direction by merely substituting the quantity $x - vt$ for x everywhere that it appears in $f(x)$. For example, if $f(x) = x^2$, then $f(x - vt) = (x - vt)^2$. Furthermore, a wave traveling in the positive x direction must depend on x and t *only* in the combination $x - vt$; thus $x^2 - (vt)^2$ does not represent such a traveling wave.

Let us follow the motion of a particular part (or *phase*) of the wave, such as that of location P of the waveform of Fig. 18-4. If the wave is to keep its shape as it travels, then the y coordinate y_P of P must not change. We see from Eq. 18-2 that the only way this can happen is for x_P, the x coordinate of P, to increase as t increases in such a way that the quantity $x_P - vt$ keeps a fixed value. That is, evaluating the quantity $x_P - vt$ gives the same result at P in Fig. 18-4*b* and at P in Fig. 18-4*a*. This remains true for any location on the waveform and for all times t. Thus for the motion of any particular phase of the wave we must have

$$x - vt = \text{constant}. \tag{18-3}$$

We can verify that Eq. 18-3 characterizes the motion of the phase of the waveform by differentiating with respect to time, which gives

$$\frac{dx}{dt} - v = 0 \quad \text{or} \quad \frac{dx}{dt} = v. \tag{18-4}$$

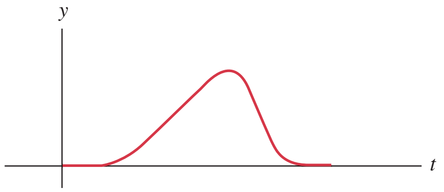


FIGURE 18-5. An observer stationed at a particular point on the x axis would record this y displacement as a function of time as the pulse of Fig. 18-4 passes. Note that the form appears to be reversed, because the leading edge of the traveling pulse arrives at the observer at the earliest times. That is, the displacements recorded by the observer at earlier times are closer to the origin here.

The velocity dx/dt describes the motion of the phase of the wave, and so it is known as the *phase velocity*. We take v to be a positive constant, independent of any property of the wave but possibly (as we shall see) depending on properties of the medium.

If the wave moves in the *negative* x direction, all we need do is replace v by $-v$. In this case we would obtain

$$y(x, t) = f(x + vt), \tag{18-5}$$

where once again $f(x)$ represents the shape at $t = 0$. That is, substituting in $f(x)$ the quantity $x + vt$ in place of x gives a wave that would move to the left in Fig. 18-4. The motion of any phase of the wave would then be characterized by the requirement that $x + vt =$ constant, and by analogy with Eq. 18-4 we can show that $dx/dt = -v$, indicating that the x component of the phase velocity in this case is indeed negative.

The function $y(x, t)$ contains the complete description of the shape of the wave and its motion. At any particular time, say t_1, the function $y(x, t_1)$ gives y as a function of x, which defines a curve; this curve represents the actual shape of the string at that time and can be regarded as a "snapshot" of the wave. On the other hand, we can consider the motion of a particular point on the string, say at the fixed coordinate x_1. The function $y(x_1, t)$ then tells us the y coordinate of that point as a function of the time. Figure 18-5 shows how a point on the x axis might move with time as the pulse of Fig. 18-4 passes, moving in the positive x direction. At times near $t = 0$, the point is not moving at all. It then begins to move gradually, as the leading edge of the pulse of Fig. 18-4 arrives. After the peak of the wave passes, the displacement of the point drops rapidly back to zero as the trailing edge passes.

Sinusoidal Waves

The above description is quite general. It holds for arbitrary wave shapes, and it holds for transverse as well as longitudinal waves. Let us consider, for example, a transverse waveform having a sinusoidal shape, which has particularly important applications. Suppose that at the time $t = 0$ we have a wavetrain along the string given by

$$y(x, 0) = y_m \sin \frac{2\pi}{\lambda} x. \tag{18-6}$$

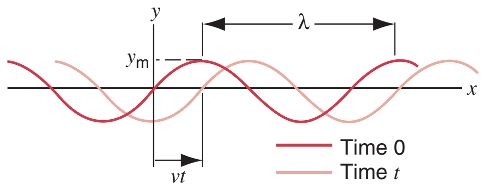


FIGURE 18-6. At $t = 0$ (darker color), the string has the sinusoidal shape given by $y = y_m \sin 2\pi x/\lambda$. At a later time t (lighter color), the wave has moved to the right a distance $x = vt$, and the string has a shape given by $y = y_m \sin 2\pi(x - vt)/\lambda$.

The wave shape is shown in Fig. 18-6. The maximum displacement y_m is called the *amplitude* of the sine curve. The value of the transverse displacement y is the same at any x as it is at $x + \lambda$, $x + 2\lambda$, and so on. The symbol λ represents the *wavelength* of the wavetrain and indicates the distance between two adjacent points in the wave having the same phase. If the wave travels in the $+x$ direction with phase speed v, then the equation of the wave is

$$y(x, t) = y_m \sin \frac{2\pi}{\lambda}(x - vt). \tag{18-7}$$

Note that this has the form $f(x - vt)$ required for a traveling wave (Eq. 18-2).

The *period* T of the wave is the time necessary for a point at any particular x coordinate to undergo one complete cycle of transverse motion. During this time T, the wave travels a distance vT that must correspond to one wavelength λ, so that

$$\lambda = vT. \tag{18-8}$$

The inverse of the period is called the *frequency* f of the wave: $f = 1/T$. Frequency has units of cycles per second, or hertz (Hz). Period and frequency were previously discussed in Chapter 17.

Putting Eq. 18-8 into Eq. 18-7, we obtain another expression for the wave:

$$y(x, t) = y_m \sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right). \tag{18-9}$$

From this form it is clear that y, at any given time, has the same value at x, $x + \lambda$, $x + 2\lambda$, and so on, and that y, at any given position, has the same value at the times t, $t + T$, $t + 2T$, and so on.

To reduce Eq. 18-9 to a more compact form, we introduce two quantities, the *wave number* k and the *angular frequency* ω. They are defined by

$$k = \frac{2\pi}{\lambda} \quad \text{and} \quad \omega = \frac{2\pi}{T} = 2\pi f. \tag{18-10}$$

The wave number k is, like ω, an angular quantity, and units for both involve radians. Units for k might be, for instance, rad/m, and for ω, rad/s. In terms of these quantities, the equation of a sine wave traveling in the positive x direction (to the right in Fig. 18-6) is

$$y(x, t) = y_m \sin (kx - \omega t). \tag{18-11}$$

The equation of a sine wave traveling in the negative x direction (to the left in Fig. 18-6) is

$$y(x, t) = y_m \sin(kx + \omega t). \quad (18\text{-}12)$$

Comparing Eqs. 18-8 and 18-10, we see that the phase speed v of the wave (which we will often call the *wave speed*) is given by

$$v = \lambda f = \frac{\lambda}{T} = \frac{\omega}{k}. \quad (18\text{-}13)$$

Transverse Velocity of a Particle

The motion of a particle in a transverse wave such as that of Fig. 18-6 is in the y direction. The wave speed describes the motion of the wave along the direction of travel (the x direction). The wave speed does *not* characterize the transverse motion of the particles of the string.

To find the transverse velocity of a particle of the string, we must find the change in the y coordinate with time. We focus our attention on a single particle of the string—that is, on a certain coordinate x. We therefore need the derivative of y with respect to t at constant x. This is represented by the symbol $\partial y/\partial t$, which indicates the *partial derivative* of y with respect to t, holding constant all other variables on which y may depend. We represent the particle velocity, which varies with x (the location of the particle) as well as with t, as $u_y(x, t)$. Assuming that we are dealing with a sinusoidal wave of the form of Eq. 18-11, we then have

$$u_y(x, t) = \frac{\partial y}{\partial t} = \frac{\partial}{\partial t}[y_m \sin(kx - \omega t]$$

$$= -y_m\omega \cos(kx - \omega t) \quad (18\text{-}14)$$

Depending on a particle's location and on the time at which it is observed, Eq. 18-14 shows that the transverse velocity can range from $-y_m\omega$ to $+y_m\omega$.

Continuing in this way, we can find the transverse acceleration of the particle at this location x according to

$$a_y(x, t) = \frac{\partial^2 y}{\partial t^2} = \frac{\partial u_y}{\partial t} = -y_m\omega^2 \sin(kx - \omega t)$$

$$= -\omega^2 y. \quad (18\text{-}15)$$

Equation 18-15 has the same form as Eq. 17-5; the transverse acceleration of any point is proportional to its transverse displacement, but oppositely directed. This shows that each particle of the string undergoes transverse simple harmonic motion as the sinusoidal wave passes.

Keep in mind the differences between the speed v of the wave and the transverse velocity u_y of a particle. The speed v represents the entire wave; all points on the phase of the wave move in the same direction with the same speed v. However, the transverse velocity u_y of a particle depends on the location of the particle and on the time. At one instant of time, one particle might have $u_y = 0$ while another particle might be moving with the maximum transverse velocity (which is $y_m\omega$ according to Eq. 18-14). At some other instant, these roles might be reversed. It is also important to note that, as we discuss in the next section, the wave speed v depends on the properties of the medium and not on the properties of the wave. The transverse particle velocity, on the other hand, depends on the properties of the wave such as amplitude and frequency, as Eq. 18-14 shows, and not on the properties of the medium.

Phase and Phase Constant

In the traveling waves of Eqs. 18-11 and 18-12 we have assumed that the displacement y is zero at the position $x = 0$ at the time $t = 0$. This, of course, need not be the case. The general expression for a sinusoidal wave traveling in the positive x direction is

$$y(x, t) = y_m \sin(kx - \omega t - \phi). \quad (18\text{-}16)$$

The quantity that appears in the argument of the sine, namely, $kx - \omega t - \phi$, is called the *phase* of the wave. Two waves with the same phase (or with phases differing by any integer multiple of 2π) are said to be "in phase"; they execute the same motion at the same time.

The angle ϕ is called the *phase constant.* The phase constant does not affect the shape of the wave; it moves the wave forward or backward in space or time. To see this, we rewrite Eq. 18-16 in two equivalent forms:

$$y(x, t) = y_m \sin\left[k\left(x - \frac{\phi}{k}\right) - \omega t\right] \quad (18\text{-}17a)$$

or

$$y(x, t) = y_m \sin\left[kx - \omega\left(t + \frac{\phi}{\omega}\right)\right]. \quad (18\text{-}17b)$$

Figure 18-7a shows a "snapshot" at any time t of the two waves represented by Eqs. 18-11 (in which $\phi = 0$) and

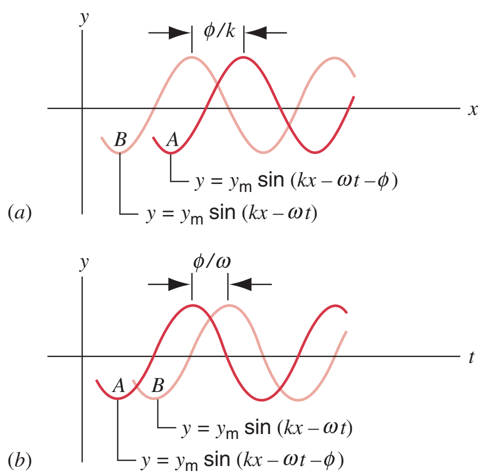


FIGURE 18-7. (a) A snapshot of two sine waves traveling in the positive x direction. Wave A has phase constant ϕ, and wave B has $\phi = 0$. Wave A is a distance of ϕ/k ahead of wave B. (b) The motion of a single point in time due to the same two waves. Wave A is a time ϕ/ω ahead of wave B. Note that, in a graph of y versus t, "ahead of" means "to the left of," whereas in a graph of y versus x, "ahead of" means "to the right of," if the waves travel in the positive x direction.

18-16. Note that any particular point on the wave described by Eq. 18-17*a* (say, a certain wave crest) is a distance ϕ/k *ahead of* the corresponding point in the wave described by Eq. 18-11.

Equivalently, if we were to observe the displacement at a fixed position x resulting from each of the two waves represented by Eqs. 18-11 and 18-16, we would obtain the result indicated by Fig. 18-7*b*. The wave described by Eq. 18-17*b* is similarly *ahead of* the wave having $\phi = 0$, in this case by a time difference ϕ/ω.

When the phase constant in Eq. 18-16 is positive, the corresponding wave is ahead of a wave described by a similar equation having $\phi = 0$. It is for this reason that we introduced the phase constant with a negative sign in Eq. 18-16. When one wave is ahead of another in time or space, it is said to "lead." On the other hand, putting a negative phase constant into Eq. 18-16 moves the corresponding wave behind the one with $\phi = 0$. Such a wave is said to "lag."

If we fix our attention on a particular point of the string, say x_1, the displacement y at that point can be written

$$y(t) = -y_m \sin(\omega t + \phi'),$$

where we have substituted a new phase constant $\phi' = \phi - kx_1$. This expression for $y(t)$ is similar to Eq. 17-6 for simple harmonic motion. Hence any particular element of the string undergoes simple harmonic motion about its equilibrium position as this wavetrain travels along the string.

SAMPLE PROBLEM 18-1. A transverse sinusoidal wave is generated at one end of a long horizontal string by a bar that moves the end up and down through a distance of 1.30 cm. The motion is continuous and is repeated regularly 125 times per second. (*a*) If the distance between adjacent wave crests is observed to be 15.6 cm, find the amplitude, frequency, speed, and wavelength of the wave motion. (*b*) Assuming the wave moves in the $+x$ direction and that, at $t = 0$, the element of the string at $x = 0$ is at its equilibrium position $y = 0$ and moving downward, find the equation of the wave.

Solution (*a*) As the bar moves a total of 1.30 cm, the end of the string moves $\frac{1}{2}(1.30 \text{ cm}) = 0.65$ cm away from the equilibrium position, first above it, then below it; therefore the amplitude y_m is 0.65 cm.

The entire motion is repeated 125 times each second, and thus the frequency is 125 vibrations per second, or $f = 125$ Hz.

The distance between adjacent wave crests, which is given as 15.6 cm, is the wavelength, as Fig. 18-6 shows. Thus $\lambda = 15.6 \text{ cm} = 0.156$ m.

The wave speed is given by Eq. 18-13:

$$v = \lambda f = (0.156 \text{ m})(125 \text{ s}^{-1}) = 19.5 \text{ m/s}.$$

(*b*) The general expression for a transverse sinusoidal wave moving in the $+x$ direction is given by Eq. 18-16,

$$y(x, t) = y_m \sin(kx - \omega t - \phi).$$

Imposing the given initial conditions ($y = 0$ and $\partial y/\partial t < 0$ for $x = 0$ and $t = 0$) yields

$$y_m \sin(-\phi) = 0 \quad \text{and} \quad -y_m\omega\cos(-\phi) < 0,$$

which means that the phase constant ϕ may be taken to be zero (or any integer multiple of 2π). Hence, for this wave

$$y(x, t) = y_m \sin(kx - \omega t),$$

and with the values just found,

$$y_m = 0.65 \text{ cm},$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.156 \text{ m}} = 40.3 \text{ rad/m},$$

$$\omega = vk = (19.5 \text{ m/s})(40.3 \text{ rad/m}) = 786 \text{ rad/s},$$

we obtain as the equation for the wave

$$y(x, t) = (0.65 \text{ cm}) \sin[(40.3 \text{ rad/m})x - (786 \text{ rad/s})t].$$

SAMPLE PROBLEM 18-2. As the wave of Sample Problem 18-1 passes along the string, each particle of the string moves up and down at right angles to the direction of the wave motion. (*a*) Find expressions for the velocity and acceleration of a particle P located at $x_P = 0.245$ m. (*b*) Evaluate the transverse displacement, velocity, and acceleration of this particle at $t = 15.0$ ms.

Solution (*a*) For a particle at $x_P = 0.245$ m in the wave of Sample Problem 18-1, we obtain, using Eq. 18-14,

$$\begin{aligned} u_y(x_P, t) &= -(0.65 \text{ cm})(786 \text{ rad/s}) \\ &\quad \times \cos[(40.3 \text{ rad/m})(0.245 \text{ m}) - (786 \text{ rad/s})t] \\ &= -(511 \text{ cm/s}) \cos[9.87 \text{ rad} - (786 \text{ rad/s})t]. \end{aligned}$$

Similarly, using Eq. 18-15, we find the magnitude of the maximum acceleration to be $\omega^2 y_m = 4.02 \times 10^5 \text{ cm/s}^2$, and so

$$a_y(x_P, t) = -(4.02 \times 10^5 \text{ cm/s}^2) \sin[9.87 \text{ rad} - (786 \text{ rad/s})t].$$

(*b*) At $t = 15.0$ ms, we evaluate the expressions for y, u_y, and a_y to give

$$y = -0.61 \text{ cm}, \quad u_y = +173 \text{ cm/s}, \quad a_y = +3.8 \times 10^5 \text{ cm/s}^2.$$

That is, the particle is close to its maximum negative displacement, it is moving in the positive y direction (away from that maximum), and it is accelerating in the positive y direction (its velocity is increasing in magnitude as the particle moves toward its equilibrium position).

18-4 WAVE SPEED ON A STRETCHED STRING

So far we have obtained a general expression for a transverse wave—for example, Eq. 18-16. The phase speed was given in Eq. 18-13: $v = \lambda f = \omega/k$. However, this expression does not tell us about the phase speed itself; it shows only how the wavelength and frequency are related to one another in terms of the wave speed.

The phase speed of a sinusoidal wave can be derived based on the mechanical properties of the medium through which the wave travels, in our case a stretched string. In this section we will obtain the phase speed by applying Newton's laws to the motion of the wave along the string. In other cases, such as sound traveling in a gas, similar methods can be used to find an expression for the wave speed.

The speed of a wave depends on the properties of the medium and is assumed to be independent of frequency and wavelength. (If the speed does depend on the frequency or wavelength of the wave, the medium is said to be *dispersive,* which we discuss later in this section.) Each element of the string pulls on its neighbors with a force given by the tension F in the string. The stronger the tension, the greater the force between neighboring elements and the more rapidly any disturbance will propagate down the string. Thus the wave speed should increase with increasing tension.

On the other hand, the inertia of each element limits how effective the tension will be in accelerating that element to move the wave along the string. Thus, for the same tension, the wave speed will be smaller in strings having more massive elements. The mass of each small element can be given in terms of the *mass density* μ (mass per unit length), which for a uniform string is equal to its mass divided by its length. On the basis of these general principles we therefore expect

$$v \propto \frac{F^a}{\mu^b},$$

where a and b are exponents that must be determined from the analysis.

It turns out that we can deduce the values of a and b based on a *dimensional analysis;* that is, there is only one combination of force and mass density that gives a quantity with the dimensions of velocity. From this type of analysis (see Exercise 5), we deduce $a = \frac{1}{2}$ and $b = \frac{1}{2}$, so that $v \propto \sqrt{F/\mu}$ or, introducing a constant of proportionality C, we have $v = C\sqrt{F/\mu}$. As we see next, the analysis using Newton's laws gives this same result and shows that $C = 1$.

Mechanical Analysis

Now let us derive an expression for the speed of a pulse in a stretched string by a mechanical analysis. In Fig. 18-8 we show a "snapshot" of a wave pulse that is moving from left to right in the string with a speed v. We can imagine instead that the entire string is moved from right to left with this same speed so that the wave pulse remains fixed in space (perhaps by pulling the string through a frictionless tube having the desired shape of the pulse). This simply means that, instead of taking our reference frame to be the walls between which the string is stretched, we choose a reference frame that is in uniform motion with respect to that one. In effect, we observe the pulse while running along the string at the same speed as the pulse. Because Newton's laws involve only accelerations, which are the same in both frames, we can use them in either frame. We just happen to choose a more convenient frame.

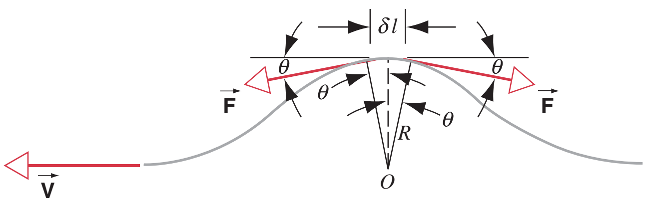


Figure 18-8. A pulse moving to the right on a stationary string is equivalent to a pulse in a fixed position on a string that is moving to the left. We consider the tension forces on a section of string of length δl on the "fixed" pulse.

We consider a small section of the pulse of length δl, as shown in Fig. 18-8. This section approximately forms an arc of a circle of radius R. The mass δm of this element is $\mu\ \delta l$, where μ is the mass density of the string. The tension F in the string is a tangential pull at each end of this small segment of the string. The horizontal components of $\vec{\mathbf{F}}$ cancel, and the vertical components are each equal to $F \sin\theta$. Hence the total vertical force F_y is $2F\sin\theta$. Because θ is small, we can take $\sin\theta \approx \theta$. From Fig. 18-8, we see that $2\theta = \delta l/R$, and so we obtain

$$F_y = 2F\sin\theta \approx 2F\theta = F\frac{\delta l}{R}. \tag{18-18}$$

This gives the force supplying the centripetal acceleration v^2/R of the string particles directed toward O. Note that the tangential velocity v of this mass element along the top of the arc is horizontal and is in magnitude equal to the wave speed. Applying Newton's second law to the string element δm, we have $\Sigma F_y = (\delta m)\, a_y$ or, using Eq. 18-18,

$$F\frac{\delta l}{R} = (\delta m)\, a_y = (\delta m)\frac{v^2}{R} = (\mu\ \delta l)\frac{v^2}{R},$$

where we have used $a_y = v^2/R$ for the centripetal acceleration and $\delta m = \mu\delta l$ for the mass of the string element. From the first and last terms of this equation, we obtain

$$v = \sqrt{\frac{F}{\mu}} \tag{18-19}$$

Equation 18-19 shows from a mechanical analysis that the constant C introduced in the dimensional analysis has the value 1.

If the amplitude of the pulse were very large compared to the length of the string, we would have been unable to use the approximation $\sin\theta \approx \theta$. Furthermore, the tension F in the string would be changed by the presence of the pulse, whereas we assumed F to be unchanged from the original tension in the stretched string. Therefore our result holds only for relatively small transverse displacements of the string, a case that is widely applicable in practice.

A periodic wave that enters a medium usually results from an external influence that disturbs the medium at a certain frequency. The wave that travels through that medium will have the same frequency as the source of the wave. The speed of the wave is determined by the properties of the medium. Given the frequency f of the wave and its speed v in the medium, the wavelength of the periodic wave *in that medium* is determined from Eq. 18-13, $\lambda = v/f$. When a wave passes from one medium to another medium of different wave speed (for example, two strings of different linear mass densities), the frequency in one medium must be the same as the frequency in the other. (Otherwise there would be a discontinuity at the point where the two strings are joined.) The wavelengths, however, will differ from one another. The relationship between the wavelengths follows from the equality of the frequencies f_1 and f_2 in the two media; that is, $f_1 = f_2$ gives

$$\frac{v_1}{\lambda_1} = \frac{v_2}{\lambda_2}. \qquad (18\text{-}20)$$

Group Speed and Dispersion (Optional)

Pure sinusoidal waves are useful mathematical devices for helping us understand wave motion. In practice, we use other kinds of waves to transport energy and information. These waves may be periodic but nonsinusoidal, such as square waves or "sawtooth" waves, or they may be nonperiodic pulses, such as that of Fig. 18-4.

We have used the phase speed to describe the motion of two kinds of waves: the pulse that preserves its shape as it travels (Fig. 18-4) and the pure sine wave (Fig. 18-6). In other cases, we must use a different speed, called the *group speed,* which is the speed at which energy or information travels in a real wave.

Figure 18-9 shows a pulse traveling through a medium. The shape of the pulse changes as it travels; the pulse spreads out, or *disperses.* (Dispersion is not the same as energy dissipation. The energy content of the pulse in Fig. 18-9 may remain constant as it travels, even though the pulse disperses. We assume that the medium is *dispersive,* but not necessarily *dissipative.*) As we see in Section 18-7, any periodic wave can be regarded as the sum or superposition of a series of sinusoidal waves of different frequencies or wavelengths. The frequencies, amplitudes, and phases of the component sinusoidal waves must be carefully chosen according to a prescribed mathematical procedure, known as *Fourier analysis,* so that the waves add to give the desired waveform. In most real media, the speed of propagation of these component waves (that is, the phase speed) depends on the frequency or wavelength of the particular component. Each component wave may travel with its own unique speed. Thus, as the wave travels, the phase relationships of the components may change, and the waveform of the sum of the components will correspondingly change as the wave travels. This is the origin of dispersion—the component waves travel at different phase speeds. There is no simple relationship between the phase speeds of the components and the group speed of the wave; the relationship depends on the dispersion of the medium.

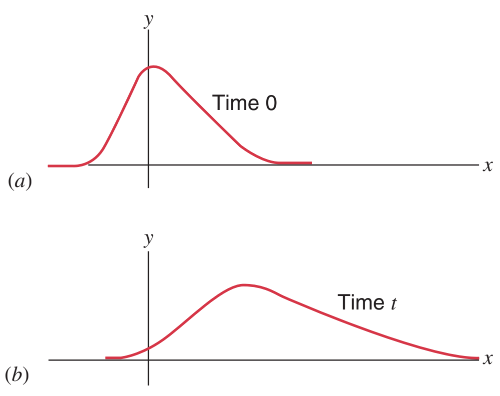


FIGURE 18-9. In a dispersive medium, the waveform changes as the wave travels.

Some real media are approximately nondispersive, in which case the wave keeps its shape, and all component waves travel with the same speed. An example is sound waves in air. If air were strongly dispersive for sound waves, conversation would be impossible, because the waveform produced by your friend's vocal cords would be jumbled by the time it reached your ears. Furthermore, the care taken by the players in an orchestra to play precisely at the same time would be to no avail, because (if air were dispersive for sound) the notes of high frequency would travel to the listener's ear at a speed different from that of the notes of low frequency, and the listener would hear the sounds at different times. Fortunately, this does not occur for sound waves. Light waves in vacuum are perfectly nondispersive; the dispersion of light waves in real media is responsible for such effects as the spectrum of colors in rainbows.

In a nondispersive medium, all the component waves in a complex waveform travel at the same phase speed, and the group speed of the waveform is equal to that common value of the phase speed. Only in this case can we speak of the phase speed of the entire waveform. In this chapter, we assume that we are dealing with mechanical waves that propagate in a nondispersive medium. ■

18-5 THE WAVE EQUATION (Optional)

In Chapter 17 we discussed the commonly encountered phenomenon of oscillation. One reason that this phenomenon is so common is that the basic equation that describes an oscillating system [$x = x_m \cos(\omega t + \phi)$, Eq. 17-6] is a solution of Eq. 17-5,

$$\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x,$$

which is an equation of a general form that can be derived from a mechanical analysis of a variety of physical situations, some of which were discussed in Section 17-5.

The situation is similar in the case of wave motion. As we demonstrate in this section, the mechanical analysis gives an equation of another commonly encountered form, the solution of which is a wave of the form of Eq. 18-2 or 18-5.

Figure 18-10 shows an element of a long string that is under tension F. A passing wave has caused the element to be displaced from its equilibrium position at $y = 0$. We

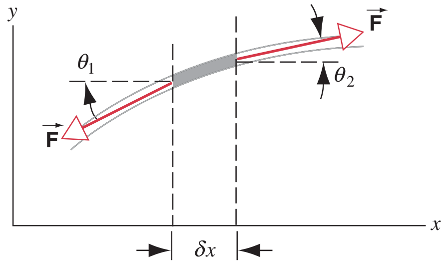


FIGURE 18-10. A small element of length δx of a long string under tension F. The figure represents a snapshot of the element at a particular time during the passage of a wave.

consider the element of the string of length δx, and we apply Newton's second law to analyze *how* this element is made to move.

The element is acted on by two forces, exerted by the portions of the string on either side of the element. These forces have equal magnitudes, because the tension is evenly distributed along the string, but they have slightly different directions, because they act tangent to the string at the endpoints of the element. The y component of the net force is

$$\sum F_y = F \sin\theta_2 - F \sin\theta_1 .$$

We consider only small displacements from equilibrium, so that the angles θ_1 and θ_2 are small, and we can write $\sin\theta \approx \tan\theta$, which gives

$$\sum F_y \approx F\tan\theta_2 - F\tan\theta_1 = F\,\delta(\tan\theta), \quad (18\text{-}21)$$

where $\delta(\tan\theta) = \tan\theta_2 - \tan\theta_1$. This resultant force must be equal to the mass of the element, $\delta m = \mu\,\delta x$, times the y component of the acceleration. If frictional or other dissipative forces can be neglected, Newton's second law gives

$$\sum F_y = \delta m\, a_y$$

$$F\delta(\tan\theta) = \mu\,\delta x\, a_y$$

$$\frac{\delta(\tan\theta)}{\delta x} = \frac{\mu}{F} a_y.$$

For the y component of the acceleration a_y, we use the transverse acceleration for a particle, $\partial^2 y/\partial t^2$. We also replace $\tan\theta$, which is the slope of the string, by the equivalent partial derivative $\partial y/\partial x$. Making these substitutions, we obtain

$$\frac{\delta(\partial y/\partial x)}{\delta x} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}. \quad (18\text{-}22)$$

We now take the limit of Eq. 18-22 as the mass element becomes very small. The left side is in the standard form for expressing the derivative with respect to x as a limit:

$$\lim_{\delta x\to 0}\frac{\delta(\partial y/\partial x)}{\delta x} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial^2 y}{\partial x^2},$$

and Eq. 18-22 becomes

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}. \quad (18\text{-}23)$$

Using Eq. 18-19 to replace μ/F with $1/v^2$, we obtain

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}. \quad (18\text{-}24)$$

Equation 18-24 is the general form of equation that describes waves: the second derivative of the wave displacement y with respect to the coordinate x in the direction of propagation is equal to $1/v^2$ times the second derivative with respect to time. This general form of equation is called the *wave equation*. It arises not only in mechanics but in other situations as well. For example, as we discuss in Chapter 38, if we use the equations of electromagnetism instead of the equations of mechanics (Newton's laws), we obtain an equation of exactly the same form as Eq. 18-24, except that the displacement y is replaced by the strength of an electric or magnetic field. The speed of propagation v for electromagnetic waves traveling in a vacuum becomes the speed of light c.

Let us see how our general formula for a traveling wave, $y(x, t) = f(x \pm vt)$, is the solution of Eq. 18-24. We make a simple change of variable and let z represent $x \pm vt$, so that $y = f(z)$. Then, repeatedly using the chain rule of calculus,

$$\frac{\partial y}{\partial x} = \frac{df}{dz}\frac{\partial z}{\partial x} = \frac{df}{dz}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{d}{dz}\left(\frac{df}{dz}\right)\frac{\partial z}{\partial x} = \frac{d^2 f}{dz^2}$$

$$\frac{\partial y}{\partial t} = \frac{df}{dz}\frac{\partial z}{\partial t} = \pm v\frac{df}{dz}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{d}{dz}\left(\pm v\frac{df}{dz}\right)\frac{\partial z}{\partial t} = (\pm v)^2\frac{d^2 f}{dz^2} = v^2\frac{d^2 f}{dz^2}.$$

Thus

$$\frac{d^2 f}{dz^2} = \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}.$$

and Eq. 18-24 is satisfied. It can be shown that *only* the combinations $x \pm vt$ in f satisfy the wave equation, so that all traveling waves must be in the form of Eq. 18-2 or 18-5.

To express these results in another way, Eq. 18-23, which was derived from Newton's laws, represents a traveling wave only when $\mu/F = 1/v^2$. This discussion thus provides an independent derivation of Eq. 18-19 for the velocity of propagation of waves along a stretched string. ■

18-6 ENERGY IN WAVE MOTION

If, as in Fig 18-1, you shake one end of a long string, your hand is doing work on the string. You are thus providing energy to the string. That energy travels along the string as a wave, and a friend at the other end of the string could extract that energy. Energy transport is an important property of waves, and in this section we examine the energy of a wave on a stretched string.

Figure 18-11a shows a wave traveling along the string at times t_1 and t_2 (a time $T/4$ later). Consider two elements on the string, each of length dx. The element at A was located on a wave crest at t_1, after which it moved downward and is crossing the axis at t_2. The element at B was crossing the axis at t_1 but is on a wave crest at t_2.

Element A is at rest at t_1, while at t_2 it has the maximum particle speed. This element therefore gains kinetic energy between t_1 and t_2. Element A has very nearly its relaxed length dx at time t_1, but at time t_2 it has been stretched to a greater length by the tension in the string. It thus gains potential energy from t_1 to t_2. On the other hand, element B loses kinetic energy between t_1 and t_2. Moreover, element B is stretched at t_1 but has its relaxed length at t_2, so its potential energy also decreases. We can thus view the travel of a wave along the string in terms of the kinetic and potential energy of each element of the string. By calculating how the energy changes with time, we can determine the power delivered by the wave.

Figure 18-11b shows an expanded view of a string element at an arbitrary point in its motion. Its length has been stretched from its relaxed length dx to dl. The element has mass $dm = \mu\, dx$ and is moving with velocity u_y given by Eq. 18-14, so its kinetic energy dK is

$$dK = \tfrac{1}{2} dm\, u_y^2 = \tfrac{1}{2}(\mu\, dx)[-y_m\omega \cos(kx - \omega t)]^2. \quad (18\text{-}25)$$

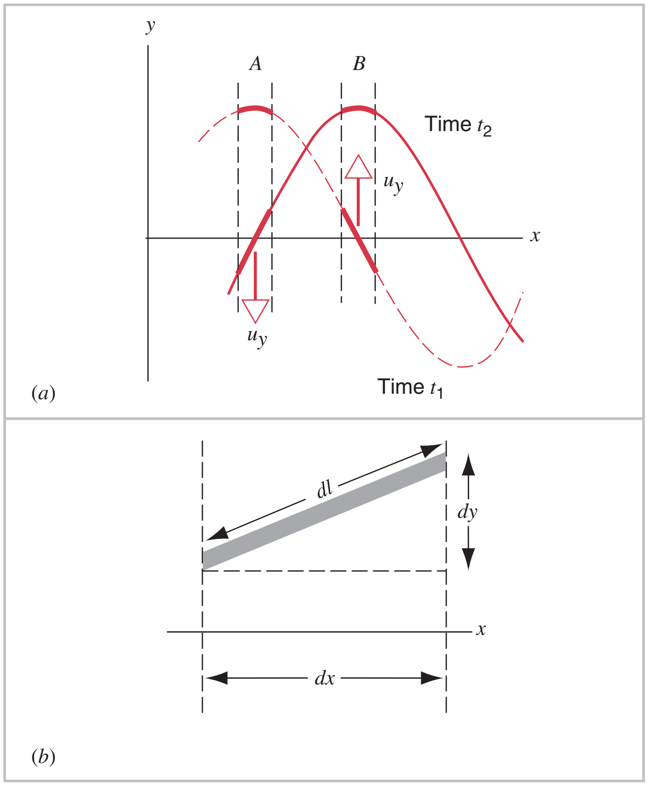


FIGURE 18-11. (a) Two small elements of string, labeled A and B, are shown on a wave at time t_1 and again at a time t_2 (one-quarter of a cycle later). The wave is moving to the right (in the direction of increasing x). (b) A magnified view of a small element of the string at an arbitrary time.

This change in kinetic energy has occurred in the time dt that it takes for the wave to move a distance along the x axis equal to the x component of the length of the element; that is, $dt = dx/v$, where v is the wave speed. The rate at which kinetic energy is transported by the wave is dK/dt, or

$$\frac{dK}{dt} = \tfrac{1}{2}\mu\omega^2 y_m^2 \frac{dx}{dt}\cos^2(kx - \omega t)$$
$$= \tfrac{1}{2}\mu\omega^2 y_m^2 v \cos^2(kx - \omega t). \quad (18\text{-}26)$$

To find the potential energy in the element, we must evaluate the work done by the tension force F as it stretches the element from length dx to length dl, or $dU = F(dl - dx)$. Approximating dl as the hypotenuse of a right triangle, as in Fig. 18-11b, we have

$$dU = F\left[\sqrt{(dx)^2 + (dy)^2} - dx\right]$$
$$= F dx\left[\sqrt{1 + (\partial y/\partial x)^2} - 1\right]. \quad (18\text{-}27)$$

The quantity $\partial y/\partial x$ gives the slope of the string, and if the amplitude of the wave is not too large this slope will be small. We can then use the binomial expansion $(1 + z)^n \approx 1 + nz + \cdots$ to write

$$dU = F dx\left[1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 - 1\right] = \tfrac{1}{2}F dx\left(\frac{\partial y}{\partial x}\right)^2. \quad (18\text{-}28)$$

With $\partial y/\partial x = -y_m k \cos(kx - \omega t)$, we find the rate at which potential energy is transported along the string to be

$$\frac{dU}{dt} = \tfrac{1}{2}F\frac{dx}{dt}[-y_m k \cos(kx - \omega t)]^2$$
$$= \tfrac{1}{2}F v y_m^2 k^2 \cos^2(kx - \omega t). \quad (18\text{-}29)$$

Using Eqs. 18-19 and 18-13 we can write $F = v^2\mu = (\omega/k)^2\mu$; substituting this result into Eq. 18-29 and comparing with Eq. 18-26, it immediately follows that $dU/dt = dK/dt$.

Note that dK and dU are both zero when the element has its maximum displacement (as in the case of element A at time t_1), and both dK and dU have their maximum values when the element crosses the x axis (as in the case of element A at time t_2). Although the motion of an element of the string reminds us of the simple harmonic oscillator, there is an important difference: the mechanical energy $dE = dU + dK$ of the mass element is *not* a constant, but instead it varies from zero at the crests and valleys to a maximum where the string crosses the axis. This should not be a surprise, because the mass element is not an isolated system—neighboring mass elements are doing work on it to change its energy.

Power and Intensity in Wave Motion

Because $dU/dt = dK/dt$, we have

$$\frac{dE}{dt} = \frac{dK}{dt} + \frac{dU}{dt} = 2\frac{dU}{dt} = \mu\omega^2 y_m^2 v \cos^2(kx - \omega t). \quad (18\text{-}30)$$

The rate at which mechanical energy is transmitted along the string is simply the power: $P = dE/dt$. This quantity varies with location along the string as well as with time. Usually we are more interested in the average power P_{av}:

$$P_{av} = \left(\frac{dE}{dt}\right)_{av} = \mu\omega^2 y_m^2 v[\cos^2(kx - \omega t)]_{av}. \quad (18\text{-}31)$$

Often we observe waves over a time that is very long compared with the period of the wave, so that we take the average over many cycles of the oscillation. The average value of the $\cos^2$ over any number of full cycles is $\frac{1}{2}$, and so

$$P_{av} = \tfrac{1}{2}\mu\omega^2 y_m^2 v. \quad (18\text{-}32)$$

The dependence of the average rate of energy transfer on the *square* of the amplitude and the *square* of the frequency is a general characteristic property of waves.

This calculation assumes that the wave transports energy with no losses due to friction or other dissipative forces. None of the mechanical energy is lost to internal energy of the string or heat transferred to the surroundings.

We have also assumed that the amplitude of the wave remains constant as it travels. This remains true (in the ideal approximation) for waves on a string, and it is strictly true for the ideal plane wave (as in Fig. 18-3*a*). However, for spherical wavefronts (as in Fig. 18-3*b*), the energy content of each wavefront remains the same, but that energy is spread over an increasing area as the wave travels. For such spherical waves, it is often more useful to describe the wave in terms of its *intensity* I, which is defined as *the average power per unit area transmitted across an area A perpendicular to the direction in which the wave is traveling,* or

$$I = \frac{P_{av}}{A}. \quad (18\text{-}33)$$

The SI unit of intensity is watts per meter squared (W/m^2).

Just as with the power of a wave, the intensity is always proportional to the square of the amplitude. However, for circular or spherical waves, the amplitude is not constant as the wavefront advances. In a spherical wave, such as might be emitted by a point source of light or sound, the surface area of a wavefront of radius r is $4\pi r^2$, so the intensity is proportional to $1/r^2$. If the distance from a source of spherical waves is doubled, the intensity becomes one-quarter as large while the amplitude of the wave becomes half as large.

18-7 THE PRINCIPLE OF SUPERPOSITION

We often observe two or more waves to travel simultaneously through the same region of space independently of one another. For example, the sound reaching our ears from a symphony orchestra is very complex, but we can pick out the sound made by individual instruments. The electrons in the antennas of our radio and TV sets are set into motion by a whole array of signals from different broadcasting centers, but we can nevertheless tune to any particular station, and the signal we receive from that station is in principle the same as that which we would receive if all other stations were to stop broadcasting.

The above examples illustrate the *principle of superposition,* which asserts that, when several waves combine at a point, the displacement of any particle at any given time is simply the sum of the displacements that each individual wave acting alone would give it. For example, suppose that two waves travel simultaneously along the same stretched string. Let $y_1(x, t)$ and $y_2(x, t)$ be the displacements that the string would experience if each wave acted alone. The displacement of the string when both waves act is then

$$y(x, t) = y_1(x, t) + y_2(x, t). \quad (18\text{-}34)$$

For mechanical waves in elastic media, the superposition principle holds whenever the restoring force varies linearly with the displacement.

Figure 18-12 shows a time sequence of "snapshots" of two pulses traveling in opposite directions in the same stretched string. When the pulses overlap, the displacement of the string is the algebraic sum of the individual displacements of the string caused by each of the two pulses alone, as Eq. 18-34 requires. The pulses simply move through one another, each moving along as if the other were not present.

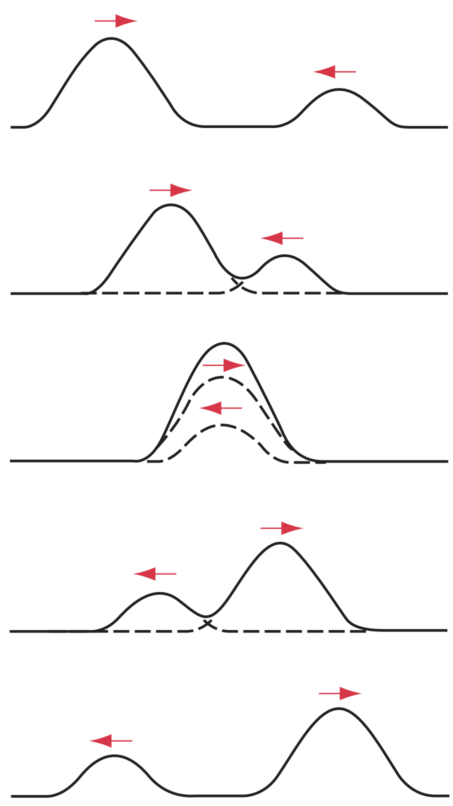


Figure 18-12. Two pulses travel in opposite directions along a stretched string. The superposition principle applies as they move through each other.

The superposition principle may seem to be an obvious result, but there are instances in which it does not hold. Suppose, for instance, that one of the waves has such a large amplitude that the elastic limit of the medium is exceeded. The restoring force is no longer directly proportional to the displacement of a particle in the medium. Then, no matter what the amplitude of the second wave (even if it is very small), its effect at a point is not a linear function of its amplitude. Furthermore, the second wave will be changed by passing through the nonlinear region, and its subsequent behavior will be altered. This situation arises only very rarely, and in most circumstances the principle of superposition is valid (as we assume throughout this text).

Fourier Analysis (Optional)

The importance of the superposition principle physically is that, where it holds, it makes it possible to analyze a complicated wave motion as a combination of simple waves. In fact, as was shown by the French mathematician J. Fourier (1768–1830), all that we need to build up the most general form of periodic wave are simple harmonic waves. Fourier showed that any periodic motion of a particle can be represented as a combination of simple harmonic motions. For example, if $y(x)$ represents the waveform (at a particular time) of a source of waves having a wavelength λ, we can analyze $y(x)$ as follows:

$$y(x) = A_0 + A_1 \sin kx + A_2 \sin 2kx + A_3 \sin 3kx + \cdots + B_1 \cos kx + B_2 \cos 2kx + B_3 \cos 3kx + \cdots, \quad (18\text{-}35)$$

where $k = 2\pi/\lambda$. This expression is called a Fourier series. The coefficients A_n and B_n have definite values for any particular periodic motion $y(x)$. For example, the so-called sawtooth wave of Fig. 18-13*a* can be described by

$$y(x) = -\frac{1}{\pi}\sin kx - \frac{1}{2\pi}\sin 2kx - \frac{1}{3\pi}\sin 3kx - \cdots .$$

If the motion is not periodic, as in the case of a pulse, the sum is replaced by an integral—the Fourier integral. Hence any motion (pulsed or continuous) of a source of waves can be represented in terms of a superposition of simple harmonic motions, and any waveform so generated can be analyzed as a combination of components that are individually simple harmonic waves. This once again illustrates the importance of harmonic motion and harmonic waves.

Only in the case of a nondispersive medium will the waveform maintain its shape as it travels. In a dispersive medium, the waveforms of the sinusoidal component waves do not change, but each may travel at a different speed. In this case, the combined waveform changes as the phase relationship between the components is altered. The wave can also change its shape if it loses mechanical energy to the medium, such as by air resistance, viscosity, or internal friction. Such dissipative forces often depend on the speed, and so the Fourier components most strongly affected are those with higher particle speeds (that is, those with high frequencies, according to Eq. 18-14 in which u_y is seen to depend on ω). Here again the wave shape may change, as the higher frequency components lose amplitude more quickly. The decay with time of the sound of piano strings is an example of this phenomenon. The vibrational motion of a piano string, immediately after it is struck by the hammer, includes a wide range of frequencies, which give it its characteristic tone. The higher frequency components of this complex motion dissipate their energy more rapidly than the lower frequency components, and thus the character of a sustained tone may change with time. ■

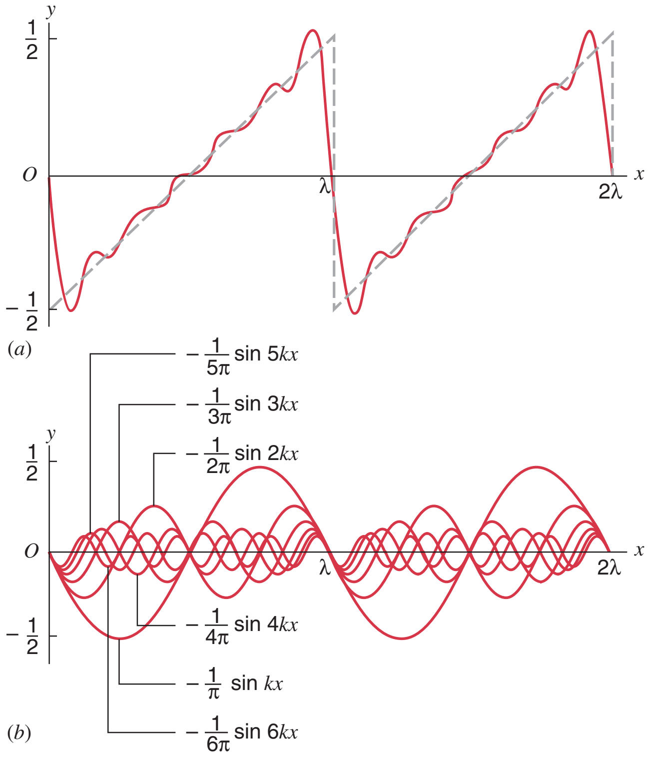


FIGURE 18-13. (*a*) The dashed line is a sawtooth wave commonly encountered in electronics. It can be represented as a Fourier series of sine waves. (*b*) The first six sine waves of the Fourier series that represents the sawtooth wave are shown, and their sum is shown as the solid curve in part (*a*). As more terms are included, the Fourier series becomes a better approximation of the wave.

18-8 INTERFERENCE OF WAVES

When two or more waves combine at a particular point, they are said to *interfere,* and the phenomenon is called *interference.* As we shall see, the resultant waveform is strongly dependent on the relative phases of the interfering waves. Figure 18-14 shows an example of interfering waves.

Let us first consider two transverse sinusoidal waves of equal amplitude and wavelength, which travel in the x direction with the same speed. We take the phase constant of one wave to be ϕ, while the other has $\phi = 0$. Figure 18-15 shows two individual waves y_1 and y_2 and their sum $y_1 + y_2$ at a particular time for the two cases of ϕ nearly 0 (the

FIGURE 18-14. Two wave trains, in this case circular ripples from two different disturbances, interfere where they overlap at particular points. The displacement at any point is the superposition of the individual displacements due to each of the two waves.

waves are nearly in phase) and ϕ nearly 180° (the waves are nearly out of phase). You can see by merely adding the individual displacements at each x that in the first case there is nearly complete reinforcement of the two waves and the resultant has nearly double the amplitude of the individual components, whereas in the second case there is nearly complete cancellation at every point and the resultant amplitude is close to zero. These cases are known, respectively, as *constructive* interference and *destructive* interference.

Let us see how interference arises from the equations for the waves. We consider a general case in which the two waves have phase constants ϕ_1 and ϕ_2, respectively. The equations of the two waves are

$$y_1(x, t) = y_m \sin(kx - \omega t - \phi_1) \qquad (18\text{-}36)$$

and

$$y_2(x, t) = y_m \sin(kx - \omega t - \phi_2). \qquad (18\text{-}37)$$

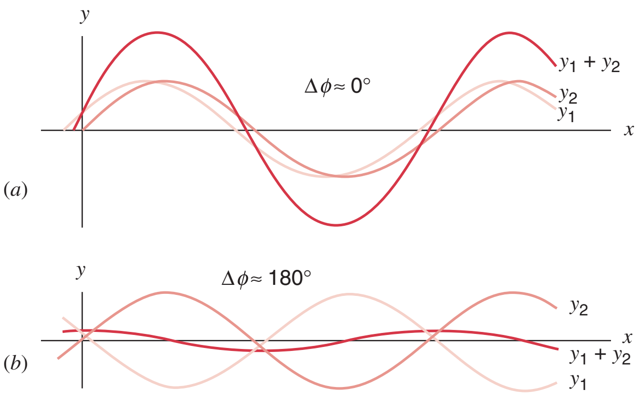


FIGURE 18-15. (*a*) The superposition of two waves of equal wavelength and amplitude that are almost in phase results in a wave of almost twice the amplitude of either component. (*b*) The superposition of two waves of equal wavelength and amplitude that are almost 180° out of phase results in a wave whose amplitude is nearly zero. Note that the wavelength of the resultant is unchanged in either case.

Now let us find the resultant wave. Using the principle of superposition, we take the sum of Eqs. 18-36 and 18-37, which gives

$$y(x, t) = y_1(x, t) + y_2(x, t) = y_m[\sin(kx - \omega t - \phi_1) + \sin(kx - \omega t - \phi_2)]. \qquad (18\text{-}38)$$

From the trigonometric identity for the sum of the sines of two angles,

$$\sin B + \sin C = 2 \sin \tfrac{1}{2}(B + C) \cos \tfrac{1}{2}(B - C), \qquad (18\text{-}39)$$

we obtain, after some rearrangement,

$$y(x, t) = [2y_m \cos(\Delta\phi/2)] \sin(kx - \omega t - \phi'), \qquad (18\text{-}40)$$

where $\phi' = (\phi_1 + \phi_2)/2$. The quantity $\Delta\phi = (\phi_2 - \phi_1)$ is called the *phase difference* between the two waves.

This resultant wave corresponds to a new wave having the same frequency but with an amplitude $2y_m|\cos(\Delta\phi/2)|$. If $\Delta\phi$ is very small (close to 0°), the resultant amplitude is nearly $2y_m$ (as shown in Fig. 18-15*a*). When $\Delta\phi$ is zero, the two waves overlap completely: the crest of one falls on the crest of the other and likewise for the valleys, which gives total constructive interference. The resultant amplitude is just twice that of either wave alone. If $\Delta\phi$ is close to 180°, on the other hand, the resultant amplitude is nearly zero (as shown in Fig. 18-15*b*). When $\Delta\phi$ is exactly 180°, the crest of one wave falls exactly on the valley of the other. The resultant amplitude is zero, corresponding to total destructive interference.

Notice that Eq. 18-40 always has the form of a sinusoidal wave. Thus adding two sine waves of the same wavelength and amplitude always gives a sine wave of the identical wavelength. We can also add components that have the same wavelength but different amplitudes. In this case the resultant again is a sine wave with the identical wavelength, but the resultant amplitude does not have the simple form given by Eq. 18-40. If the individual amplitudes are y_{1m} and y_{2m}, then if the waves are in phase ($\Delta\phi = 0$) the resultant amplitude is $y_{1m} + y_{2m}$ (Fig. 18-16*a*), whereas if they

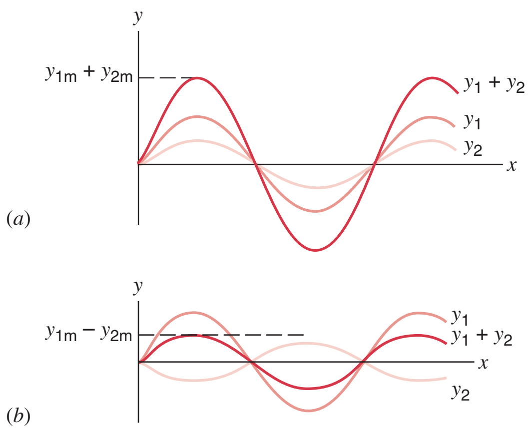


FIGURE 18-16. The addition of two waves of the same wavelength and phase but differing amplitudes (lighter color) gives a resultant of the same wavelength and phase. (*a*) The amplitudes add if the waves are in phase, and (*b*) they subtract if the waves are 180° out of phase.

are out of phase ($\phi = 180°$) the resultant amplitude is $|y_{1m} - y_{2m}|$ (Fig. 18-16*b*). There can be no complete destructive interference in this case, although there is partial destructive interference.

SAMPLE PROBLEM 18-3. Two waves travel in the same direction along a string and interfere. The waves have the same wavelength and travel with the same speed. The amplitude of each wave is 9.7 mm, and there is a phase difference of 110° between them. (*a*) What is the amplitude of the combined wave resulting from the interference of the two waves? (*b*) To what value should the phase difference be changed so that the combined wave will have an amplitude equal to that of one of the original waves?

Solution (*a*) The amplitude of the combined wave (always a positive quantity) was given in Eq. 18-40:

$$2y_m|\cos(\Delta\phi/2)| = 2(9.7\text{ mm})|\cos(110°/2)| = 11.1\text{ mm}.$$

(*b*) If the quantity $2y_m|\cos(\Delta\phi/2)|$ is to equal y_m, then we must have

$$2|\cos(\Delta\phi/2)| = 1,$$

or

$$\Delta\phi = 2\cos^{-1}(\tfrac{1}{2}) = 120° \quad\text{or}\quad -120°.$$

Either wave can be leading the other by 120° (plus or minus any integer multiple of 360°) to produce the desired combination wave.

18-9 STANDING WAVES

In the previous section we considered the effect of superposing two component waves of equal amplitude and frequency moving in the same direction on a string. What is the effect if the waves are moving along the string in *opposite* directions?

Figure 18-17 is a graphical indication of the effect of adding the component waveforms to obtain the resultant. Two traveling waves are shown in the figure, one moving to the left and the other to the right. "Snapshots" are shown of the two component waves and their resultant at intervals of one-quarter period.

One particular feature results from this superposition: there are certain points along the string, called *nodes,* at which the displacement is zero *at all times.* (Figure 18-16 also showed some points in which the resultant had zero displacement, but that figure represented a snapshot of traveling waves *at a particular time.* If we took another snapshot an instant later, we would find that those points no longer had zero displacement, because the wave is traveling. In Fig. 18-17*c*, the zeros remain zeros at all times.) Between the nodes are the *antinodes,* where the displacement oscillates with the largest amplitude. Such a pattern of nodes and antinodes is known as a *standing wave.*

To analyze the standing wave mathematically, we represent the two waves by

$$y_1(x, t) = y_m \sin(kx - \omega t),$$

$$y_2(x, t) = y_m \sin(kx + \omega t).$$

Hence the resultant may be written

$$y(x, t) = y_1(x, t) + y_2(x, t)$$

$$= y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t) \qquad (18\text{-}41)$$

or, making use of the trigonometric relation of Eq. 18-39,

$$y(x, t) = [2y_m \sin kx]\cos \omega t. \qquad (18\text{-}42)$$

Equation 18-42 is the equation of a standing wave. It cannot represent a traveling wave, because x and t do *not* appear in the combination $x - vt$ or $x + vt$ required for a traveling wave.

Note that a particle at any particular location x undergoes simple harmonic motion, and that all particles vibrate with the same angular frequency ω. In a traveling wave each particle of the string vibrates with the same amplitude. In a standing wave, however, *the amplitude is not the same for different particles but varies with the location x of the particle.* In fact, the amplitude, $|2y_m \sin kx|$, has a *maximum* value of $2y_m$ at positions where $kx = \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \ldots$. That is,

$$kx = \left(n + \frac{1}{2}\right)\pi \qquad n = 0, 1, 2, \ldots$$

or, substituting $k = 2\pi/\lambda$,

$$x = \left(n + \frac{1}{2}\right)\frac{\lambda}{2} \qquad n = 0, 1, 2, \ldots \qquad (18\text{-}43)$$

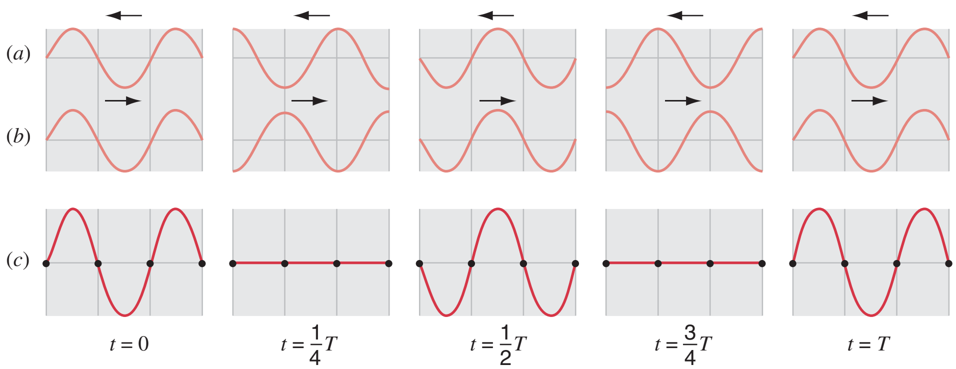


FIGURE 18-17. (*a*, *b*) Two traveling waves of the same wavelength and amplitude, moving in opposite directions. (*c*) The superposition of the two waves at different instants of time. The nodes in the standing wave pattern are indicated by dots. Note that the traveling waves have no nodes.

These points are the antinodes and are spaced one-half wavelength apart.

The amplitude has a *minimum* value of zero at positions where $kx = 0, \pi, 2\pi, 3\pi, \ldots$, so

$$kx = n\pi \qquad n = 0, 1, 2, \ldots$$

or

$$x = n\frac{\lambda}{2} \qquad n = 0, 1, 2, \ldots. \tag{18-44}$$

These points are the nodes and are also spaced one-half wavelength apart. The separation between a node and an adjacent antinode is one-quarter wavelength.

It is clear that energy is not transported along the string to the right or to the left, for energy cannot flow past the nodes in the string, which are permanently at rest. Hence the energy remains "standing" in the string, although it alternates between vibrational kinetic energy and elastic potential energy. When the antinodes are all at their maximum displacements, the energy is stored entirely as potential energy, in particular as the elastic potential energy associated with the stretching of the string. When all parts of the string are simultaneously passing through equilibrium (as in the second and fourth snapshots of Fig. 18-17*c*), the energy is stored entirely as kinetic energy. Figure 18-18 shows a more detailed description of the shifting of energy between kinetic and potential forms during one cycle of oscillation. Compare Fig. 18-18 with Fig. 12-5 for the oscillating block–spring system. How are these systems similar?

We can equally well regard the motion as an oscillation of the string as a whole, each particle undergoing simple harmonic motion of angular frequency ω and with an amplitude that depends on its location. Each small part of the string has inertia and elasticity, and the string as a whole can be thought of as a collection of coupled oscillators. Hence the vibrating string is the same in principle as the block–spring system, except that the block–spring system has only one natural frequency, and a vibrating string has a large number of natural frequencies (see Section 18-10.)

Reflection at a Boundary

To set up a standing wave in a string, we want to superimpose two waves traveling in opposite directions. One way to achieve this is to send a wave along a string so that it meets its reflection coming back. Here we consider the reflection process in more detail.

By way of illustration we consider a pulse rather than a sinusoidal wave. Suppose a pulse travels along a string that is fixed at one end, as shown in Fig. 18-19*a*. When the

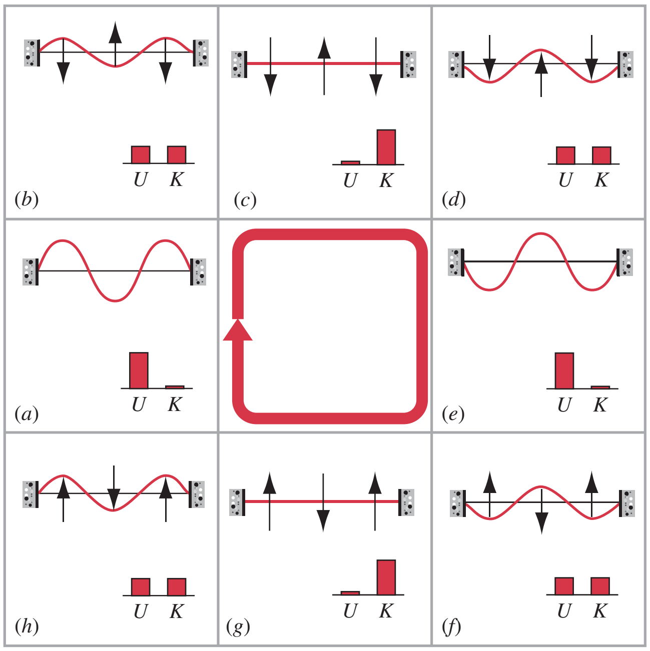


Figure 18-18. A standing wave on a stretched string, showing one cycle of oscillation. At (*a*) the string is momentarily at rest with the antinodes at their maximum displacement. The energy of the string is all elastic potential energy. (*b*) One-eighth of a cycle later, the displacement is reduced and the energy is partly potential and partly kinetic. The vectors show the instantaneous velocities of particles of the string at certain locations. (*c*) The displacement is zero; there is no potential energy, and the kinetic energy is maximum. The particles of the string have their maximum velocities. (*d*–*h*) The motion continues through the remainder of the cycle, with the energy being continually exchanged between potential and kinetic forms.

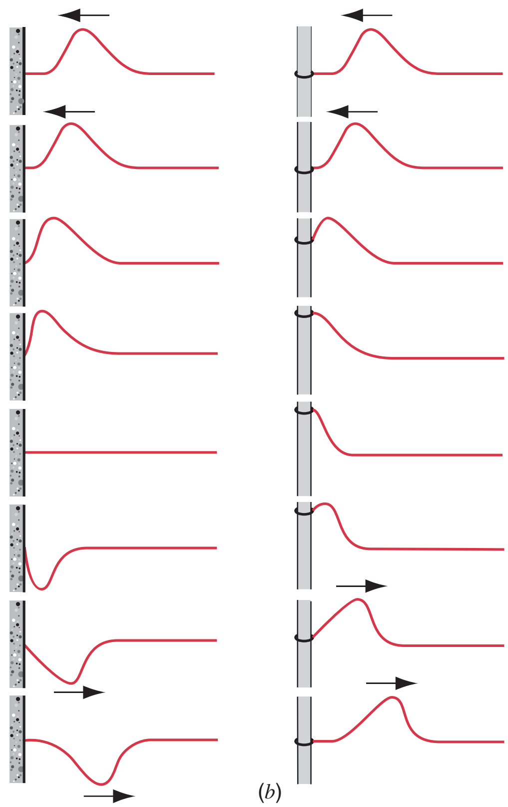


Figure 18-19. (*a*) A transverse pulse incident from the right is reflected by a rigid wall. Note that the phase of the reflected pulse is inverted, or changed by 180°. (*b*) Here the end of the string is free to move, the string being attached to a loop that can slide freely along the rod. The phase of the reflected pulse is unchanged.

pulse arrives at that end, it exerts an upward force on the support. Because the support is rigid, it does not move, and by Newton's third law it must exert an equal but oppositely directed force on the string. That force would be downward in Fig. 18-19*a* and causes an inverted pulse to travel in the opposite direction along the string. The incident and reflected pulses must tend to produce opposite displacements at the fixed end of the string, in order to keep that point fixed. We can consider this to be a situation of total destructive interference—the incident and reflected waves must be 180° out of phase. *On reflection from a fixed end, a transverse wave undergoes a phase change of 180°.*

The reflection of a pulse at a free end of a stretched string—that is, at an end that is free to move transversely—is represented in Fig. 18-19*b*. The end of the string is attached to a very light ring that is free to slide without friction along a transverse rod. When the pulse arrives at the free end, it exerts a force on the element of string there. This element is accelerated, and (as in the case of a pendulum) its motion carries it past the equilibrium point; it "overshoots" and exerts a reaction force on the string. This generates a pulse that travels back along the string in a direction opposite to that of the incident pulse. Once again we get reflection, but now at a free end. The free end will obviously suffer the maximum displacement of the particles on the string; an incident and a reflected wavetrain must interfere constructively at that point if we are to have a maximum there. Hence the reflected wave is always in phase with the incident wave at that point. *At a free end, a transverse wave is reflected without change of phase.*

So far we have assumed that the wave reflects at the boundary with no loss of intensity. In practice, we always find that at any boundary between two media there is partial reflection and partial transmission; for example, looking at a piece of ordinary window glass, you can see some light reflected back toward you and some transmitted through the glass. We can demonstrate this effect with transverse waves on strings by tying together two strings of different mass densities. When a wave traveling along one of the strings reaches the point where the strings are joined, part of the wave energy is transmitted to the other string and part is reflected back. The amplitude of the reflected wave is less than the amplitude of the original incident wave, because the wave transmitted to the second string carries away some of the incident energy.

If the second string has a greater mass density than the first, the wave reflected back into the first string still suffers a phase shift of 180° on reflection. However, because its amplitude is less than the incident wave, the boundary point is not a node and moves. Thus a net energy transfer occurs along the first string into the second. If the second string has a smaller mass density than the first, partial reflection occurs without change of phase, but once again energy is transmitted to the second string. In practice, the best way to realize a "free end" for a string is to attach it to a long and very much lighter string. The energy transmitted is negligible, and the second string serves to maintain the tension in the first one.

Note that the transmitted wave travels with a speed different from that of the incident and reflected waves. The wave speed is determined by the relation $v = \sqrt{F/\mu}$; the tension is the same in both strings, but their densities are different. Hence the wave travels more slowly in the denser string. The frequency of the transmitted wave is the same as that of the incident and reflected waves. (If this were not true, there would be a discontinuity at the point where the strings are joined.) Waves having the same frequency but traveling with different speeds have different wavelengths. From the relation $\lambda = v/f$, we conclude that in the denser string, where v is smaller, the wavelength is shorter. This phenomenon of change of wavelength as a wave passes from one medium to another will be encountered frequently in our study of light waves. It also occurs for sound waves: a string, such as on a guitar, vibrates with a certain frequency and wavelength; the wave transmitted to the air has the same frequency as that of the string, but a different wavelength, because the speed of waves on the string differs from their speed in air.

18-10 STANDING WAVES AND RESONANCE

Consider a string of length L that is fixed at both ends, such as we might find on a guitar or a violin. If we pluck the string near the middle and then examine its motion, we might find that it looks like Fig. 18-20*a*. A standing wave is established with a node at each end and an antinode in the middle.

How is it possible that by plucking the string we set up standing waves? The initial shape of the string, just as we release it, might have a triangular shape that we can analyze as a sum of sine and cosine terms using the method of

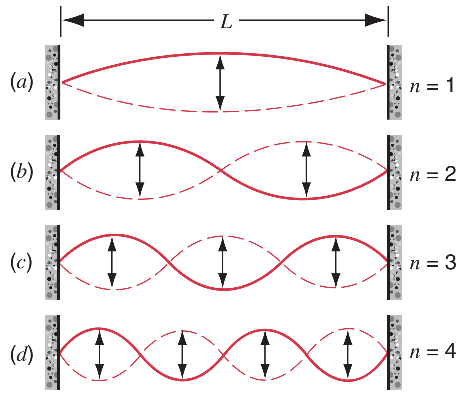


FIGURE 18-20. Standing wave patterns on a string of length L stretched between two fixed supports. Four different patterns are shown, corresponding to different wavelengths and frequencies.

Fourier analysis described in Section 18-7. Each of these waves travels along the string, reflects from the ends, and interferes with all of the waves traveling on the string. The higher frequencies tend to damp out more rapidly, leaving us with only the standing wave corresponding to the lowest possible frequency, which is shown in Fig. 18-20*a*. The spacing between nodes is always $\lambda/2$, so for the standing wave pattern shown in Fig. 18-20*a* we have $L = \lambda/2$.

We can produce a different standing wave on the string by placing a finger lightly near the center to keep it from moving and plucking about 1/4 of the way from either end. This procedure will produce a standing wave that looks like Fig. 18-20*b*. For this wave $L = \lambda$. By damping and plucking the string in suitably chosen locations, we can produce the standing wave patterns shown in Figs. 18-20*c* and 18-20*d*, for which $L = 3\lambda/2$ and $L = 2\lambda$, respectively.

You can see that the condition for a standing wave to be set up in a string of length L fixed at both ends is

$$L = n\frac{\lambda}{2} \qquad (n = 1, 2, 3, \ldots)$$

or

$$\lambda_n = \frac{2L}{n} \qquad (n = 1, 2, 3, \ldots), \qquad (18\text{-}45)$$

where λ_n is the nth wavelength in this infinite series. Note that n is the number of half-wavelengths or "loops" that appear in the patterns of Fig. 18-20. Using Eq. 18-13 ($v = \lambda f$), we can write Eq. 18-45 as

$$f_n = \frac{v}{\lambda_n} = n\frac{v}{2L} \qquad (n = 1, 2, 3, \ldots). \qquad (18\text{-}46)$$

These are the allowed frequencies of the standing waves on the string.

If we consider the similarity between a vibrating spring and a simple harmonic oscillator, we may wonder why the simple oscillator such as the block–spring system has only one allowed frequency whereas the string has an infinite number. In the block–spring system, the inertia is concentrated ("lumped") in a single element of the system (the block), but in the string the inertia is distributed throughout the system. Similarly, the elasticity of the block–spring system is lumped in one element (the spring) but in the string it is distributed throughout the system. Although there is only one way for the block–spring system to store kinetic and potential energy, the vibrating spring has an infinite number of ways to store its energy.

In general, a lumped system of N elements has N different oscillating frequencies, each of which corresponds to a different pattern of oscillation. Figure 18-21 shows an example of a lumped system with one, two, or three elements. The limit as N tends to infinity leads us to the completely distributed system of the stretched string, with its infinite number of vibrational frequencies.

Resonance in the Stretched String

Figure 18-22 shows time exposures of a student shaking one end of a string that is fixed at the other end. The resulting patterns of oscillation look just like the standing waves of Fig. 18-20. Careful examination would show that the student's hand is moving back and forth at small amplitude with one of the frequencies given by Eq. 18-46. We can regard those frequencies as the natural frequencies of the vibrating system. The student's hand is the driving force that sets the string into oscillation, and when the driving frequency matches one of the natural frequencies we get an oscillation at large amplitude, in exact analogy with our discussion of resonance of the forced oscillator in Section 17-8.

As the student shakes the string, his hand is doing work on it to pump energy into the vibrating system. Energy is lost by the system, perhaps to internal energy of the string, to air resistance, or to the support at the fixed end. As in the case of the forced oscillator, eventually a steady state is reached in which the energy supplied by the student exactly balances the energy lost by the string to dissipative forces.

If the student shakes the string at a frequency that differs from one of the natural frequencies, the reflected wave returns to the student's hand out of phase with the motion of the hand. In this case the string does work on the hand, in addition to the hand doing work on the string. No fixed standing wave pattern is produced; the amplitude of the resulting motion of the string is small and not much different

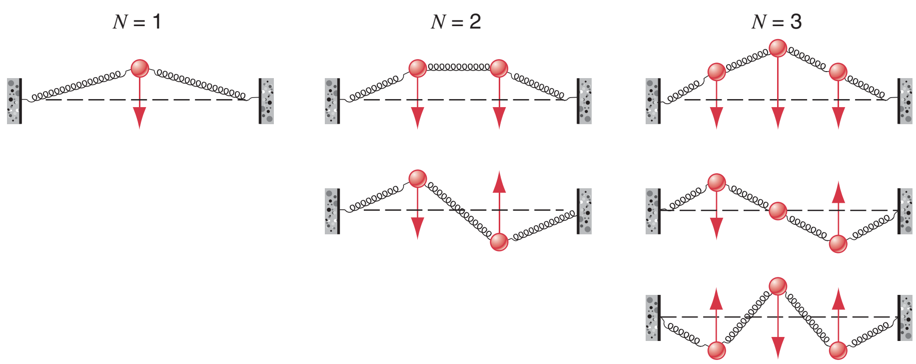


FIGURE 18-21. Some patterns of oscillation of an oscillator having lumped elements—in this case oscillating bodies connected by springs of negligible mass. Each different pattern of motion has a different natural frequency, the number of natural frequencies being equal to the number of oscillating bodies.

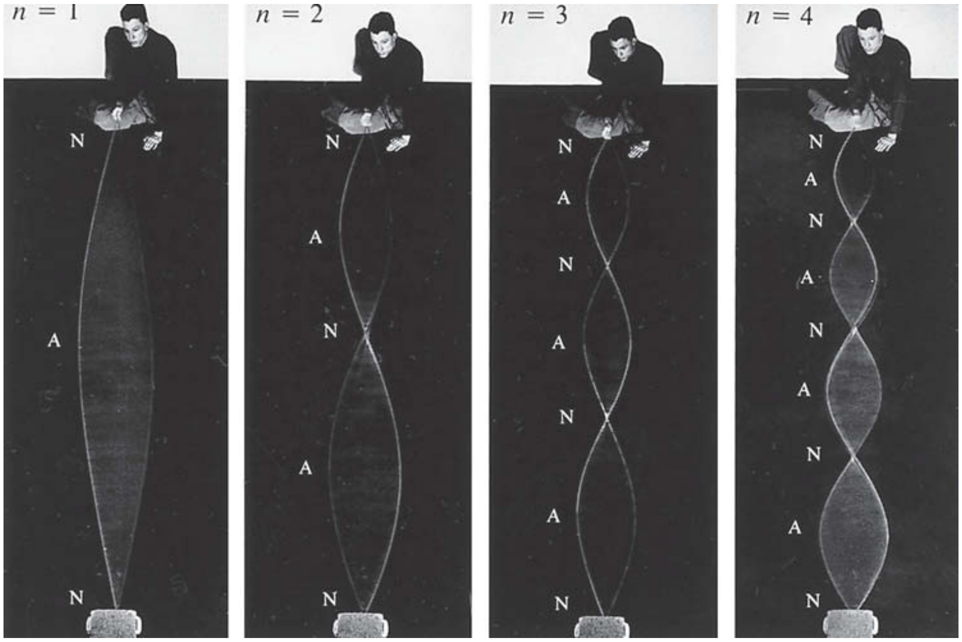


FIGURE 18-22. A student shakes a stretched string (actually a rubber tube) at four resonant frequencies, producing four different patterns of standing waves. The letters N and A indicate the nodes and antinodes, respectively.

from the motion of the student's hand. This situation is analogous to the erratic, small-amplitude motion of a swing being pushed with a frequency other than its natural one. At resonance, the motion of the student's hand is in phase with that of the string, so no energy is lost by the string through work done on the student's hand.

In actuality, the motion of the string is a very good approximation to the standing wave patterns of Fig. 18-20, but not quite an exact one. The resonant frequency is almost, but not exactly, a natural frequency of the system. The apparent nodes are not true nodes, because some energy must be flowing past them along the string to compensate for losses due to damping. If there were no damping, the resonant frequency would be exactly a natural frequency, and the amplitude would increase without limit as energy continued to be supplied to the string by the student's hand. Eventually the elastic limit would be exceeded and the string would break.

If it were possible to shake the string with an assortment of frequencies, the motion of the string would select those frequencies that were equal to its natural frequencies. Motion at those frequencies would be reinforced and would occur at large amplitude, whereas motion at the other frequencies would be damped or suppressed. This principle governs the production of sound by musical instruments, as we discuss in the next chapter.

SAMPLE PROBLEM 18-4. In the arrangement of Fig. 18-23, a motor sets the string into motion at a frequency of 120 Hz. The string has a length of $L = 1.2$ m, and its linear mass density is 1.6 g/m. To what value must the tension be adjusted (by increasing the hanging weight) to obtain the pattern of motion having four loops?

Solution To find the tension, we can substitute Eq. 18-19 into Eq. 18-46 and obtain

$$F = \frac{4L^2 f_n^2 \mu}{n^2}.$$

The tension corresponding to $n = 4$ (for 4 loops) is found to be

$$F = \frac{4(1.2\ \text{m})^2(120\ \text{Hz})^2(0.0016\ \text{kg/m})}{4^2} = 8.3\ \text{N}.$$

This corresponds to a hanging weight of about 2 lb.

SAMPLE PROBLEM 18-5. A violin string tuned to concert A (440 Hz) has a length of 0.34 m. (*a*) What are the three longest wavelengths of the resonances of the string? (*b*) What are the corresponding wavelengths that reach the ear of the listener?

Solution (*a*) The resonant wavelengths of a string of length $L = 0.34$ m can be found directly from Eq. 18-45:

$$\lambda_1 = 2L/1 = 2(0.34\ \text{m}) = 0.68\ \text{m},$$

$$\lambda_2 = 2L/2 = 0.34\ \text{m},$$

$$\lambda_3 = 2L/3 = 0.23\ \text{m}.$$

FIGURE 18-23. Sample Problem 18-4. A string under tension is connected to a vibrator. For a fixed vibrator frequency, standing wave patterns will occur for certain discrete values of the tension in the string.

(*b*) When a wave passes from one medium (the string) to another (the air) of differing wave speed, the frequency remains the same, but the wavelength changes. Equation 18-20 gives the relationship between the wavelengths. To find the wave speed on the string, we note that in the lowest resonant mode $f = 440$ Hz and $\lambda = 0.68$ m, so that

$$v = f\lambda = (440 \text{ Hz})(0.68 \text{ m}) = 299 \text{ m/s}.$$

In air, the wave speed is 343 m/s, and from Eq. 18-20 we obtain

$$\lambda_{air} = \lambda_{string}\frac{v_{air}}{v_{string}} = \lambda_{string}\frac{343 \text{ m/s}}{299 \text{ m/s}} = 1.15\lambda_{string}$$

We thus find the wavelengths in air:

$$\lambda_1 = 0.78 \text{ m}, \qquad \lambda_2 = 0.39 \text{ m}, \qquad \lambda_3 = 0.26 \text{ m}.$$

MULTIPLE CHOICE

18-1 Mechanical Waves

18-2 Types of Waves

18-3 Traveling Waves

1. A disturbance can be written

$$y(x, t) = (e^{-(x/b)^2}e^{2xt/b}e^{-t^2})^a.$$

This disturbance is
   (A) not a traveling wave.
   (B) a traveling wave with speed $v = a$.
   (C) a traveling wave with speed $v = a/b$.
   (D) a traveling wave with speed $v = b$.

2. A traveling wave is of the form

$$y(x, t) = A\cos(kx - \omega t) + B\sin(kx - \omega t),$$

which can also be written as

$$y(x, t) = D\sin(kx - \omega t - \phi),$$

where

(*a*)
   (A) $D = A + B$ (B) $D = |A| + |B|$
   (C) $D^2 = A^2 + B^2$ (D) $D = A - B$

and

(*b*)
   (A) $\phi = \tan^{-1}(A/B)$. (B) $\phi = \tan^{-1}(B/A)$.
   (C) $\phi = \tan^{-1}(-A/B)$. (D) $\phi = 0$.

3. Consider the maximum transverse speed u_{max} of a particle in a wave and the wave speed v. Which of the following statements is most true?
   (A) u_{max} is always greater than v.
   (B) u_{max} is always equal to v.
   (C) u_{max} is always less than v.
   (D) u_{max} is unrelated to v.

18-4 Wave Speed on a Stretched Spring

4. A string is stretched horizontally between a fixed point and a frictionless pulley; the string passes over the pulley and an object of mass m is hanging from the end of the string. The tension in this string is T_0; the speed of a wave on this string is v_0. A second string is connected beside the first, passes over the same pulley, and then is attached to the same object. Assuming both strings support the object equally,

(*a*) the tension in the first string is now
   (A) $T_0/2$. (B) T_0 (C) $2T_0$.

(*b*) The speed of a wave on the first string would now be
   (A) $\sqrt{v_0/2}$. (B) $v_0/\sqrt{2}$. (C) $v_0/2$.
   (D) v_0. (E) $\sqrt{2}v_0$.

The two strings are now twisted together to make one string with twice the mass density. This new string is still attached to the same hanging object.

(*c*) The speed of a wave on this new string would now be
   (A) $\sqrt{v_0/2}$. (B) $v_0/\sqrt{2}$. (C) $v_0/2$.
   (D) v_0. (E) $\sqrt{2}v_0$.

5. Dispersion happens as a wave pulse travels through a medium because
   (A) different wave frequencies lose energy at different rates.
   (B) different wave amplitudes lose energy at different rates.
   (C) different wave frequencies travel through the medium at different wave speeds.
   (D) different wave amplitudes travel through the medium with different wavelengths.

18-5 The Wave Equation

6. Which of the following functions is *not* a solution to the wave equation (Eq. 18-24)?
   (A) $y = \sin x\cos t$ (B) $y = \tan(x + t)$
   (C) $y = x^3 - 6x^2t + 12xt^2 - 8t^3$
   (D) $y = \sin(x + t)\cos(x - t)$

7. Which of the following functions *is* a solution to the wave equation (Eq. 18-24)?
   (A) $y = x^2 - t^2$ (B) $y = \sin x^2 \sin t$
   (C) $y = \log(x^2 - t^2) - \log(x - t)$
   (D) $y = e^x \sin t$

18-6 Energy in Wave Motion

8. A certain wave on a string with amplitude A_0 and frequency f_0 transfers energy at an average rate of P_0. If the amplitude and frequency are both doubled, the new wave would transfer energy at an average rate of
   (A) P_0. (B) $4P_0$. (C) π^2P_0.
   (D) $4\pi^2P_0$. (E) $16P_0$.

9. A wave on a string passes the point $x = 0$ with amplitude A_0, angular frequency ω_0, and average rate of energy transfer P_0. As the wave travels down the string it gradually loses energy; at the point $x = l$ the average rate of energy transfer is now $P_0/2$.

(*a*) At the point $x = l$ the angular frequency of the wave
   (A) is still ω_0.
   (B) can be less than ω_0 but is more than $\omega_0/\sqrt{2}$.
   (C) can be less than ω_0 but is more than $\omega_0/2$.
   (D) is equal to $\omega_0/\sqrt{2}$.
   (E) is equal to $\omega_0/2$.

(*b*) At the point $x = l$ the amplitude of the wave

(A) is still A_0.
(B) can be less than A_0 but is more than $A_0/\sqrt{2}$.
(C) can be less than A_0 but is more than $A_0/2$.
(D) is equal to $A_0/\sqrt{2}$.
(E) is equal to $A_0/2$.

18-7 The Principle of Superposition

18-8 Interference of Waves

10. Two waves travel down the same string. The waves have the same velocity, frequency (f_0), and wavelength but different phase constants ($\phi_1 > \phi_2$) and amplitudes ($A_1 > A_2$).
(*a*) According to the principle of superposition, the resultant wave has an amplitude A such that

(A) $A = A_1 + A_2$. (B) $A = A_1 - A_2$.
(C) $A_2 \leq A \leq A_1$. (D) $A_1 - A_2 \leq A \leq A_1 + A_2$.

(*b*) According to the principle of superposition, the resultant wave has a frequency f such that

(A) $f = f_0$. (B) $f_0/2 < f < f_0$.
(C) $0 < f < f_0$. (D) $f = 2f_0$.

11. Two waves moving along the same string are defined by $y_1 = 2\sin(kx - \omega t + 0)$ and $y_2 = 2\sin(kx - \omega t + 2\pi)$. The amplitude of the resultant wave is

(A) 0. (B) 2. (C) $2\sqrt{2}$. (D) 4.

18-9 Standing Waves

12. In the equation for the standing wave (Eq. 18-42), what does the quantity ω/k represent?

(A) The transverse speed of the particles of the string.
(B) The speed of either of the component waves.
(C) The speed of the standing wave.
(D) A quantity that is independent of the properties of the string.

13. A standing wave occurs on a string when two waves of equal amplitude, frequency, and wavelength move in opposite directions on a string. If the wavelength of the two waves is decreased to one-half the original length while the wave speed remains unchanged, then the angular frequency of oscillation of the standing wave will

(A) decrease to one-half. (B) remain the same.
(C) double.

14. Assume that one of the components of the standing wave (as written in Eq. 18-41) has an additional phase constant $\Delta\phi$. How will this affect the standing wave?

(A) The standing wave will have a different frequency.
(B) The standing wave will have a different amplitude.
(C) The standing wave will have a different spacing between the nodes.
(D) None of these things will happen.

15. In a standing wave on a string, the spacing between nodes is Δx. If the tension in the string is doubled but the frequency of the standing waves is fixed, then the spacing between the nodes will change to

(A) $2\Delta x$. (B) $\sqrt{2}\Delta x$.
(C) $\Delta x/2$. (D) $\Delta x/\sqrt{2}$.

18-10 Standing Waves and Resonance

16. A string is stretched between fixed points. The string has a mass density μ, is under a tension F, and has a length L. The string is vibrating at the lowest allowed frequency.
(*a*) The wave speed on this string is a function of

(A) μ. (B) F. (C) L.
(D) μ and F. (E) μ, F, and L.

(*b*) The lowest allowed standing wave frequency is a function of

(A) μ. (B) F. (C) L.
(D) μ and F. (E) μ, F, and L.

(*c*) The lowest allowed standing wave wavelength is a function of

(A) μ. (B) F. (C) L.
(D) μ and F. (E) μ, F, and L.

17. A 10-cm-long rubber band obeys Hooke's law. When the rubber band is stretched to a total length of 12 cm the lowest resonant frequency is f_0. The rubber band is then stretched to a length of 13 cm. The lowest resonant frequency will now be

(A) higher than f_0.
(B) the same as f_0.
(C) lower than f_0.
(D) changed, but the direction of the change depends on the elastic constant and the original tension.

QUESTIONS

1. How could you prove experimentally that energy is associated with a wave?

2. Energy can be transferred by particles as well as by waves. How can we experimentally distinguish between these methods of energy transfer?

3. Can a wave motion be generated in which the particles of the medium vibrate with angular simple harmonic motion? If so, explain how and describe the wave.

4. In analyzing the motion of an elastic wave through a material medium, we often ignore the molecular structure of matter. When is this justified and when is it not?

5. How do the amplitude and the intensity of surface water waves vary with the distance from the source?

6. How can one create plane waves? Spherical waves?

7. A passing motor boat creates a wake that causes waves to wash ashore. As time goes on, the period of the arriving waves grows shorter and shorter. Why?

8. The following functions in which A is a constant are of the form $y = f(x \pm vt)$:

$$y = A(x - vt), \qquad y = A\sqrt{x - vt},$$
$$y = A(x + vt)^2, \qquad y = A\ln(x + vt).$$

Explain why these functions are not useful in wave motion.

9. Can one produce on a string a waveform that has a discontinuity in slope at a point—that is, a sharp corner? Explain.

10. The inverse-square law does not apply exactly to the decrease in intensity of sounds with distance. Why not?

11. When two waves interfere, does one alter the progress of the other?

12. When waves interfere, is there a loss of energy? Explain your answer.

13. Why do we not observe interference effects between the light beams emitted from two flashlights or between the sound waves emitted by two violins?

14. As Fig. 18-17 shows, twice during the cycle the configuration of standing waves in a stretched string is a straight line, exactly what it would be if the string were not vibrating at all. Discuss from the point of view of energy conservation.

15. Two waves of the same amplitude and frequency are traveling on the same string. At a certain instant the string looks like a straight line. Are the two waves necessarily traveling in the same direction? What is the phase relationship between the two waves?

16. If two waves differ only in amplitude and are propagated in opposite directions through a medium, will they produce standing waves? Is energy transported? Are there any nodes?

17. The partial reflection of wave energy by discontinuities in the path of transmission is usually wasteful and can be minimized by insertion of "impedance matching" devices between sections of the path bordering on the discontinuity. For example, a megaphone helps match the air column of mouth and throat to the air outside the mouth. Give other examples and explain qualitatively how such devices minimize reflection losses.

18. Consider the standing waves in a string to be a superposition of traveling waves and explain, using superposition ideas, why there are no true nodes in the resonating string of Fig. 18-23, even at the "fixed" end. (Hint: Consider damping effects.)

19. Standing waves in a string are demonstrated by an arrangement such as that of Fig. 18-23. The string is illuminated by a fluorescent light and the vibrator is driven by the same electric outlet that powers the light. The string exhibits a curious color variation in the transverse direction. Explain.

20. In the discussion of transverse waves on a string, we have dealt only with displacements in a single plane, the xy plane. If all displacements lie in one plane, the wave is said to be plane polarized. Can there be displacements in a plane other than the plane dealt with? If so, can two different plane polarized waves be combined? What appearance would such a combined wave have?

21. A wave transmits energy. Does it transfer momentum? Can it transfer angular momentum? (See "Energy and Momentum Transport in String Waves," by D. W. Juenker, *American Journal of Physics,* January 1976, p. 94.)

22. In the Mexico City earthquake of September 19, 1985, areas with high damage alternated with areas of low damage. Also, buildings between 5 and 15 stories high sustained the most damage. Discuss these effects in terms of standing waves and resonance.

EXERCISES

18-1 Mechanical Waves

18-2 Types of Waves

18-3 Traveling Waves

1. A wave has a wave speed of 243 m/s and a wavelength of 3.27 cm. Calculate (*a*) the frequency and (*b*) the period of the wave.

2. By rocking a boat, a child produces surface water waves on a previously quiet lake. It is observed that the boat performs 12 oscillations in 30 s and also that a given wave crest reaches shore 15 m away in 5.0 s. Find (*a*) the frequency, (*b*) the speed, and (*c*) the wavelength of the waves.

3. A sinusoidal wave travels along a string. The time for a particular point to move from maximum displacement to zero displacement is 178 ms. The wavelength of the wave is 1.38 m. Find (*a*) the period, (*b*) the frequency, and (*c*) the speed of the wave.

4. Write an expression describing a transverse wave traveling along a string in the $+x$ direction with wavelength 11.4 cm, frequency 385 Hz, and amplitude 2.13 cm.

18-4 Wave Speed on a Stretched Spring

5. Assuming that the wave speed on a stretched string depends on the tension F and linear mass density μ as $v \propto F^a/\mu^b$, use dimensional analysis to show that $a = \frac{1}{2}$ and $b = \frac{1}{2}$.

6. The equation of a transverse wave traveling along a string is given by

$$y = (2.30 \text{ mm}) \sin [(1822 \text{ rad/m})x - (588 \text{ rad/s})t].$$

Find (*a*) the amplitude, (*b*) the frequency, (*c*) the velocity, (*d*) the wavelength of the wave, and (*e*) the maximum transverse speed of a particle in the string.

7. The equation of a transverse wave traveling along a very long string is given by

$$y = (6.0 \text{ cm}) \sin [(2.0\pi \text{ rad/m})x + (4.0\pi \text{ rad/s})t].$$

Calculate (*a*) the amplitude, (*b*) the wavelength, (*c*) the frequency, (*d*) the speed, (*e*) the direction of propagation of the wave, and (*f*) the maximum transverse speed of a particle in the string.

8. Calculate the speed of a transverse wave in a string of length 2.15 m and mass 62.5 g under a tension of 487 N.

9. The speed of a wave on a string is 172 m/s when the tension is 123 N. To what value must the tension be increased in order to raise the wave speed to 180 m/s?

10. The equation of a particular transverse wave on a string is

$$y = (1.8 \text{ mm}) \sin [(23.8 \text{ rad/m})x + (317 \text{ rad/s})t].$$

The string is under a tension of 16.3 N. Find the linear mass density of the string.

11. A simple harmonic transverse wave is propagating along a string toward the left (or $-x$) direction. Figure 18-24 shows a plot of the displacement as a function of position at time $t = 0$. The string tension is 3.6 N and its linear density is 25 g/m. Calculate (*a*) the amplitude, (*b*) the wavelength, (*c*) the wave speed, (*d*) the period, and (*e*) the maximum speed of a particle in the string. (*f*) Write an equation describing the traveling wave.

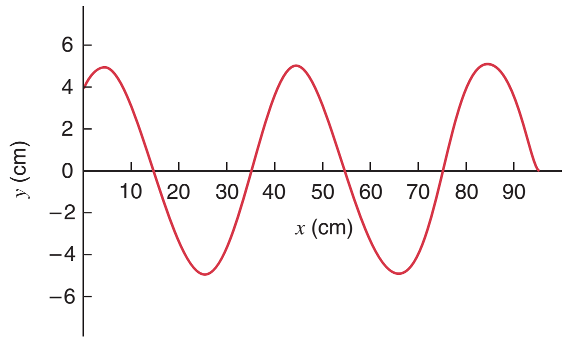


FIGURE 18-24. Exercise 11.

12. In Fig. 18-25*a*, string 1 has a linear mass density of 3.31 g/m, and string 2 has a linear mass density of 4.87 g/m. They are under tension due to the hanging block of mass $M = 511$ g. (*a*) Calculate the wave speed in each string. (*b*) The block is now divided into two blocks (with $M_1 + M_2 = M$) and the apparatus is rearranged as shown in Fig. 18-25*b*. Find M_1 and M_2 such that the wave speeds in the two strings are equal.

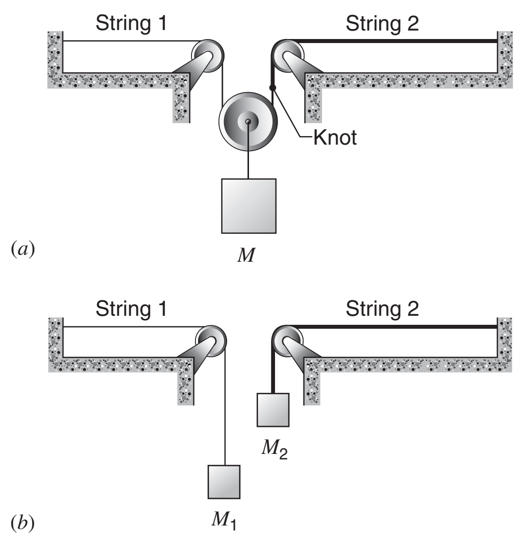


FIGURE 18-25. Exercise 12.

13. A wire 10.3 m long and having a mass of 97.8 g is stretched under a tension of 248 N. If two pulses, separated in time by 29.6 ms, are generated one at each end of the wire, where will the pulses meet?

18-5 The Wave Equation

14. In a spherically symmetric system, the three-dimensional wave equation is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial y}{\partial r}\right) = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}.$$

(*a*) Show that

$$y(r, t) = \frac{A}{r}\sin(kr - \omega t)$$

is a solution to this wave equation. (*b*) What are the dimensions of the constant A?

18-6 Energy in Wave Motion

15. A string 2.72 m long has a mass of 263 g. The tension in the string is 36.1 N. What must be the frequency of traveling waves of amplitude 7.70 mm in order that the average transmitted power be 85.5 W?

16. A line source emits a cylindrical expanding wave. Assuming that the medium absorbs no energy, find how (*a*) the intensity and (*b*) the amplitude of the wave depend on the distance from the source.

17. An observer measures an intensity of 1.13 W/m^2 at an unknown distance from a source of spherical waves whose power output is also unknown. The observer walks 5.30 m closer to the source and measures an intensity of 2.41 W/m^2 at this new location. Calculate the power output of the source.

18. (*a*) Show that the intensity I is the product of the energy density u (energy per unit volume) and the speed of propagation v of a wave disturbance; that is, show that $I = uv$. (*b*) Calculate the energy density in a sound wave 4.82 km from a 47.5-kW siren, assuming the waves to be spherical, the propagation isotropic with no atmospheric absorption, and the speed of sound to be 343 m/s.

18-7 The Principle of Superposition

18-8 Interference of Waves

19. What phase difference between two otherwise identical traveling waves, moving in the same direction along a stretched string, will result in the combined wave having an amplitude 1.65 times that of the common amplitude of the two combining waves? Express your answer in both degrees and radians.

20. Determine the amplitude of the resultant wave when two sinusoidal waves having the same frequency and traveling in the same direction are combined, if their amplitudes are 3.20 cm and 4.19 cm and they differ in phase by $\pi/2$ rad.

21. For the case in which the component waves in Eq. 18-38 have different amplitudes y_{m1} and y_{m2}, show that the quantity in square brackets in Eq. 18-40 becomes $[y_{m1}^2 + y_{m2}^2 + 2y_{m1}y_{m2}\cos\Delta\phi]^{1/2}$ and the phase constant ϕ' becomes

$$\phi' = \sin^{-1}\left[\frac{y_{m1}\sin\phi_1 + y_{m2}\sin\phi_2}{(y_{m1}^2 + y_{m2}^2 + 2y_{m1}y_{m2}\cos\Delta\phi)^{1/2}}\right].$$

Check that both expressions reduce to the expected results when $y_{m1} = y_{m2} = y_m$.

22. Two pulses are traveling along a string in opposite directions, as shown in Fig. 18-26. (*a*) If the wave speed is 2.0 m/s and the pulses are 6.0 cm apart, sketch the patterns after 5.0, 10, 15, 20, and 25 ms. (*b*) What has happened to the energy at $t = 15$ ms?

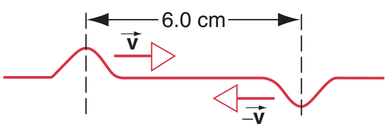


FIGURE 18-26. Exercise 22.

23. Three sinusoidal waves travel in the positive x direction along the same string. All three waves have the same frequency. Their amplitudes are in the ratio $1:\frac{1}{2}:\frac{1}{3}$ and their phase angles are 0, $\pi/2$, and π, respectively. Plot the resultant waveform and discuss its behavior as t increases.

24. Four sinusoidal waves travel in the positive x direction along the same string. Their frequencies are in the ratio $1:2:3:4$ and their amplitudes are in the ratio $1:\frac{1}{2}:\frac{1}{3}:\frac{1}{4}$, respectively. When $t = 0$, at $x = 0$, the first and third waves are 180° out of phase with the second and fourth. Plot the resultant waveform when $t = 0$ and discuss its behavior as t increases.

18-9 Standing Waves

25. A string fixed at both ends is 8.36 m long and has a mass of 122 g. It is subjected to a tension of 96.7 N and set vibrating. (*a*) What is the speed of the waves in the string? (*b*) What is the wavelength of the longest possible standing wave? (*c*) Give the frequency of that wave.

26. A nylon guitar string has a linear mass density of 7.16 g/m and is under a tension of 152 N. The fixed supports are 89.4 cm apart. The string is vibrating in the standing wave pattern shown in Fig. 18-27. Calculate the (*a*) speed, (*b*) wavelength, and (*c*) frequency of the component waves whose superposition gives rise to this vibration.

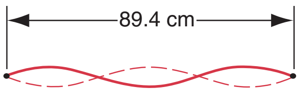


FIGURE 18-27. Exercise 26.

27. The equation of a transverse wave traveling in a string is given by

$$y = (0.15\text{ m})\sin[(0.79\text{ rad/m})x - (13\text{ rad/s})t].$$

(*a*) What is the displacement at $x = 2.3$ m, $t = 0.16$ s? (*b*) Write down the equation of a wave that, when added to the given one, would produce standing waves on the string. (*c*) What is the displacement of the resultant standing wave at $x = 2.3$ m, $t = 0.16$ s?

28. A string vibrates according to the equation

$$y = (0.520\text{ cm})\sin[(1.14\text{ rad/cm})x]\cos[(137\text{ rad/s})t].$$

(*a*) What are the amplitude and speed of the component waves whose superposition can give rise to this vibration? (*b*) Find the distance between nodes. (*c*) What is the velocity of a particle of the string at the position $x = 1.47$ cm at time $t = 1.36$ s?

29. Vibrations from a 622-Hz tuning fork set up standing waves in a string clamped at both ends. The wave speed for the string is 388 m/s. The standing wave has four loops and an amplitude of 1.90 mm. (*a*) What is the length of the string? (*b*) Write an equation for the displacement of the string as a function of position and time.

18-10 Standing Waves and Resonance

30. A 15.0-cm violin string, fixed at both ends, is vibrating in its $n = 1$ mode. The speed of waves in this wire is 250 m/s, and the speed of sound in air is 348 m/s. What are (*a*) the frequency and (*b*) the wavelength of the emitted sound wave?

31. What are the three lowest frequencies for standing waves on a wire 9.88 m long having a mass of 0.107 kg, which is stretched under a tension of 236 N?

32. A 1.48-m-long wire has a mass of 8.62 g and is held under a tension of 122 N. The wire is held rigidly at both ends and set into vibration. Calculate (*a*) the speed of waves on the wire, (*b*) the wavelengths of the waves that produce one- and two-loop standing waves on the wire, and (*c*) the frequencies of the waves in (*b*).

33. One end of a 120-cm string is held fixed. The other end is attached to a weightless ring that can slide along a frictionless rod as shown in Fig. 18-28. What are the three longest possible wavelengths for standing waves in this string? Sketch the corresponding standing waves.

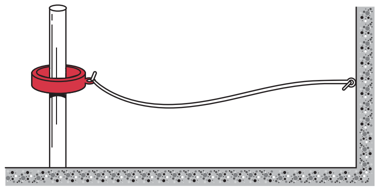


FIGURE 18-28. Exercise 33.

34. A 75.6-cm string is stretched between fixed supports. It is observed to have resonant frequencies of 420 and 315 Hz, and no other resonant frequencies between these two. (*a*) What is the lowest resonant frequency for this string? (*b*) What is the wave speed for this string?

PROBLEMS

1. A wave of frequency 493 Hz has a speed of 353 m/s. (*a*) How far apart are two points differing in phase by 55.0°? (*b*) Find the difference in phase between two displacements at the same point but at times differing by 1.12 ms.

2. Write the equation for a wave traveling in the negative direction along the x axis and having an amplitude of 1.12 cm, a frequency of 548 Hz, and a speed of 326 m/s.

3. The tensile stress S in a wire is defined as the tension force per unit cross-sectional area. Show that the speed of transverse waves in a wire is $v = (S/\rho)^{1/2}$ where ρ is the mass density of the wire. (*b*) Allowing for a reasonable safety factor, the maximum tensile stress to which steel should be subject is 720 MPa. The density of steel is 7.8 g/cm^3. Find the maximum speed of a transverse wave in a steel wire.

4. A continuous sinusoidal wave is traveling on a string with speed 82.6 cm/s. The displacement of the particles of the string at $x = 9.60$ cm is found to vary with time according to the equation $y = (5.12 \text{ cm}) \sin [(1.16 \text{ rad}) - (4.08 \text{ rad/s})t]$. The linear mass density of the string is 3.86 g/cm. (*a*) Find the frequency of the wave. (*b*) Find the wavelength of the wave. (*c*) Write the general equation giving the transverse displacement of the particles of the string as a function of position and time. (*d*) Calculate the tension in the string.

5. Prove that the slope of a string at any point is numerically equal to the ratio of the particle speed to the wave speed at that point.

6. For a wave on a stretched cord, find the ratio of the maximum particle speed (the maximum speed with which a single particle in the cord moves transverse to the wave) to the wave speed. If a wave having a certain frequency and amplitude is imposed on a cord, would this speed ratio depend on the material of which the cord is made, such as wire or nylon?

7. The type of rubber band used inside some baseballs and golf balls obeys Hooke's law over a wide range of elongation of the band. A segment of this material has an unstretched length L and a mass m. When a force F is applied, the band stretches an additional length ΔL. (*a*) What is the speed (in terms of m, ΔL, and the force constant k) of transverse waves on this rubber band? (*b*) Using your answer to (*a*), show that the time required for a transverse pulse to travel the length of the rubber band is proportional to $1/\sqrt{\Delta L}$ if $\Delta L \ll L$ and is constant if $\Delta L \gg L$.

8. A uniform rope of mass m and length L hangs from a ceiling. (*a*) Show that the speed of a transverse wave in the rope is a function of y, the distance from the lower end, and is given by $v = \sqrt{gy}$. (*b*) Show that the time it takes a transverse wave to travel the length of the rope is given by $t = 2\sqrt{L/g}$. (*c*) Does the actual mass of the rope affect the results of (*a*) and (*b*)?

9. A nonuniform wire of length L and mass M has a variable linear mass density given by $\mu = kx$, where x is the distance from one end of the wire and k is a constant. (*a*) Show that $M = kL^2/2$. (*b*) Show that the time t required for a pulse generated at one end of the wire to travel to the other end is given by $t = \sqrt{8ML/9F}$ where F is the tension in the wire.

10. A uniform circular hoop of string is rotating clockwise in the absence of gravity (see Fig. 18-29). The tangential speed is v. Find the speed of waves on this string. (Note that the answer is independent of the radius of the hoop and the linear mass density of the string!)

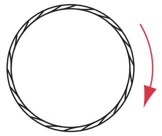


FIGURE 18-29. Problem 10.

11. Violins in Handel's time were constructed to play an "A" at 422.5 Hz. (How could we know this?) Modern orchestras, however, are tuned to play an "A" at 440 Hz. Assuming that all other things are equal, by what percentage does a player need to increase the tension on the strings to get a Handel era violin to play in tune today?

12. A transverse sinusoidal wave is generated at one end of a long, horizontal string by a bar that moves up and down through a distance of 1.12 cm. The motion is continuous and is repeated regularly 120 times per second. The string has linear density 117 g/m and is kept under a tension of 91.4 N. Find (*a*) the maximum magnitude of the transverse speed u_y and (*b*) the maximum magnitude of the transverse component of the tension. (*c*) Show that the two maximum values calculated above occur at the same phase values for the wave. What is the transverse displacement y of the string at these phases? (*d*) What is the maximum power transferred along the string? (*e*) What is the transverse displacement y for conditions under which this maximum power transfer occurs? (*f*) What is the minimum power transfer along the string? (*g*) What is the transverse displacement y for conditions under which this minimum power transfer occurs?

13. Consider two point sources S_1 and S_2 in Fig. 18-30, which emit waves of the same frequency f and amplitude A. The waves start in the same phase, and this phase relation at the sources is maintained throughout time. Consider point P at which r_1 is nearly equal to r_2. (*a*) Show that the superposition of these two waves gives a wave whose amplitude y_m varies with the position P approximately according to

$$y_m = \frac{2A}{r} \cos \frac{k}{2} (r_1 - r_2),$$

in which $r = (r_1 + r_2)/2$. (*b*) Then show that total cancellation occurs when $r_1 - r_2 = (n + \frac{1}{2})\lambda$, n being any integer, and that total reinforcement occurs when $r_1 - r_2 = n\lambda$. The locus of points whose difference in distance from two fixed points is a constant is a hyperbola, the fixed points being the foci. Hence each value of n gives a hyperbolic line of constructive interference and a hyperbolic line of destructive interference. At points at which r_1 and r_2 are not approximately equal (as near the sources), the amplitudes of the waves from S_1 and S_2 differ and the cancellations are only partial. (This is the basis of the OMEGA navigation system.)

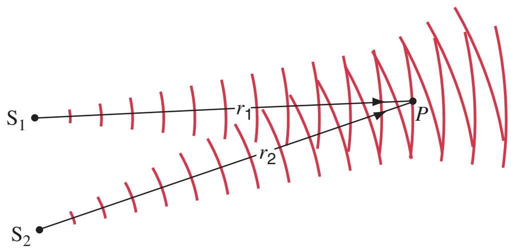


FIGURE 18-30. Problem 13.

14. A source S and a detector D of high-frequency waves are a distance d apart on the ground. The direct wave from S is found to be in phase at D with the wave from S that is reflected from a horizontal layer at an altitude H (Fig. 18-31). The incident and reflected rays make the same angle with the reflecting layer. When the layer rises a distance h, no signal is detected at D. Neglect absorption in the atmosphere and find the relation between d, h, H, and the wavelength λ of the waves.

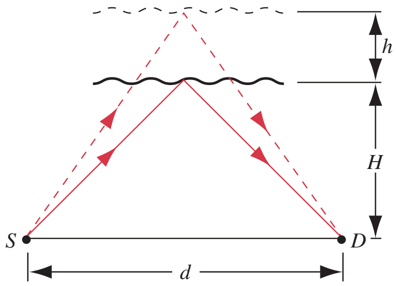


FIGURE 18-31. Problems 14 and 15.

15. Refer to Problem 14 and Fig. 18-31. Suppose that $d =$ 230 km and $H = 510$ km. The waves are 13.0 MHz radio waves ($v = 3.00 \times 10^8$ m/s). At the detector D the combined signal strength varies from a maximum to zero and back to a maximum again six times in 1 min. At what vertical speed is the reflecting layer moving? (The layer is moving slowly, so that the vertical distance moved in 1 min is small compared to H and d.)

16. Consider a standing wave that is the sum of two waves traveling in opposite directions but otherwise identical. Show that the maximum kinetic energy in each loop of the standing wave is $2\pi^2\mu y_m^2 fv$.

17. An incident traveling wave, amplitude A_i, is only partially reflected from a boundary, with the amplitude of the reflected wave being A_r. The resulting superposition of two waves with different amplitudes and traveling in opposite directions gives a standing wave pattern of waves whose envelope is shown in Fig. 18-32. The standing wave ratio (SWR) is defined as $(A_i + A_r)/(A_i - A_r) = A_{max}/A_{min}$, and the percent reflection is defined as the ratio of the average power in the reflected wave to the average power in the incident wave, times 100. (*a*) Show that for 100% reflection SWR $= \infty$ and that for no reflection SWR $= 1$. (*b*) Show that a measurement of the SWR just before the boundary reveals the percent reflection occurring at the boundary according to the formula

$$\% \text{ reflection} = [(\text{SWR} - 1)^2/(\text{SWR} + 1)^2](100).$$

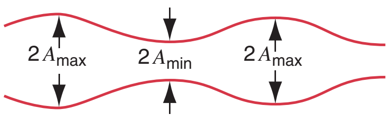


FIGURE 18-32. Problems 17 and 18.

18. Estimate (*a*) the SWR (standing wave ratio) and (*b*) the percent reflection at the boundary for the envelope of the standing wave pattern shown in Fig. 18-32.

19. Two strings of linear mass density μ_1 and μ_2 are knotted together at $x = 0$ and stretched to a tension F. A wave $y = A \sin k_1(x - v_1 t)$ in the string of density μ_1 reaches the junction between the two strings, at which it is partly transmitted into the string of density μ_2 and partly reflected. Call these waves $B \sin k_2(x - v_2 t)$ and $C \sin k_1(x + v_1 t)$, respectively. (*a*) Assuming that $k_2 v_2 = k_1 v_1 = \omega$ and that the displacement of the knot arising from the incident and reflected waves is the same as that arising from the transmitted wave, show that $A = B + C$. (*b*) If it is assumed that both strings near the knot have the same slope (why?)—that is, dy/dx in string 1 $=$ dy/dx in string 2—show that

$$C = A\frac{k_2 - k_1}{k_2 + k_1} = A\frac{v_1 - v_2}{v_1 + v_2}.$$

Under what conditions is C negative?

20. Wave interference can occur for waves with different frequencies. (*a*) Show that the resultant of the two waves

$$y_1(x, t) = y_m \sin(k_1 x - \omega_1 t)$$

$$y_2(x, t) = y_m \sin(k_2 x - \omega_2 t)$$

can be written as

$$y(x, t) = 2y_m \cos\left[\frac{1}{2}(\Delta k x - \Delta\omega t)\right] \sin(k'x - \omega' t).$$

(*b*) What is ω'/k'? (*c*) Qualitatively describe the motion of this wave.

21. In an experiment on standing waves, a string 92.4 cm long is attached to the prong of an electrically driven tuning fork, which vibrates perpendicular to the length of the string at a frequency of 60.0 Hz. The mass of the string is 44.2 g. How much tension must the string be under (weights are attached to the other end) if it is to vibrate with four loops?

22. An aluminum wire of length $L_1 = 60.0$ cm and of cross-sectional area 1.00×10^{-2} cm^2 is connected to a steel wire of the same cross-sectional area. The compound wire, loaded with a block m of mass 10.0 kg, is arranged as shown in Fig. 18-33 so that the distance L_2 from the joint to the supporting pulley is 86.6 cm. Transverse waves are set up in the wire by using an external source of variable frequency. (*a*) Find the lowest frequency of excitation for which standing waves are observed such that the joint in the wire is a node. (*b*) What is the total number of nodes observed at this frequency, excluding the two at the ends of the wire? The density of aluminum is 2.60 g/cm^3 and that of steel is 7.80 g/cm^3.

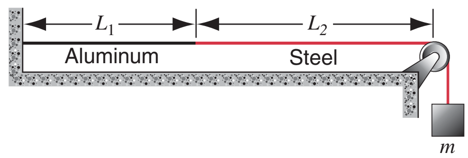


FIGURE 18-33. Problem 22.

COMPUTER PROBLEM

1. A simple function is given by $y(x) = x(\pi - x)$ in the region $0 < x < \pi$. It is desired that this function be approximated by a series of sine functions in the form $y(x) \approx a_1 \sin x + a_3 \sin 3x + a_5 \sin 5x + \cdots$. (*a*) Use a graphing program and estimate the values for a_1, a_3, and a_5 that give the best visual fit. (*b*) Use a symbolic math program (such as Maple or Mathematica to evaluate the integrals

$$I_n = \int_0^{\pi} \sin^2 nx \, dx$$

and

$$I_0 = \int_0^{\pi} \sin nx \sin mx \, dx,$$

where n and m are integers, but not equal to each other. (*c*) Find the *exact* values of the coefficients a_n for $n \in \{1, 2, 3, 4, 5\}$ by evaluating

$$a_n = \frac{1}{I_n} \int_0^{\pi} x(\pi - x) \sin nx \, dx.$$

Why does this work? Compare your answers to the visual inspection process.

CHAPTER 19

SOUND WAVES

In Chapter 18 we studied transverse mechanical waves, in particular the vibrations of a stretched string. Now we turn our attention to longitudinal mechanical waves, in particular sound waves. What we call sound is a longitudinal mechanical vibration with frequencies from about 20 Hz to about 20,000 Hz, which is the typical range of human hearing. Longitudinal waves of higher frequency, which are called ultrasonic waves, are used in locating underwater objects and in medical imaging. Longitudinal (and transverse) mechanical waves of lower frequency, called infrasonic, occur as seismic waves in earthquakes.

In this chapter we discuss the properties of sound waves, their propagation, and their production by vibrating systems.

19-1 PROPERTIES OF SOUND WAVES

Like the transverse wave on the string, sound is a mechanical wave, meaning that the disturbance propagates due to the mechanical (elastic) forces between particles in the medium. Mechanical waves can travel through any material medium (solid, liquid, or gas). In solids, mechanical waves can be longitudinal or transverse, but in fluids (which cannot support shearing forces) the waves are only longitudinal, which means that the particles of the medium oscillate along the same direction that the wave is traveling.

When we discuss sound waves, we normally mean longitudinal waves in the frequency range 20 Hz to 20,000 Hz, the normal range of human hearing. However, the branch of physics and engineering that deals with the study of sound waves, called *acoustics,* generally includes the study of mechanical waves of all frequencies, with transverse as well as longitudinal vibrations in the case of solids. In this chapter we consider mainly sound waves in air, which are strictly longitudinal.

Although a small source of sound in an open area emits waves that are three-dimensional, we will simplify the problem by considering one-dimensional waves. Figure 19-1 shows how a one-dimensional sound wave might be established in a long tube filled with air. At one end of the tube is a piston, which might represent the moving cone of a loud-

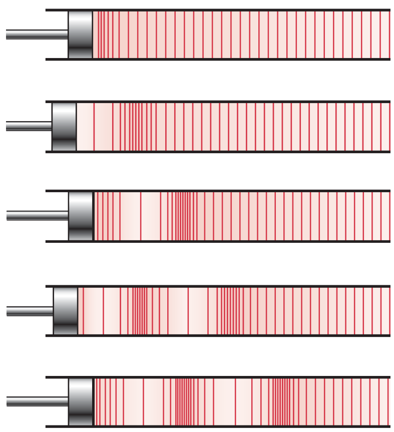


FIGURE 19-1. Sound waves generated in a tube by a moving piston, which might represent the moving cone of a loudspeaker. The vertical lines divide the compressible medium in the tube into layers of equal mass.

speaker. As the piston moves back and forth, it alternately compresses and expands (rarefies) the air next to it. This disturbance travels down the tube as a sound wave. As the wave passes any point, air molecules at that location move back and forth about their equilibrium positions parallel to the direction that the wave is traveling.

As we will see, we can describe the sound wave either in terms of changes in the local pressure in the medium or else in terms of the displacement of the air molecules from their equilibrium positions. These two descriptions convey the same information, but they have somewhat different mathematical forms.

19-2 TRAVELING SOUND WAVES

As the piston in Fig. 19-1 oscillates, it causes variations in the density of the air in the tube from one place to another and also from one instant of time to another, as Fig. 19-1 shows. The regions of high density are called *compressions,* and the regions of low density are called *rarefactions.* As the sound wave travels, the compressions and rarefactions travel along the tube.

We express the density in the tube as $\rho(x, t)$, a function both of location and time. The undisturbed density of the air in the tube is ρ_0, and the sound wave causes fluctuations in the density $\Delta\rho(x, t)$ that are very small compared with ρ_0. That is $\rho(x, t) = \rho_0 + \Delta\rho(x, t)$, where $\Delta\rho(x, t)$ can be positive or negative but $|\Delta\rho(x, t)| \ll \rho_0$.

We can also describe the sound wave in terms of the variation in the pressure in the tube. The pressure variations travel along the tube in phase with the density variations: when the density reaches its maximum at a particular location, the pressure also reaches its maximum at that location. The undisturbed pressure in the absence of the sound wave is p_0, and the pressure fluctuations $\Delta p(x, t)$ can be positive or negative but are very small in comparison with p_0. The total pressure at the point with coordinate x and at time t is $p(x, t) = p_0 + \Delta p(x, t)$.

Let us assume that the piston is driven so that its position can be described by a sine or cosine function; then the density and pressure of air in the tube will also vary sinusoidally. Figure 19-2*a* shows a "snapshot" of the density of the air in the tube at a particular instant of time, and Figs. 19-2*b* and *c* show the corresponding variations in the density and pressure of the air. The density fluctuates with amplitude $\Delta\rho_m$ about the value ρ_0, and the pressure with amplitude Δp_m about p_0. For sinusoidal waves, neglecting for now any phase constant, we can write

$$\Delta\rho(x, t) = \Delta\rho_m \sin(kx - \omega t). \qquad (19\text{-}1)$$

The pressure variations can be written similarly as

$$\Delta p(x, t) = \Delta p_m \sin(kx - \omega t). \qquad (19\text{-}2)$$

Usually it is most convenient to describe the sound wave in terms of its pressure variation.

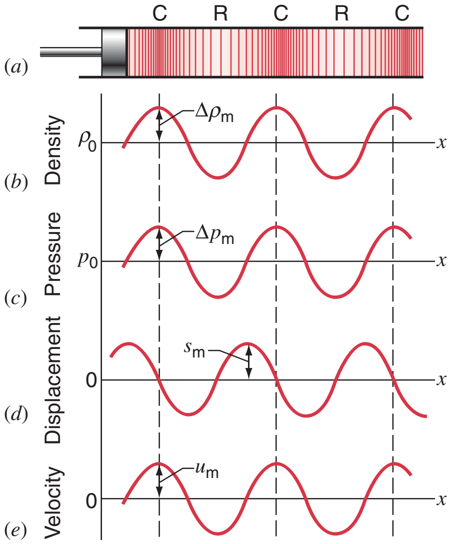


FIGURE 19-2. (*a*) Compressions C and rarefactions R in a sound wave traveling along a tube. (*b*) A snapshot of the density, which varies with amplitude $\Delta\rho_m$ about the value ρ_0. (*c*) A snapshot of the pressure, which varies with amplitude Δp_m about p_0. (*d*) The longitudinal displacement, which shows at every location x how a small element of air has been displaced from its equilibrium position. (*e*) The longitudinal velocity of the small elements of air.

The relationship between the pressure amplitude Δp_m and the density amplitude $\Delta\rho_m$ depends on the mechanical properties of the medium. In Eq. 15-5 we introduced the bulk modulus $B = -\Delta p/(\Delta V/V)$, which describes the relative change in volume of an element of fluid in response to a change in pressure. With $\rho = m/V$, we have $d\rho = -(m/V^2)dV = -(\rho/V)dV$. Replacing the differentials with differences, we can write $\Delta\rho = -\rho(\Delta V/V)$. Using Eq. 15-5 to replace $\Delta V/V$ with $-\Delta p/B$, and replacing ρ with its undisturbed value ρ_0 (because the fluctuations $\Delta\rho$ are small), we have

$$\Delta\rho = -\rho\frac{\Delta V}{V} = -\rho_0\frac{-\Delta p}{B} = \Delta p\frac{\rho_0}{B}$$

or, in terms of the density and pressure amplitudes,

$$\Delta\rho_m = \Delta p_m\frac{\rho_0}{B}. \qquad (19\text{-}3)$$

Because Eq. 15-5 applies to all fluids (not only to air), Eq. 19-3 similarly applies to sound waves in all fluids.

There is one special caution that must be observed in using Eq. 19-3. As a sound wave moves along the tube, it alternately compresses and expands each small volume of air. When a volume element of gas is compressed, work is done on it, so that its internal energy and its temperature increase. In fluids, the rate at which energy can flow (as heat) from one volume element to another is generally rather small, so that the internal energy increase associated with the compression does not have sufficient time at sound-wave frequencies to be transferred as heat to the adjacent cooler regions associated with the rarefactions. We call this an *adiabatic* (no heat transfer) process, and in Eq. 19-3 we must therefore use the adiabatic bulk modulus. Otherwise, if heat could flow so that the adjacent regions could reach a common temperature, we would use the *isothermal* (constant temperature) bulk modulus. For a gas such as air, the adiabatic bulk modulus is about 1.4 times the isothermal bulk modulus. Isothermal and adiabatic processes are considered in greater detail in Chapter 23.

Sound as a Displacement Wave

Just as we considered the displacement and velocity of "particles" of the medium for transverse mechanical waves on a string in Chapter 18, it is also instructive to consider the displacement and velocity associated with the longitudinal sound wave. In this case, "particles" of the medium refer to elements of volume of the fluid.

Figure 19-3 shows an element of gas inside the tube through which the sound wave is traveling. The element has mass δm, and in the equilibrium state it is located at the coordinate x and has thickness δx and cross-sectional area A. The undisturbed density of this fluid element is

$$\rho_0 = \frac{\delta m}{A\ \delta x}. \qquad (19\text{-}4)$$

As the sound wave passes, this element of fluid oscillates about its equilibrium position. We represent the displacement of the fluid element from this equilibrium position by the function $s(x, t)$. The value of this function varies from place to place and also from time to time, as different fluid elements are moved in different directions by the sound wave. The function $s(x, t)$ is similar to the transverse wave function $y(x, t)$, with one important exception: the displacement s is measured along the direction of propagation for a longitudinal wave, instead of at right angles to the direction of propagation (in the case of the transverse wave of Chapter 18).

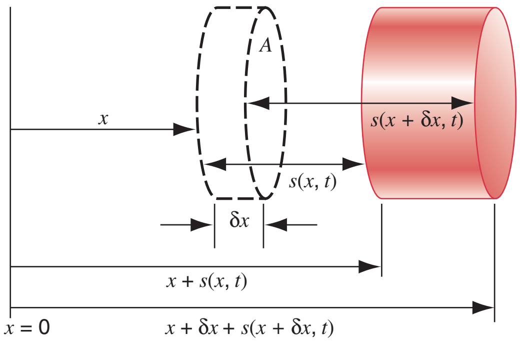


FIGURE 19-3. An element of air thickness δx and cross-sectional area A is originally located at x. The sound wave displaces the element by an amount $s(x, t)$.

In response to the sound wave, the left-hand edge of the fluid element moves from coordinate x to coordinate $x + s(x, t)$, as shown in Fig. 19-3. The right-hand edge of the element moves from coordinate $x + \delta x$ to $x + \delta x + s(x + \delta x, t)$. That is, in response to the sound wave, the fluid element not only moves, but its thickness changes from δx to $[x + \delta x + s(x + \delta x, t)] - [x + s(x, t)] = \delta x + s(x + \delta x, t) - s(x, t)$, which we can write as

$$\delta x\left[1 + \frac{s(x + \delta x, t) - s(x, t)}{\delta x}\right].$$

In the limit of a very thin volume element, in which $\delta x \to 0$, the quotient in this expression can be written as $\partial s/\partial x$, the partial derivative of s with respect to x, so that the thickness of the element is $\delta x[1 + \partial s/\partial x]$. The density of this fluid element is now

$$\rho = \frac{\delta m}{A\ \delta x(1 + \partial s/\partial x)} = \frac{\rho_0}{1 + \partial s/\partial x} \qquad (19\text{-}5)$$

using Eq. 19-4 for ρ_0. We have assumed that the density variations are very small, so that $\partial s/\partial x \ll 1$. Using the binomial expansion (see Appendix I), $(1 + z)^{-1} = 1 - z + \cdots$, we keep only the first term in the expansion and obtain $\rho = \rho_0(1 - \partial s/\partial x)$. The change in density is $\Delta\rho(x, t) = \rho - \rho_0$, or

$$\Delta\rho(x, t) = -\rho_0\frac{\partial s}{\partial x}. \qquad (19\text{-}6)$$

For sinusoidal waves, we can use Eq. 19-1 for $\Delta\rho(x, t)$, and so obtain

$$\frac{\partial s}{\partial x} = -\frac{\Delta\rho(x, t)}{\rho_0} = -\frac{\Delta\rho_m}{\rho_0}\sin(kx - \omega t).$$

Integrating with respect to x gives

$$s(x, t) = \frac{\Delta\rho_m}{k\rho_0}\cos(kx - \omega t) = s_m\cos(kx - \omega t). \qquad (19\text{-}7)$$

The displacement amplitude is

$$s_m = \frac{\Delta\rho_m}{k\rho_0} = \frac{\Delta p_m}{kB}, \qquad (19\text{-}8)$$

where we have used Eq. 19-3 to relate $\Delta\rho_m$ to Δp_m.

Note that the displacement variation is expressed as a cosine function when the pressure and density variations are expressed as sine functions. Thus the displacement variations are 90° out of phase with the pressure and density variations. The displacement waveform is shown in Fig. 19-2*d*. At locations where the pressure and density are a maximum or minimum, the displacement of the elements of fluid is zero; the displacement is a maximum or minimum where the pressure and density variations are zero. For example, at a compression, fluid elements to the left must have positive displacements and fluid elements to the right must have negative displacements, consistent with fluid elements at the center of the compression having zero displacements.

As a fluid element oscillates about its equilibrium position, its longitudinal velocity is

$$u_x(x, t) = \frac{\partial s}{\partial t} = \omega s_m \sin(kx - \omega t) = u_m \sin(kx - \omega t), \tag{19-9}$$

where the amplitude of the velocity variations is

$$u_m = \omega s_m = \frac{\omega \Delta p_m}{kB} = \frac{v \Delta p_m}{B} \tag{19-10}$$

using Eq. 19-8 and Eq. 18-13 ($v = \omega/k$). Once again, it is important to note that this is a longitudinal velocity, directed along the axis of the tube (parallel to the direction of propagation of the wave). The velocity variations are in phase with the pressure and density variations, as shown in Fig. 19-2*e*.

Although we have described a sound wave in terms of either a pressure wave or a displacement wave, the two descriptions are in general not equivalent. Only when a single longitudinal wave is propagating in a single direction can we easily shift back and forth between the two descriptions. When we consider the reflection of a sound wave at the end of a tube, or when we superimpose two sound waves that interfere at a point, using the displacement wave description can lead to serious errors.* For example, consider two sound waves from different sources (two loudspeakers, for instance) that travel along different directions and interfere at a point, such that one wave gives a pressure change Δp and the other $-\Delta p$. On the basis of the pressure description, we expect complete destructive interference at that point, because the pressures add like scalars. However, the displacements (which are along the directions of travel of the two waves) do not add to zero, because they are vectors in different directions. *It is usually preferable to describe a sound wave as a pressure wave* to avoid such difficulties. Moreover, as we shall see in the next section, it is the pressure change, not the displacement, that is detected by ears and microphones.

SAMPLE PROBLEM 19-1. The maximum pressure variation Δp_m that the ear can tolerate in loud sounds is about 28 Pa at 1000 Hz. The faintest sound that can be heard at 1000 Hz has a pressure amplitude of about 2.8×10^{-5} Pa. Find the corresponding density and displacement amplitudes. The bulk modulus for air under standard conditions is 1.4×10^5 Pa, and the speed of sound in air is 343 m/s at room temperature.

Solution The wave number is

$$k = \frac{2\pi}{\lambda} = \frac{2\pi f}{v} = \frac{2\pi \times 10^3\ \text{Hz}}{343\ \text{m/s}} = 18.3\ \text{rad/m}.$$

The density of air under these conditions is 1.21 kg/m³. Hence for $\Delta p_m = 28$ Pa, we obtain, using Eq. 19-3,

$$\Delta \rho_m = \Delta p_m \frac{\rho_0}{B} = (28\ \text{Pa}) \frac{1.21\ \text{kg/m}^3}{1.4 \times 10^5\ \text{Pa}}$$

$$= 2.4 \times 10^{-4}\ \text{kg/m}^3$$

and, using Eq. 19-8,

$$s_m = \frac{\Delta p_m}{kB} = \frac{28\ \text{Pa}}{(18.3\ \text{rad/m})(1.4 \times 10^5\ \text{Pa})}$$

$$= 1.1 \times 10^{-5}\ \text{m}.$$

The displacement amplitudes for the loudest sounds are about 10^{-5} m, a very small value indeed. For the faintest sounds, we have similarly

$$\Delta \rho_m = (2.8 \times 10^{-5}\ \text{Pa}) \frac{1.21\ \text{kg/m}^3}{1.4 \times 10^5\ \text{Pa}}$$

$$= 2.4 \times 10^{-10}\ \text{kg/m}^3$$

and

$$s_m = \frac{2.8 \times 10^{-5}\ \text{Pa}}{(18.3\ \text{rad/m})(1.4 \times 10^5\ \text{Pa})} = 1.1 \times 10^{-11}\ \text{m}.$$

This is about one-tenth of the radius of a typical atom and suggests how sensitive the ear must be to detect vibrations of such a small amplitude.

19-3 THE SPEED OF SOUND

As in the case of the transverse mechanical wave, the speed of a sound wave depends on the ratio of an elastic property of the medium and an inertial property. In analogy with Section 18-4, we might guess that the elastic property is the undisturbed pressure and the inertial property is the undisturbed density and try to do a dimensional analysis based on

$$v \propto \frac{p_0^a}{\rho_0^b}$$

to determine the exponents a and b. The dimensional analysis gives $a = b = \frac{1}{2}$, as you should verify. However, in this case we are not as fortunate as we were in the case of transverse waves—the expression $v = (p_0/\rho_0)^{1/2}$ does not give the correct value for the speed of sound. For example, in air at 20°C we would obtain $v = 289$ m/s from this expression, which is in poor agreement with the measured value, $v = 343$ m/s. You will recall that dimensional analysis gives us only the functional dependence and is unable to provide the values of any dimensionless constants that may be part of the equation. In this case we are clearly missing an important constant.

Let us turn instead to a mechanical analysis. Rather than a sinusoidal wave, it is simpler to consider only a single compressional pulse that is traveling down the tube. The pulse is traveling at speed v, the wave speed. Within the

* For a careful discussion of this point, see "Pressure and Displacement in Sound Waves," by C. T. Tindle, *American Journal of Physics,* September 1984, p. 749.

pulse, which has a width L, the pressure increase is a constant Δp.

Figure 19-4*a* shows the pulse as it is just about to enter a fluid element of length L_0 and cross-sectional area A where the undisturbed pressure is p_0. After a time t, the leading edge of the pulse has reached the end of the fluid element, which has now been compressed into a length L (Fig. 19-4*b*).

During the time t two horizontal forces act on the element: the compressional pulse exerts a force $(p_0 + \Delta p)A$ to the right, while the undisturbed fluid on the right of the element exerts a force $p_0 A$ that acts to the left. The net external force on the element during this interval is $A\ \Delta p$, acting to the right.

We cannot apply the particle form of Newton's laws to this element, because it does not behave like a particle; all parts of the element do not move in the same way. Instead, we can treat the element as a system of particles and use Newton's law in the form of Eq. 7-17 ($\Sigma F_{\text{ext},x} = Ma_{\text{cm},x}$). Measured from the right-hand end of the element, the center of mass moves from a location $-L_0/2$ to $-L/2$ in a time t. The acceleration of the center of mass is, using Eq. 2-28 in the form $x - x_0 = \frac{1}{2}a_x t^2$,

$$a_{\text{cm},x} = \frac{2(x - x_0)}{t^2} = \frac{2[(-L/2) - (-L_0/2)]}{t^2} = -\frac{\Delta L}{t^2}, \qquad (19\text{-}11)$$

where $\Delta L = L - L_0$ is the change in length of the fluid element. The mass of the fluid element is $M = \rho_0 A L_0$ and so Newtons' second law gives

$$A\ \Delta p = (\rho_0 A L_0)(-\Delta L/t^2). \qquad (19\text{-}12)$$

If the pulse moves with speed v, the time needed for it to move through the fluid element is $t = L_0/v$. Making this substitution into Eq. 19-12 and rearranging terms, we obtain

$$v^2 = \frac{1}{\rho_0}\frac{-\Delta p}{A\ \Delta L/AL_0}. \qquad (19\text{-}13)$$

The quantity $A\ \Delta L$ is the change in volume ΔV of the fluid element, and AL_0 is its original volume V. With these substitutions, the second factor on the right of Eq. 19-13 becomes $-\Delta p/(\Delta V/V)$, which is just the bulk modulus B. Taking the square root on both sides, we can thus write Eq. 19-13 as

$$v = \sqrt{\frac{B}{\rho_0}}. \qquad (19\text{-}14)$$

Equation 19-14 gives the speed of sound in fluids in terms of the bulk modulus and the density. Note that, as in the case of transverse waves on a string, the speed of sound in a fluid depends only on the properties of the medium and not on the frequency or wavelength of the wave.

In gases, the bulk modulus can be written γp_0, where γ is a constant (called the *specific heat ratio* and discussed in Chapter 23) that depends on the type of gas and typically has values between 1.3 and 1.7. For air, $\gamma = 1.4$; this factor accounts for the discrepancy in the result for the speed of sound obtained from the dimensional analysis at the beginning of this section.

Equation 19-14 applies to fluids (gases and liquids) but not to solids. In solids, a shearing modulus as well as a compressional modulus may be present, and the analysis is often more complicated than the simple one-dimensional case presented here.

Table 19-1 gives some representative values for the speed of sound in various materials.

Finally, we note that in this section we have treated the fluid as a continuous medium. In a gas, however, the spaces between molecules are large (compared with the size of the molecules), and the molecules move with a random thermal motion. The oscillations produced by a sound wave are superimposed on these random thermal motions. An impulse given to one molecule is passed on to another molecule only after the first has moved through the empty space between them and collided with the second. There is thus a close connection between the average molecular speed in a fluid and the speed of sound in that fluid. In particular, as we in-

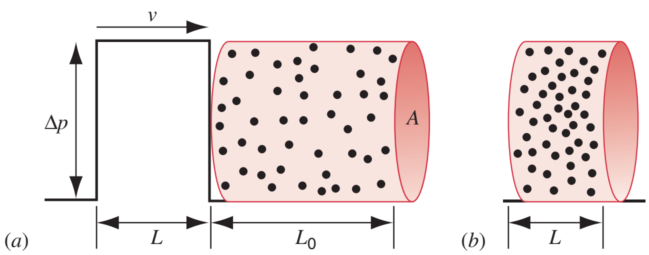


FIGURE 19-4. (*a*) A compressional pulse is about to enter a fluid element of undisturbed length L_0. (*b*) A time t later, the pulse has reached the right-hand end of the element and compressed the element to a length L.

TABLE 19-1 The Speed of Sound[a]

Medium	*Speed* (m/s)
Gases	
Air (0°C)	331
Air (20°C)	343
Helium	965
Hydrogen	1284
Liquids	
Water (0°C)	1402
Water (20°C)	1482
Seawater[b]	1522
Solids[c]	
Aluminum	6420
Steel	5941
Granite	6000

[a] At 0°C and 1 atm pressure, unless indicated otherwise.
[b] At 20°C and 3.5% salinity.
[c] Longitudinal waves; speeds of transverse waves are about half as large as those for longitudinal waves.

crease the temperature, the average molecular speed and the speed of sound in a gas increase in exactly the same manner.

19-4 POWER AND INTENSITY OF SOUND WAVES

In Section 18-6 we found the intensity of transverse waves on a string by considering the kinetic and potential energies associated with the wave motion. We can do the same type of analysis for sound waves by calculating the kinetic energy due to the motion of each fluid element and the internal energy stored in each fluid element (which is very similar in form to potential energy) as the wave passes. However, the internal energy calculation requires details of the dynamic behavior of gases that we shall not discuss until Chapter 23. As a result, we will use a different method to find the power transferred by a sound wave.

As the wave travels, each fluid element exerts a force on the fluid element ahead of it. If the pressure increase in the fluid element is Δp, the force it exerts on the next element is $F_x = A\,\Delta p$, where A is the cross-sectional area of the fluid element. Using Eq. 19-2 for the pressure, we find that the force is

$$F_x = A\,\Delta p_m \sin(kx - \omega t). \qquad (19\text{-}15)$$

The velocity u_x of the thin slice of fluid is given by Eq. 19-9. The power delivered to the fluid element is

$$P = u_x F_x = A\,\Delta p_m u_m \sin^2(kx - \omega t). \qquad (19\text{-}16)$$

Using Eq. 19-10 we can write this as

$$P = \frac{Av(\Delta p_m)^2}{B}\sin^2(kx - \omega t). \qquad (19\text{-}17)$$

As we did in Section 18-6, we assume that we observe the waves during a time that is very long compared with the period of oscillation of the wave. Since the average value of $\sin^2\theta$ over any number of full cycles is $\frac{1}{2}$, the average power is

$$P_{av} = \frac{Av(\Delta p_m)^2}{2B} = \frac{A(\Delta p_m)^2}{2\rho v} \qquad (19\text{-}18)$$

using Eq. 19-14 to replace B with ρv^2.

As in the case of the transverse wave, the power depends on the *square* of the amplitude, in this case the pressure amplitude. Note also that the frequency does not appear explicitly in Eq. 19-18 (although it would appear if we instead expressed the average power in terms of the displacement amplitude). Hence, by measuring pressure amplitudes, we can directly compare the intensities of sounds having *different* frequencies. For this reason, instruments that measure pressure changes are preferable to those that measure displacements; moreover, as we learned from Sample Problem 19-1, displacements from the weakest audible sounds are very small and would be difficult to measure directly.

When we are comparing different sounds, it is more useful to use the *intensity* (average power per unit area) of the wave. From Eq. 19-18, we can immediately obtain the intensity I:

$$I = \frac{P_{av}}{A} = \frac{(\Delta p_m)^2}{2\rho v}. \qquad (19\text{-}19)$$

The response of the ear to sound of increasing intensity is approximately logarithmic, so it is convenient to introduce a logarithmic scale of intensity called the *sound level* SL:

$$SL = 10\log\frac{I}{I_0}. \qquad (19\text{-}20)$$

The SL is defined with respect to a reference intensity I_0, which is chosen to be 10^{-12} W/m^2 (a typical value for the threshold of human hearing). Sound levels defined in this way are measured in units of *decibels* (dB). A sound of intensity I_0 has a sound level of 0 dB, whereas sound at the upper range of human hearing, called the threshold of pain, has an intensity of 1 W/m^2 and a SL of 120 dB. Multiplication of the intensity I by a factor of 10 corresponds to adding 10 dB to the SL.

We can also use dB as a *relative* measure to compare different sounds with one another, rather than with the reference intensity. Suppose we wish to compare two sounds of intensities I_1 and I_2:

$$SL_1 - SL_2 = 10\log\frac{I_1}{I_0} - 10\log\frac{I_2}{I_0}$$
$$= 10\log\frac{I_1}{I_2}. \qquad (19\text{-}21)$$

For example, two sounds whose intensity ratio is 2 differ in SL by $10\log 2 = 3$ dB.

The sensitivity of the human ear varies with frequency. The threshold of 10^{-12} W/m^2 applies only to the midrange

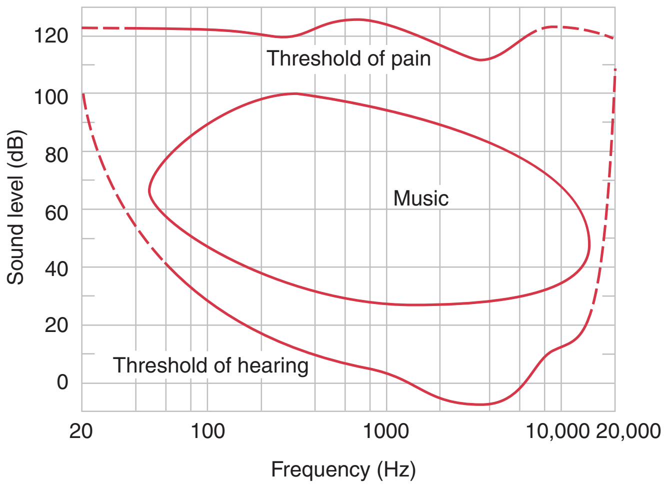


FIGURE 19-5. The average range of sound levels for human hearing. Note the dependence of the threshold levels on frequency. A sound that we can just hear at 100 Hz must have 1000 times the acoustic power (30 dB greater sound level) than one we can just hear at 1000 Hz, because our ear is that much less sensitive at 100 Hz.

TABLE 19-2 Some Intensities and Sound Levels

Sound	*Intensity I* (W/m^2)	*Sound Level* (dB)
Threshold of hearing	1×10^{-12}	0
Rustle of leaves	1×10^{-11}	10
Whisper (at 1 m)	1×10^{-10}	20
City street, no traffic	1×10^{-9}	30
Office, classroom	1×10^{-7}	50
Normal conversation (at 1 m)	1×10^{-6}	60
Jackhammer (at 1 m)	1×10^{-3}	90
Rock group	1×10^{-1}	110
Threshold of pain	1	120
Jet engine (at 50 m)	10	130
Space shuttle engine (at 50 m)	1×10^{8}	200

frequencies around 1000 Hz. At the higher frequencies, say 10,000 Hz, the threshold rises to about 10 dB (10^{-11} W/m^2), whereas at a lower frequency of 100 Hz the threshold is about 30 dB (10^{-9} W/m^2). It takes 1000 times the sound intensity at 100 Hz to produce the same physiological response as a given sound intensity at 1000 Hz. Figure 19-5 shows the variation with frequency of the thresholds of hearing and of pain, and Table 19-2 shows some representative sound levels and their corresponding intensities.

SAMPLE PROBLEM 19-2. Spherical sound waves are emitted uniformly in all directions from a point source, the radiated power P being 25 W. What are the intensity and the sound level of the sound wave a distance $r = 2.5$ m from the source?

Solution All the radiated power P must pass through a sphere of radius r centered on the source. Thus

$$I = \frac{P}{4\pi r^2}.$$

We see that the intensity of the sound drops off as the inverse square of the distance from the source. Numerically, we have

$$I = \frac{25\ \text{W}}{(4\pi)(2.5\ \text{m})^2} = 0.32\ \text{W/m}^2$$

and

$$SL = 10 \log \frac{I}{I_0}$$

$$= 10 \log \frac{0.32\ \text{W/m}^2}{10^{-12}\ \text{W/m}^2} = 115\ \text{dB}.$$

A comparison of this result with the list in Table 19-2 shows this sound level to be dangerous to a person's hearing.

19-5 INTERFERENCE OF SOUND WAVES

In Section 18-8 we discussed the interference that can occur when two different waves exist at the same point in space at the same time. The principle of superposition, which we used to add transverse waves on a string, also applies to sound waves.

Figure 19-6 shows two loudspeakers driven from a common source. At point P, the pressure variation due only to speaker S_1 is Δp_1, and that due to S_2 alone is Δp_2. (Note that these are spherical waves and are therefore not described by Eq. 19-2; in particular, the pressure amplitude does not remain constant as a spherical wave travels but instead decreases like $1/r$). The total pressure disturbance at point P is $\Delta p = \Delta p_1 + \Delta p_2$.

The type of interference that occurs at point P depends on the *phase difference* $\Delta\phi$ between the waves. In contrast to Section 18-8, in which the phase difference between the waves was due to their different phase constants, in this case the phase difference comes about because the waves may travel different distances from the speakers to arrive at point P. The phase difference $\Delta\phi$ between the two waves arriving at P depends on the path difference $\Delta L = |r_1 - r_2|$ from the speakers to point P. Path difference and phase difference are related by

$$\frac{\Delta\phi}{2\pi} = \frac{\Delta L}{\lambda}. \qquad (19\text{-}22)$$

That is, a phase difference of π corresponds to a path difference of $\lambda/2$, a phase difference of 2π corresponds to a path difference of λ, and so forth.

For some locations of point P, the pressure variations arrive in phase ($\Delta\phi = 0, 2\pi, 4\pi, \ldots$) and interfere constructively. For other locations of P, the waves arrive out of phase ($\Delta\phi = \pi, 3\pi, 5\pi, \ldots$) and interfere destructively. Using the condition for constructive interference, $\Delta\phi = m(2\pi)$ with $m = 0, 1, 2, \ldots$, Eq. 19-22 shows that the corresponding path difference for constructive interference is

$$\Delta L = m\lambda \qquad (m = 0, 1, 2, \ldots). \qquad (19\text{-}23)$$

That is, at locations where $|r_1 - r_2| = 0, \lambda, 2\lambda, \ldots$, the intensity reaches a maximum value. If the speakers are dri-

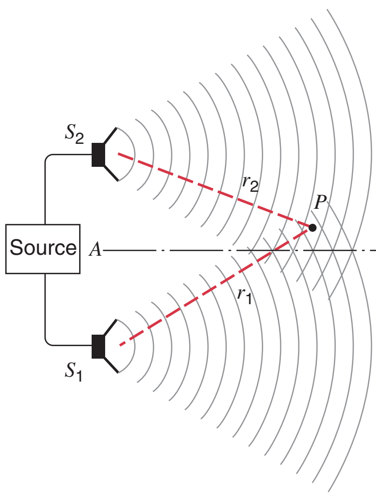


FIGURE 19-6. Two loudspeakers S_1 and S_2, driven by a common source, send signals to point P, where the signals interfere.

ven in phase, then at all points equidistant from the speakers (the line AB, which represents the entire midplane) there is constructive interference.

For destructive interference, the phase difference is $\Delta\phi = (m + \frac{1}{2})2\pi$ with $m = 0, 1, 2, \ldots$, so the path difference for destructive interference is

$$\Delta L = (m + \tfrac{1}{2})\lambda \qquad (m = 0, 1, 2, \ldots). \quad (19\text{-}24)$$

That is, at locations where $|r_1 - r_2| = \lambda/2, 3\lambda/2, 5\lambda/2, \ldots$, the intensity has a minimum value (not necessarily zero, because in general the two waves arrive at point P with difference amplitudes). Locations of destructive interference correspond to "dead spots" in the listening environment of the speakers.

If the speakers emit a mixture of many different wavelengths, some points P might show destructive interference for one wavelength and constructive interference for another.

SAMPLE PROBLEM 19-3. In the geometry of Fig. 19-6, a listener is seated at a point a distance of 1.2 m directly in front of one speaker. The two speakers, which are separated by a distance D of 2.3 m, emit pure tones of wavelength λ. The waves are in phase when they leave the speakers. For what wavelengths will the listener hear a minimum in the sound intensity?

Solution The minimum sound intensity occurs when the waves from the two speakers interfere destructively, according to the criteria of Eq. 19-24. If the listener is seated in front of speaker 2, then $r_2 = 1.2$ m, and r_1 can be found from the Pythagorean formula,

$$r_1 = \sqrt{r_2^2 + D^2} = \sqrt{(1.2\text{ m})^2 + (2.3\text{ m})^2} = 2.6\text{ m}.$$

Thus $r_1 - r_2 = 2.6\text{ m} - 1.2\text{ m} = 1.4\text{ m}$, and, according to Eq. 19-24, we have

$$1.4\text{ m} = \lambda/2, 3\lambda/2, 5\lambda/2, \ldots,$$

corresponding to

$$\lambda = 2.8\text{ m}, 0.93\text{ m}, 0.56\text{ m}, \ldots.$$

Complete destructive interference will not occur at this location, because the two waves arriving at the observation point have different amplitudes, if they leave the speakers with equal amplitudes.

19-6 STANDING LONGITUDINAL WAVES

We now consider what happens when a sound wave such as that shown in Fig. 19-1 reaches the end of the tube. In analogy with the transverse wave on the string (see Fig. 18-19), a reflection occurs, and the reflected wave travels back down the tube in the opposite direction. The behavior of the wave at the reflecting end depends on whether the end of the tube is open or closed.

Let us first consider a tube that is closed at the end. As the wave travels down the tube and reaches the end, it can compress the layers of air at the closed end against the fixed barrier. At that end, the pressure can therefore vary with its maximum amplitude, and *the closed end is a pressure antinode.* A pressure wave reflects from a closed end in a manner similar to the reflection of a transverse displacement wave at the free end of a string (Fig. 18-19*b*). If a compression, for example, is incident on the closed end, it reflects back along the tube as a compression. In analogy with our discussion of transverse waves on strings, we say that *a longitudinal pressure wave is reflected from a closed end with no change of phase.* The same effect occurs in the case of a longitudinal wave traveling on a spring such as a Slinky toy and reflecting from a *fixed* end: a compression is reflected as a compression.

Now consider what happens if the end of the tube is open. The pressure at the end of the open tube is the same as the ambient pressure p_0 in the surrounding room. We cannot change the pressure at that end of the tube unless we change the pressure in the entire room. The pressure at the open end therefore remains at the value p_0, and *the open end is a pressure node.* Comparison with Fig. 18-19*a* shows that this case is analogous to the transverse displacement wave reflecting from the fixed end of the string. The attempt by the wave incident on the open end to compress the air at that end causes a rarefaction, which travels back down the tube in the opposite direction. Thus *a longitudinal pressure wave is reflected from an open end with a phase change of 180°.* The same effect can again be observed with a coiled spring: a compression is reflected as a rarefaction.

Let us now assume that we have a train of sinusoidal waves traveling down the tube. The waves are reflected at the end, which will behave either as a pressure node (if the end is open) or a pressure antinode (if the end is closed). We assume the source of the wavetrain is a speaker at the opposite end. The movement of the speaker sends a compressional wave down the tube, and the superposition of the original and reflected waves gives a pattern of standing waves, just as was the case for the transverse waves on the string. Within the tube will be a pattern of pressure nodes and antinodes (which are not points, as in the case of transverse waves on a string, but planes).

If the frequency (or wavelength) of the source of the waves is selected to be a particular value that depends on the length of the tube, then a standing wave pattern is established along the entire tube, in analogy with the case of the standing wave patterns shown in Fig. 18-20. If there is a node of pressure at the speaker end, then little energy is given back to the speaker from the standing wave pattern in the tube, and we have a condition of resonance. The driving frequency must be equal to one of the natural frequencies of the system, which are determined by the length of the tube.

Figure 19-7*a* shows a tube that is driven by a speaker at one end and is open at the other end. As we have discussed previously, the speaker end is a pressure node at resonance and the open end is likewise a pressure node. In Fig. 19-7*a*

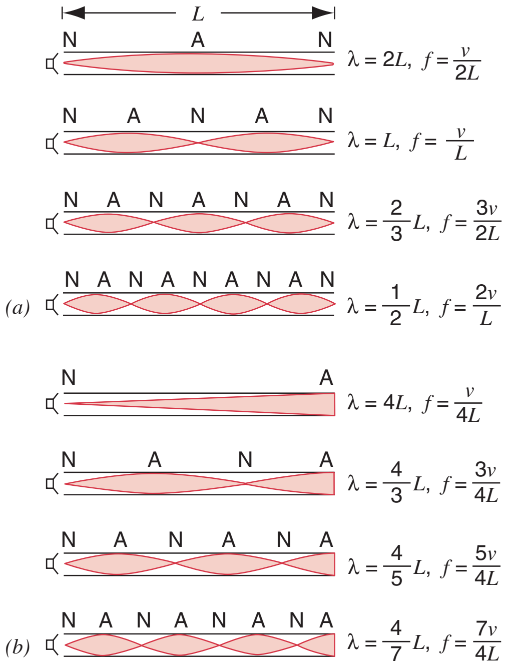


FIGURE 19-7. (*a*) The pressure waves of the first four resonant modes of a tube driven by a speaker and open at the other end. There is a pressure node N at each end, and antinodes A are located between the nodes. The curves suggest the sinusoidal variation of pressure within the tube. (*b*) The pressure waves of the first four resonant modes of a tube that is closed at one end. The closed end is a pressure antinode. Note the differences in vibrational patterns and wavelengths between the open and closed tubes.

are shown the resulting variations in the pressure amplitude of the standing waves.* These patterns look very similar to those of Fig. 18-20. In the first mode of oscillation, the length L of the tube is equal to $\lambda/2$, where λ is the wavelength of the wave produced by the speaker for that particular resonant condition. The wavelength is therefore $2L$, and the corresponding frequency is $f = v/\lambda = v/2L$. The other resonances shown in Fig. 19-7*a* have successively smaller wavelengths, which can be written in general as

$$\lambda_n = \frac{2L}{n}, \qquad n = 1, 2, 3, \ldots . \qquad (19\text{-}25)$$

The corresponding resonant frequencies, determined using the expression $f = v/\lambda$ with the above wavelengths, are

$$f_n = n\frac{v}{2L}, \qquad n = 1, 2, 3, \ldots \qquad \text{(open tube).} \qquad (19\text{-}26)$$

Here v represents the speed of the wave in the medium that fills the tube, usually air.

Figure 19-7*b* shows the case in which the tube is closed at one end and open at the other end. In this case, the closed end must be a pressure antinode. In the first resonant mode, the length L of the tube is $\frac{1}{4}\lambda$, and so the source must be producing a wave whose wavelength is $4L$. In the next mode, the wavelength changes so that now L is $\frac{3}{4}\lambda$, and thus $\lambda = \frac{4}{3}L$. Continuing the series, we see that in this case the general expression for the wavelengths of the resonant modes is

$$\lambda_n = \frac{4L}{n}, \qquad n = 1, 3, 5, \ldots . \qquad (19\text{-}27)$$

Note that only odd values of the integer n appear in this case. The corresponding resonant frequencies are

$$f_n = n\frac{v}{4L}, \qquad n = 1, 3, 5, \ldots \qquad \text{(closed tube).} \qquad (19\text{-}28)$$

As we discuss in the next section, the resonant frequencies given by Eq. 19-26 or 19-28 determine the musical notes played by the wind instruments.

The actual location of the pressure node at an open end is not *exactly* at the end of the tube. The wave extends slightly into the medium beyond the tube, so the effective length of the tube is a bit greater than its actual length and the resonant frequencies are a bit smaller. For narrow tubes of cylindrical shape, the length correction is roughly equal to $0.6R$, where R is the radius of the tube. For a tube open at both ends, the length correction must be applied at each end. For a tube of length 0.6 m and radius 1 cm (typical values for the smaller wind instruments such as the clarinet or flute), the lowest frquency without the end correction would be 286 Hz if the tube were open and 143 Hz if the tube were closed. With the end correction, the corresponding values would be 280 Hz and 142 Hz. The corrections are small, but nevertheless quite important.

SAMPLE PROBLEM 19-4. Figure 19-8 shows an apparatus that can be used to measure the speed of sound in air by using the resonance condition. A small speaker is held above a cylindrical tube partially filled with water. By adjusting the water level, the length of the air column can be changed until the tube is in resonance, at which point an increase in the sound intensity can be heard. In an experiment, the speaker is driven at a fixed frequency of 1080 Hz, and three resonances are observed when the water level is at distances of $x_1 = 6.5$ cm, $x_2 = 22.2$ cm, and $x_3 = 37.7$ cm below the top of the tube. Find the value of the speed of sound from these data.

Solution The air column acts like a tube of variable length closed at one end. The pattern of standing waves shows a pressure node near the speaker and a pressure antinode at the surface of the water. Since we do not know the end correction, we cannot use the given data directly to find the speed of sound from Eq. 19-28. However, we note from the resonance conditions shown in Fig. 19-7*b* that the distance between adjacent pressure nodes is $\frac{1}{2}\lambda$; the same is true for the distance between adjacent antinodes. From the

* A beautiful demonstration of the locations of the pressure nodes and antinodes can be obtained with a Rubens flame-tube. See "Rubens Flame-tube Demonstration," by George W. Ficken and Francis C. Stephenson, *The Physics Teacher,* May 1979, p. 306.

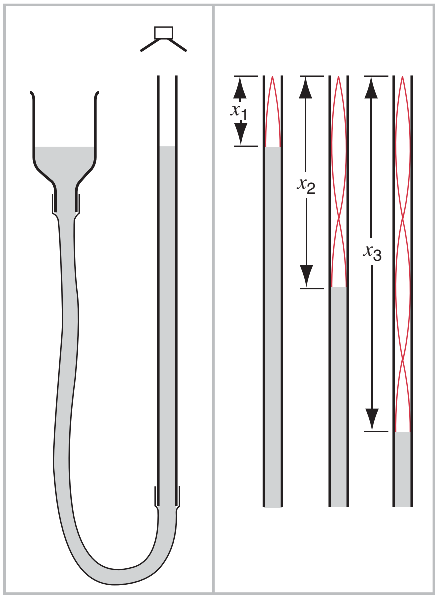


FIGURE 19-8. Sample Problem 19-4. An apparatus for measuring the speed of sound in air. The water level can be adjusted by raising and lowering the reservoir on the left, which is connected through a hose to the tube. At right are shown the pressure waveforms of the first three resonant modes for a fixed wavelength.

data given, we therefore conclude from the first two resonances that

$$\tfrac{1}{2}\lambda = x_2 - x_1 = 22.2\ \text{cm} - 6.5\ \text{cm} = 15.7\ \text{cm},$$

and similarly, from the second and third resonances,

$$\tfrac{1}{2}\lambda = x_3 - x_2 = 37.7\ \text{cm} - 22.2\ \text{cm} = 15.5\ \text{cm}.$$

The average of these two values, which we take as the best value from this measurement, is 15.6 cm, corresponding to a wavelength of $2(15.6\ \text{cm}) = 31.2\ \text{cm} = 0.312\ \text{m}$. We therefore deduce the speed of sound to be

$$v = \lambda f = (0.312\ \text{m})(1080\ \text{Hz}) = 337\ \text{m/s}.$$

Other than the end correction, what physical factors in this experiment (including the properties of the air) might influence the measured value?

19-7 VIBRATING SYSTEMS AND SOURCES OF SOUND*

A vibrating system transmits a wave through the air to the ears of the listener. This is the basic principle of the production of sound by voice or by musical instruments. We have already studied the propagation of the sound wave, and now to understand the nature of the sound we must study the vibrating system that produces it.

As we discussed in Section 18-10 in the case of the vibrating string and in the previous section in the case of a column of air, a distributed system has a large (perhaps infinite) number of natural vibrational frequencies. These are the frequencies at which it *can* vibrate. Which of the frequencies *will* be present in the vibration depends on how the system is set into vibration.

Suppose the system is capable of vibrating at a number of frequencies $f_1, f_2, f_3, \ldots$. We write these in ascending order, so that $f_1 < f_2 < f_3 < \cdots$. The lowest frequency, f_1, is called the *fundamental* frequency, and the corresponding mode of oscillation is called the fundamental mode. The higher frequencies are called *overtones,* with f_2 being the first overtone, f_3 the second overtone, and so on.

In certain systems, the overtones are all integer multiples of the fundamental:

$$f_n = nf_1, \tag{19-29}$$

where n is an integer. In such a case, the overtones are called *harmonics.* The first member of a harmonic sequence is the fundamental, the second harmonic is the first overtone, and so on.

Why do some vibrating systems produce pleasant sounds while others produce harsh or discordant sounds? When several frequencies are heard simultaneously, a pleasant sensation results if the frequencies are in the ratio of small whole numbers, such as 3:2 or 5:4. If a system produces overtones that are harmonics, its vibrations will include frequencies that have these ratios, and it would produce a pleasing sound. If the overtones are not harmonics, it is likely that the sound will be discordant. Much of the effort in the design of musical instruments is devoted to the production of harmonic sequences of overtones. Some instruments, such as those based on vibrating strings, produce overtones that are automatically harmonics when the vibrations have small amplitude. In other cases, the shape of the instrument must be carefully designed to make it harmonic; a bell is an example of such an instrument. The harmonics that an instrument produces give it its richness and diversity of tone, and they are critical to the beauty of the sound of the instrument. If instruments produced only fundamentals, they would all sound exactly alike.

We can classify musical insturments into three categories: those based on vibrating strings, those based on vibrating columns of air, and more complex systems including vibrating plates, rods, and membranes.

Vibrating Strings

These instruments include the bowed strings (violins, for example), plucked strings (guitar, harpsichord), and struck strings (piano).

If a string fixed at both ends is bowed, struck, or plucked, transverse vibrations travel along the string; these disturbances are reflected at the fixed ends, and a standing

* For a listing of references on the physics of musical instruments and related topics, see "Resource Letter MA-2: Musical Acoustics," by Thomas D. Rossing, *American Journal of Physics,* July 1987, p. 589.

wave pattern is formed. The natural modes of vibration of the string are excited, and these vibrations give rise to longitudinal waves in the surrounding air, which transmits them to our ears as a musical sound.

We have seen (Section 18-10) that a string of length L, fixed at both ends, can resonate at frequencies given by Eq. 18-46:

$$f_n = n\frac{v}{2L}, \qquad n = 1, 2, 3, \ldots . \tag{19-30}$$

Here v is the speed in the string of the transverse traveling waves whose superposition can be thought of as giving rise to the vibrations; the speed v $(= \sqrt{F/\mu})$ is the same for all frequencies. (Note that v is *not* the speed of sound in air; even though Eq. 19-30 looks exactly like Eq. 19-26, v stands for different quantities in the two equations.) At any one of these frequencies the string contains a whole number n of loops between its ends; it has nodes at each end and $n - 1$ additional nodes equally spaced along its length (Fig. 19-9).

If the string is initially distorted so that its shape is the same as *any one* of the possible harmonics, it vibrates only at the frequency of that particular harmonic. The initial conditions usually arise from striking or bowing the string, however, and in such cases not only the fundamental but many of the overtones are present in the resulting vibration. We have a superposition of several natural modes of oscillation. The actual displacement is the sum of the several harmonics with various amplitudes. The impulses that are sent through the air to the ear and brain give rise to one net effect, which is characteristic of the particular stringed instrument. The quality of the sound of a note of any particular frequency played by an instrument is determined by the number of overtones present and their respective intensities. Figure 19-10 shows the sound spectra and corresponding waveforms for the violin and piano.

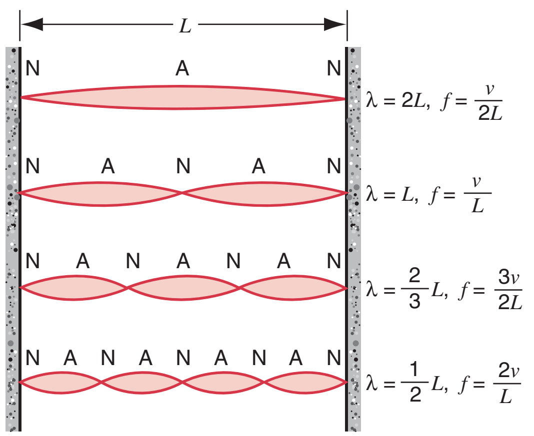


FIGURE 19-9. The first four resonant modes of a vibrating string fixed at both ends. Nodes and antinodes of displacement are denoted by N and A.

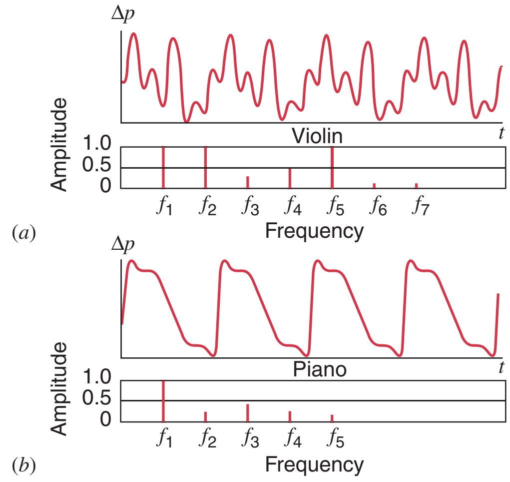


FIGURE 19-10. Waveforms and sound spectra for two stringed instruments, (*a*) violin and (*b*) piano, each playing a note of fundamental frequency $f_1 = 440$ Hz (concert A). The sound spectrum below each waveform shows the harmonics that are present in the complex tone and their relative amplitudes.

Vibrating Air Columns

An organ pipe is a simple example of a sound originating in a vibrating air column. If both ends of a pipe are open and a stream of air is directed against an edge at one end, standing longitudinal waves can be set up in the tube. The air column then resonates at its natural frequencies of vibration, given by Eq. 19-26. As with the bowed string, the fundamental and the overtones (which are harmonics) are produced at the same time. If one end of the tube is closed, the fundamental frequency is reduced by half, relative to its value for an open tube of the same length, and only the odd harmonics are present, which changes the quality of the sound. That is, an open pipe produces the same fundamental tone as a closed pipe of half the length, but because the mixture of harmonics is different in the two pipes, the quality of the tones differs.

Reed instruments, such as the clarinet, produce their tones differently. Air is forced through a narrow opening, one side of which is covered by a reed that has elastic properties. According to Bernoulli's equation, the high-speed air passing through the narrow opening causes a local region of low pressure inside the mouthpiece. The outside pressure exceeds the inside pressure, which forces the reed inward so that it covers the opening. As soon as the opening is covered, the air flow is interrupted, the dynamic low-pressure region is eliminated, and the reed pops open, allowing the air flow to start again. This repeated opening and closing of the air passage causes maximum variations in pressure at the mouthpiece end of the instrument, which therefore behaves like an antinode of pressure. In a clarinet, the other end of the instrument is open, and therefore the resonances of the instrument are those given by Eq. 19-28 for a tube closed at one end and open at the other. Some wind instruments, such as the flute, use a method similar to the organ

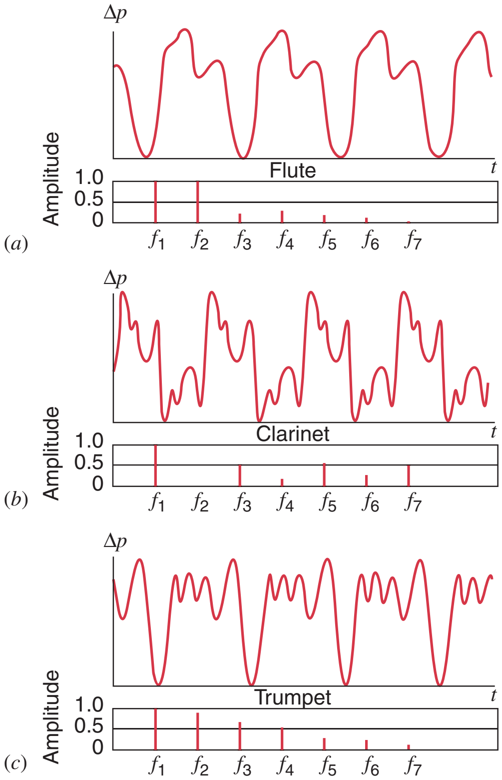


Figure 19-11. Waveforms of some wind instruments: (*a*) flute, (*b*) clarinet, and (*c*) trumpet, and their sound spectra, as in Fig. 19-10. Note that the clarinet spectrum shows mostly the odd harmonics, whereas the flute and trumpet have odd as well as even harmonics.

pipe to produce the tone, such that the mouthpiece behaves like an open end; their resonant frequencies are given by Eq. 19-26. Still others (oboe, saxophone), which use a reed to produce their tone, have a conical (that is, tapered) rather than a cylindrical bore, which gives them overtones that are approximately harmonics, odd as well as even. The brass instruments (trumpet or trombone, for example) are also called *lip reed* instruments, because the player's lip acts like a reed, but again the bore is slightly tapered, and as a result the overtones contain all the harmonics. Figure 19-11 shows waveforms of some wind instruments.

Other Vibrating Systems

Vibrating rods, plates, and stretched membranes also give rise to sound waves. Consider a stretched flexible membrane, such as a drumhead. If it is struck a blow, a two-dimensional pulse travels outward from the struck point and is reflected again and again at the boundary of the membrane. If some point of the membrane is forced to vibrate periodically, continuous trains of waves travel out along the membrane. Just as in the one-dimensional case of the string, so here too standing waves can be set up in the two-dimensional membrane. Each of these standing waves has a certain frequency natural to (or characteristic of) the membrane. Again the lowest frequency is called the fundamental, and the others are overtones. Generally, many overtones are present along with the fundamental when the membrane is vibrating. These vibrations may excite sound waves of the same frequency.

The nodes of a vibrating membrane are lines rather than points (as in a vibrating string) or planes (as in a tube). Since the boundary of the membrane is fixed, it must be a nodal line. For a circular membrane fixed at its edge, possible modes of vibration together with their nodal lines are shown in Fig. 19-12. The natural frequency of each mode is

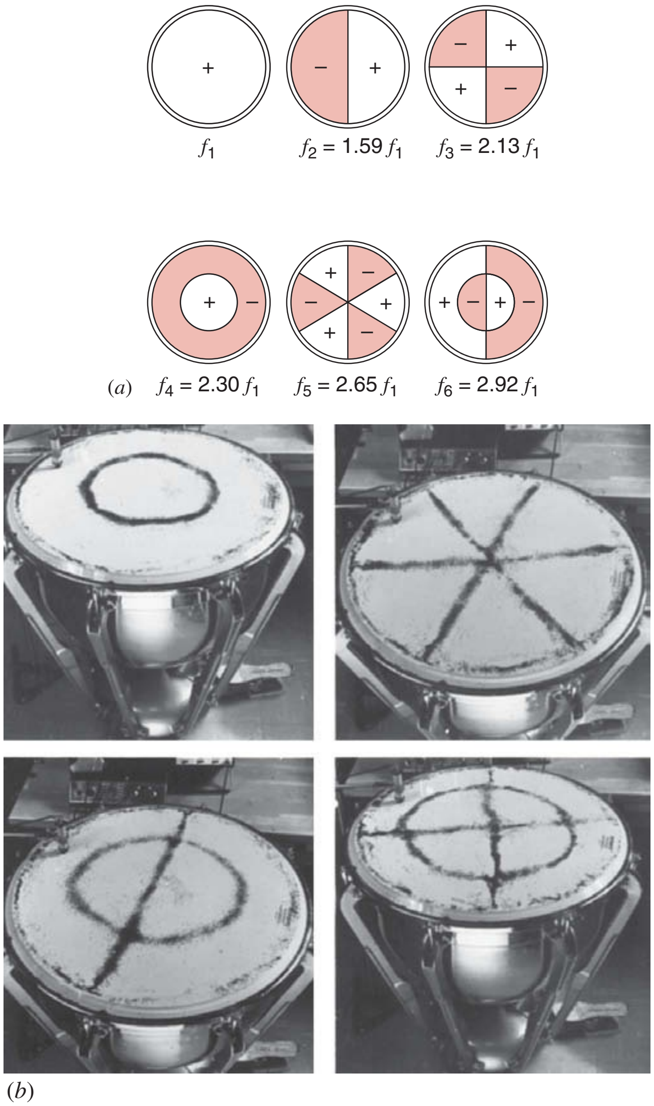


Figure 19-12. (*a*) The lowest six resonant modes of a circular drumhead clamped at its rim. The lines represent nodes; the rim is also a nodal line. The + or − signs indicate that, at a given instant, a particular region is moving up out of the page or down into the page. In this case the overtones are not integral multiples of the fundamental and are thus not harmonics. (*b*) The vibrational patterns of a kettledrum in the modes numbered 4, 5, and 6, and one additional mode not illustrated in (*a*). They are made visible by sprinkling dark powder on the drumhead and setting it into vibration at the proper frequency using a mechanical vibrator. As the drumhead vibrates, the powder is shaken and eventually settles on the nodal lines, where there is no motion.

given in terms of the fundamental f_1. The frequencies of the overtones are not harmonics; that is, they are not integral multiples of f_1. Vibrating rods also have a nonharmonic set of natural frequencies. Rods and plates have limited use as musical instruments for this reason. In instruments such as the xylophone and the marimba, small bars of wood or metal are set into vibration by striking. The shape of the bars is carefully modified, making them thinner in the center, in such a way that the overtones become approximately harmonics.

19-8 BEATS

We have previously considered the effect of waves that are superimposed to give regions of maximum and minimum (zero) intensity, such as in the case of a standing wave in a tube. This illustrates a type of interference that we can call *interference in space.*

The same principle of superposition leads us to another type of interference, which we can call *interference in time.* In this case we examine the superposition of two waves at a given point as a function of the time. This superposition, which in general can result in quite complex waveforms, takes a particularly simple form when the two waves have nearly the same frequency. With sound such a condition exists when, for example, two instruments or two guitar strings are being tuned to one another.

Consider a point in space through which the waves are passing. Figure 19-13*a* shows the pressure produced at that point by the two waves separately as a function of time. For simplicity we have assumed that the two waves have equal amplitude, although this is not necessary. The resultant pressure at that point as a function of time is the sum of the individual pressures and is plotted in Fig. 19-13*b*. We see that the amplitude of the resultant wave is not constant but

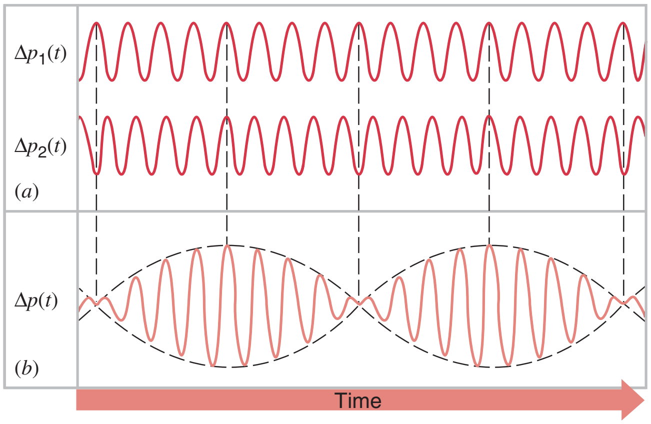


FIGURE 19-13. (*a*) Two sinusoidal waveforms of nearly equal frequencies. (*b*) The superposition of the two waveforms. Note that the two waves in part (*a*) go from being in phase, giving a resultant of large amplitude, to being out of phase, giving a resultant of zero amplitude. The dashed curves show the sinusoidal variation of the modulating envelope with angular frequency ω_{amp}.

varies with time. In the case of sound the varying amplitude gives rise to variations in loudness, which are called *beats.*

Let us represent the variation in pressure with time (for constant x) produced by one wave as

$$\Delta p_1(t) = \Delta p_m \sin \omega_1 t,$$

where we have chosen the phase constant to enable us to write the wave in this simple form. The pressure variation at the same point produced by the other wave of equal amplitude is represented as

$$\Delta p_2(t) = \Delta p_m \sin \omega_2 t.$$

By the superposition principle, the resultant pressure is

$$\Delta p(t) = \Delta p_1(t) + \Delta p_2(t)$$
$$= \Delta p_m(\sin \omega_1 t + \sin \omega_2 t). \qquad (19\text{-}31)$$

Using the trigonometric identity

$$\sin A + \sin B = 2 \cos \frac{A - B}{2} \sin \frac{A + B}{2},$$

Eq. 19-31 can be written

$$\Delta p(t) = \left[2\Delta p_m \cos \left(\frac{\omega_1 - \omega_2}{2} \right) t \right] \sin \left(\frac{\omega_1 + \omega_2}{2} \right) t. \qquad (19\text{-}32)$$

So far everything we have done applies to any two waves, no matter what their frequencies. When the frequencies are nearly the same, Eq. 19-32 can be simplified by writing the second factor in terms of the average angular frequency ω_{av} of the two waves,

$$\omega_{av} = \frac{\omega_1 + \omega_2}{2}. \qquad (19\text{-}33)$$

The first factor, contained in the brackets of Eq. 19-32, gives a time-varying amplitude to the sinusoidal variation of the second factor. This amplitude factor varies with an angular frequency

$$\omega_{amp} = \frac{|\omega_1 - \omega_2|}{2}. \qquad (19\text{-}34)$$

In terms of ω_{av} and ω_{amp}, we can write Eq. 19-32 as

$$\Delta p(t) = [2\Delta p_m \cos \omega_{amp} t] \sin \omega_{av} t. \qquad (19\text{-}35)$$

If ω_1 and ω_2 are nearly equal, the amplitude frequency ω_{amp} is small, and the amplitude fluctuates slowly. Figure 19-13 shows the superposition of the two waves according to Eq. 19-32. Notice that in the case of nearly equal frequencies, the rapid variation of the resultant wave occurs with a frequency that is approximately that of either of the two added waves. The overall amplitude of the resultant varies slowly with the amplitude frequency ω_{amp}, which defines an "envelope" within which the more rapid variation occurs. This phenomenon is a form of *amplitude modulation,* which has a counterpart (side bands) in AM radio receivers.

In the case shown in Fig. 19-13*b*, the ear would perceive a tone at a frequency f_{av} ($= \omega_{av}/2\pi$) which is approximately the same as the frequencies f_1 ($= \omega_1/2\pi$) or f_2 ($= \omega_2/2\pi$) of the two component waves. The tone grows alternately loud and soft as the amplitude of the resultant varies with time, having maxima and minima as shown in Fig. 19-13*b*.

A beat—that is, a maximum of intensity—occurs whenever $\cos \omega_{\text{amp}}t$ equals $+1$ or -1, since the intensity depends on the *square* of the amplitude. Each of these values occurs once in each cycle of the envelope (see Fig. 19-13), so the number of beats per second is twice the number of cycles per second of the envelope. The beat angular frequency ω_{beat} is then

$$\omega_{\text{beat}} = 2\omega_{\text{amp}} = |\omega_1 - \omega_2|. \qquad (19\text{-}36)$$

Using $\omega = 2\pi f$, we can rewrite this expression as

$$f_{\text{beat}} = |f_1 - f_2|. \qquad (19\text{-}37)$$

Hence *the number of beats per second equals the difference of the frequencies of the component waves.* Beats between two tones can be detected by the ear up to a frequency of about 15 Hz. At higher frequencies individual beats cannot be distinguished in the sound produced. Musicians often listen for beats when tuning certain instruments. The tuning is changed until the beat frequency decreases and the beats disappear.

SAMPLE PROBLEM 19-5. A violin string that should be tuned to concert A (440 Hz) is slightly mistuned. When the violin string is played in its fundamental mode along with a concert A tuning fork, 3 beats per second are heard. (*a*) What are the possible values of the fundamental frequency of the string? (*b*) Suppose the string were played in its first overtone simultaneously with a tuning fork one octave above concert A (880 Hz). How many beats per second would be heard? (*c*) When the tension of the string is increased slightly, the number of beats per second in the fundamental mode increases. What was the original frequency of the fundamental?

Solution (*a*) From Eq. 19-37, we know that the frequency f_1 of the string differs by the beat frequency (3 Hz) from the frequency f_2 of the tuning fork (440 Hz), but from the number of beats per second alone, we cannot tell whether the string has a higher or lower frequency. Thus the possible frequencies are

$$f_1 = 440 \text{ Hz} \pm 3 \text{ Hz} = 443 \text{ Hz} \quad \text{or} \quad 437 \text{ Hz}.$$

(*b*) In the first overtone, the frequency of the string is twice its fundamental frequency, and thus either 886 Hz or 874 Hz. When played against a 880-Hz tuning fork, the frequency difference in either case is 6 Hz, and thus 6 beats per second would be heard.
(*c*) Increasing the tension in the string raises the speed of transverse waves and therefore raises the fundamental frequency (see Eq. 19-30). Since we are given that this increases the beat frequency, we conclude that the frequency of the fundamental mode was previously greater than 440 Hz, since increasing the frequency made the difference from 440 Hz even greater. Thus the string was originally tuned to 443 Hz, and to bring it into proper tuning the tension must be reduced.

19-9 THE DOPPLER EFFECT

When a listener is in motion toward a stationary source of sound, the pitch (frequency) of the sound heard is higher than when the listener is at rest. If the listener is in motion away from the stationary source, a lower pitch is heard. We obtain similar results when the source is in motion toward or away from a stationary listener. The pitch of the whistle of a locomotive or the siren of a fire engine is higher when the source is approaching the hearer than when it has passed and is receding.

In a paper written in 1842, Christian Johann Doppler (1803–1853, Austrian) called attention to the fact that the color of a luminous body must be changed by relative motion of the body and the observer. This *Doppler effect,* as it is called, applies to waves in general. Doppler himself mentions the application of his principle to sound waves. An experimental test was carried out in Holland in 1845 by Buys Ballot, "using a locomotive drawing an open car with several trumpeters."

Moving Observer, Source at Rest

We now consider the Doppler effect for sound waves, treating only the special case in which the source and observer move along the line joining them. Let us adopt a reference frame at rest in the medium through which the sound travels. Figure 19-14 shows a source of sound S at rest in this frame and an observer O moving toward the source at a speed v_O. The circles represent wavefronts, spaced one wavelength apart, traveling through the medium. An observer at rest in the medium would receive vt/λ waves in time t, where v is the speed of sound in the medium and λ is the wavelength. Because of the motion toward the source, however, the observer receives v_Ot/λ additional waves in

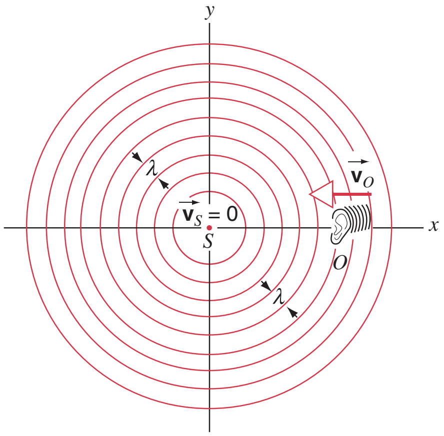


FIGURE 19-14. A stationary source of sound S emits spherical wavefronts, shown one wavelength apart. An observer O, represented by the ear, moves with speed v_O toward the source. The moving observer encounters more waves per second than an observer at rest and therefore measures a higher frequency. The observer would measure a *lower* frequency for motion away from the source.

this same time t. The frequency f' that is actually heard is the number of waves received per unit time, or

$$f' = \frac{vt/\lambda + v_O t/\lambda}{t} = \frac{v + v_O}{\lambda} = \frac{v + v_O}{v/f}.$$

That is,

$$f' = f\frac{v + v_O}{v} = f\left(1 + \frac{v_O}{v}\right). \qquad (19\text{-}38)$$

The frequency f' heard by the observer is the frequency f heard at rest plus the increase $f(v_O/v)$ arising from the motion of the observer. When the observer is in motion *away from* the stationary source, there is a *decrease* in frequency $f(v_O/v)$ corresponding to the waves that do not reach the observer in each unit of time because of the receding motion. Then

$$f' = f\frac{v - v_O}{v} = f\left(1 - \frac{v_O}{v}\right). \qquad (19\text{-}39)$$

Hence the general relation holding when the *source is at rest* with respect to the medium but the *observer is moving* through it is

$$f' = f\frac{v \pm v_O}{v}, \qquad (19\text{-}40)$$

where the plus sign holds for motion toward the source and the minus sign holds for motion away from the source. Note that the change in frequency occurs because the observer intercepts more or fewer waves each second as a result of the motion through the medium.

Moving Source, Observer at Rest

When the source is in motion *toward* a stationary observer, the effect is a shortening of the wavelength (see Fig. 19-15), because the source is following after the approaching

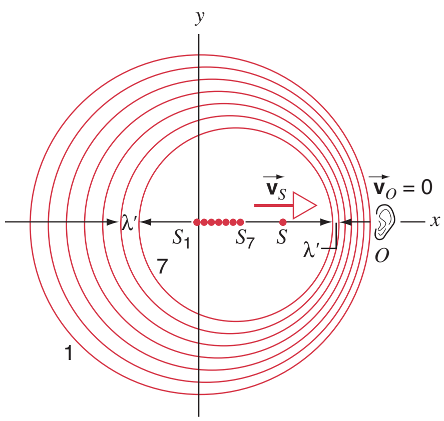


Figure 19-15. Here the observer O is at rest, with the source moving toward it at speed v_S. Wavefront 1 was emitted when the source was at S_1, wavefront 7 when the source was at S_7, and so on. At the instant of this drawing, the source is at S. The observer measures a shorter wavelength because of the "bunching up" of the wavefronts along the motion. An observer on the negative x axis, from whom the source would be moving away, would measure a longer wavelength.

waves, and the crests therefore come closer together. If the frequency of the source is f and its speed is v_S, then during each vibration it travels a distance v_S/f, and each wavelength is shortened by this amount. Hence the wavelength of the sound arriving at the observer is not $\lambda = v/f$ but $\lambda' = v/f - v_S/f$. The frequency of the sound heard by the observer is increased and is given by

$$f' = \frac{v}{\lambda'} = \frac{v}{(v - v_S)/f} = f\frac{v}{v - v_S}. \qquad (19\text{-}41)$$

If the source moves *away from* the observer, the wavelength emitted is v_S/f greater than λ, so that the observer hears a decreased frequency—namely,

$$f' = \frac{v}{(v + v_S)/f} = f\frac{v}{v + v_s}. \qquad (19\text{-}42)$$

Hence the general relation holding when the *observer is at rest* with respect to the medium but the *source is moving* through it is

$$f' = f\frac{v}{v \pm v_S}, \qquad (19\text{-}43)$$

where the minus sign holds for motion toward the observer and the plus sign holds for motion away from the observer. Note that the change here is the shortening or increasing of the wavelength transmitted through the medium due to the motion of the source through the medium.

If *both source and observer move* through the transmitting medium, you should be able to show that the observer hears a frequency

$$f' = f\frac{v \pm v_O}{v \mp v_S}, \qquad (19\text{-}44)$$

where the upper signs (+ numerator, − denominator) correspond to the source and observer moving along the line joining the two in the direction toward the other, and the lower signs in the direction away from the other. Equation 19-44 incorporates all four different possibilities, as Sample Problem 19-6 shows. Note that Eq. 19-44 reduces to Eq. 19-40 when $v_S = 0$ and Eq. 19-43 when $v_O = 0$, as it must.

If a source of sound is moved away from an observer and toward a wall, the observer hears two notes of different frequency. The note heard directly from the receding source is lowered in pitch by the motion. The other note is due to the waves reflected from the wall, and this is raised in pitch (because the source is moving *toward* the wall, and the wall "hears" the higher frequency). The superposition of these two wave trains produces beats. A similar effect occurs if a wave from a stationary source is reflected from a moving object. The beat frequency can be used to deduce the speed of the object. This is the basic principle of radar speed monitors, and it is also used to track satellites.

The discussion in this section applies to the Doppler shift for sound waves and other similar mechanical waves. Light waves also show the Doppler effect; however, because there is no medium of propagation for light, the formulas developed in this section do not apply. See Chapter 39 for a discussion of the Doppler effect for light waves.

SAMPLE PROBLEM 19-6. The siren of a police car emits a pure tone at a frequency of 1125 Hz. Find the frequency that you would perceive in your car under the following circumstances: (*a*) your car at rest, police car moving toward you at 29 m/s (65 mi/h); (*b*) police car at rest, your car moving toward it at 29 m/s; (*c*) you and the police car moving toward one another at 14.5 m/s; (*d*) you moving at 9 m/s, police car chasing behind you at 38 m/s.

Solution All four parts of this problem can be solved using Eq. 9-44. (*a*) Here $v_O = 0$ (your car is at rest) and $v_S = 29$ m/s. We choose the upper (minus) sign in the denominator of Eq. 19-44, because the police car is moving toward you. We thus obtain, using $v = 343$ m/s for the speed of sound in still air,

$$f' = f\frac{v}{v - v_S} = (1125\text{ Hz})\frac{343\text{ m/s}}{343\text{ m/s} - 29\text{ m/s}} = 1229\text{ Hz}.$$

(*b*) In this case $v_S = 0$ (the police car is at rest) and $v_O = 29$ m/s. We choose the upper (plus) sign in the numerator of Eq. 19-44, because you are moving toward the police car, and we find

$$f' = f\frac{v + v_O}{v} = (1125\text{ Hz})\frac{343\text{ m/s} + 29\text{ m/s}}{343\text{ m/s}} = 1220\text{ Hz}.$$

(*c*) In this case $v_S = 14.5$ m/s and $v_O = 14.5$ m/s. We choose the upper signs in both the numerator and denominator of Eq. 19-44, because you and the police car are moving toward each other. We thus obtain

$$f' = f\frac{v + v_O}{v - v_S} = (1125\text{ Hz})\frac{343\text{ m/s} + 14.5\text{ m/s}}{343\text{ m/s} - 14.5\text{ m/s}} = 1224\text{ Hz}.$$

(*d*) Here $v_O = 9$ m/s and $v_S = 38$ m/s. You are moving away from the police car, so we choose the lower (minus) sign in the numerator, but the police car is moving *toward* you, so we choose the upper (minus) sign in the denominator. The result is

$$f' = f\frac{v - v_O}{v - v_S} = (1125\text{ Hz})\frac{343\text{ m/s} - 9\text{ m/s}}{343\text{ m/s} - 38\text{ m/s}} = 1232\text{ Hz}.$$

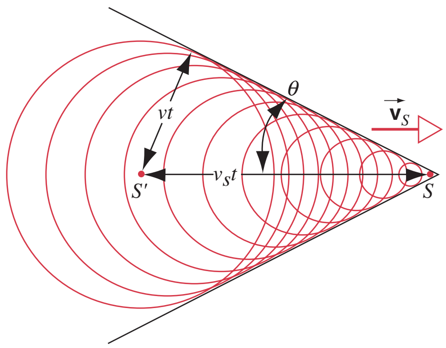
Note that in all four cases in this sample problem, the relative speed between you and the police car is the same—namely, 29 m/s—but the perceived frequencies are different in the four cases. The Doppler shift for sound is determined not only by the relative speed between source and observer, but also by both of their speeds relative to the medium that carries the sound.

Effects at High Speed (Optional)

When v_O or v_S becomes comparable in magnitude to v, the formulas just given for the Doppler effect usually must be modified. One modification is required because the linear relation between restoring force and displacement assumed up until now may no longer hold in the medium. The speed of wave propagation is then no longer the normal phase speed, and the wave shapes change in time. Components of the motion at right angles to the line joining source and observer also contribute to the Doppler effect at these high speeds. When v_O or v_S exceeds v, the Doppler formula does not apply; for example, if $v_S > v$, the source gets ahead of the wave in one direction; if $v_O > v$ and the observer moves away from the source, the wave never catches up with the observer.

There are many instances in which the source moves through a medium at a speed greater than the phase speed of the wave in that medium. In such cases the wavefront takes the shape of a cone with the moving body at its apex. Some examples are the bow wave from a speedboat on the water and the "shock wave" from an airplane or projectile moving through the air at a speed greater than the speed of sound in that medium (supersonic speeds). Another example is the so-called Cerenkov radiation, which consists of light waves emitted by charged particles that move through a medium with a speed greater than the phase speed of light in that medium. The blue glow of the water that often surrounds the core of a nuclear reactor is one type of Cerenkov radiation.

In Fig. 19-16*a* we show the present positions of the spherical waves that originated at various positions of a source during its motion. The radius of each sphere at this



(*a*)

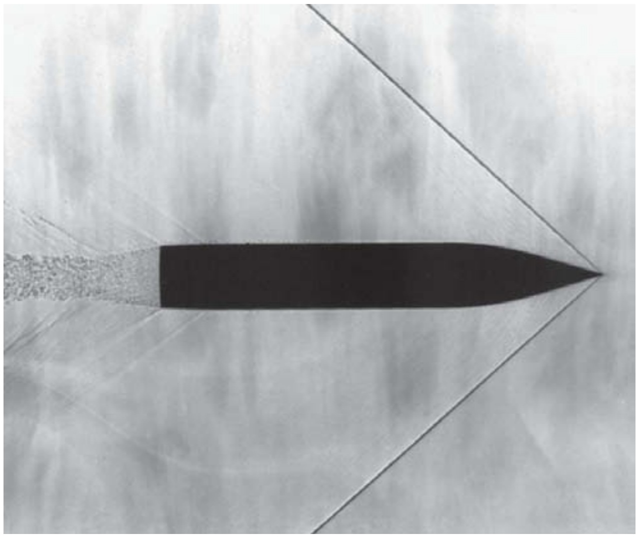

(*b*)

FIGURE 19-16. (*a*) Wavefronts of a source moving at supersonic speed. The wavefronts are spherical and their envelope is a cone. Compare this figure with Fig. 19-15. (*b*) A photograph of a projectile fired from a gun at Mach 2. Note the Mach cone.

time is the product of the wave speed v and the time t that has elapsed since the source was at its center. The envelope of these waves is a cone whose surface makes an angle θ with the direction of motion of the source. From the figure we obtain the result

$$\sin\theta = \frac{v}{v_S}. \qquad (19\text{-}45)$$

For surface water waves the cone reduces to a pair of intersecting lines. In aerodynamics the ratio v_S/v is called the *Mach number.* An aircraft flying at supersonic speed generates a *Mach cone* similar to that shown in Fig. 19-16. When the edge of that cone intercepts the ground below, we hear a "sonic boom," which (contrary to common belief) is *not* associated with an aircraft "breaking the sound barrier." The sonic boom is merely the total effect of the concentration on one surface of the aircraft's radiated sound energy, which would normally radiate in all directions at subsonic speeds. As the photograph of Fig. 19-16*b* shows, it might be possible to hear two sonic booms from the same aircraft, one from the leading edge and another from the trailing edge. (Note also that the Mach cone never intercepts the projectile itself; thus the aircraft's passengers do not hear the sonic boom.) ■

MULTIPLE CHOICE

19-1 Properties of Sound Waves

19-2 Traveling Sound Waves

1. Which is larger for a sound wave in a fluid, the relative density variations, $\Delta\rho_m/\rho_0$, or the relative pressure variations, $\Delta p_m/p_0$?
   (A) $\Delta\rho_m/\rho_0 > \Delta p_m/p_0$, always.
   (B) $\Delta\rho_m/\rho_0 = \Delta p_m/p_0$, always.
   (C) $\Delta\rho_m/\rho_0 < \Delta p_m/p_0$, always.
   (D) Which is larger varies, depending on the pressure and the bulk modulus.

19-3 The Speed of Sound

2. A thin steel rod of length 12 km is suspended in a frictionless tube. A researcher gently taps one end with a hammer. The researcher on the other end hears the tap.
   (A) at the same instant.
   (B) almost, but not quite, instantaneously.
   (C) approximately 2 seconds later.
   (D) approximately 30 seconds later.
   (E) approximately 1/2 minute later.

3. Which is larger, the velocity of a sound wave v or the amplitude of the velocity variations u_m of the oscillating sound particles?
   (A) v is always greater than u_m.
   (B) v and u_m are equal.
   (C) v must be less than u_m.
   (D) The two velocities are unrelated to each other.

19-4 Power and Intensity of Sound Waves

4. Spherical sound waves are emitted uniformly in all directions from a point source. The variation in sound level SL as a function of distance r from the source can be written as
   (A) $SL = -b \log r^a$
   (B) $SL = a - b(\log r)^2$
   (C) $SL = a - b \log r$
   (D) $SL = a - b/r^2$

   where a and b are *positive* constants.

5. If the average power of a sound wave is expressed in terms of displacement amplitudes s_m and frequencies f, then
   (A) $P_{av} \propto f^2 s_m^2$. (B) $P_{av} \propto f s_m^2$.
   (C) $P_{av} \propto f^{-1} s_m^2$. (D) $P_{av} \propto f^{-2} s_m^2$.

19-5 Interference of Sound Waves

6. One way to improve the performance of bass speakers on a sound system is to attach a curved tube to the back of the speaker that passes near the front so that the sound waves from the back of the speaker are allowed to interfere constructively with the sound waves from the front (Fig. 19-17). If the average wavelength of the sound coming from the speaker is λ, then the length of the tube to produce constructive interference should be
   (A) $\lambda/4$. (B) $\lambda/2$. (C) $3\lambda/4$. (D) λ.

FIGURE 19-17. Multiple-choice question 6.

19-6 Standing Longitudinal Waves

7. What is the pattern for resonant frequencies of a tube closed at both ends?
   (A) The same as a tube open at both ends, $f_n = nv/2L$, $n = 1, 2, 3, 4, 5, \ldots$
   (B) The same as a tube closed at one end, $f_n = nv/4L$, $n = 1, 3, 5, 7, 9, \ldots$
   (C) $f_n = nv/8L$, $n = 1, 5, 9, 13, 17, \ldots$
   (D) A tube closed at both ends does not have any resonant frequencies.

19-7 Vibrating Systems and Sources of Sound

8. The octave thumb key on a clarinet forces the resonance mode from the fundamental to the first overtone. Pressing the

key opens a small hole on the back of the clarinet. Where should this hole be located?

(A) Near a pressure node for a typical fundamental
(B) Near a pressure antinode for a typical fundamental
(C) Near a pressure node for a typical first overtone
(D) Near a pressure antinode for a typical first overtone

9. A violin string of length L is bowed so that its sound is a mixture of the fundamental and the first three overtones. How far from the end of the string should a small pick-up microphone be placed so that it transmits all of these tones?

(A) $L/2$ (B) $L/3$ (C) $L/4$ (D) $L/8$

19-8 Beats

10. You are provided with three similar, but slightly different, tuning forks. When A and B are both struck, a beat frequency of f_{AB} is heard. When A and C are both struck, a beat frequency of f_{AC} is heard. It was noticed that $f_{AB} < f_{AC}$.

(*a*) Which tuning fork has the highest frequency?

(A) A (B) B (C) C
(D) The answer cannot be determined from the information given.

(*b*) Which tuning fork has the middle frequency?

(A) A (B) B (C) C
(D) The answer cannot be determined from the information given.

(*c*) B and C are simultaneously struck. What will be the observed beat frequency?

(A) $|f_{AB} + f_{BC}|$ (B) $|f_{AB} - f_{BC}|$
(C) Either $|f_{AB} + f_{BC}|$ or $|f_{AB} - f_{BC}|$ will be heard.
(D) Both $|f_{AB} + f_{BC}|$ and $|f_{AB} - f_{BC}|$ will simultaneously be heard.

19-9 The Doppler Effect

11. A source of sound is moving toward an observer. The source passes an identical sound source, which is at rest. The observer can hear the sound produced by both sources.

(*a*) Before the moving source passes the stationary source, the observer hears

(A) a higher pitch from the moving source.
(B) a higher pitch from the stationary source.
(C) the same pitch from both sources.

(*b*) At the instant the moving source passes the stationary source, the observer hears

(A) a higher pitch from the moving source.
(B) a higher pitch from the stationary source.
(C) the same pitch from both sources.

(*c*) After the moving source passes the stationary source, the observer hears

(A) a higher pitch from the moving source.
(B) a higher pitch from the stationary source.
(C) the same pitch from both sources.

12. Three musicians experiment with the Doppler effect. Musician A rides in a car at a speed u directly away from musician B who is stationary. Musician C rides in a car directly toward B and travels at the same speed as A (Fig. 19-18). Musician A plays a note at frequency f_A on his trumpet. B hears the note, adjusts his trumpet, and plays *the same note he heard.* C hears the only note played by B.

(*a*) Assume that all three musicians are always in a straight line. Compared to the original note as played by musician A, the final note heard by C will be

(A) the same pitch. (B) higher in pitch.
(C) lower in pitch.

(*b*) Assume instead that A is moving due north away from B, while C is moving due west toward B. Compared to the original note as played by musician A, the final note heard by C will be

(A) the same pitch. (B) higher in pitch.
(C) lower in pitch.

FIGURE 19-18. Multiple-choice question 12.

QUESTIONS

1. Why will sound not travel through a vacuum?

2. List some sources of infrasonic waves and of ultrasonic waves.

3. Ultrasonic waves can be used to reveal internal structures of the body. They can, for example, distinguish between liquid and soft human tissues far better than can x rays. How? Why do we still use x rays?

4. What experimental evidence is there for assuming that the speed of sound in air is the same for all wavelengths?

5. Give a qualitative explanation why the speed of sound in lead is less than that in copper.

6. Transverse waves on a string can be plane polarized. Can sound waves be polarized?

7. Bells frequently sound less pleasant than pianos or violins. Why?

8. A bell is rung for a short time in a school. After a while its sound is inaudible. Trace the sound waves and the energy they transfer from the time of emission until they become inaudible.

9. The pitch of the wind instruments rises and that of the string instruments falls as an orchestra warms up. Explain why.

10. Explain how a stringed instrument is tuned.

11. Is resonance a desirable feature of every musical instrument? Give examples.

12. When you strike one prong of a tuning fork, the other prong also vibrates, even if the bottom end of the fork is clamped firmly in a vise. How can this happen? That is, how does the second prong “get the word” that somebody has struck the first prong?

13. How can a sound wave travel down an organ pipe and be reflected at its open end? It would seem that there is nothing there to reflect it.

14. How can we experimentally locate the positions of nodes and antinodes on a string, in an air column, and on a vibrating surface?

15. Explain how a note is produced when you blow across the top of a test tube. What would be the effect of blowing harder? Of raising the temperature of the air in the test tube?

16. How might you go about reducing the noise level in a machine shop?
17. Foghorns emit sounds of very low pitch. For what purpose?
18. Are longitudinal waves in air always audible as sound, regardless of frequency or intensity? What frequencies would give a person the greatest sensitivity, the greatest tolerance, and the greatest range?
19. What is the common purpose of the valves of a cornet and the slide of a trombone? The bugle has no valves. How then can we sound different notes on it? To what notes is the bugler limited? Why?
20. Explain how bowing a violin string gets it to vibrate.
21. What is the meaning of zero decibels? Could the reference intensity for audible sound be set so as to permit negative sound levels in decibels? If so, how?
22. Discuss the factors that determine the range of frequencies in your voice and the quality of your voice.
23. Explain the origin of the sound in ordinary whistling.
24. What physical properties of a sound wave correspond to the human sensations of pitch, loudness, and tone quality?
25. What is the difference between a violin note and the same note sung by a human voice that enables us to distinguish between them?
26. Does your singing really sound better in a shower? If so, what are the physical reasons?
27. Explain the audible sound produced by drawing a wet finger around the rim of a wine glass.
28. Would a plucked violin string oscillate for a longer or shorter time if the violin had no sounding board? Explain.
29. Is a bowed violin string an example of forced damped oscillations? How would the string sound if it were not damped?
30. A tube can act like an acoustic filter, discriminating against the passage through it of sounds of frequencies different from the natural frequencies of the tube. The muffler of an automobile is an example. (*a*) Explain how such a filter works. (*b*) How can we determine the cut-off frequency, below which sound is not transmitted?
31. Discuss factors that improve the acoustics in music halls.
32. What is the effect of using a megaphone or cupping your hands in front of your mouth to project your voice over a distance?
33. A lightning flash dissipates an enormous amount of energy and is essentially instantaneous. How is that energy transformed into the sound waves of thunder? (See "Thunder," by Arthur A. Few, *Scientific American,* July 1975, p. 80.)
34. Sound waves can be used to measure the speed at which blood flows in arteries and veins. Explain how.
35. Suppose that George blows a whistle and Gloria hears it. She will hear an increased frequency whether she is running toward George or George is running toward her. Are the increases in frequency the same in each case? Assume the same speeds of running.
36. Suppose that, in the Doppler effect for sound, the source and receiver are at rest in some reference frame but the transmitting medium (air) is moving with respect to this frame. Will there be a change in wavelength, or in frequency, received?
37. You are standing in the middle of the road and a bus is coming toward you at constant speed, with its horn sounding. Because of the Doppler effect is the pitch of the horn rising, falling, or constant?
38. How might the Doppler effect be used in an instrument to detect the fetal heart beat? (Such measurements are routinely made; see "Ultrasound in Medical Diagnosis," by Gilbert B. Devey and Peter N. T. Wells, *Scientific American,* May 1978, p. 98.)
39. Bats can examine the characteristics of objects—such as size, shape, distance, direction, and motion—by sensing the way the high-frequency sounds they emit are reflected off the objects back to the bat. Discuss qualitatively how each of these features affects the reflected sound waves. (See "Information Content of Bat Sonar Echoes," by J. A. Simmons, D. J. Howell, and N. Suga, *American Scientist,* March–April 1975, p. 204.)
40. Assume that you can detect an object by bouncing waves off it (such as in sonar or radar, for instance) as long as the object is larger than the wavelength of the waves. Then consider that bats and porpoises each can emit sound waves of frequency 100 kHz; however, bats can detect objects as small as insects but porpoises only small fish. Why the difference?
41. The natural C-trumpet is a brass instrument without valves, which can play only the notes C_4, G_4, C_5, E_5, G_5, $B_5^{\flat\flat}$, C_6, . . . Is this sequence characteristic of a pipe open on both ends, or one closed on one and open on the other? To which type does the trumpet really belong? Explain.
42. Is there a Doppler effect for sound when the observer or the source moves at right angles to the line joining them? How then can we determine the Doppler effect when the motion has a component at right angles to this line?
43. Two ships with steam whistles of the same pitch sound off in the harbor. Would you expect this to produce an interference pattern with regions of high and low intensity? If not, why not?

EXERCISES

Where needed in the problems, use speed of sound in air = 343 m/s and density of air = 1.21 kg/m^3 unless otherwise specified.

19-1 Properties of Sound Waves

19-2 Traveling Sound Waves

1. A continuous sinusoidal longitudinal wave is sent along a coiled spring from a vibrating source attached to it. The frequency of the source is 25 Hz, and the distance between successive rarefactions in the spring is 24 cm. (*a*) Find the wave speed. (*b*) If the maximum longitudinal displacement of a particle in the spring is 0.30 cm and the wave moves in the $-x$ direction, write the equation for the wave. Let the source be at $x = 0$ and the displacement $s = 0$ at the source when $t = 0$.
2. The pressure in a traveling sound wave is given by the equation

$$\Delta p = (1.48\ \text{Pa}) \sin [(1.07\pi\ \text{rad/m})x - (334\pi\ \text{rad/s})t].$$

Find (*a*) the pressure amplitude, (*b*) the frequency, (*c*) the wavelength, and (*d*) the speed of the wave.

19-3 The Speed of Sound

3. Diagnostic ultrasound of frequency 4.50 MHz is used to examine tumors in soft tissue. (*a*) What is the wavelength in air of such a sound wave? (*b*) If the speed of sound in tissue is 1500 m/s, what is the wavelength of this wave in tissue?

4. If the wavelength of sound is large, by a factor of about 10, relative to the mean free path of the molecules, then sound waves can propagate through a gas. For air at room temperature the mean free path is about 0.1 pm. Calculate the frequency above which sound waves could not propagate.

5. Figure 19-19 shows a remarkably detailed image of a transistor in a microelectronic circuit, formed by an acoustic microscope. The sound waves have a frequency of 4.2 GHz. The speed of such waves in the liquid helium in which the specimen is immersed is 240 m/s. (*a*) What is the wavelength of these ultrahigh-frequency acoustic waves? (*b*) The ribbon-like conductors in the figure are ≈ 2 pm wide. To how many wavelengths does this correspond?

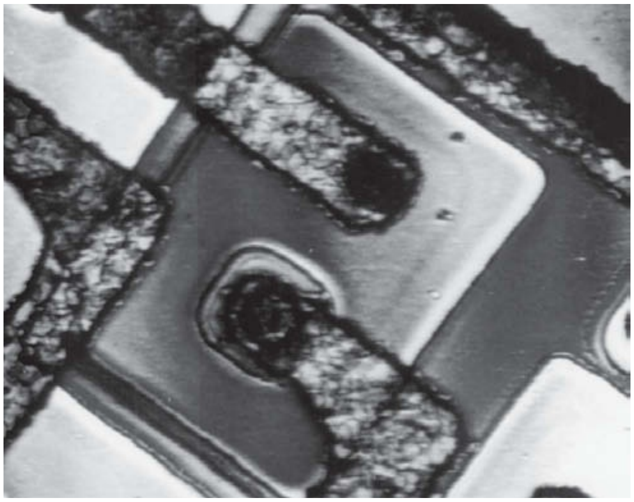


FIGURE 19-19. Exercise 5.

6. (*a*) A rule for finding your distance from a lightning flash is to count seconds from the time you see the flash until you hear the thunder and then divide the count by 5. The result is supposed to give the distance in miles. Explain this rule and determine the percent error in it at 0°C and 1 atm pressure. (*b*) Devise a similar rule for obtaining the distance in kilometers.

7. A column of soldiers, marching at 120 paces per minute, keeps in step with the music of a band at the head of the column. It is observed that the men at the rear of the column are striding forward with the left foot when those in the band are advancing with the right. What is the length of the column approximately?

8. You are at a large outdoor concert, seated 300 m from the stage microphone. The concert is also being broadcast live, in stereo, around the world via satellite. Consider a listener 5000 km away. Who hears the music first and by what time difference?

9. Earthquakes generate sound waves in the Earth. Unlike in a gas, there are both transverse (S) and longitudinal (P) sound waves in a solid. Typically, the speed of S waves is about 4.5 km/s and that of P waves 8.2 km/s. A seismograph records P and S waves from an earthquake. The first P waves arrive 3 min before the first S waves; see Fig. 19-20. How far away did the earthquake occur?

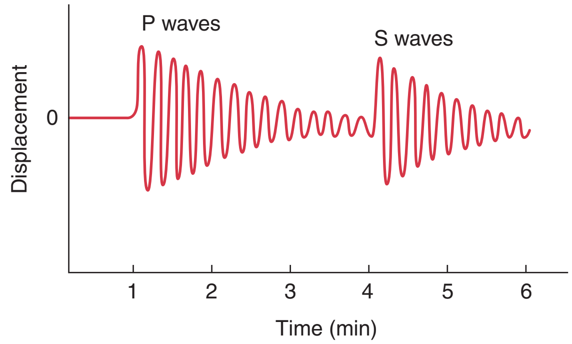


FIGURE 19-20. Exercise 9.

19-4 Power and Intensity of Sound Waves

10. Show that the sound wave intensity I can be written in terms of the frequency f and displacement amplitude s_m in the form

$$I = 2\pi^2 \rho v f^2 s_m^2.$$

11. A source emits spherical waves isotropically (that is, with equal intensity in all directions). The intensity of the wave 42.5 m from the source is 197 μW/m^2. Find the power output of the source.

12. A sound wave of frequency 313 Hz has an intensity of 1.13 μW/m^2. What is the amplitude of the air vibrations caused by this sound?

13. A sound wave of intensity 1.60 μW/cm^2 passes through a surface of area 4.70 cm^2. How much energy passes through the surface in 1 h?

14. Find the intensity ratio of two sounds whose sound levels differ by 1.00 dB.

15. A certain sound level is increased by an additional 30 dB. Show that (*a*) its intensity increases by a factor of 1000 and (*b*) its pressure amplitude increases by a factor of 32.

16. A salesperson claimed that a stereo system would deliver 110 W of audio power. Testing the system with several speakers set up so as to simulate a point source, the consumer noted that she could get as close as 1.3 m with the volume full on before the sound hurt her ears. Should she report the firm to the Consumer Protection Agency?

17. Find the energy density in a sound wave 4.82 km from a 5.20-kW emergency siren, assuming the waves to be spherical and the propagation isotropic with no atmospheric absorption.

18. You are standing at a distance D from an isotropic source of sound waves. You walk 51.4 m toward the source and observe that the intensity of these waves has doubled. Calculate the distance D.

19. Estimate the maximum possible sound level in decibels of sound waves in air. (Hint: Set the pressure amplitude equal to 1 atm.)

20. Suppose that the average sound level of human speech is 65 dB. How many persons in a room speaking at the same time each at 65 dB are needed to produce a sound level of 80 dB?

21. Suppose that a rustling leaf generates 8.4 dB of sound. Find the sound level from a tree with 2.71×10^5 rustling leaves.

22. In a test, a subsonic jet flies overhead at an altitude of 115 m. The sound level on the ground as the jet passes overhead is 150 dB. At what altitude should the plane fly so that the ground noise is no greater than 120 dB, the threshold of pain? Ignore the finite time required for the sound to reach the ground.

19-5 Interference of Sound Waves

23. A sound wave of 42.0-cm wavelength enters the tube shown in Fig. 19-21. What must be the smallest radius r such that a minimum will be heard at the detector?

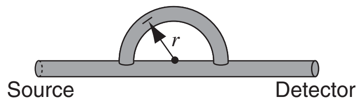


FIGURE 19-21. Exercise 23.

24. Two stereo loudspeakers are separated by a distance of 2.12 m. Assume that the amplitude of the sound from each speaker is approximately the same at the position of a listener, who is 3.75 m directly in front of one of the speakers; see Fig. 19-22. (*a*) For what frequencies in the audible range (20–20,000 Hz) will there be a minimum signal? (*b*) For what frequencies is the sound a maximum?

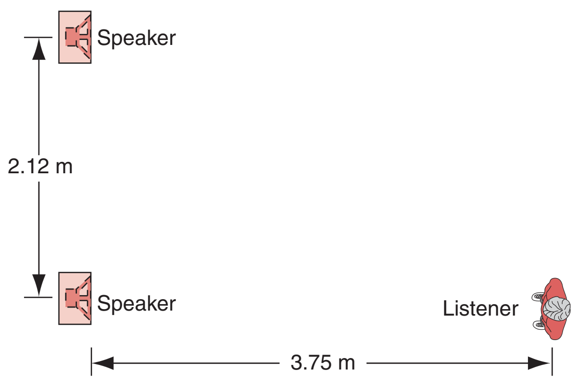


FIGURE 19-22. Exercise 24.

25. A spherical sound source is placed at P_1 near a reflecting wall AB and a microphone is located at point P_2, as shown in Fig. 19-23. The frequency of the sound source is variable. Find the two lowest frequencies for which the sound intensity, as observed at P_2, will be a maximum. There is no phase change on reflection; the angle of incidence equals the angle of reflection.

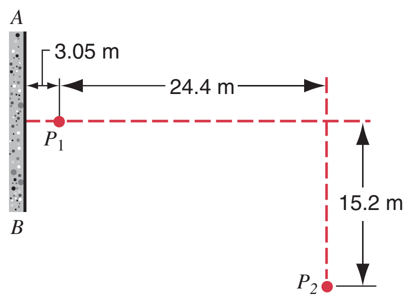


FIGURE 19-23. Exercise 25.

26. Two sources of sound are separated by a distance of 5.00 m. They both emit sound at the same amplitude and frequency, 300 Hz, but they are 180° out of phase. At what points along the line connecting them will the sound intensity be the largest?

19-6 Standing Longitudinal Waves

27. The strings of a cello have a length L. (*a*) By what length ΔL must they be shortened by fingering to change to the pitch by a frequency ratio r? (*b*) Find ΔL, if $L = 80.0$ cm and $r = \frac{6}{5}, \frac{5}{4}, \frac{4}{3}$, and $\frac{3}{2}$.

28. A sound wave in a fluid medium is reflected at a barrier so that a standing wave is formed. The distance between nodes is 3.84 cm and the speed of propagation is 1520 m/s. Find the frequency.

29. A well with vertical sides and water at the bottom resonates at 7.20 Hz and at no lower frequency. The air in the well has a density of 1.21 kg/m^3 and a bulk modulus of 1.41×10^5 Pa. How deep is the well?

30. S in Fig. 19-24 is a small loudspeaker driven by an audio oscillator and amplifier, adjustable in frequency from 1000 to 2000 Hz only. D is a piece of cylindrical sheetmetal pipe 45.7 cm long and open at both ends. (*a*) At what frequencies will resonance occur when the frequency emitted by the speaker is varied from 1000 to 2000 Hz? (*b*) Sketch the displacement nodes for each resonance. Neglect end effects.

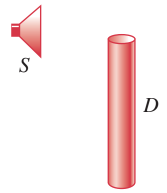


FIGURE 19-24. Exercise 30.

31. The width of the terraces in an amphitheater in Los Angeles, Fig. 19-25, is 36 in. (= 0.914 m). A single hand-clap occurring at the center of the stage will reflect back to the stage as a tone of what frequency?

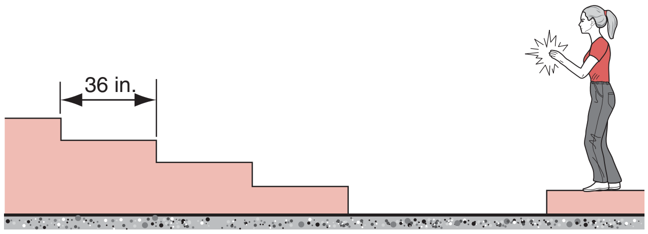


FIGURE 19-25. Exercise 31.

32. A tunnel leading straight through a hill greatly amplifies tones at 135 and 138 Hz. Find the shortest length the tunnel could have.

19-7 Vibrating Systems and Sources of Sound

33. (*a*) Find the speed of waves on an 820-mg violin string 22.0 cm long if the frequency of the fundamental is 920 Hz. (*b*) Calculate the tension in the string.

34. If a violin string is tuned to a certain note, by what factor must the tension in the string be increased if it is to emit a note of double the original frequency (that is, a note one octave higher in pitch)?

35. A certain violin string is 30 cm long between its fixed ends and has a mass of 2.0 g. The string sounds an A note (440 Hz) when played without fingering. Where must one put one's finger to play a C (528 Hz)?

36. An open organ pipe has a fundamental frequency of 291 Hz. The first overtone ($n = 3$) of a closed organ pipe has the same frequency as the second harmonic of the open pipe. How long is each pipe?

19-8 Beats

37. A tuning fork of unknown frequency makes three beats per second with a standard fork of frequency 384 Hz. The beat frequency decreases when a small piece of wax is put on a prong of the first fork. What is the frequency of this fork?

38. The A string of a violin is a little too taut. Four beats per second are heard when it is sounded together with a tuning fork that is vibrating accurately at the pitch of concert A (440 Hz). What is the period of the violin string vibration?

39. You are given four tuning forks. The fork with the lowest frequency vibrates at 500 Hz. By using two tuning forks at a time, the following beat frequencies are heard: 1, 2, 3, 5, 7, and 8 Hz. What are the possible frequencies of the other three tuning forks?

19-9 The Doppler Effect

40. A source S generates circular waves on the surface of a lake, the pattern of wave crests being shown in Fig. 19-26. The speed of the waves is 5.5 m/s and the crest-to-crest separation is 2.3 m. You are in a small boat heading directly toward S at a constant speed of 3.3 m/s with respect to the shore. What frequency of the waves do you observe?

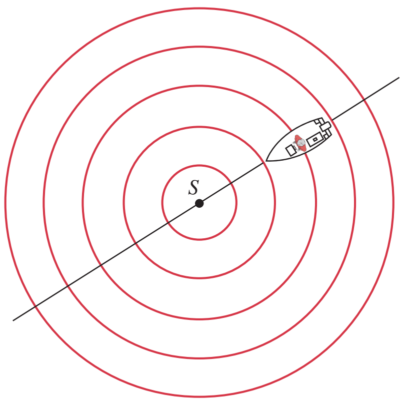


FIGURE 19-26. Exercise 40.

41. The 15.8-kHz whine of the turbines in the jet engines of an aircraft moving with speed 193 m/s is heard at what frequency by the pilot of a second craft trying to overtake the first at a speed of 246 m/s?

42. An ambulance emitting a whine at 1602 Hz overtakes and passes a cyclist pedaling a bike at 2.63 m/s. After being passed, the cyclist hears a frequency of 1590 Hz. How fast is the ambulance moving?

43. A whistle of frequency 538 Hz moves in a circle of radius 71.2 cm at an angular speed of 14.7 rad/s. What are (*a*) the lowest and (*b*) the highest frequencies heard by a listener a long distance away at rest with respect to the center of the circle?

44. In 1845, Buys Ballot first tested the Doppler effect for sound. He put a trumpet player on a flatcar drawn by a locomotive and another player near the tracks. If each player blows a 440-Hz note, and if there are 4.0 beat/s as they approach each other, what is the speed of the flatcar?

45. Estimate the speed of the projectile illustrated in the photograph in Fig. 19-16*b*. Assume the speed of sound in the medium through which the projectile is traveling to be 380 m/s.

46. A sonar device sends 148-kHz sound waves from a hiding police car toward a truck approaching at a speed of 44.7 m/s. Calculate the frequency of the reflected waves detected at the police car.

47. An acoustic burglar alarm consists of a source emitting waves of frequency 28.3 kHz. What will be the beat frequency of waves reflected from an intruder walking at 0.95 m/s directly away from the alarm?

48. A siren emitting a sound of frequency 1000 Hz moves away from you toward a cliff at a speed of 10.0 m/s. (*a*) What is the frequency of the sound you hear coming directly from the siren? (*b*) What is the frequency of the sound you hear reflected off the cliff? (*c*) Find the beat frequency. Could you hear the beats? Take the speed of sound in air as 330 m/s.

49. A person in a car blows a trumpet sounding at 438 Hz. The car is moving toward a wall at 19.3 m/s. Calculate (*a*) the frequency of the sound as received at the wall and (*b*) the frequency of the reflected sound arriving back at the source.

50. In a discussion of Doppler shifts of ultrasonic (high-frequency) waves used in medical diagnosis, the authors remark: "For every millimeter per second that a structure in the body moves, the frequency of the incident ultrasonic wave is shifted approximately 1.3 Hz/MHz." What speed of the ultrasonic waves in tissue do you deduce from this statement?

51. A bat is flitting about in a cave, navigating very effectively by the use of ultrasonic bleeps (short emissions of high-frequency sound lasting a millisecond or less and repeated several times a second). Assume that the sound emission frequency of the bat is 39.2 kHz. During one fast swoop directly toward a flat wall surface, the bat is moving at 8.58 m/s. Calculate the frequency of the sound the bat hears reflected off the wall.

PROBLEMS

1. The speed of sound in a certain metal is v. One end of a long pipe of that metal of length L is struck a hard blow. A listener at the other end hears two sounds, one from the wave that has traveled along the pipe and the other from the wave that has traveled through the air. (*a*) If v_{air} is the speed of sound in air, what time interval t elapses between the arrival of the two sounds? (*b*) A hammer strikes a long aluminum rod at one end. A listener, whose ear is close to the other end of the rod, hears the sound of the blow twice, with a 120-ms interval between. How long is the rod?

2. A stone is dropped into a well. The sound of the splash is heard 3.00 s later. What is the depth of the well?

3. A certain loudspeaker produces a sound with a frequency of 2.09 kHz and an intensity of 962 μW/m^2 at a distance of 6.11 m. Presume that there are no reflections and that the loudspeaker emits the same in all directions. (*a*) Find the intensity at 28.5 m. (*b*) Find the displacement amplitude at 6.11 m. (*c*) Calculate the pressure amplitude at 6.11 m.

4. (*a*) If two sound waves, one in air and one in water, are equal in intensity, what is the ratio of the pressure amplitude of the wave in water to that of the wave in air? (*b*) If the pressure amplitudes are equal instead, what is the ratio of the intensities of the waves? Assume that the water is at 20°C.

5. A line source (for instance, a long freight train on a straight track) emits a cylindrical expanding wave. Assuming that the air absorbs no energy, find how (*a*) the intensity and (*b*) the amplitude of the wave depend on the distance from the source. Ignore reflections and consider points near the center of the train.

6. In Fig. 19-27 we show an acoustic interferometer, used to demonstrate the interference of sound waves. *S* is a source of sound (a loudspeaker, for instance), and *D* is a sound detector, such as the ear or a microphone. Path *SBD* can be varied in length, but path *SAD* is fixed. The interferometer contains air, and it is found that the sound intensity has a minimum value of 10 μW/cm^2 at one position of *B* and continuously climbs to a maximum value of 90 μW/cm^2 at a second position 1.65 cm from the first. Find (*a*) the frequency of the sound emitted from the source and (*b*) the relative amplitudes of the waves arriving at the detector for each of the two positions of *B*. (*c*) How can it happen that these waves have different amplitudes, considering that they originate at the same source?

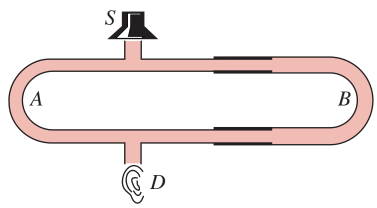


FIGURE 19-27. Problem 6.

7. A certain loudspeaker (assumed to be a point source) emits 31.6 W of acoustic power. A small microphone of effective cross-sectional area 75.2 mm^2 is located 194 m from the loudspeaker. Calculate (*a*) the sound intensity at the microphone, (*b*) the power incident on the microphone, and (*c*) the amount of energy that impinges on the microphone in 25.0 min.

8. The *reverberation time* of an auditorium or concert hall is the time required for the sound intensity (in W/m^2) to decrease by a factor of 10^6. The reverberation time depends on the frequency of the sound. Suppose that in a particular concert hall, the reverberation time for a note of a certain frequency is 2.6 s. If the note is sounded at a sound level of 87 dB, how long will it take for the sound level to fall to 0 dB (the threshold of human hearing)?

9. A large parabolic reflector having a circular opening of radius 0.50 m is used to focus sound. If the energy is delivered from the focus to the ear of a listening detective through a tube of diameter 1.0 cm with 12% efficiency, how far away can a whispered conversation be understood? (Assume that the sound level of a whisper is 20 dB at 1.0 m from the source, considered to be a point, and that the threshold for hearing is 0 dB.)

10. The period of a pulsating variable star may be estimated by considering the star to be executing radial longitudinal pulsations in the fundamental standing wave mode; that is, the radius varies periodically with the time, with a displacement antinode at the surface. (*a*) Would you expect the center of the star to be a displacement node or antinode? (*b*) By analogy with the open organ pipe, show that the period of pulsation *T* is given by

$$T = \frac{4R}{v_s},$$

where *R* is the equilibrium radius of the star and v_s is the average sound speed. (*c*) Typical white dwarf stars are composed of material with a bulk modulus of 1.33×10^{22} Pa and a density of 1.0×10^{10} kg/m^3. They have radii equal to 0.009 solar radius. What is the approximate pulsation period of a white dwarf? (See “Pulsating Stars,” by John R. Percy, *Scientific American,* June 1975, p. 66.)

11. In Fig. 19-28, a rod *R* is clamped at its center; a disk *D* at its end projects into a glass tube that has cork filings spread over its interior. A plunger *P* is provided at the other end of the tube. The rod is set into longitudinal vibration and the plunger is moved until the filings form a pattern of nodes and antinodes (the filings form well-defined ridges at the pressure antinodes). If we know the frequency *f* of the longitudinal vibrations in the rod, a measurement of the average distance *d* between successive antinodes determines the speed of sound *v* in the gas in the tube. Show that

$$v = 2fd.$$

This is Kundt’s method for determining the speed of sound in various gases.

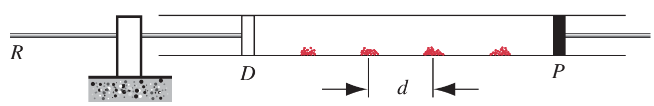


FIGURE 19-28. Problem 11.

12. A tube 1.18 m long is closed at one end. A stretched wire is placed near the open end. The wire is 33.2 cm long and has a mass of 9.57 g. It is fixed at both ends and vibrates in its fundamental mode. It sets the air column in the tube into vibration at its fundamental frequency by resonance. Find (*a*) the frequency of oscillation of the air column and (*b*) the tension in the wire.

13. A 30.0-cm violin string with linear mass density 0.652 g/m is placed near a loudspeaker that is fed by an audio oscillator of variable frequency. It is found that the string is set into oscillation only at the frequencies 880 and 1320 Hz as the frquency of the oscillator is varied continuously over the range 500–1500 Hz. What is the tension in the string?

14. You are given five tuning forks, each of which has a different frequency. By trying every pair of tuning forks, (*a*) what maximum number of different beat frequencies might be obtained? (*b*) What minimum number of different beat frequencies might be obtained?

15. The speed of light in water is 2.25×10^8 m/s (about three-fourths the speed in a vacuum). A beam of high-speed electrons from a betatron emits Cerenkov radiation in water, the wavefront being a cone of angle 58.0°. Find the speed of the electrons in the water.

16. Two identical tuning forks oscillate at 442 Hz. A person is located somewhere on the line between them. Calculate the beat frequency as measured by this individual if (*a*) she is standing still and the tuning forks both move to the right at 31.3 m/s, and (*b*) the tuning forks are stationary and the listener moves to the right at 31.3 m/s.

17. A plane flies at 396 m/s at constant altitude. The sonic boom reaches an observer on the ground 12.0 s after the plane flies overhead. Find the altitude of the plane. Assume the speed of sound to be 330 m/s.

18. Figure 19-29 shows a transmitter and receiver of waves contained in a single instrument. It is used to measure the speed V of a target object (idealized as a flat plate) that is moving directly toward the unit, by analyzing the waves reflected from it. (*a*) Apply the Doppler equations twice, first with the target as observer, and then with the target as a source, and show that the frequency f_r of the reflected waves at the receiver is related to their source frequency f_s by

$$f_r = f_s\left(\frac{v + V}{v - V}\right),$$

where v is the speed of the waves. (*b*) In a great many practical situations, $V \ll v$. In this case, show that the equation above becomes

$$\frac{f_r - f_s}{f_s} \approx \frac{2V}{v}.$$

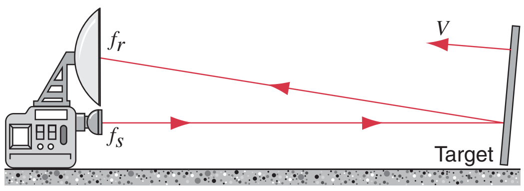


FIGURE 19-29. Problem 18.

19. Two submarines are on a head-on collision course during maneuvers in the North Atlantic. The first sub is moving at 20.2 km/h and the second sub at 94.6 km/h. The first submarine sends out a sonar signal (sound wave in water) at 1030 Hz. Sonar waves travel at 5470 km/h. (*a*) The second sub picks up the signal. What frequency does the second sonar detector hear? (*b*) The first sub picks up the reflected signal. What frequency does the first sonar detector hear? See Fig. 19-30. The ocean is calm; assume no currents.

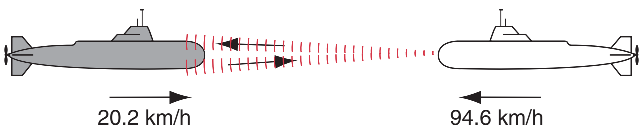


FIGURE 19-30. Problem 19.

20. A submarine moving north with a speed of 75.2 km/h with respect to the ocean floor emits a sonar signal (sound waves in water used in ways similar to radar; see Table 19-1) of frequency 989 Hz. If the ocean at that point has a current moving north at 30.5 km/h relative to the land, what frequency is observed by a ship drifting with the current north of the submarine? (Hint: All speeds in the Doppler equations must be taken with respect to the medium.)

21. A 2000-Hz siren and a civil defense official are both at rest with respect to the Earth. What frequency does the official hear if the wind is blowing at 12 m/s (*a*) from source to observer and (*b*) from observer to source?

22. Two trains on parallel tracks are traveling toward each other at 34.2 m/s relative to the ground. One train is blowing a whistle at 525 Hz. (*a*) What frequency will be heard on the other train in still air? (*b*) What frequency will be heard on the other train if the wind is blowing at 15.3 m/s parallel to the tracks and toward the whistle? (*c*) What frequency will be heard if the wind direction reverses?

COMPUTER PROBLEMS

1. Write a computer program for a Doppler sonar. The program should request the speed of sound, the frequency of the output pulse (or "ping"), the frequency of the reflected pulse, and the time delay between the output ping and the return ping. The program should then inform the user of the probable distance to the target and the target's possible speed(s) toward or away from the source. Try the program with the following data: the speed of sound is 340 m/s; the output pulse frequency is 20 kHz; the frequency of the reflected pulse is 20.612 kHz; and the time delay between the output and reflected pings is 0.230 s.

2. Generalize the previous program so that the data from two consecutive pings can be used to determine both the distance to the target and the velocity of the target. The program will also need to request the rate at which outgoing pings are sent. Assume that the outgoing pings are omnidirectional, but the direction of incoming pings can be resolved. Try the program with the following data: the speed of sound is 340 m/s; the output pulse frequency is 20 kHz and the pulses are sent once per second; a 20.921 kHz reflected pulse coming from 40° E of N is received 0.288 s after the first pulse is sent; and a second 20.921 kHz reflected pulse coming from 36.5° E of N is received 0.311 s after the second pulse is sent.

CHAPTER 20

THE SPECIAL THEORY OF RELATIVITY*

The special theory of relativity has an undeserved reputation as a difficult subject. It is not mathematically complicated; most of its details can be understood using techniques well known to readers of this text. Perhaps the most challenging aspect of special relativity is its insistence that we replace some of our ideas about space and time, which we have acquired through years of "common-sense" experiences, with new ideas.

The essential ideas of special relativity were formally presented in a paper written by Albert Einstein and published in 1905.† In this chapter we present the basic postulates of Einstein's theory and their consequences, introduce the mathematical techniques that allow measurements made in one frame of reference to be transformed into another, and study some of the consequences for both kinematics and dynamics.

20-1 TROUBLES WITH CLASSICAL PHYSICS

The kinematics developed by Galileo and the mechanics developed by Newton, which form the basis of what we call *classical physics,* had many triumphs. Particularly noteworthy are the understanding of the motion of the planets and the use of kinetic theory to explain certain observed properties of gases. However, a number of experimental phenomena cannot be understood with these otherwise successful classical theories. Let us consider a few of these difficulties. We will consider examples of experiments specifically designed to reveal the limitations of classical physics and—as we shall see—to serve as tests of Einstein's special theory of relativity.

Troubles with Our Ideas about Time

The pion (π^+ or π^-) is a particle that can be created in a high-energy particle accelerator. It is a very unstable particle; pions created at rest are observed to decay (to other particles) with an average lifetime of only 26.0 ns (26.0×10^{-9} s). In one particular experiment, pions were created in motion at a speed of $v = 0.913c$ (where c is the speed of light, 3.00×10^8 m/s). In this case they were observed to travel in the laboratory an average distance of $D = 17.4$ m before decaying, from which we conclude that they decay in a time given by $D/v = 63.7$ ns, much larger than the lifetime measured for pions at rest (26.0 ns). This effect, called *time dilation,* suggests that something about the relative motion between the pion and the laboratory has stretched the measured time interval by a factor of about 2.5. Such an effect cannot be explained by Newtonian physics, in which time is a universal coordinate having identical values for all observers.

* Some instructors may wish to delay covering relativity until after the treatment of electromagnetic waves in Chapter 38. Relativistic effects in wave motion are discussed in Chapter 39. An abbreviated coverage of this chapter can be done by postponing Sections 20-4 through 20-7.

† In that year he also published his papers on Brownian motion and on the photoelectric effect. It was for this latter paper (and not specifically for his theory of relativity) that he was awarded the 1921 Nobel prize in physics. Einstein also proposed a *general* theory of relativity in 1917. The general theory deals with the effect of gravity on space and time, some consequences of which were discussed in Section 14-9. In this chapter we consider only the *special* theory, in which gravity plays no role.

Troubles with Our Ideas about Length

Suppose an observer in the above laboratory placed one marker at the location of the pion's formation and another at the location of its decay. The distance between the markers is measured to be 17.4 m. Now consider the situation according to a different observer who is traveling along with the pion at a speed of $u = 0.913c$. This observer, to whom the pion appears to be at rest, measures its lifetime to be 26.0 ns, characteristic of pions at rest. To this observer, the distance between the markers showing the formation and decay of the pion is $(0.913c)(26.0 \times 10^{-9}\text{ s}) = 7.1$ m. Thus two observers who are in relative motion measure different values for the same length interval. This is likewise inconsistent with Newtonian physics, in which spatial coordinates are absolute and give identical readings for all observers.

Troubles with Our Ideas about Speed

Figure 20-1 shows a game between *A* and *B*, as seen by an observer *O*. All three observers are at rest in this reference frame. *A* throws a ball at superluminal (faster than light) speed toward *B*, who catches it. The light signal carrying the view of *A* throwing the ball travels to observer *O*, as does the light signal carrying the view of *B* catching the ball. Both light signals travel at speed *c*, which is less than the speed of the ball thrown by *A*. At the location of observer *O*, as shown in Fig. 20-1, the light signal from *B* arrives before the light signal from *A*. Therefore, according to *O*, *B* catches the ball before *A* throws it! Newtonian physics permits us to accelerate projectiles to unlimited speeds and therefore allows such apparent violations of cause and effect to be observed.

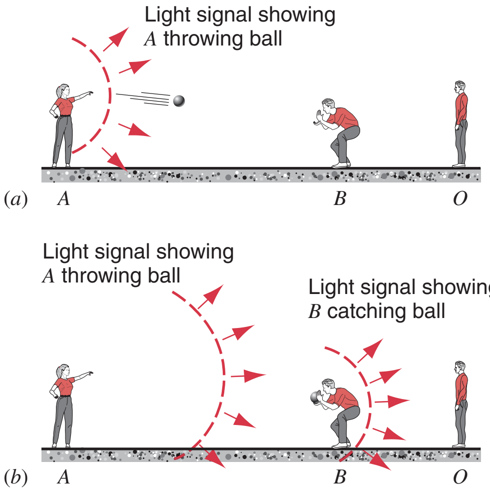


FIGURE 20-1. (*a*) *A* throws a ball to *B*. The ball moves faster than light and so is ahead of the light signal that shows *A* throwing the ball. (*b*) The light signal showing *B* catching the ball will reach the observer *O* before the light signal showing *A* throwing the ball. Such logical inconsistencies argue against the possiblity of accelerating particles to speeds faster than light.

Troubles with Our Ideas about Energy

The positron (e^+) is the *antiparticle* of the electron (e^-). The particles have the same mass but opposite electric charges, the electron having a negative charge and the positron a positive charge. Positrons are emitted in a common type of radioactive decay process. When these positrons encounter electrons in ordinary matter, we observe the process known as *electron–positron annihilation,* in which both particles disappear and in their place we find only electromagnetic radiation (gamma rays, which are very similar in character to ordinary light but of much shorter wavelength). Symbolically, we can represent this process as

$$e^+ + e^- \rightarrow \text{radiation.}$$

Figure 20-2*a* shows a system consisting of an electron and a positron, which are initially moving toward one another at very low speed. The particles are in a container, which is completely within our system. In Fig. 20-2*b*, the electron and positron have annihilated one another, and in their place we find radiation. In Fig. 20-2*c*, the radiation has been absorbed by the walls of the container, in the process increasing its internal energy.

Can we apply the law of conservation of energy, as we discussed in Chapter 13, to this system? Clearly no energy enters or leaves the system boundary; no external work is done on this system, and there is no heat transfer to or from the system. In this case, we expect that the total energy inside the system boundary should remain constant. Yet there is energy in the radiation of Fig. 20-2*b* and in the internal energy of the walls of the container in Fig. 20-2*c* that has no obvious counterpart in Fig. 20-2*a*. It appears that the law of conservation of energy is violated in this process. If we wish to preserve conservation of energy as a fundamental law of physics, we need to account for this apparent violation of the law.

Troubles with Our Ideas about Light

Einstein proposed his special theory of relativity in 1905, based on a thought experiment that he had devised. As a 16-year-old student, Einstein had learned the theory of electromagnetism and had thought about a paradox: If you were to move at the speed of light parallel to a light beam traveling in empty space, you would observe "static" electric and magnetic field patterns. (In a similar way, we showed in Fig. 18-8 a "static" disturbance on a string, which would be seen by an observer moving along the string at the same speed as waves on the string.) However, Einstein knew that such static electric and magnetic field patterns in empty space violated the theory of electromagnetism.

Einstein was faced with two choices to resolve this paradox: either electromagnetic theory was wrong or else the classical kinematics that permits an observer to travel along with a light beam was wrong. With the intuition that was perhaps his greatest attribute, Einstein put his faith in electromagnetic theory and sought an alternative to the

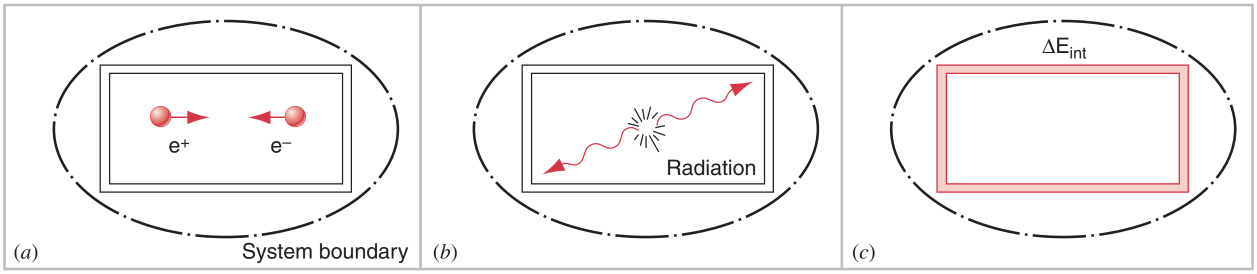


Figure 20-2. (*a*) An electron and a positron slowly approach one another inside a container in our system. (*b*) After annihilation, radiation appears. (*c*) The radiant energy is absorbed by the walls of the container, increasing its internal energy by an amount ΔE_{int}.

kinematics of Galileo and Newton. Later in this chapter we show how this new kinematics, which forms the basis of special relativity, prevents any observer from catching a light beam. We also show how it solves the other problems with time, length, speed, and energy discussed previously.

The critical test of any theory is of course how well it agrees with experiment. Einstein's special theory of relativity has been subjected to exhaustive tests over the past 95 years and has passed every one. Where classical physics and relativity theory predict different results, experiment has always been found to agree with relativity theory.

20-2 THE POSTULATES OF SPECIAL RELATIVITY

A scientific theory usually begins with general statements called *postulates,* which attempt to provide a basis for the theory. From these postulates we can obtain a set of mathematical laws in the form of equations that relate physical variables. Finally, we test the predictions of the equations in the laboratory. The theory stands until contradicted by experiment, after which the postulates may be modified or replaced, and the cycle is repeated.

For about two centuries, the mechanics of Galileo and Newton withstood all experimental tests. In this case the postulates concern the absolute nature of space and time. Based on his thought experiment about catching a light beam, Einstein realized the need to replace the Galilean laws of relative motion. In his 1905 paper, entitled "On the Electrodynamics of Moving Bodies," Einstein offered two postulates that form the basis of his special theory of relativity. We can rephrase his postulates as follows:

> The principle of relativity: *The laws of physics are the same in all inertial reference frames.*

> The principle of the constancy of the speed of light: *The speed of light in free space has the same value c in all inertial reference frames.*

The first postulate declares that the laws of physics are absolute, universal, and the same for all inertial observers. Laws that hold for one inertial observer cannot be violated for *any* inertial observer.

The second postulate is much more difficult to accept, because it violates our "common sense," which is firmly grounded in the Galilean kinematics that we have learned from everyday experiences. Consider three observers A, B, and C, each of whom is at rest in a different inertial reference frame. A flash of light is emitted by observer A, who observes the light to travel at speed c. The frame of observer B is moving away from A at a speed of $c/4$; Galilean kinematics predicts that B measures the value $c - c/4 = 3c/4$ for the speed of the light emitted by A. Observer C is in a frame that is moving *toward* A with speed $c/4$; according to Galileo, observer C measures a speed of $c + c/4 = 5c/4$ for the speed of the light emitted by A. Einstein's second postulate, on the other hand, asserts that *all three observers measure the same speed c for the light pulse!*

This is of course not the way ordinary objects behave. A projectile fired from a moving car has a velocity relative to the ground determined by the vector sum of the velocity of the projectile relative to the car and the velocity of the car relative to the ground. However, the velocities of light waves and particles moving at speeds close to c do not behave in this way. We discuss the relativistic law for velocity addition in Section 20-6 and show that it reduces to the "common-sense" Galilean law at low speeds.

Einstein put forth these postulates at a time when experimental tests were difficult or impossible. During the following decades, the development of high-energy particle accelerators made possible the study of the motions of particles at speeds close to c. In 1964, for example, an experiment was performed at CERN, the European high-energy particle physics laboratory near Geneva, Switzerland. The proton accelerator at CERN was used to produce a beam of particles called neutral pions (π^0), which decay rapidly (with an average lifetime of about 10^{-16} s) to two gamma rays:

$$\pi^0 \rightarrow \gamma + \gamma.$$

Gamma rays are electromagnetic radiations that travel at the speed of light. The experimenters measured directly the speed of the gamma rays emitted by the decaying pions, which were moving at a speed of $0.99975c$. According to Galileo, gamma rays emitted in the direction of motion of the pions should have a speed of $c + 0.99975c = 1.99975c$ in the laboratory frame of reference. According to Einstein,

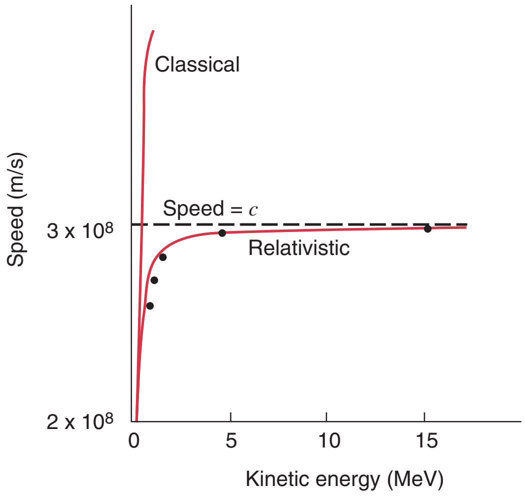


FIGURE 20-3. The points represent measurements of the speed of electrons accelerated through a large voltage difference to a known kinetic energy. The measurements show that, no matter how great the kinetic energy, the speed of the electrons does not exceed c. (See "Speed and Kinetic Energy of Relativistic Electrons," by William Bertozzi, *American Journal of Physics,* May 1964, p. 551.)

they should have a speed of c. The measured speed was 2.9977×10^8 m/s, equal to c to within 1 part in 10^4, thus providing direct verification of the second postulate.

The two postulates taken together have another consequence: they imply that *it is impossible to accelerate a particle to a speed greater than c, no matter how much kinetic energy we give it.* This is also a prediction that can be tested in the laboratory, and one that brings out another difference between the postulates of relativity and those of classical physics. Classical physics places no upper limit on the speed that an object may attain; relativity does impose such a limiting speed, which, by the first postulate, must be the same for all frames of reference.

In another experiment done in 1964, electrons were accelerated by a large voltage difference (up to 15 million volts), and the speed of the electrons was directly determined. Figure 20-3 shows the measured speeds at a function of the kinetic energy acquired by the electrons. No matter how much the accelerating voltage is increased, the speed never quite reaches or exceeds c. Once again, experiments at high speeds are inconsistent with predictions based on the kinematics of Galileo and Newton but instead confirm the postulates of special relativity.

20-3 CONSEQUENCES OF EINSTEIN'S POSTULATES

In Section 20-1 we discussed difficulties in interpreting certain measurements of time, length, and velocity based on classical physics. Let us see how Einstein's postulates can resolve those difficulties.

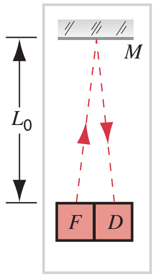


FIGURE 20-4. The clock ticks at intervals Δt_0 determined by the time necessary for a light flash to travel the distance $2L_0$ from the flashing bulb F to the mirror M and back to the detector D. (The lateral distance between F and D is assumed to be negligible in comparison with L_0.)

The Relativity of Time

We consider two observers: S is at rest on the ground, and S' is in a train moving on a long, straight track at constant speed u relative to S. The observers carry identical timing devices, illustrated in Fig. 20-4, consisting of a flashing lightbulb F attached to a detector D and separated by a distance L_0 from a mirror M. The bulb emits a flash of light that travels to the mirror. When the reflected light returns to D, the clock ticks and another flash is triggered. The time interval Δt_0 between ticks is simply the distance $2L_0$ traveled by the light divided by the speed of light c:

$$\Delta t_0 = 2L_0/c. \tag{20-1}$$

The interval Δt_0 is observed by either S or S' when the clock is at rest with respect to that observer.

We now consider the situation when one observer looks at a clock carried by the other. Figure 20-5 shows a representation of the sequence of events that S observes* on the clock carried by S' on the moving train. According to S, the flash is emitted at A, reflected at B, and detected at C. In this interval Δt, according to S the clock moves forward a horizontal distance of $u\,\Delta t$ from the location where the flash was emitted.

According to S, the light beam travels a distance $2L$, where $L = \sqrt{L_0^2 + (u\,\Delta t/2)^2}$, as shown in Fig. 20-5. The time interval measured by S for the light to travel this distance at a speed c (the same speed measured by S'!) is

$$\Delta t = \frac{2L}{c} = \frac{2\sqrt{L_0^2 + (u\,\Delta t/2)^2}}{c}. \tag{20-2}$$

Substituting for L_0 from Eq. 20-1 and solving Eq. 20-2 for Δt gives

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - u^2/c^2}}. \tag{20-3}$$

* We assume that S has a row of synchronized clocks, which S can use to make time measurements at points A, B, and C. Establishment of a synchronized array of clocks is discussed in Section 20-5.

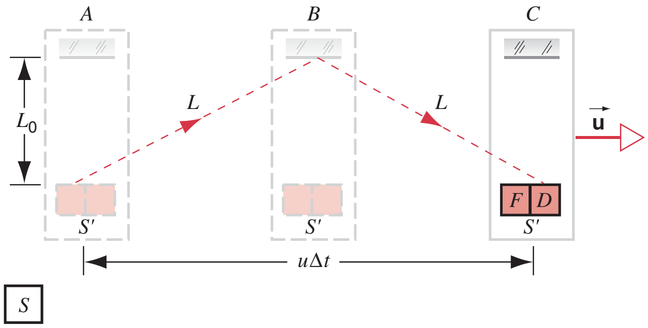


FIGURE 20-5. In the frame of reference of S, the clock carried by S' on the train moves with speed u. The dashed line, of length $2L$, shows the path of the light beam according to S.

The factor in the denominator of Eq. 20-3 is always less than or equal to 1, and thus $\Delta t \geq \Delta t_0$. That is, the observer relative to whom the clock is in motion (observer S) measures a greater interval between ticks. This effect is called *time dilation*. The time interval Δt_0 measured by an observer (S' in this case) relative to whom the clock is at rest is called the *proper time*. The proper time interval between events is the smallest interval between them that any observe can measure; all observers in motion relative to the clock measure *longer* intervals.

Equation 20-3 enables us to understand the difficulty with the pion decay experiments discussed in Section 20-1. A pion at rest decays in a time interval of 26.0 ns; this interval is a proper time interval and is designated as Δt_0. (The pion is in effect a clock, and the interval from formation to decay of the pion can be regarded as a tick of the clock.) An observer in the laboratory, relative to whom the pion is in motion at a speed of $u = 0.913c$, would be expected to measure a time interval of

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - u^2/c^2}} = \frac{26.0 \text{ ns}}{\sqrt{1 - (0.913)^2}} = 63.7 \text{ ns},$$

in agreement with the measured value.

Equation 20-3, which is deduced from Einstein's postulates, gives the relationship between time intervals according to special relativity for observers in relative motion. Note that the factor in the denominator differs appreciably from 1 only at speeds that approach the speed of light. Even at a speed of $0.1c$, Eq. 20-3 gives $\Delta t = 1.005\Delta t_0$. At ordinary speeds, we can take $\Delta t = \Delta t_0$ to a very high precision. This is the classical result (which is obtained directly from Eq. 20-3 in the limit $u \ll c$) and is in accord with our "common-sense" experience.

Equation 20-3 is valid for any direction of the relative motion of S and S'. It is also valid for any type of clock, not just the special one we used in its derivation. It has been verified experimentally not only with decaying elementary particles (such as the pion) moving at high speed, but also with precise atomic clocks moving relative to one another at ordinary (jet airliner) speeds. Even biological clocks such as human aging are expected to be affected by time dilation. An interesting aspect of this effect, called the *twin paradox,* is discussed later in this chapter.

The Relativity of Length

We now consider the effect of Einstein's postulates on the measurement of length intervals. Suppose that S' turns the clock on the train sideways, so that the light now travels along the direction of motion of the train. Figure 20-6 shows the sequence of events as observed by S for the moving clock. According to S the length of the clock is L; as we shall see, this length is different from the length L_0 measured by S', relative to whom the clock is at rest.

A flash of light is emitted at position A in Fig. 20-6 and reaches the mirror (position B) a time Δt_1 later. The total distance traveled by the light in this interval is $c\,\Delta t_1$, which can also be written as the length L of the clock plus the additional distance $u\,\Delta t_1$ that the mirror moves forward in this interval due to the motion of the train. That is,

$$c\,\Delta t_1 = L + u\,\Delta t_1. \tag{20-4}$$

During the return trip from the mirror to the detector (position C in Fig. 20-6), which takes an interval Δt_2 according to S, the light travels a distance $c\,\Delta t_2$, which must equal the length L less the distance $u\,\Delta t_2$ that the train moves forward in this interval, or

$$c\,\Delta t_2 = L - u\,\Delta t_2. \tag{20-5}$$

After solving Eqs. 20-4 and 20-5 for Δt_1 and Δt_2, we add to find the total time interval Δt, which gives

$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{L}{c - u} + \frac{L}{c + u}$$

$$= \frac{2L}{c}\,\frac{1}{1 - u^2/c^2}. \tag{20-6}$$

From Eq. 20-3, setting $\Delta t_0 = 2L_0/c$,

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - u^2/c^2}} = \frac{2L_0}{c}\,\frac{1}{\sqrt{1 - u^2/c^2}}. \tag{20-7}$$

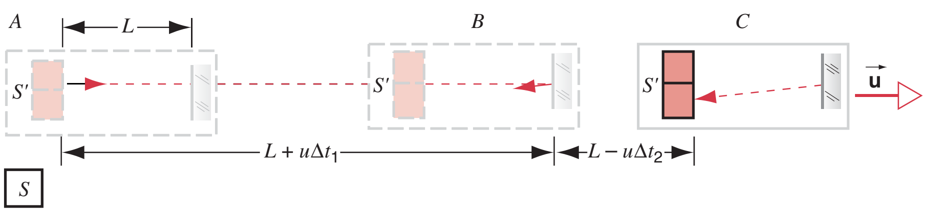


FIGURE 20-6. Here the clock carried by S' on the train emits its light flash in the direction of motion of the train. The figure at C has been displaced to the right for clarity.

Setting Eqs. 20-6 and 20-7 equal to one another and solving, we obtain

$$L = L_0\sqrt{1 - u^2/c^2}. \qquad (20\text{-}8)$$

Equation 20-8 summarizes the effect known as *length contraction.* The length L_0 measured by an observer (such as S') who is at rest with respect to the object being measured is called the *rest length* (also known as the *proper length,* in analogy with the proper time). All observers in motion relative to S' measure a shorter length, but only for dimensions along the direction of motion; length measurements transverse to the direction of motion are unaffected. In the situation shown in Fig. 20-5, the length L_0 is unaffected by the relative motion.

Equation 20-8 can help us resolve the difficulties with the classical concept of length discussed in Section 20-1. The two markers placed in the laboratory at the locations of the formation and decay of the pion are separated by a distance of 17.4 m. Since the markers are at rest in the laboratory, the distance between them is a rest length. To an observer traveling with the pion, the entire laboratory is in motion at $u = 0.913c$, and the distance between the markers is measured, according to Eq. 20-8, to have a contracted length

$$L = (17.4 \text{ m})\sqrt{1 - (0.913)^2} = 7.1 \text{ m},$$

consistent with the discussion of Section 20-1.

Under ordinary circumstances, $u \ll c$ and the effects of length contraction are far too small to be observed. For example, a rocket of length 100 m launched from Earth with the high speed sufficient to escape the Earth's gravity ($u = 11.2$ km/s) would be measured to contract, according to an observer on the Earth, by an amount roughly equivalent to only 2 atomic diameters!

Length contraction suggests that objects in motion are measured to have a shorter length than they do at rest. No actual shrinkage is implied, merely a difference in measured results, just as two observers in relative motion measure a different frequency for the same source of sound (the Doppler effect).

The Relativistic Addition of Velocities

Let us now modify our timing device, as shown in Fig. 20-7. The flashing bulb F is moved to the mirror end and is replaced by a device P that emits particles at a speed v_0, as measured by an observer at rest with respect to the device.

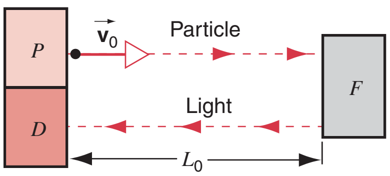


FIGURE 20-7. In this timing device, a particle is emitted by P at a speed v_0. When the particle reaches F, it triggers the emission of a flash of light that travels to the detector D.

The bulb is triggered to flash when it is struck by a particle, and a light beam makes the return trip to the detector D. Thus the time interval Δt_0, measured by an observer (such as S') who is at rest with respect to the device, consists of two parts: one due to the particle traveling the distance L_0 at speed v_0 and another due to the light beam traveling the same distance at speed c:

$$\Delta t_0 = L_0/v_0 + L_0/c. \qquad (20\text{-}9)$$

The sequence of events observed by S as the timing device is carried by S' on the train is identical with that of Fig. 20-6. The emitted particle, which travels at speed v according to S, reaches F after an interval Δt_1, during which time it travels a distance $v\,\Delta t_1$, which is equal to the (contracted) length L plus the additional distance $u\,\Delta t_1$ moved by the train in that interval:

$$v\,\Delta t_1 = L + u\,\Delta t_1. \qquad (20\text{-}10)$$

In the interval Δt_2, the light beam travels a distance $c\,\Delta t_2$ equal to the length L less the distance $u\,\Delta t_2$ moved forward by the train in that interval:

$$c\,\Delta t_2 = L - u\,\Delta t_2. \qquad (20\text{-}11)$$

Solving Eqs. 20-10 and 20-11 for Δt_1 and Δt_2, we can then find the total time interval $\Delta t = \Delta t_1 + \Delta t_2$ between ticks according to S, and we substitute that result along with Eq. 20-9 into Eq. 20-3, which gives (after using Eq. 20-8 to relate L_0 and L)

$$v = \frac{v_0 + u}{1 + v_0 u/c^2}. \qquad (20\text{-}12)$$

Equation 20-12 gives one form of the velocity addition law consistent with Einstein's postulates; here we are concerned only with adding velocities that are along the direction of relative motion (the direction of $\vec{\mathbf{u}}$). Later in this chapter we derive more general results.

According to Galileo and Newton, a projectile fired forward at speed v_0 in a train that is moving at speed u should have a speed $v_0 + u$ relative to an observer on the ground. This clearly permits speeds in excess of c to be realized. The difference between the classical result and the relativistic result is the denominator of Eq. 20-12, which can certainly be replaced by 1 in ordinary circumstances when the speeds are much smaller than c. This important factor, as we see in Sample Problem 20-2, prevents the relative speed from ever exceeding c.

If the projectile is a light beam ($v_0 = c$ according to S'), then Eq. 20-12 immediately gives $v = c$ for all observers, no matter what their speed relative to S' (that is, independent of u). Thus Eq. 20-12 is consistent with Einstein's second postulate.

SAMPLE PROBLEM 20-1. Muons are elementary particles with a (proper) lifetime of 2.2 μs. They are produced with very high speeds in the upper atmosphere when cosmic rays (high-energy particles from space) collide with air molecules. Take the height L_0 of the atmosphere (its rest length) to be 100 km in the reference frame of the Earth, and find the minimum speed that will enable the muons to survive the journey to the surface of the Earth. Solve this problem in two ways: (*a*) in the Earth's frame of reference and (*b*) in the muon's frame of reference.

Solution (*a*) In the Earth's frame of reference (Fig. 20-8*a*), the decay of the moving muon is slowed by the time dilation effect. If the muon is moving at a speed that is very close to c, the time necessary for it to travel from the top of the atmosphere to the Earth is

$$\Delta t = \frac{L_0}{c} = \frac{100 \text{ km}}{3.00 \times 10^8 \text{ m/s}} = 333\ \mu\text{s}.$$

The muon must survive for at least 333 μs in the Earth's frame of reference. We now find the speed that dilates the lifetime from its proper value Δt_0 (= 2.2 μs) to this value, according to the time dilation formula (Eq. 20-3):

$$333\ \mu\text{s} = \frac{2.2\ \mu\text{s}}{\sqrt{1 - u^2/c^2}}.$$

Solving, we find

$$u = 0.999978c.$$

(*b*) In the muon's frame of reference, the atmosphere is rushing by at high speed. In this frame of reference the entire atmosphere must rush by in a time equal to the (proper) lifetime of the muon, and thus the height of the atmosphere can be no greater than

$$L = c\,\Delta t_0 = (3.00 \times 10^8 \text{ m/s})(2.2 \times 10^{-6} \text{ s}) = 660 \text{ m}.$$

This is of course the measured contracted length in the muon's frame of reference (see Fig. 20-8*b*). The relationship between the rest length L_0 (= 100 km), measured in the Earth's frame of reference, and the contracted length, measured in the muon's frame of reference, is given by Eq. 20-8, and so

$$660 \text{ m} = (100 \text{ km})\sqrt{1 - u^2/c^2}.$$

Solving for the speed u, we obtain the same result given in part (*a*).

Note that a *time dilation* in one frame of reference can be observed as a *length contraction* in another. This interrelationship of time and space is a fundamental part of special relativity.

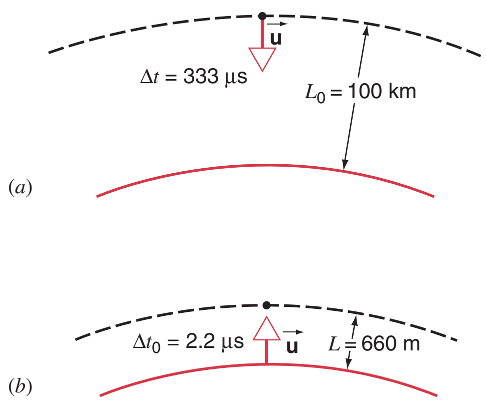


FIGURE 20-8. Sample Problem 20-1. (*a*) In the reference frame of the Earth, a muon takes 333 μs to travel a distance of 100 km through the atmosphere. (*b*) In the reference frame of the muon, the atmosphere is only 660 m high, and the journey takes 2.2 μs.

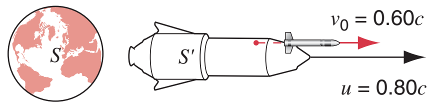


FIGURE 20-9. Sample Problem 20-2. A spaceship moves away from Earth at a speed of 0.80c. An observer S' on the spaceship fires a missile and measures its speed to be 0.60c relative to the ship.

SAMPLE PROBLEM 20-2. A spaceship is moving away from the Earth at a speed of 0.80c when it fires a missile parallel to the direction of motion of the ship. The missile moves at a speed of 0.60c relative to the ship (Fig. 20-9). What would be the speed of the missile as measured by an observer on the Earth? Compare with the predictions of Galilean kinematics.

Solution This problem is similar to that of the observer and the train. Here S' is on the ship and S is on Earth, and S' moves with a velocity of $u = 0.80c$ relative to S. The missile moves at velocity $v_0 = 0.60c$ relative to S', and we seek its velocity relative to S. Using Eq. 20-12, we obtain

$$v = \frac{v_0 + u}{1 + v_0 u/c^2} = \frac{0.60c + 0.80c}{1 + (0.60c)(0.80c)/c^2}$$

$$= \frac{1.40c}{1.48} = 0.95c.$$

According to classical kinematics (the numerator of Eq. 20-12), an observer on the Earth would see the missile moving at $0.60c + 0.80c = 1.40c$, thereby exceeding the maximum relative speed of c permitted by relativity. You can see how Eq. 20-12 brings about this speed limit. Even if v_0 were $0.9999\cdots c$ and u were $0.9999\cdots c$, the relative velocity v measured by S would remain less than c.

20-4 THE LORENTZ TRANSFORMATION

Einstein's posulates provide a first step in the resolution of the difficulties we presented in Section 20-1, but a more formal mathematical basis is needed to give the theory its full power to calculate the expected outcomes of a wider variety of physical processes. For example, we might wish to compute how the results of measurements of an energy or a magnetic field strength differ for observers in relative motion.

We require a set of relationships called *transformation equations* that relate observations of a single event by two different observers. The transformation equations have three ingredients: (1) an observer S at rest in one inertial frame, (2) another observer S' at rest in a different inertial frame that is in motion at constant velocity with respect to S, and (3) a single event that is observed by *both* S and S'. The event occurs, according to each observer, at a particular set of coordinates in three-dimensional space *and* at a particular time. Knowing the relative velocity of S and S', we

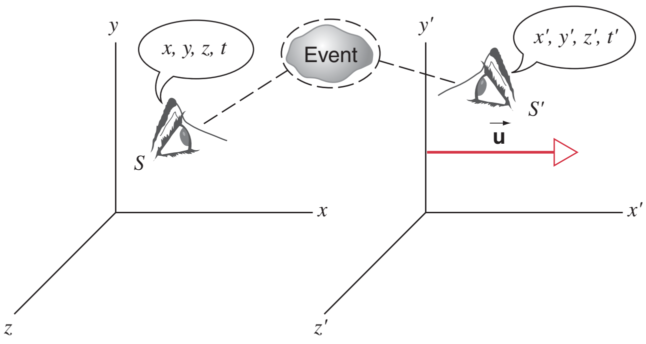


FIGURE 20-10. Two observers, whose frames of reference are represented by S and S', observe the same event. S' moves relative to S with velocity $\vec{u}$ along the common xx' direction. S measures the coordinates x, y, z, t of the event, while S' measures the coordinates x', y', z', t' of the same event.

wish to calculate the coordinates x', y', z', t' of an event as observed by S' from the coordinates of x, y, z, t of the *same event* according to S. We simplify this problem somewhat, without losing generality, by always choosing the x and x' axes to be along the direction of $\vec{u}$ (see Fig. 20-10).

This problem can be solved using the classical kinematics of Galileo, and the resulting *Galilean transformation* equations are

$$\begin{aligned} x' &= x - ut, \\ y' &= y, \\ z' &= z, \\ t' &= t. \end{aligned} \tag{20-13}$$

The first of these equations is consistent with our "common-sense" experience. For instance, suppose S is at rest on the ground and measures the location x of a fencepost. S', who is in a car moving at speed u relative to S, does indeed find the fencepost at the location $x' = x - ut$ (Fig. 20-11). The fourth equation, $t' = t$, was simply taken for granted in classical physics (as exemplified by Newton's universal time coordinate).

The relativistic relationships we seek have come to be known as the *Lorentz transformation equations.* They are named for the Dutch physicist H. A. Lorentz, who proposed them (before Einstein) for quite a different reason and who was not fully aware of their implications about the nature of space and time. The equations can be derived directly from Einstein's postulates, if we invoke certain reasonable assumptions about the symmetry and the homogeneity of space and time. As an example of this latter property, consider an observer S who measures the length of a rod held by observer S' in a different inertial frame. The result of the measurement carried out by S should not depend on where S' is located in the reference frame or on the time of day at which S makes the measurement.

The Lorentz transformation equations, derived on these assumptions, are*

$$\begin{aligned} x' &= \frac{x - ut}{\sqrt{1 - u^2/c^2}} = \gamma(x - ut), \\ y' &= y, \\ z' &= z, \\ t' &= \frac{t - ux/c^2}{\sqrt{1 - u^2/c^2}} = \gamma(t - ux/c^2). \end{aligned} \tag{20-14}$$

Note that an object located initially at the origin according to S (that is, $x = 0$ at $t = 0$) is also initially located at the origin according to S' (that is, $x' = 0$ and $t' = 0$).

In these equations, we have used the *Lorentz factor* γ, defined as

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}. \tag{20-15}$$

It is also convenient in relativity equations to introduce the *speed parameter* β, defined as the ratio between the relative speed u of the two coordinate systems and the speed of light:

$$\beta = u/c. \tag{20-16}$$

Some sample values of β and γ are given in Table 20-1, and the relationship between β and γ is shown in Fig. 20-12. The range of γ is from 1 (at low speed, where $u \ll c$ or $\beta \ll 1$) to ∞ (at high speed, where $u \rightarrow c$ or $\beta \rightarrow 1$).

Note that the Lorentz transformation equations reduce to those of the Galilean transformation (Eqs. 20-13) when $u \ll c$. One convenient way to show this is to let $c \rightarrow \infty$, so that $u/c \rightarrow 0$. In this case, as you should prove, the relativistic Eqs. 20-14 reduce directly to the classical Eqs. 20-13. All classical results derived in previous chapters agree

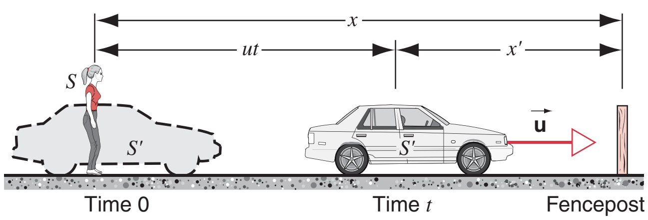


FIGURE 20-11. According to S, the fencepost is at the coordinate x. According to S', who is at coordinate ut relative to S at time t, the fencepost is at the coordinate $x' = x - ut$. Note that the origins of S and S' coincide at $t = 0$.

* See *Basic Concepts in Relativity,* by Robert Resnick and David Halliday (Macmillan, 1992), for a derivation of these equations.

TABLE 20-1 Sample Values of the Speed Parameter and the Lorentz Factor

β	γ	β	γ
0.00	1.000	0.90	2.29
0.10	1.005	0.99	7.09
0.30	1.048	0.999	22.4
0.60	1.25	0.9999	70.7

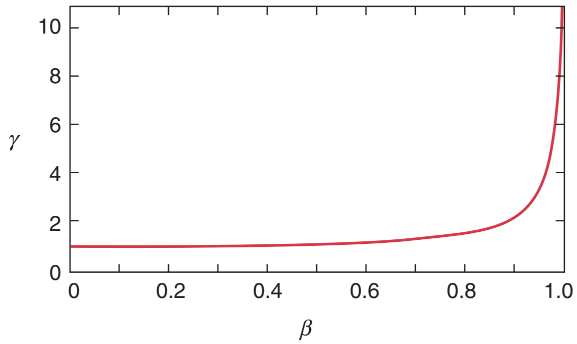


FIGURE 20-12. The Lorentz factor γ as a function of the speed parameter β.

with experiment when $u \ll c$. Only at high speeds must we take relativistic effects into account.

Equations 20-14 permit us to find the space and time coordinates in S' if we know those in S. Suppose, however, that we wish to know the coordinates in S, given those in S'. From the point of view of S' in Fig. 20-10, S appears to move in the *negative* x (or x') direction. We can obtain the *inverse Lorentz transformation* by merely switching primed and unprimed coordinates in Eqs. 20-14 and substituting $-u$ for u. This gives

$$\begin{aligned} x &= \gamma(x' + ut'), \\ y &= y', \\ z &= z', \\ t &= \gamma(t' + ux'/c^2). \end{aligned} \tag{20-17}$$

We can use a different method of inverting the Lorentz transformation (see Exercise 14) by solving Eqs. 20-14 algebraically for x and t (treating the first and last equations as a system of two equations in two unknowns). When we do, we obtain exactly the inverse transformation given by Eqs. 20-17, which were obtained directly from a symmetry argument.

Table 20-2 summarizes the equations of the Lorentz transformation when the relative velocity between the coordinate systems is in the common xx' direction. The equations are shown in four forms: the Lorentz transformation (Eqs. 20-14), the inverse Lorentz transformation (Eqs. 20-17), and both corresponding *interval* transformations, which are useful when we wish to transform not a coordinate but a space or time interval, such as $\Delta x' = x_2' - x_1'$ (the distance between two events, as measured by S') or $\Delta t' = t_2' - t_1'$ (the time between two events, as measured by S').

SAMPLE PROBLEM 20-3. In inertial frame S, a red light and a blue light are separated by a distance $\Delta x = 2.45$ km, with the red light at the larger value of x. The blue light flashes, and 5.35 μs later the red light flashes. Frame S' is moving in the direction of increasing x with a speed of $u = 0.855c$. What is the distance between the two flashes and the time between them as measured in S'?

Solution The Lorentz parameter is

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}} = \frac{1}{\sqrt{1 - (0.855)^2}} = 1.928.$$

We are given the intervals in S as $\Delta x = 2450$ m and $\Delta t = 5.35 \times 10^{-6}$ s. From Table 20-2, we have the interval transformations

$$\begin{aligned} \Delta x' &= \gamma(\Delta x - u\,\Delta t) \\ &= 1.928[2450\text{ m} - (0.855)(3.00 \times 10^8\text{ m/s})(5.35 \times 10^{-6}\text{ s})] \\ &= 2078\text{ m} = 2.08\text{ km} \end{aligned}$$

and

$$\begin{aligned} \Delta t' &= \gamma(\Delta t - u\,\Delta x/c^2) \\ &= 1.928[5.35 \times 10^{-6}\text{ s} - (0.855)(2450\text{ m})/(3.00 \times 10^8\text{ m/s})] \\ &= -3.147 \times 10^{-6}\text{ s} = -3.15\ \mu\text{s}. \end{aligned}$$

In S', the red flash is also located at the more distant coordinate, but the distance is 2.08 km rather than 2.45 km. Also, in S' the red flash comes *before* the blue flash (in contrast to what is observed in S); the time between flashes is 3.15 μs according to S'.

20-5 MEASURING THE SPACE–TIME COORDINATES OF AN EVENT

So far, we have said little about how observers S and S' go about measuring the coordinates x, y, z, t and x', y', z', t' of an event, as in the case of the light flashes in Sample Problem 20-3. The procedure we now describe forms a conceptual foundation on which actual laboratory procedures can be based.

We assume that S has a large team of assistants available to help in the setting up of a coordinate system. Each assistant is given a clock and a measuring rod of a certain

TABLE 20-2 The Lorentz Transformation Equations[a]

Lorentz Transformation	*Inverse Transformation*	*Interval Transformation*	*Inverse Interval Transformation*
$x' = \gamma(x - ut)$	$x = \gamma(x' + ut')$	$\Delta x' = \gamma(\Delta x - u\,\Delta t)$	$\Delta x = \gamma(\Delta x' + u\,\Delta t')$
$y' = y$	$y = y'$	$\Delta y' = \Delta y$	$\Delta y = \Delta y'$
$z' = z$	$z = z'$	$\Delta z' = \Delta z$	$\Delta z = \Delta z'$
$t' = \gamma(t - ux/c^2)$	$t = \gamma(t' + ux'/c^2)$	$\Delta t' = \gamma(\Delta t - u\,\Delta x/c^2)$	$\Delta t = \gamma(\Delta t' + u\,\Delta x'/c^2)$

[a] Apply these equations only in the case of relative motion in the xx' direction. The Lorentz factor is $\gamma = 1/\sqrt{1 - u^2/c^2}$.

length. For example, three assistants have measuring rods 1 m in length. They are instructed to lay out their rods, each along one of the three coordinate axes, and to wait at the position determined by the end of the rod until they see a flash of light at the origin, at which time they are to start their clocks at the preset reading of 3.33×10^{-9} s (3.33 ns, the time necessary for light to travel the distance of 1 m from the origin to the assistant's location). Three other assistants, who are similarly each assigned one of the coordinate axes, are given rods of length 2 m and are instructed to start their clocks, when they see the flash of light, at the preset time of 6.67 ns (the time for light to travel 2 m). Each assistant is sent to a post with a rod of some length L and a clock preset at $t = L/c$.

When all the assistants are at their posts, S sets off a flash of light at the origin and simultaneously starts the clock at the origin, which is preset to zero. As the light signal reaches the other clocks, each is started in turn at the preset reading. Thus the clock on the x axis at $x = 1$ m is started at the preset reading of 3.33 ns when the clock at the origin reads 3.33 ns; the clock on the x axis at $x = 2$ m starts at the preset reading of 6.67 ns when the clock at the origin and the clock at $x = 1$ m both read 6.67 ns; and so on for all the clocks in the coordinate system. All clocks in the entire system are thus perfectly synchronized. The resulting system of rods and clocks is represented in Fig. 20-13.

Suppose that S wishes to chart the progress of a particle as it moves through the coordinate system. All that S and the assistants must do is watch the particle as it travels and, as it passes each point, write down the coordinate and the reading on the clock at that coordinate.

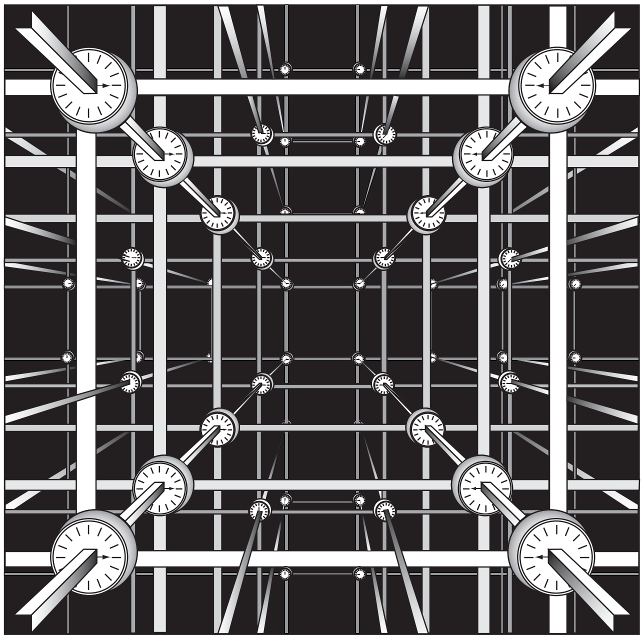


FIGURE 20-13. A framework of measuring rods and clocks that might be used by an observer in a particular reference frame to determine the space–time coordinates of an event.

Of course, this calibration holds only for observer S. Observer S' and all other inertial observers must carry out a similar procedure to define a coordinate system and synchronize its clocks. With such a scheme the measuring rods and clocks of each observer (which of course are at rest in the frame of that observer) are unique to that inertial frame and are independent of the rods and clocks of observers in other inertial frames.

This procedure suggests that space and time are not independent coordinates, but that the description of an event must include its coordinates in both space and time. (That is, we cannot use a clock at one location to record the passage of a particle through another location.) For this reason, special relativity usually is formulated in terms of combined *space–time coordinates* x, y, z, t. Space and time are treated as equivalent coordinates in special relativity.

20-6 THE TRANSFORMATION OF VELOCITIES

In this section we use the equations of the Lorentz transformation to relate the velocity $\vec{v}$ of a particle measured by an observer in the S frame to the velocity $\vec{v}'$ of the same particle measured by an observer in the S' frame, who is in turn moving with velocity $\vec{u}$ relative to S. In this discussion, it is important to keep in mind the meanings of these three velocities.

Suppose observer S finds the particle to move from coordinates x_1, y_1, z_1, t_1 to x_2, y_2, z_2, t_2. Observer S', on the other hand, records the observations of the initial and final coordinates of the same particle as x_1', y_1', z_1', t_1' and x_2', y_2', z_2', t_2'.

Let us calculate v_x' $(= \Delta x'/\Delta t')$, the x' component of the velocity measured by S'. From Table 20-2, we obtain the transformation equations for the intervals $\Delta x'$ and $\Delta t'$. Dividing these two equations, we obtain

$$v_x' = \frac{\Delta x'}{\Delta t'} = \frac{\gamma(\Delta x - u\,\Delta t)}{\gamma(\Delta t - u\,\Delta x/c^2)} = \frac{\Delta x/\Delta t - u}{1 - u(\Delta x/\Delta t)/c^2},$$

or, replacing $\Delta x/\Delta t$ by v_x,

$$v_x' = \frac{v_x - u}{1 - uv_x/c^2}. \tag{20-18}$$

In similar fashion, we obtain the transformation equations for the y and z components of the velocities:

$$v_y' = \frac{v_y}{\gamma(1 - uv_x/c^2)} \quad \text{and} \quad v_z' = \frac{v_z}{\gamma(1 - uv_x/c^2)}. \tag{20-19}$$

Note that $v_y' \neq v_y$, even though $\Delta y = \Delta y'$, because $\Delta t \neq \Delta t'$. Similar considerations hold for v_z'. This is another example of the difference between the way the Galilean and Lorentz transformations deal with the time coordinate. Be sure to note that the denominators of all three equations include the factor v_x.

TABLE 20-3 The Lorentz Velocity Transformation

Velocity Transformation	*Inverse Velocity Transformation*
$v'_x = \dfrac{v_x - u}{1 - uv_x/c^2}$	$v_x = \dfrac{v'_x + u}{1 + uv'_x/c^2}$
$v'_y = \dfrac{v_y}{\gamma(1 - uv_x/c^2)}$	$v_y = \dfrac{v'_y}{\gamma(1 + uv'_x/c^2)}$
$v'_z = \dfrac{v_z}{\gamma(1 - uv_x/c^2)}$	$v_z = \dfrac{v'_z}{\gamma(1 + uv'_x/c^2)}$

Equations 20-18 and 20-19 give the *Lorentz velocity transformation.* They are analogous to the equations of the Lorentz coordinate transformation: they relate observations in one coordinate frame to observations in another. Table 20-3 summarizes these equations, along with the corresponding inverse velocity transformation. Note that the inverse transformation equation for v_x is identical with Eq. 20-12, which we derived in quite a different way. In Eq. 20-12, the speed v_0 is the same as the speed v'_x measured by S'.

Let us examine Eqs. 20-18 and 20-19 in the nonrelativistic limit. Do they reduce to the classical Galilean transformation when $u \ll c$ (or equivalently, when $c \to \infty$)? In this case Eqs. 20-18 and 20-19 reduce to

$$v'_x = v_x - u, \quad v'_y = v_y, \quad \text{and} \quad v'_z = v_z, \tag{20-20}$$

which are indeed the Galilean results, as given by Eq. 4-32 or by differentiating Eqs. 20-13, the Galilean coordinate transformation equations.

We now show directly that the Lorentz velocity transformation gives the result demanded by Einstein's second postulate (the constancy of the speed of light): a speed of c measured by one observer must also be measured to be c by any other observer. Suppose that the common event being observed by S and S' is the passage of a light beam along the x direction. Observer S measures $v_x = c$ and $v_y = v_z = 0$. What velocity does observer S' measure? Using Eqs. 20-18 and 20-19, we find the velocity components measured by S' to be

$$v'_x = \frac{v_x - u}{1 - uv_x/c^2} = \frac{c - u}{1 - uc/c^2} = \frac{c - u}{(c - u)/c} = c,$$

$$v'_y = v'_z = 0.$$

Note that this result follows *independent of the relative speed u between S and S′*. A speed of c measured in one inertial reference frame leads to a speed of c measured in *all* frames. Thus the speed of light is indeed the same for all observers. The same conclusion holds for any direction of travel of the light beam; see Exercise 15.

SAMPLE PROBLEM 20-4. A particle is accelerated from rest in the laboratory until its velocity is $0.60c$. As viewed from a frame that is moving with the particle at a speed of $0.60c$ relative to the laboratory, the particle is then given an additional increment of velocity amounting to $0.60c$. Find the final velocity of the particle as measured in the laboratory frame.

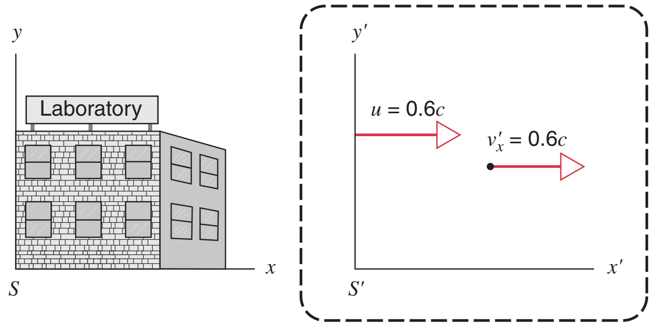


FIGURE 20-14. Sample Problem 20-4. S', the frame of reference of the particle after the first acceleration, moves with speed $u = 0.60c$ relative to the laboratory (frame S). Relative to S', the particle moves at speed $v'_x = 0.60c$ after its second acceleration.

Solution Once again, the problem becomes a direct application of the Lorentz velocity transformation once we have clearly specified the reference frames S and S' and the system being observed. Clearly the particle is the system being observed, and if we seek its velocity measured in the laboratory frame it is natural to associate the laboratory with the S frame. The S' frame is then the inertial reference frame occupied by the particle after the first acceleration and before the second (see Fig. 20-14). Relative to this frame, the velocity of the particle after the second acceleration is $v'_x = 0.60c$. The velocity of the S' frame with respect to the S frame is $u = 0.60c$. We know v'_x and u, and we seek v_x, which is given by the inverse velocity transformation from Table 20-3:

$$v_x = \frac{v'_x + u}{1 + uv'_x/c^2} = \frac{0.60c + 0.60c}{1 + (0.60c)(0.60c)/c^2} = \frac{1.20c}{1.36} = 0.88c.$$

The speed is less than c, in contradiction to the prediction of the Galilean transformation, which gives $v_x = 1.20c$.

Suppose we now let the S' frame be that of the particle after the second acceleration, so that $u = 0.88c$ relative to the original S frame (the laboratory). Let there now be a *third* acceleration, so that, relative to the new S' frame, the particle again moves with velocity $v'_x = 0.60c$. By repeating the above procedure, you should show that an observer in the laboratory (S) frame will measure a speed of $v'_x = 0.97c$ in this case.

No matter how many times we accelerate the particle in a reference frame moving with the particle, its velocity measured in the original laboratory frame (or in any other frame) will never exceed c.

20-7 CONSEQUENCES OF THE LORENTZ TRANSFORMATION

We have already shown that some unexpected consequences follow from applying Einstein's postulates to physical situations. Now we use the more mathematical basis of the Lorentz transformation to show that these same consequences and others can be obtained.

The Relativity of Time

In Section 20-3 we showed that the time dilation effect follows directly from applying Einstein's postulates to measurements of time intervals by two observers in motion relative to one another. Figure 20-15 shows a different view of the time dilation effect. Clock C' is at rest in the frame of S', who moves at speed u relative to S. S' measures the time interval $\Delta t' = t'_2 - t'_1$ in which the hand of the clock moves between two marks, passing the first mark at time t'_1 and the second at time t'_2.

The hand of clock C' passing the two marks can be regarded as two events, which occur at the same location x'_0 according to S' (because clock C' is at rest in that frame). However, S (whose reference frame contains a stationary set of synchronized clocks such as that described in Section 20-5) observes the hand of clock C' to pass the first mark at the location x_1 (where the local stationary clock reads time t_1) and to pass the second mark at location x_2 (where a *different* stationary clock reads the time t_2). We can find the relationship between the time intervals Δt and $\Delta t'$ directly from the inverse Lorentz transformation. From Table 20-2, we have

$$\Delta t = \gamma(\Delta t' + u\,\Delta x'/c^2). \qquad (20\text{-}21)$$

This general expression gives the time interval Δt measured by S corresponding to the time interval $\Delta t'$ measured by S' for events that are separated by a distance $\Delta x'$. According to S', relative to whom clock C' is at rest, the two events (the hand passing the two marks) take place at the same location x'_0, so $\Delta x' = 0$. Because S' is at rest relative to clock C', the time interval $\Delta t'$ measured by S' is a proper time interval, which we represent as Δt_0. Substituting $\Delta x' = 0$ and $\Delta t' = \Delta t_0$ into Eq. 20-21, we obtain

$$\Delta t = \gamma\,\Delta t_0 = \frac{\Delta t_0}{\sqrt{1 - u^2/c^2}},$$

which is identical to Eq. 20-3, the time dilation equation.

The time dilation effect is completely symmetric. If a clock C at rest in S is observed by S', then S' concludes that clock C is running slow. Each observer believes that the other's clock is running slower than the ones at rest in the reference frame of the observer. Time dilation is often summarized by the phrase, "moving clocks run slow." It is useful to remember this phrase, but do so with caution. The phrase indicates that a clock moving relative to a frame containing an array of synchronized clocks will be found to run slow *when timed by those clocks*. That is, only in the sense of comparing a single moving clock with two separated synchronized stationary clocks can we declare that "moving clocks run slow."

Consider three other consequences of the Lorentz transformation that are related to measurements of time:

1. *The relativity of simultaneity.* Suppose S' has two clocks at rest, located at x'_1 and x'_2, and separated by the interval $\Delta x' = x'_2 - x'_1$. A flash of light emitted from a point midway between the clocks reaches the two clocks simultaneously, according to S' (see Fig. 20-16*a*). That is, a measurement by S' of the interval between the arrival of the light signals at the two clocks gives $\Delta t' = 0$. Now consider the situation from the point of view of S, relative to whom the frame of S' (including the clocks) moves with speed u (Fig. 20-16*b*). Clearly the light signal reaches clock 1 before it reaches clock 2, and thus the arrival of the light signals at the locations of the two clocks is *not* simultaneous to S. We therefore reach the following conclusion:

If two observers are in relative motion, in general they do not agree on whether two events at different locations are simultaneous. If one observer finds the two events to be simultaneous, the other does not.

This conclusion also follows directly from Eq. 20-21: if $\Delta t' = 0$ and $\Delta x' \neq 0$, then $\Delta t \neq 0$. Note that this occurs only when the two events occur at different locations according to S'. If the two events occur at the same location and are simultaneous according to S', they are simultaneous to S as well.

2. *The Doppler shift.* In Section 19-9 we considered the Doppler effect for sound waves, in which the motion of a

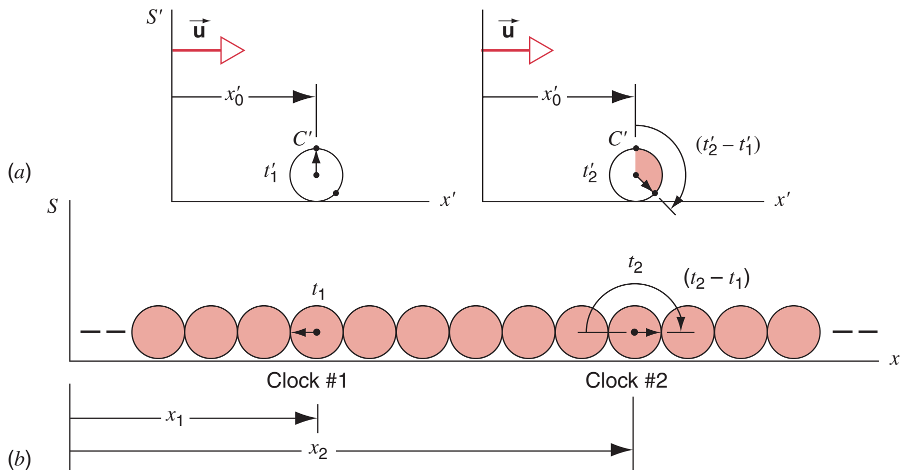


FIGURE 20-15. Clock C' is fixed at position x'_0 in reference frame S'. Observer S, relative to whom clock C' is in motion at velocity $\vec{\mathbf{u}}$, compares the reading of C' with two different stationary clocks from the array of synchronized clocks (numbered 1 and 2) established in the frame of S. As shown, the interval $t_2 - t_1$ measured by S is greater than the interval $t'_2 - t'_1$. Observer S therefore declares that, by comparison with the clocks in S, the moving clock is running slow.

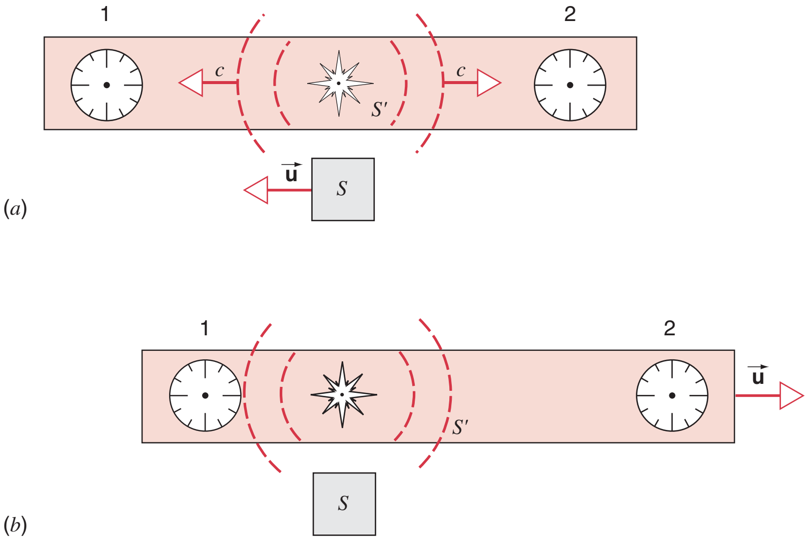


FIGURE 20-16. (*a*) In the frame of reference of *S*′, a flash of light at a point midway between two clocks reaches the clocks at the same instant. (*b*) In the frame of reference of *S*, the flash of light reaches clock 1 before it reaches clock 2.

source or an observer of waves relative to the medium carrying the waves causes a change in the frequency measured by the observer.

In the case of light waves, "motion relative to the medium" is not a valid concept. Special relativity gives a Doppler shift for light that depends only on the relative speed between the source and the observer; in contrast to the case of sound waves, in which we used different formulas to account for source motion and observer motion, in the case of light waves one formula, involving only the *relative* motion, is sufficient. The relativistic Doppler formula is thus simpler to apply than the classical one.

Another aspect of the Doppler effect in special relativity has no classical counterpart. This is the *transverse Doppler effect,* which, in contrast to the cases considered in Section 19-9, occurs when the source or the observer moves perpendicular to the line connecting them. The transverse Doppler effect is actually another result of time dilation, and the precise measurements of the transverse Doppler effect provide some of our most sensitive experimental tests of time dilation. We consider the Doppler effect for light in more detail in Chapter 39.

3. *The twin paradox.* Time dilation applies not only to elementary particles but to all naturally occurring time intervals including pulse rates and human lifetimes. This fact has been used to propose an apparent puzzle that has become known as the *twin paradox.**

Suppose two twins, Fred and Ethel, are on a platform coasting in space. Ethel embarks on a journey in a high-speed spaceship to a distant star while Fred remains on the platform. During this journey, Fred is able to monitor Ethel's heart beat and average respiration rate, and he finds them to be slower by the time dilation effect; thus Ethel's entire aging process has been slowed. Fred therefore expects that, upon Ethel's return to the platform after the journey to the star, she will be younger than he.

The paradox seemingly occurs when we analyze the situation from the frame of reference of Ethel, thereby regarding Fred and the platform as the ones making the journey. According to this analysis, Fred is the traveling twin and should be the younger at the end of the journey. Here is the paradox: When they get together at the end of the journey, it cannot be true that Ethel is younger than Fred and also that Fred is younger than Ethel.

The resolution of the paradox comes when we realize that Fred and Ethel are not really in symmetric situations. For the two twins to get back together again, one of them must decelerate and reverse directions, resulting in an easily measurable acceleration of one of them. Put another way, Ethel must change from one inertial reference frame (the one moving away from Fred) to another (one moving toward Fred). Fred, on the other hand, experiences no acceleration and remains in the same inertial reference frame for the entire duration of the journey. It is indeed Ethel who is the traveler and who will be younger upon her return.

Although we have not yet been able to do an experiment of this sort with actual twins, it has been done with atomic clocks.† Two identical clocks were carefully calibrated; one was then flown on a commerical airliner around the world and compared with its stay-at-home twin upon its return. The speed during such a trip was of course far less than c, but atomic clocks are capable of sufficient precision that the small resulting asymmetry in the "aging" of the two clocks, amounting to about 10^{-7} s, could easily and precisely be determined. It was found that the clock in the airliner, which was the one subject to an acceleration and therefore

* For more details about the twin paradox, see *Basic Concepts in Relativity* by Robert Resnick and David Halliday (Macmillan, 1992), p. 156.

† See "Around-the-World Atomic Clocks: Observed Relativistic Time Gains," by J. C. Hafele and Richard E. Keating, *Science,* July 14, 1972, p. 166.

the true traveler, was indeed "younger" (that is, running slower) after the journey.

The reading of the clock in the airliner must also be corrected for the time it spends at a different gravitational potential, an effect of *general* relativity. Corrections for special and general relativity are thus of important practical concern when such precise clocks are transported from one location to another.

The Relativity of Length

Length contraction, which we discussed in Section 20-3, also follows directly from the equations of the Lorentz transformation. Let us first realize that to measure the length of an object we must make a *simultaneous* determination of the coordinates of the ends of the object (see Fig. 20-17). *It does no good to measure the coordinate of one end of a moving object at one time and the coordinate of the other end at a different time.*

Suppose (see Fig. 20-18) that a measuring rod of rest length L_0 is carried by S'. Observer S wishes to measure its length. According to S', in whose frame of reference the rod is at rest, the ends of the rod are at coordinates x_2' and x_1', such that $\Delta x' = x_2' - x_1' = L_0$, the rest length of the rod. Observer S, using the calibrated and synchronized coordinates established according to the procedure described in Section 20-5, makes a simultaneous determination of the coordinates x_2 and x_1 of the ends of the rod. The interval $\Delta x = x_2 - x_1$ gives the length L of the rod according to S. From the interval equation in Table 20-2, we have

$$\Delta x' = \gamma(\Delta x - u\,\Delta t). \tag{20-22}$$

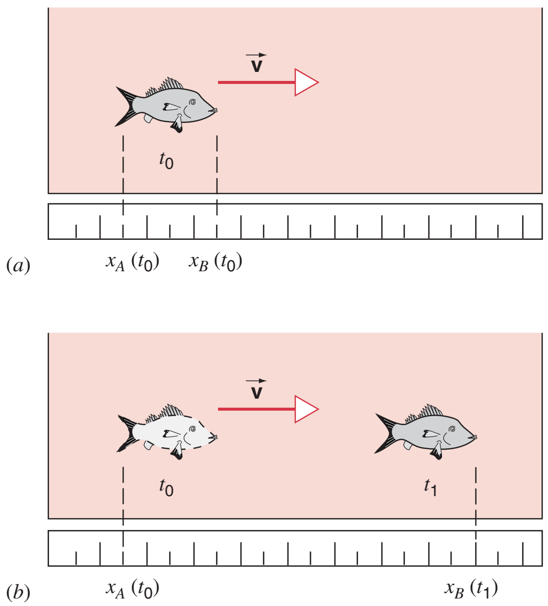


FIGURE 20-17. (*a*) To measure the length of a moving fish, you must determine *simultaneously* the positions of its head and tail. (*b*) If the determination is not simultaneous, the measurement does not give the length.

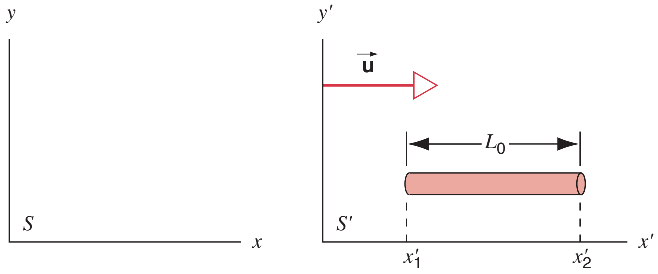


FIGURE 20-18. The ends of a measuring rod are determined to be at coordinates x_1' and x_2' according to S', relative to whom the rod is at rest. To determine the length of the rod, S must make a *simultaneous* determination of the coordinates x_1 and x_2 of its endpoints.

Putting $\Delta t = 0$ (because S made a *simultaneous* determination of x_2 and x_1), we solve for Δx (= L) and obtain

$$L = \Delta x = \frac{\Delta x'}{\gamma} = L_0\sqrt{1 - u^2/c^2},$$

which is identical to Eq. 20-8.

We have deduced time dilation and length contraction both from the postulates (Section 20-3) and from the Lorentz transformation (this section). These are not independent derivations, however, because the Lorentz transformation itself is obtained from the postulates. Ultimately, all of special relativity follows directly from Einstein's postulates.

Like time dilation, length contraction is an effect that holds for all observers in relative motion. Questions such as "Does a moving measuring rod *really* shrink?" have meaning only in the sense that they refer to measurements by observers in relative motion. The essence of relativity is that results of measurements of length and time intervals are subject to the state of motion of the observer relative to the events being measured and refer only to measurements by a particular observer in a particular frame of reference. If different observers were to bring the rod to rest in their individual inertial frames, each would measure the same value for the length of the rod. In this respect, special relativity is a theory of measurement that simply says "motion affects measurement."

SAMPLE PROBLEM 20-5. An observer S is standing on a platform of length $D_0 = 65$ m on a space station. A rocket passes at a relative speed of 0.80c moving parallel to the edge of the platform. The observer S notes that the front and back of the rocket simultaneously line up with the ends of the platform at a particular instant (Fig. 20-19*a*). (*a*) According to S, what is the time necessary for the rocket to pass a particular point on the platform? (*b*) What is the rest length L_0 of the rocket? (*c*) According to an observer S' on the rocket, what is the length D of the platform? (*d*) According to S', how long does it take for observer S to pass the entire length of the rocket? (*e*) According to S, the ends of the rocket simultaneously line up with the ends of the platform. Are these events simultaneous to S'?

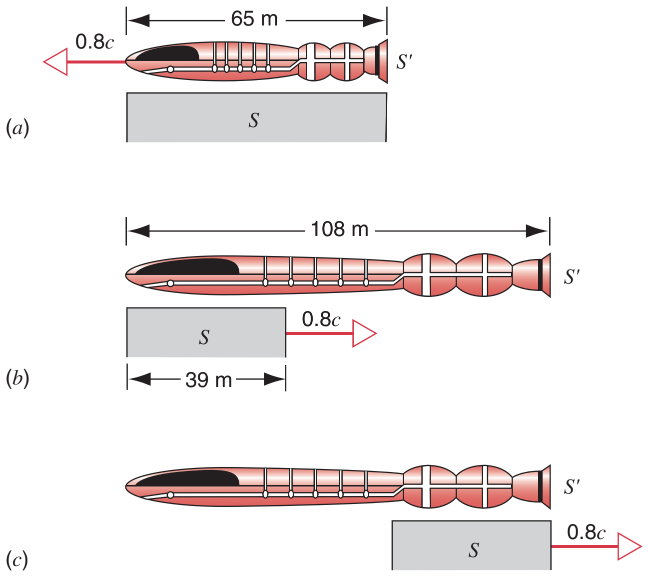


FIGURE 20-19. Sample Problem 20-5. (*a*) From the reference frame of S at rest on the platform, the passing rocket lines up simultaneously with the front and back of the platform. (*b*, *c*) From the reference frame of the rocket, the passing platform lines up first with the front of the rocket and later with the rear. Note the differing effects of length contraction in the two reference frames.

Solution (*a*) According to S, the length L of the rocket matches the length D_0 of the platform. The time for the rocket to pass a particular point is measured by S to be

$$\Delta t_0 = \frac{L}{0.80c} = \frac{65\text{ m}}{2.40 \times 10^8\text{ m/s}} = 0.27\ \mu\text{s}.$$

This is a proper time interval, because S is measuring the time interval between two events that occur at the same point in the frame of reference of S (the front of the rocket passes a point, and then the back of the rocket passes the same point).

(*b*) S measures the contracted length L of the rocket. We can find its rest length L_0 using Eq. 20-8:

$$L_0 = \frac{L}{\sqrt{1 - u^2/c^2}} = \frac{65\text{ m}}{\sqrt{1 - (0.80)^2}} = 108\text{ m}.$$

(*c*) According to S the platform is at rest, so 65 m is its rest length D_0. According to S', the contracted length of the platform is therefore

$$D = D_0\sqrt{1 - u^2/c^2} = (65\text{ m})\sqrt{1 - (0.80)^2} = 39\text{ m}.$$

(*d*) For S to pass the entire length of the rocket, S' concludes that S must move a distance equal to its rest length, or 108 m. The time needed to do this is

$$\Delta t' = \frac{108\text{ m}}{0.80c} = 0.45\ \mu\text{s}.$$

Note that this is *not* a proper time interval for S', who determines this time interval using one clock at the front of the rocket to measure the time at which S passes the front of the rocket, and another clock on the rear of the rocket to measure the time at which S passes the rear of the rocket. The two events therefore occur at different points in S' and so cannot be separated by a proper time in S'. The corresponding time interval measured by S for the same two events, which we calculated in part (*a*), *is* a proper time interval for S, because the two events *do* occur at the same point in S. The time intervals measured by S and S' should be related by the time dilation formula:

$$\Delta t' = \gamma\,\Delta t = \frac{0.27\ \mu\text{s}}{\sqrt{1 - (0.80)^2}} = 0.45\ \mu\text{s},$$

in agreement with the value calculated above from the proper length of the rocket in S'.

(*e*) According to S', the rocket has a rest length of $L_0 = 108$ m and the platform has a contracted length of $D = 39$ m. There is thus no way that S' could observe the two ends of both to align simultaneously. The sequence of events according to S' is illustrated in Figs. 20-19*b* and 20-19*c*. The time interval $\Delta t'$ in S' between the two events that are simultaneous in S can be calculated from the interval equation for $\Delta t'$ in Table 20-2 with $\Delta t = 0$, which gives

$$\Delta t' = -\gamma u\,\Delta x/c^2 = \frac{-(0.80c)(-65\text{ m})}{c^2\sqrt{1 - (0.80)^2}} = 0.29\ \mu\text{s}.$$

We can check this result by noting that, according to S', the time interval between the situations shown in Figs. 20-19*b* and 20-19*c* must be that necessary for the platform to move a distance of 108 m − 39 m = 69 m, which takes a time

$$\Delta t' = \frac{69\text{ m}}{0.80c} = 0.29\ \mu\text{s},$$

in agreement with the value calculated from the interval transformation. This last result illustrates the relativity of simultaneity: two events that are simultaneous to S (the lining up of the two ends of the rocket with the two ends of the platform) *cannot* be simultaneous to S'.

20-8 RELATIVISTIC MOMENTUM

So far we have investigated the effect of Einstein's two postulates on the kinematical variables time, displacement, and velocity as viewed from two different inertial frames. In this section and the next, we broaden our efforts to include the dynamical variables momentum and energy. Here we discuss the relativistic view of linear momentum.

Consider the collision shown in Fig. 20-20*a*, viewed from the S frame of reference. Two particles, each of mass m, move with equal and opposite velocities v and $-v$ along the x axis. They collide at the origin, and the distance between their lines of approach has been adjusted so that after the collision the particles move along the y axis with equal and opposite final velocities (Fig. 20-20*b*). We assume the collision to be perfectly elastic, so that no kinetic energy is lost. The final velocities must then be v and $-v$.

Using the classical formula ($\vec{\mathbf{p}} = m\vec{\mathbf{v}}$), the components of the total momentum $\vec{\mathbf{P}}$ of the two-particle system in the S frame are

$$\begin{aligned} \text{Initial:}\quad & P_{xi} = mv + m(-v) = 0, \\ & P_{yi} = 0. \\ \text{Final:}\quad & P_{xf} = 0, \\ & P_{yf} = mv + m(-v) = 0. \end{aligned}$$

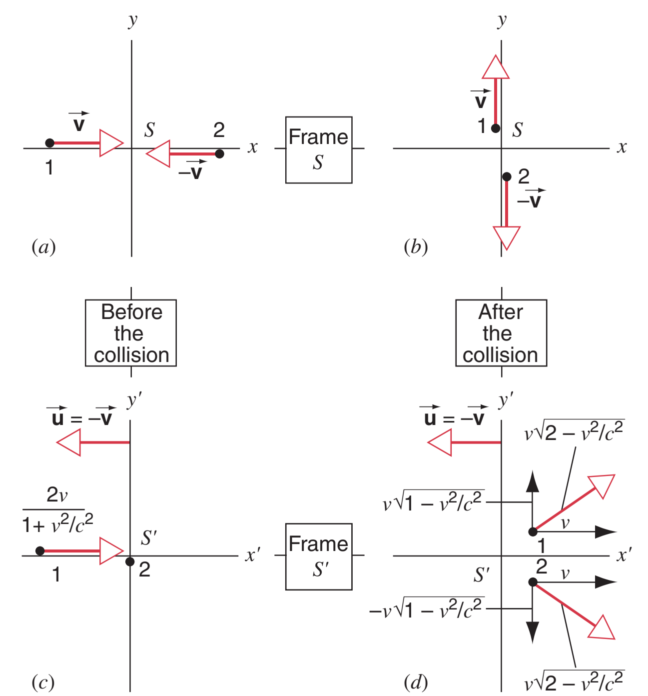


FIGURE 20-20. A collision between two particles of the same mass is shown (*a*) before the collision in the reference frame of *S*, (*b*) after the collision in the reference frame of *S*, (*c*) before the collision in the reference frame of *S*′, and (*d*) after the collision in the reference frame of *S*′.

Thus, $P_{xi} = P_{xf}$ and $P_{yi} = P_{yf}$; the initial (vector) momentum is equal to the final momentum, and momentum is conserved in the *S* frame.

Let us now view the same collision from the *S*′ frame, which moves relative to the *S* frame with speed $u = -v$ (Fig. 20-20*c*). Note that in the *S*′ frame, particle 2 is at rest before the collision. We use the Lorentz velocity transformation, Eqs. 20-18 and 20-19, to find the transformed x' and y' components of the initial and final velocities, as they would be observed by *S*′. These values, which you should calculate, are shown in Figs. 20-20*c* and 20-20*d*.

We now use those velocities to find the components of the total momentum of the system in the *S*′ frame:

$$P'_{xi} = m\left(\frac{2v}{1 + v^2/c^2}\right) + m(0) = \frac{2mv}{1 + v^2/c^2},$$

$$P'_{yi} = 0,$$

$$P'_{xf} = mv + mv = 2mv,$$

$$P'_{yf} = mv\sqrt{1 - v^2/c^2} + m(-v\sqrt{1 - v^2/c^2}) = 0.$$

We see that P'_{xi} is *not* equal to P'_{xf}, and *S*′ will conclude that momentum is *not* conserved.

It is clear from the above calculation that the law of conservation of linear momentum, which we have found useful in a variety of applications, does not satisfy Einstein's first postulate (the law must be the same in all inertial frames) if we calculate momentum as $p = mv$. Therefore, *if we are to retain the conservation of momentum as a general law consistent with Einsteins' first postulate, we must find a new definition of momentum.* This new definition of momentum must have two properties. (1) It must yield a law of conservation of momentum that satisfies the principle of relativity; that is, if momentum is conserved according to an observer in one inertial frame, then it is conserved according to observers in *all* inertial frames. (2) At low speeds, the new definition must reduce to $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$, which we know works perfectly well in the nonrelativistic case.

The relativistic formula for the momentum of a particle of mass *m* moving with velocity $\vec{\mathbf{v}}$ is

$$\vec{\mathbf{p}} = \frac{m\vec{\mathbf{v}}}{\sqrt{1 - v^2/c^2}}. \tag{20-23}$$

In terms of components, we can write Eq. 20-23 as

$$p_x = \frac{mv_x}{\sqrt{1 - v^2/c^2}} \quad \text{and} \quad p_y = \frac{mv_y}{\sqrt{1 - v^2/c^2}}. \tag{20-24}$$

The speed *v* that appears in the denominator of these expressions is *always* the speed of the particle as measured in a particular inertial frame. It is *not* the speed of an inertial frame. The velocity in the numerator can be any of the components of the velocity vector.

Let us see how this new definition restores conservation of momentum in the collision we considered. In the *S* frame, the velocities before and after are equal and opposite, and thus Eq. 20-23 again gives zero for the initial and final momenta. In the *S*′ frame, we can use the magnitudes of the velocities as given in Figs. 20-20*c* and 20-20*d* to obtain, as you should verify

$$P'_{xi} = P'_{xf} = \frac{2mv}{1 - v^2/c^2}, \qquad P'_{yi} = P'_{yf} = 0. \tag{20-25}$$

Thus the initial and final momenta are equal in the *S*′ frame. Momentum is conserved in both the *S* and *S*′ frames. In fact, the definition of momentum given in Eq. 20-23 gives conservation of momentum in *all* inertial frames, as required by the principle of relativity.

Note also that, in the limit of low speeds, the denominator of Eq. 20-23 is nearly equal to 1; at low speeds Eq. 20-23 reduces to the familiar classical formula $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$. Equation 20-23 thus also satisfies this necessary criterion of relativistic formulas.

Of course, the ultimate test is agreement with experiment. Figure 20-21 shows a collection of data, based on independent determinations of the momentum and velocity of electrons. The data are plotted as p/mv, which should have the constant value 1 according to classical physics. The results agree with the relativistic equation and not with the classical one. Note that the classical and relativistic predictions agree for low speeds, and in fact the difference between the two is not at all apparent until the speed exceeds $0.1c$, which accounts for our failure to observe the relativistic corrections in experiments with ordinary laboratory objects.

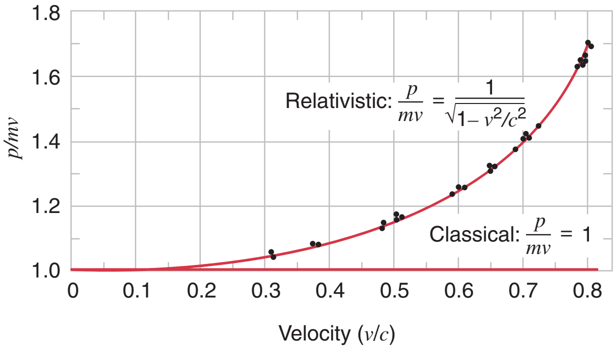


FIGURE 20-21. The ratio p/mv is plotted for electrons of various speeds. According to classical physics, $p = mv$, and thus the classical equations predict $p/mv = 1$. The data clearly agree with the relativistic result and not with the classical result. At low speeds, the classical and relativistic predictions are indistinguishable.

SAMPLE PROBLEM 20-6. What is the momentum of a proton moving at a speed of $v = 0.86c$?

Solution Using Eq. 20-23, we obtain

$$\begin{aligned} p &= \frac{mv}{\sqrt{1 - v^2/c^2}} \\ &= \frac{(1.67 \times 10^{-27}\ \text{kg})(0.86)(3.00 \times 10^8\ \text{m/s})}{\sqrt{1 - (0.86)^2}} \\ &= 8.44 \times 10^{-19}\ \text{kg} \cdot \text{m/s}. \end{aligned}$$

The units of kg · m/s are generally not convenient in solving problems of this type. Instead, we evaluate the quantity pc:

$$\begin{aligned} pc &= (8.44 \times 10^{-19}\ \text{kg} \cdot \text{m/s})(3.00 \times 10^8\ \text{m/s}) \\ &= 2.53 \times 10^{-10}\ \text{J} \\ &= 1580\ \text{MeV}, \end{aligned}$$

where we have used the conversion factor 1 MeV = 1.60×10^{-13} J. The momentum is obtained from this result by dividing by the symbol c (not its numerical value), which gives

$$p = 1580\ \text{MeV}/c.$$

The units of MeV/c for momentum are often used in relativistic calculations because, as we show in the next section, the quantity pc often appears in these calculations.

20-9 RELATIVISTIC ENERGY

In analogy with our discussion of momentum in the previous section, special relativity gives us a different approach to kinetic energy. Let us first indicate the difficulty by reconsidering the collision shown in Fig. 20-20. If we use the classical expression $\frac{1}{2}mv^2$, the collision does not conserve kinetic energy in the S' frame. (We chose the final velocities in the S frame so that kinetic energy would be conserved.) Using the velocities shown in Figs. 20-20c and 20-20d, you can show (see Exercise 38) that, with $K = \frac{1}{2}mv^2$, the total initial and final kinetic energies are

$$K'_\text{i} = \frac{2mv^2}{(1 + v^2/c^2)^2}, \qquad K'_\text{f} = mv^2(2 - v^2/c^2). \tag{20-26}$$

Thus K'_i is not equal to K'_f, and the collision, which conserves kinetic energy according to S ($K_\text{i} = K_\text{f}$) and is therefore elastic, does not conserve kinetic energy according to S' and is therefore inelastic. This situation violates the relativity postulate; the type of collision (elastic versus inelastic) should depend on the properties of the colliding objects and not on the particular reference frame from which we happen to be viewing the collision. As was the case with momentum, we require a new definition of kinetic energy if we are to preserve the law of conservation of energy and the relativity postulate.

The classical expression for kinetic energy also violates the second relativity postulate by allowing speeds in excess of the speed of light. There is no limit (in either classical or relativistic dynamics) to the energy we can give to a particle. Yet, if we allow the kinetic energy to increase without limit, the classical expression $K = \frac{1}{2}mv^2$ implies that the velocity must correspondingly increase without limit, thereby violating the second postulate. We must therefore find a way to redefine kinetic energy, so that the kinetic energy of a particle can be increased without limit while its speed remains less than c.

The relativistic expression for the kinetic energy of a particle can be derived using essentially the same procedure we used to derive the classical expression, starting with the particle form of the work–energy theorem (see Problem 16). The result of this calculation is

$$K = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2. \tag{20-27}$$

Equation 20-27 looks very different from the classical expression $K = \frac{1}{2}mv^2$, but we can show (see Exercise 35) that Eq. 20-27 does reduce to the classical expression in the limit of low speeds ($v \ll c$). You can also see from Eq. 20-27 that the relativistic expression for kinetic energy allows a particle to have unlimited energy even though its speed remains less than the speed of light. Figure 20-3 showed a comparison of the classical and relativistic forms of the dependence of kinetic energy on speed; clearly the relativistic form is in much better agreement with the data than the classical form, and the relativistic expression allows K to become very large while v remains less than c. Similar results are obtained indirectly today at every large accelerator facility in the world. Particles are accelerated to speeds very close to c, and the design parameters of the accelerators must be based on relativistic dynamics. Thus every modern accelerator is in effect a laboratory for testing special relativity. Needless to say, the success of these accelerators is a dramatic confirmation of special relativity.

Using Eq. 20-27 for the kinetic energy, you can show that kinetic energy is conserved in the S' frame of the collision of Fig. 20-20 (see Exercise 39).

SAMPLE PROBLEM 20-7. In the Stanford Linear Collider* electrons are accelerated to a kinetic energy of 50 GeV. Find the speed of such an electron as (*a*) a fraction of c and (*b*) a difference from c. For the electron, $mc^2 = 0.511$ MeV $= 0.511 \times 10^{-3}$ GeV.

Solution (*a*) First we solve Eq. 20-27 for v, obtaining

$$v = c\sqrt{1 - \frac{1}{(1 + K/mc^2)^2}}\,, \tag{20-28}$$

and thus

$$v = c\sqrt{1 - \frac{1}{(1 + 50\text{ GeV}/0.511 \times 10^{-3}\text{ GeV})^2}}$$
$$= 0.999\,999\,999\,948c.$$

Calculators cannot be trusted to 12 significant digits. Here is a way to avoid this difficulty. We can write Eq. 20-28 as $v = c(1 + x)^{1/2}$, where $x = -1/(1 + K/mc^2)^2$. Because $K \gg mc^2$, we have $x \ll 1$, and we can use the binomial expansion to write $v \approx c(1 + \frac{1}{2}x)$, or

$$v \approx c\left[1 - \frac{1}{2(1 + K/mc^2)^2}\right], \tag{20-29}$$

which gives

$$v = c(1 - 5.2 \times 10^{-11}).$$

This leads to the value of v given above.

(*b*) From the above result, we have

$$c - v = 5.2 \times 10^{-11}c = 0.016\text{ m/s} = 1.6\text{ cm/s}.$$

Energy and Mass in Special Relativity

We can also express Eq. 20-27 as

$$K = E - E_0, \tag{20-30}$$

where the *total relativistic energy* E is defined as

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}\,, \tag{20-31}$$

and the *rest energy* E_0 is defined as

$$E_0 = mc^2. \tag{20-32}$$

The rest energy is in effect the total relativistic energy of a particle measured in a frame of reference in which the particle is at rest.†

The rest energy can be regarded as the internal energy of a particle or a system of particles at rest. According to Eq. 20-32, whenever we add energy ΔE to a material object that remains at rest, we increase its mass by an amount $\Delta m = \Delta E/c^2$. If we compress a spring and increase its potential energy by an amount ΔU, then its mass increases by $\Delta U/c^2$. If we raise the temperature of an object, increasing its internal energy by ΔE_{int} in the process, we increase its mass by $\Delta E_{\text{int}}/c^2$. These mass changes are very small and normally beyond our ability to measure in the case of ordinary objects (because c^2 is a very large number), but in the case of decays and reactions of nuclei and subnuclear particles, the relative mass change can be large enough to be measurable.

The inclusion of rest energy as another form of energy allows us to interpret the situation of Fig. 20-2 from the viewpoint of conservation of energy. In Fig. 20-2*a*, the total energy within the system boundary is $E_i = 2m_e c^2$, where m_e is the mass of an electron or a positron (we neglect the small kinetic energies of the particles). In Fig. 20-2*b*, we have radiation whose total energy E_R is equal to E_i, and the increase in internal energy in Fig. 20-2*c* is again equal to E_i. Thus total energy is conserved, and the process in effect represents the transformation of one form of energy (rest energy) into another (energy of the radiation or internal energy of the walls of the container).

Equation 20-32 suggests that we must include rest energy among the kinds of energy that can characterize a system. The sum of all possible kinds of energy, that is, the *total relativistic energy,* must be conserved in any interaction. There is no requirement that, for example, rest energy and kinetic energy be conserved separately and, indeed, they are not.

Consider the radioactive decay process in which a heavy nucleus splits into two smaller fragments, a process called *spontaneous fission.* To conserve momentum, the two fragments must fly away from each other. If the initial nucleus is originally at rest, it is clear that kinetic energy is not conserved in this process. Measurement shows that the total rest energy of the two fragments is smaller than the rest energy of the original nucleus; thus rest energy is not conserved either. However, the sum of rest energy plus kinetic energy is conserved, the decrease in rest energy accounting precisely for the increase in kinetic energy.

We can apply Eq. 20-32 to other isolated systems consisting of particles and radiation. Let us consider a star such as the Sun as our system. The Sun radiates an energy of 4×10^{26} J every second. As we did in the case of electron–positron annihilation, we regard this radiant energy as a decrease in the rest energy of the system, and the corresponding change in the mass is

$$\Delta m = \frac{\Delta E_0}{c^2} = \frac{-4 \times 10^{26}\text{ J}}{(3 \times 10^8\text{ m/s})^2} = -4 \times 10^9\text{ kg}$$

in every second. This decrease in mass is quite significant by ordinary standards but quite small compared with the total mass of the Sun (2×10^{30} kg). In one year, the Sun's mass decreases by a fraction of only 6×10^{-14}.

SAMPLE PROBLEM 20-8. Two 35-g putty balls are thrown toward each other, each with a speed of 1.7 m/s. The balls strike each other head-on and stick together. By how much does

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* See "The Stanford Linear Collider," by John R. Rees, *Scientific American,* October 1989, p. 58.

† Often m in Eq. 20-32 is called the *rest mass* m_0 and is distinguished from the "relativistic mass," which is defined as $m_0/\sqrt{1 - v^2/c^2}$. We choose not to use relativistic mass, because it can be a misleading concept. Whenever we refer to mass, we always mean rest mass.

the mass of the combined ball differ from the sum of the masses of the two original balls?

Solution We treat the two putty balls as an isolated system. No external work is done on this isolated system, so we can write conservation of energy as $\Delta K + \Delta E_0 = 0$. With $\Delta K = K_f - K_i$, where $K_f = 0$ and K_i is the total kinetic energy of the two balls before the collision, we have

$$\Delta K + \Delta E_0 = (0 - K_i) + \Delta E_0 = 0,$$

or

$$\Delta E_0 = K_i = 2(\tfrac{1}{2}mv^2) = (0.035 \text{ kg})(1.7 \text{ m/s})^2 = 0.101 \text{ J}.$$

This increase in rest energy might be in the form of internal energy, perhaps resulting in an increase in the temperature of the combined system. The corresponding increase in mass is

$$\Delta m = \frac{\Delta E_0}{c^2} = \frac{0.101 \text{ J}}{(3.00 \times 10^8 \text{ m/s})^2} = 1.1 \times 10^{-18} \text{ kg}.$$

Such a tiny increase in mass is hopelessly beyond our ability to measure.

Sample Problem 20-9. In a 1989 experiment at the Stanford Linear Collider, Z^0 particles were produced when a beam of electrons collided head-on with a beam of positrons of the same kinetic energy. Find the kinetic energy of the two beams needed to produce the Z^0, which has a rest energy of 91.2 GeV (1 GeV = 10^9 eV).

Solution As in the collision between the putty balls considered in Sample Problem 20-8, let us assume that the system consisting of the initial e^+ and e^- is isolated, and that no energy is exchanged with the surroundings in the process of forming the Z^0 particle. The change in rest energy between the initial state (an electron and a positron of rest energy 0.511 MeV each) and the final state (the Z^0) is

$$\Delta E_0 = 91.2 \text{ GeV} - 2(0.511 \text{ MeV}) = 91.2 \text{ GeV},$$

the total rest energy of the electron and positron (1.022 MeV = 0.001022 GeV) being quite neglible here. From conservation of energy in this isolated system, we have $\Delta K + \Delta E_0 = 0$, so

$$\Delta K = -\Delta E_0 = -91.2 \text{ GeV} = K_f - K_i.$$

If we assume that the Z^0 is produced at rest, then $K_f = 0$ and the energies of the positron and electron must each be $\frac{1}{2}(91.2 \text{ GeV}) = 45.6$ GeV. In contrast with the previous sample problem, the relative change in rest energy (or in mass) within the system is substantial in this case, the final mass being about 100,000 times the initial mass.

Conservation of Total Relativistic Energy

The total relativistic energy is given by Eq. 20-30 as

$$E = K + E_0. \qquad (20\text{-}33)$$

In interactions of particles at relativistic speeds, we can replace our previous principle of conservation of energy with one based on the total relativistic energy:

In an isolated system of particles, the total relativistic energy remains constant.

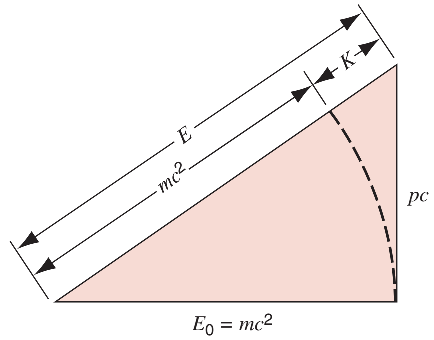


Figure 20-22. A useful mnemonic device for recalling the relationships between E_0, p, K, and E. Note that to put all variables in energy units, the quantity pc must be used.

Manipulation of Eqs. 20-23 and 20-31 gives a useful relationship among the total energy, momentum, and rest energy:

$$E = \sqrt{(pc)^2 + (mc^2)^2}. \qquad (20\text{-}34)$$

Figure 20-22 shows a useful mnemonic device for remembering this relationship, which has the form of the Pythagorean theorem for the sides of a right triangle.

Because the rest energies of the initial and final particles were equal in the collision shown in Fig. 20-20, conservation of total relativistic energy is equivalent to conservation of kinetic energy for that collision. In general, collisions of particles at high energies can result in the production of new particles, and thus the final rest energy may not be equal to the initial rest energy (see Sample Problem 20-11). Such collisions must be analyzed using conservation of total relativistic energy E; kinetic energy is not conserved when the rest energy changes in a collision.

Sample Problem 20-10. A certain accelerator produces a beam of neutral kaons ($m_K c^2 = 498$ MeV) with kinetic energy 325 MeV. Consider a kaon that decays in flight into two pions ($m_\pi c^2 = 140$ MeV). Find the kinetic energy of each pion in the special case in which the pions travel parallel or antiparallel to the direction of the kaon beam.

Solution The energy of the particles that remain after the decay can be found by applying principles of conservation of total relativistic energy and momentum. The initial total relativistic energy is, from Eq. 20-33,

$$E_K = K + m_K c^2 = 325 \text{ MeV} + 498 \text{ MeV} = 823 \text{ MeV}.$$

The initial momentum can be found from Eq. 20-34:

$$p_K c = \sqrt{E_K^2 - (m_K c^2)^2} = \sqrt{(823 \text{ MeV})^2 - (498 \text{ MeV})^2} = 655 \text{ MeV}.$$

The total energy of the final system consisting of the two pions is

$$E = E_1 + E_2 = \sqrt{(p_1 c)^2 + (m_\pi c^2)^2} + \sqrt{(p_2 c)^2 + (m_\pi c^2)^2} = 823 \text{ MeV}, \qquad (20\text{-}35)$$

which, by conservation of total relativistic energy, we have equated to the initial total energy of 823 MeV. Thus we have one equation in the two unknowns p_1 and p_2.

To find a second equation in the two unknowns we apply conservation of momentum. The final momentum of the two-pion system along the beam direction is $p_1 + p_2$, and setting this equal to the initial momentum p_K gives

$$p_1c + p_2c = p_Kc = 655 \text{ MeV}. \qquad (20\text{-}36)$$

We now have two equations (Eqs. 20-35 and 20-36) in the two unknowns p_1 and p_2. Solving Eq. 20-36 for p_2c and substituting this result into Eq. 20-35, we obtain (after some algebraic manipulation) a quadratic equation for p_1c, which can be solved by standard algebraic techniques to give

$$p_1c = 668 \text{ MeV or } -13 \text{ MeV}.$$

Since the labels 1 and 2 of the two pions are arbitrary, the solution gives one pion traveling parallel to the beam with momentum $p_1 = 668$ MeV/c, while the other pion travels in the opposite direction with momentum $p_2 = -13$ MeV/c. The corresponding kinetic energies are found using Eqs. 20-30 and 20-34, which give

$$K = \sqrt{(pc)^2 + (m_\pi c^2)^2} - m_\pi c^2$$

$$K_1 = \sqrt{(668 \text{ MeV})^2 + (140 \text{ MeV})^2} - 140 \text{ MeV} = 543 \text{ MeV},$$

$$K_2 = \sqrt{(-13 \text{ MeV})^2 + (140 \text{ MeV})^2} - 140 \text{ MeV} = 0.6 \text{ MeV}.$$

This problem can also be solved in a different way by making a Lorentz transformation to a reference frame in which the kaons are at rest. The two pions are emitted in this frame in opposite directions (because the total momentum must be zero), and so they share the decay energy equally. Transforming back to the lab frame then gives the solution for the momenta and energies (see Exercise 43). The next sample problem demonstrates another application of this technique.

SAMPLE PROBLEM 20-11. The discovery of the antiproton $\bar{\text{p}}$ (a particle with the same rest energy as a proton, 938 MeV, but with the opposite electric charge) took place in 1956 at Berkeley through the following reaction:

$$\text{p} + \text{p} \rightarrow \text{p} + \text{p} + \text{p} + \bar{\text{p}},$$

in which accelerated protons were incident on a target of protons at rest in the laboratory. The minimum incident kinetic energy needed to produce the reaction is called the *threshold* kinetic energy, for which the final particles move together as if they were a single unit. Find the threshold kinetic energy to produce antiprotons in this reaction.

Solution This problem is conceptually the reverse case of the previous sample problem. Here particles are coming together to form a composite. We demonstrate an alternate method by solving in the center-of-mass reference frame, in which the two protons come together with equal and opposite momenta to form a new particle at rest (Fig. 20-23).

The final total relativistic energy in the center-of-mass frame S' is the rest energy of the products, which are produced at rest in this frame, so

$$E'_f = 4m_pc^2.$$

The initial energy is simply the sum of the total energies of the two original reacting protons:

$$E'_i = E'_1 + E'_2.$$

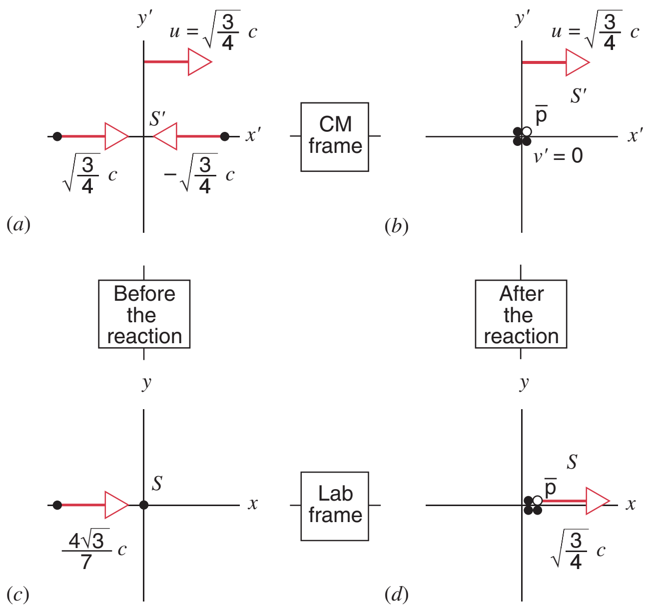


FIGURE 20-23. Sample Problem 20-11. The production of an antiproton, viewed from (a, b) the center-of-mass frame and (c, d) the laboratory frame. Compare with Fig. 20-20.

Conservation of energy requires $E'_i = E'_f$, and since the energies E'_1 and E'_2 are equal in the S' frame, we have

$$E'_1 = E'_2 = 2m_pc^2.$$

The corresponding magnitude of the velocity of either reacting proton in the S' frame is found by solving Eq. 20-31 for v/c, which gives

$$\frac{v'_1}{c} = \sqrt{1 - \left(\frac{m_pc^2}{E'_1}\right)^2} = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}}.$$

We now make a Lorentz transformation back to the laboratory using this as the transformation speed, which brings one of the protons to rest and gives the other a velocity v that can be found from the inverse velocity transformation expression for v_x from Table 20-3. Using $v' = c\sqrt{3/4}$ and $u = c\sqrt{3/4}$, and dropping the x subscript, we have

$$v = \frac{v' + u}{1 + uv'/c^2} = \frac{2c\sqrt{3/4}}{1 + (\sqrt{3/4})^2} = \frac{4\sqrt{3}}{7}c.$$

This is the speed of the incident proton in the laboratory frame. Its total energy can be found from Eq. 20-31:

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \frac{m_pc^2}{\sqrt{1 - (4\sqrt{3}/7)^2}} = 7m_pc^2,$$

and the threshold kinetic energy is

$$K = E - m_pc^2 = 6m_pc^2 = 6(938 \text{ MeV}) = 5628 \text{ MeV} = 5.628 \text{ GeV}.$$

The Bevatron accelerator at Berkeley was designed with this experiment in mind, so that it could produce a beam of protons whose energy exceeded 5.6 GeV. The discovery of the antiproton in this reaction was honored with the award of the 1959 Nobel prize to the experimenters, Emilio Segrè and Owen Chamberlain.

20-10 THE COMMON SENSE OF SPECIAL RELATIVITY

We have reached a point where we can look back at our presentation of special relativity and think about its common sense. We must first of all note that relativity affects every aspect of physics; we have concentrated in this chapter on mechanics, and later in this text we consider the effect of relativity on electromagnetism. Indeed, we must carefully reexamine every subfield of physics from the perspective of special relativity, verifying that each is consistent with the two postulates. We must also note that relativity has passed every experimental test without the slightest discrepancy. It is a theory that is of great aesthetic value, providing us with a view more satisfying than that of classical physics about the validity of different perspectives and symmetries. It is also a theory of great practical value, providing engineers with the proper guidance to construct large particle accelerators and providing those concerned about maintaining standards with the proper procedures for correcting the readings of atomic clocks when they are transported from one location to another.

The first postulate of relativity is really an outgrowth of Newton's first law, the law of inertia, which defined the concept of inertial frames and gave us the first notion that inertial observers would draw identical conclusions from observing an experiment in which no net force acts. It is not too great a leap to extend that view to assert that inertial observers should also draw identical conclusions from observing an experiment in which there *is* a net force. Finally, why should we single out the laws of mechanics for this equivalence? By extending it to an equivalence for inertial observers of *all* the laws of physics, we arrive at the first postulate.

The second postulate is also a reasonable one. It seems unrealistic to be able to transmit a signal at an infinite speed, thereby providing instantaneous communication throughout the universe. Moreover, experiments on the relativity of time show that such instant communication between distant points is not consistent with observation. If there is a limiting speed, then surely (by the first postulate) it must be the same for all observers, regardless of their state of motion.

For some, the first exposure to the relativity of simultaneity, the apparent shrinking of moving rods, and the slowing down of time may be disturbing. However, a bit of thought will persuade you that the classical alternatives are even more disturbing. For example, a classical rigid rod of definite length is not a concept that is consistent with relativity; a signal (say, a quick movement) at one end cannot be transmitted instantly to the other end. We must give up the idea of all observers being able to use the *same* measuring rod. We replace this idea with one that gives each observer a measuring rod and permits that observer to use that rod to make measurements within a particular frame of reference. No observer's measuring instruments or results are preferred over any other's. Finally, relativity gives us a wonderful symmetry between these observers; it does not assert the *reality* of slowing clocks, but rather that, from their two differing perspectives, two observers in relative motion each observe that the other's clock is slow. There is no necessity to grant preferred status to either of them, or to any other inertial observer.

According to classical physics, space and time are absolute. This leads to the result that the laws of physics must be different for different observers. Relativity, on the other hand, tells us that the laws of physics must be the same for all observers, and as a consequence space and time become relative concepts. Clearly, relativity is "more absolute" than classical physics. The arbitrary and complex physical world of classical physics, in which each observer must use a different set of physical laws, becomes the more uniform and simple physical world of relativity.

Relativity broadens our view of the universe by placing us among the many inertial observers of that universe. It brings together concepts that, according to classical physics, were treated separately: for instance, space and time into space–time, or mass and energy into rest energy. It points the way toward a single unifying theory that includes all possible interactions between particles: electricity and magnetism into electromagnetism; electromagnetism and the so-called weak forces (those responsible for certain radioactive decay processes) into the electroweak interaction; the electroweak and the strong nuclear interactions into one of the proposed Grand Unified Theories (GUTs); and finally GUTs and gravity into the hypothetical Theory of Everything. Einstein, who knew about only the first of these unifications, would surely be very pleased at these developments.

MULTIPLE CHOICE

20-1 Troubles with Classical Physics

20-2 The Postulates of Special Relativity

20-3 Consequences of Einstein's Postulates

1. A pilot on a space ship moving at $0.86c$ away from the Earth sends a laser beam signal to Earth.

(*a*) The pilot measures the speed v of the laser beam signal to be

(A) $v < c$. (B) $v = c$. (C) $v > c$.

(*b*) The people on Earth measure the speed v of the laser beam signal to be

(A) $v < c$. (B) $v = c$. (C) $v > c$.

2. A 20-m-long spaceship (as measured by a pilot on the spaceship) travels at constant speed past a 40-m-long space dock (as measured by a worker on the dock). The worker measures the length of the ship as it passes and finds it to be 18 m long.

(*a*) How fast is the ship moving past the dock?

(A) $v \approx c/100$ (B) $v \approx c/10$ (C) $v \approx c/2$
(D) $v = c$ (E) $v > c$

(*b*) The pilot measures the length of the dock as

(A) 36 m. (B) 38 m. (C) 42 m. (D) 44 m.

(*c*) The pilot observes a clock on the dock for one minute (according to a clock on the ship). The clock on the dock, however, will show an elapsed time of

(A) 49 s. (B) 54 s. (C) 60 s. (D) 67 s.

(*d*) The dock worker observes a clock on the ship for one minute (according to a clock on the dock). The clock on the ship, however, will show an elapsed time of

(A) 54 s. (B) 60 s. (C) 67 s. (D) 78 s.

(*e*) The pilot launches a missile with a speed of 0.9c relative to the ship. The speed of the missile relative to the dock is

(A) greater than (B) equal to (C) less than

the speed of light, as measured by the dock worker.

20-4 The Lorentz Transformation

20-5 Measuring the Space–Time Coordinates of an Event

20-6 The Transformation of Velocities

3. A spaceship is moving at 0.90c away from the Earth. The speed of the Earth, as measured by the space ship, is

(A) 0. (B) 0.45c. (C) 0.90c. (D) 1.9c.

4. A spaceship moving at 0.60c away from the Earth fires two missiles, one directly forward and one directly backward. Both missiles are fired with a speed of 0.80c relative to the ship.

(*a*) The missile fired away from the Earth has a speed v_1 relative to the Earth, where

(A) $v_1 < 0.6c$. (B) $0.6c < v_1 < 0.8c$.
(C) $0.8c < v_1 < c$. (D) $v_1 = 1.4c$.
(E) $v_1 > 1.4c$.

(*b*) The missile fired toward the Earth has a speed v_2 relative to the Earth, where

(A) $v_2 < 0.2c$. (B) $v_2 = 0.2c$.
(C) $0.2c < v_2 < 0.6c$. (D) $0.6c < v_2 < 0.8c$.
(E) $0.8c < v_2 < c$.

(*c*) The speed of the missile fired away from the Earth as measured by the other missile is v_3, where

(A) $v_3 < 0.6c$. (B) $0.6c < v_3 < 0.8c$.
(C) $0.8c < v_3 < c$. (D) $v_3 = 1.6c$.
(E) $v_3 > 1.6c$.

20-7 Consequences of the Lorentz Transformation

5. Two events A and B are simultaneous and occur at the same location in frame S. In any other frame S'

(A) the events will occur at the same location, but could occur at different times.
(B) the events will occur at different locations, but will still be simultaneous.
(C) the events will be both simultaneous and at the same location.
(D) the events will be neither simultaneous nor at the same location.

6. Two events A and B are simultaneous but occur at different locations in frame S. In a different frame S'

(A) the events could be simultaneous and occur at the same location.
(B) the events could either be simultaneous or occur at the same location, but not both.
(C) the events could be simultaneous but cannot occur at the same location.
(D) the events cannot be simultaneous but could occur at the same location.
(E) the events cannot be simultaneous and cannot occur at the same location.

7. Two events A and B are not simultaneous but occur at the same location in frame S. In a different frame S'

(A) the events could be simultaneous and occur at the same location.
(B) the events could either be simultaneous or occur at the same location, but not both.
(C) the events could be simultaneous but cannot occur at the same location.
(D) the events cannot be simultaneous but could occur at the same location.
(E) the events cannot be simultaneous and cannot occur at the same location.

20-8 Relativistic Momentum

8. A particle of mass m and momentum of magnitude $2mc$ strikes a particle of mass m, which is at rest. The two particles stick together after collision.

(*a*) The speed of the moving particle *before* the collision is

(A) less than $c/2$. (B) between $c/2$ and c.
(C) between c and $2c$. (D) $2c$.

(*b*) The magnitude of the total momentum after the collision is

(A) less than $2mc$. (B) equal to $2mc$.
(C) between $2mc$ and $3mc$. (D) greater than $3mc$.

(*c*) The speed of the two particles after collision is

(A) less than $c/2$. (B) equal to $c/2$.
(C) between $c/2$ and c. (D) greater than c.

20-9 Relativistic Energy

9. Which of the following decays is prohibited by energy conservation? (See Appendix F.)

(A) $\pi^0 \rightarrow e^+ + e^-$
(B) $\pi^+ \rightarrow e^+ + \pi^0$
(C) $p \rightarrow n + e^+ + \bar{\nu}_e$
(D) $\rho^+ \rightarrow \pi^+ + \pi^+ + \pi^-$

10. (*a*) The speed of an electron that has a kinetic energy of $K \gg m_e c^2$ is v_1, where

(A) $v_1 \ll c$. (B) $v_1 \approx c$.
(C) $v_1 > c$. (D) $v_1 \gg c$.

(*b*) What would be the speed of an electron that has a kinetic energy of $4K$?

(A) Between v_1 and c (B) Less than $2v_1$
(C) Equal to $2v_1$ (D) Greater than $2v_1$
(E) Both (A) and (C) are correct.

11. An energetic proton emitted from the Sun has a total energy in excess of 100 GeV. As measured from the Earth, how much time elapses from when the proton leaves the Sun to when it strikes the Earth?

(A) ≈ 6 s (B) ≈ 1 min
(C) ≈ 9 min (D) ≈ 20 min
(E) The question cannot be answered without more knowledge of the proton's energy.

20-10 The Common Sense of Special Relativity

QUESTIONS

1. The speed of light in a vacuum is a true constant of nature, independent of the wavelength of the light or the choice of an (inertial) reference frame. Is there any sense, then, in which Einstein's second postulate can be viewed as contained within the scope of his first postulate?

2. Discuss the problem that young Einstein grappled with; that is, what would be the appearance of an electromagnetic wave to a person running along with it at speed c?

3. Is the concept of an incompressible fluid valid in relativity? What about perfectly rigid bodies?

4. A quasar (quasi-stellar object) travels away from the Earth at half the speed of light. What is the speed, with respect to the Earth, of the light we detect coming from it?

5. Quasars are the most intrinsically luminous objects in the universe. Many of them fluctuate in brightness, often on a time scale of a day or so. How can the rapidity of these brightness changes be used to estimate an upper limit to the size of these objects? (Hint: Separated points cannot change in a coordinated way unless information is sent from one to the other.)

6. The sweep rate of the tail of a comet can exceed the speed of light. Explain this phenomenon and show that there is no contradiction with relativity.

7. Consider a spherical light wavefront spreading out from a point source. As seen by an observer at the source, what is the difference in velocity of portions of the wavefront traveling in opposite directions? What is the relative velocity of one of these portions of the wavefront with respect to the other?

8. Borrowing two phrases from Herman Bondi, we can catch the spirit of Einstein's two postulates by labeling them: (1) the principle of "the irrelevance of velocity" and (2) the principle of "the uniqueness of light." In what senses are velocity irrelevant and light unique in these two statements?

9. A beam from a laser falls at right angles on a plane mirror and reflects from it. What is the speed of the reflected beam if the mirror is (*a*) fixed in the laboratory and (*b*) moving directly toward the laser with speed v?

10. Give an example from classical physics in which the motion of a clock affects its rate—that is, the way it runs. (The magnitude of the effect may depend on the detailed nature of the clock.)

11. Although in relativity (where motion is relative and not absolute) we find that "moving clocks run slow," this effect has nothing to do with the motion altering the way a clock works. With what does it have to do?

12. We have seen that if several observers watch two events, labeled A and B, one of them may say that event A occurred first but another may claim that it was event B that did so. What would you say to a friend who asked you which event really did occur first?

13. Let event A be the departure of an airplane from San Francisco and event B be its arrival in New York. Is it possible to find two observers who disagree about the time order of these events? Explain.

14. Two observers, one at rest in S and one at rest in S', each carry a meter stick oriented parallel to their relative motion. Each observer finds upon measurement that the other observer's meter stick is the shorter of the two sticks. Does this seem like a paradox to you? Explain. (Hint: Compare with the following situation. Harry waves goodbye to Walter who is in the rear of a station wagon driving away from Harry. Harry says that Walter gets smaller. Walter says that Harry gets smaller. Are they measuring the same thing?)

15. How does the concept of simultaneity enter into the measurement of the length of an object?

16. In relativity the time and space coordinates are intertwined and treated on a more or less equivalent basis. Are time and space fundamentally of the same nature, or is there some essential difference between them that is preserved even in relativity?

17. In the "twin paradox," explain (in terms of heartbeats, physical and mental activities, and so on) why the younger returning twin has not lived any longer than her own proper time even though her stay-at-home brother may say that she has. Hence explain the remark: "You age according to your own proper time."

18. If zero-mass particles have a speed c in one reference frame, can they be found at rest in any other frame? Can such particles have any speed other than c?

19. A particle with zero mass (a neutrino, possibly) can transport momentum. However, by Eq. 20-23, $p = mv/\sqrt{1 - v^2/c^2}$, the momentum is directly proportional to the mass and therefore should be zero if the mass is zero. Explain.

20. How many relativistic expressions can you think of in which the Lorentz factor γ enters as a simple multiplier?

21. Is the mass of a stable, composite particle (a gold nucleus, for example) greater than, equal to, or less than the sum of the masses of its constituents? Explain.

22. Sometimes the masses of elementary particles are given in units of MeV/c^2. For example, the mass of an electron is 0.511 MeV/c^2. Is this really a unit of mass? Explain.

23. "The relation $E_0 = mc^2$ is essential to the operation of a power plant based on nuclear fission but has only a negligible relevance for a fossil-fuel plant." Is this a true statement? Explain why or why not.

24. A hydroelectric plant generates electricity because water falls under gravity through a turbine, thereby turning the shaft of a generator. According to the mass–energy concept, must the appearance of energy (the electricity) be identified with a mass decrease somewhere? If so, where?

25. Some say that relativity complicates things. Give examples to the contrary, wherein relativity simplifies matters.

EXERCISES

20-1 Troubles with Classical Physics
20-2 The Postulates of Special Relativity
20-3 Consequences of Einstein's Postulates

1. Quite apart from effects due to the Earth's rotational and orbital motions, a laboratory frame is not strictly an inertial frame because a particle placed at rest there will not, in general, remain at rest; it will fall under gravity. Often, however, events happen so quickly that we can ignore free fall and treat the frame as inertial. Consider, for example, a 1.0-MeV electron (for which $v = 0.941c$) projected horizontally into a laboratory test chamber and moving through a distance of 20 cm. (*a*) How long would it take, and (*b*) how far would the electron fall during this interval? What can you conclude about the suitability of the laboratory as an inertial frame in this case?

2. A 100-MeV electron, for which $v = 0.999987c$, moves along the axis of an evacuated tube that has a length of 2.86 m as measured by a laboratory observer S with respect to whom the tube is at rest. An observer S' moving with the electron, however, would see this tube moving past with speed v. What length would this observer measure for the tube?

3. A rod lies parallel to the x axis of reference frame S, moving along this axis at a speed of $0.632c$. Its rest length is 1.68 m. What will be its measured length in frame S?

4. The mean lifetime of muons stopped in a lead block in the laboratory is measured to be 2.20 μs. The mean lifetime of high-speed muons in a burst of cosmic rays observed from the Earth is measured to be 16.0 μs. Find the speed of these cosmic ray muons.

5. An unstable high-energy particle enters a detector and leaves a track 1.05 mm long before it decays. Its speed relative to the detector was $0.992c$. What is its proper lifetime? That is, how long would it have lasted before decay had it been at rest with respect to the detector?

6. A particle moves along the x' axis of frame S' with a speed of $0.43c$. Frame S' moves with a speed of $0.587c$ with respect to frame S. What is the measured speed of the particle in frame S?

7. A spaceship of rest length 130 m drifts past a timing station at a speed of $0.740c$. (*a*) What is the length of the spaceship as measured by the timing station? (*b*) What time interval between the passage of the front and back end of the ship will the station monitor record?

8. A pion is created in the higher reaches of the Earth's atmosphere when an incoming high-energy cosmic-ray particle collides with an atomic nucleus. A pion so formed descends toward Earth with a speed of $0.99c$. In a reference frame in which they are at rest, pions have a lifetime of 26 ns. As measured in a frame fixed with respect to the Earth, how far will such a typical pion move through the atmosphere before it decays?

9. To circle the Earth in low orbit a satellite must have a speed of about 7.91 km/s. Suppose that two such satellites orbit the Earth in opposite directions. (*a*) What is their relative speed as they pass? Evaluate using the classical Galilean velocity transformation equation. (*b*) What fractional error was made because the (correct) relativistic transformation equation was not used?

20-4 The Lorentz Transformation

10. What must be the value of the speed parameter β if the Lorentz factor γ is to be (*a*) 1.01? (*b*) 10.0? (*c*) 100? (*d*) 1000?

11. Find the speed parameter of a particle that takes 2 years longer than light to travel a distance of 6.0 ly.

12. Observer S assigns to an event the coordinates $x = 100$ km, $t = 200\ \mu$s. Find the coordinates of this event in frame S', which moves in the direction of increasing x with speed $0.950c$. Assume that $x = x'$ at $t = t' = 0$.

13. Observer S reports that an event occurred on the x axis at $x = 3.20 \times 10^8$ m at a time $t = 2.50$ s. (*a*) Observer S' is moving in the direction of increasing x at a speed of $0.380c$. What coordinates would S' report for the event? (*b*) What coordinates would S'' report if S'' were moving in the direction of decreasing x at this same speed?

14. Derive Eqs. 20-17 for the inverse Lorentz transformation by algebraically inverting the equations for the Lorentz transformation, Eqs. 20-14.

20-6 The Transformation of Velocities

15. Suppose that observer S fires a light beam in the y direction ($v_x = 0$, $v_y = c$). Observer S' is moving at speed u in the x direction. (*a*) Find the components v'_x and v'_y of the velocity of the light beam according to S', and (*b*) show that S' measures a speed of c for the light beam.

16. One cosmic-ray proton approaches the Earth along its axis with a velocity of $0.787c$ toward the north pole and another, with velocity $0.612c$, toward the south pole. See Fig. 20-24. Find the relative speed of approach of one particle with respect to the other. (Hint: It is useful to consider the Earth and one of the particles as the two inertial reference frames.)

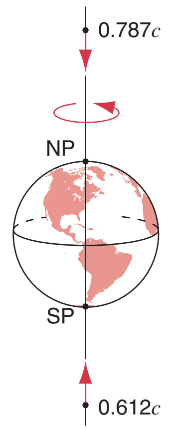


FIGURE 20-24. Exercise 16.

17. Galaxy A is reported to be receding from us with a speed of $0.347c$. Galaxy B, located in precisely the opposite direction, is also found to be receding from us at this same speed. What recessional speed would an observer on galaxy A find (*a*) for our galaxy and (*b*) for galaxy B?

18. It is concluded from measurements of the red shift of the emitted light that quasar Q_1 is moving away from us at a speed of 0.788c. Quasar Q_2, which lies in the same direction in space but is closer to us, is moving away from us at speed 0.413c. What velocity for Q_2 would be measured by an observer on Q_1?

19. In Fig. 20-25, A and B are trains on perpendicular tracks, shown radiating from station S. The velocities are in the station frame (S frame). (*a*) Find v_{AB}, the velocity of train B with respect to train A. (*b*) Find v_{BA}, the velocity of train A with respect to train B. (*c*) Comment on the fact that these two relative velocities do not point in opposite directions.

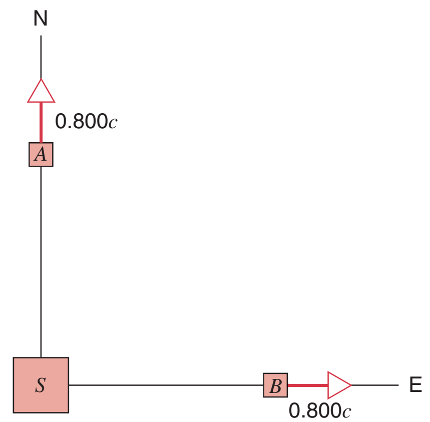


FIGURE 20-25. Exercise 19.

20-7 Consequences of the Lorentz Transformation

20. An electron is moving at a speed such that it could circumnavigate the Earth at the equator in 1 s. (*a*) What is its speed, in terms of the speed of light? (*b*) What is its kinetic energy K? (*c*) What percent error do you make if you use the classical formula to calculate K?

21. The rest radius of the Earth is 6370 km and its orbital speed about the Sun is 29.8 km/s. By how much would the Earth's diameter appear to be shortened to an observer stationed so as to be able to watch the Earth move past at this speed?

22. An airplane whose rest length is 42.4 m is moving with respect to the Earth at a constant speed of 522 m/s. (*a*) By what fraction of its rest length will it appear to be shortened to an observer on Earth? (*b*) How long would it take by Earth clocks for the airplane's clock to fall behind by 1 μs? (Assume that only special relativity applies.)

23. A spaceship whose rest length is 358 m has a speed of 0.728c with respect to a certain reference frame. A micrometeorite, with a speed of 0.817c in this frame, passes the spaceship on an antiparallel track. How long does it take this micrometeorite to pass the spaceship?

24. A clock moves along the x axis at a speed of 0.622c and reads zero as it passes the origin. (*a*) Calculate the Lorentz factor. (*b*) What time does the clock read as it passes $x = 183$ m?

25. A space traveler takes off from Earth and moves at speed 0.988c toward the star Vega, which is 26.0 light-years distant. How much time will have elapsed by Earth clocks (*a*) when the traveler reaches Vega and (*b*) when the Earth observers receive word from him that he has arrived? (*c*) How much older will the Earth observers calculate the traveler to be when he reaches Vega than he was when he started the trip?

26. You wish to make a round trip from Earth in a spaceship, traveling at constant speed in a straight line for 6 months and then returning at the same constant speed. You wish further, on your return, to find the Earth as it will be 1000 years in the future. (*a*) How fast must you travel? (*b*) Does it matter whether or not you travel in a straight line on your journey? If, for example, you traveled in a circle for 1 year, would you still find that 1000 years had elapsed by Earth clocks when you returned?

20-8 Relativistic Momentum

27. Show that 1 kg·m/s $= 1.875 \times 10^{21}$ MeV/c.

28. A particle has a momentum equal to mc. Calculate its speed.

29. Calculate the speed parameter β of a particle with a momentum of 12.5 MeV/c if the particle is (*a*) an electron and (*b*) a proton.

20-9 Relativistic Energy

30. Find the speed parameter β and the Lorentz factor γ for an electron whose kinetic energy is (*a*) 1.0 keV, (*b*) 1.0 MeV, and (*c*) 1.0 GeV.

31. Find the speed parameter β and the Lorentz factor γ for a particle whose kinetic energy is 10 MeV if the particle is (*a*) an electron, (*b*) a proton, and (*c*) an alpha particle.

32. A particle has a speed of 0.990c in a laboratory reference frame. What are its kinetic energy, its total energy, and its momentum if the particle is (*a*) a proton or (*b*) an electron?

33. Quasars are thought to be the nuclei of active galaxies in the early stages of their formation. A typical quasar radiates energy at the rate of 1.20×10^{41} W. At what rate is the mass of this quasar being reduced to supply this energy? Express your answer in solar mass units per year, where one solar mass unit (smu) is the mass of our Sun.

34. Calculate the speed of a particle (*a*) whose kinetic energy is equal to twice its rest energy and (*b*) whose total energy is equal to twice its rest energy.

35. (*a*) Using the binomial expansion (see Appendix I), show that Eq. 20-27 reduces to the classical expression $K = \frac{1}{2}mv^2$ when $v \ll c$. (*b*) By evaluating the second term in the expansion, find the value of v/c for which the error in using the classical expression is at most 1%.

36. A 1000-kg automobile is moving at 20 m/s. Calculate the kinetic energy using both the nonrelativistic equation and the relativistic equation. What is the relative difference between these results?

37. Find the momentum of a particle of mass m in order that its total energy be three times its rest energy.

38. Use the velocities given in Fig. 20-20 in the S' frame and show that, according to S', the kinetic energies before and after the collision, computed classically, are given by Eqs. 20-26.

39. Reconsider the collision shown in Fig. 20-20. Using Eq. 20-27 for the relativistic kinetic energy, calculate the initial and final kinetic energies in frame S' and thereby show that kinetic energy is conserved in this frame as in frame S.

40. Consider the following, all moving in free space: a 2.0-eV photon, a 0.40-MeV electron, and a 10-MeV proton. (*a*) Which is moving the fastest? (*b*) The slowest? (*c*) Which has the greatest momentum? (*d*) The least? (Note: A photon is a light particle of zero mass.)

41. How much work must be done to increase the speed of an electron from (*a*) $0.18c$ to $0.19c$ and (*b*) $0.98c$ to $0.99c$? Note that the speed increase ($= 0.01c$) is the same in each case.

42. Two identical particles, each of mass 1.30 mg, moving with equal but opposite velocities of $0.580c$ in the laboratory reference frame, collide and stick together. Find the mass of the resulting particle.

43. (*a*) Consider the decay of the kaon described in Sample Problem 20-10, but use a frame of reference (the center-of-mass frame) in which the kaons are initially at rest. Show that the two pions emitted in the decay travel in opposite directions with equal speeds of $0.827c$. (*b*) What is the velocity of the original kaons as observed in the laboratory frame? (*c*) Assume that the two pions are emitted in the center-of-mass frame with velocities of $v'_x = +0.827c$ and $v'_x = -0.827c$. By calculating the corresponding velocities in the laboratory frame, show that the kinetic energies in the laboratory frame are identical with those found in the solution to Sample Problem 20-10.

44. An alpha particle with kinetic energy 7.70 MeV strikes a ^{14}N nucleus at rest. An ^{17}O nucleus and a proton are produced, the proton emitted at 90° to the direction of the incident alpha particle and carrying kinetic energy 4.44 MeV. The rest energies of the various particles are: alpha particle, 3730.4 MeV; ^{14}N, 13,051 MeV; proton, 939.29 MeV; ^{17}O, 15,843 MeV. (*a*) Find the kinetic energy of the ^{17}O nucleus. (*b*) At what angle with respect to the direction of the incident alpha particle does the ^{17}O nucleus move?

PROBLEMS

1. The length of a spaceship is measured to be exactly half its rest length. (*a*) What is the speed of the spaceship relative to the observer's frame? (*b*) By what factor do the spaceship's clocks run slow, compared to clocks in the observer's frame?

2. Frame S' moves relative to frame S at $0.620c$ in the direction of increasing x. In frame S' a particle is measured to have a velocity of $0.470c$ in the direction of increasing x'. (*a*) What is the velocity of the particle with respect to frame S? (*b*) What would be the velocity of the particle with respect to S if it moved (at $0.470c$) in the direction of decreasing x' in the S' frame? In each case, compare your answers with the predictions of the classical velocity transformation equation.

3. An experimenter arranges to trigger two flashbulbs simultaneously, a blue flash located at the origin of his reference frame and a red flash at $x = 30.4$ km. A second observer, moving at a speed $0.247c$ in the direction of increasing x, also views the flashes. (*a*) What time interval between them does she find? (*b*) Which flash does she say occurs first?

4. Inertial frame S' moves at a speed of $0.60c$ with respect to frame S in the direction of increasing x. In frame S, event 1 occurs at the origin at $t = 0$ and event 2 occurs on the x axis at $x = 3.0$ km and at $t = 4.0$ ps. What times of occurrence does observer S' record for these same events? Explain the reversal of the time order.

5. Show that

$$(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - c^2(\Delta t)^2 = (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 - c^2(\Delta t')^2,$$

independent of γ and u.

6. A radioactive nucleus moves with a constant speed of $0.240c$ along the x axis of a reference frame S fixed with respect to the laboratory. It decays by emitting an electron whose speed, measured in a reference frame S' moving with the nucleus, is $0.780c$. Consider first the cases in which the emitted electron travels (*a*) along the common xx' axis and (*b*) along the y' axis and find, for each case, its velocity (magnitude and direction) as measured in frame S. (*c*) Suppose, however, that the emitted electron, viewed now from frame S, travels along the y axis of that frame with a speed of $0.780c$. What is its velocity (magnitude and direction) as measured in frame S'?

7. A spaceship, at rest in a certain reference frame S, is given a speed increment of $0.500c$. It is then given a further $0.500c$ increment in this new frame, and this process is continued until its speed with respect to its original frame S exceeds $0.999c$. How many increments does it require?

8. An observer S sees a flash of red light 1210 m away and a flash of blue light 730 m closer and on the same straight line. S measures the time interval between the occurrence of the flashes to be 4.96 μs, the red flash occurring first. (*a*) Find the relative velocity, magnitude, and direction of a second observer S' who would record these flashes as occurring at the same place. (*b*) From the point of view of S', which flash occurs first and what is the measured time interval between the flashes?

9. Consider the previous problem. Suppose now that observer S sees the two flashes in the same positions as in that problem but occurring closer together in time. How close together in time can they be and still have it possible to find a frame S' in which they occur at the same place?

10. (*a*) Can a person, in principle, travel from Earth to the galactic center (which is about 23,000 ly distant) in a normal lifetime? Explain, using both time-dilation and length-contraction arguments. (*b*) What constant speed would be needed to make the trip in 30 y (proper time)?

11. Observers S and S' stand at the origins of their respective frames, which are moving relative to each other with a speed $0.600c$. Each has a standard clock, which, as usual, they set to zero when the two origins coincide. Observer S keeps the S' clock visually in sight. (*a*) What time will the S' clock record when the S clock records 5.00 μs? (*b*) What time will observer S actually read on the S' clock when the S clock reads 5.00μs?

12. Show that two events A and B separated by a distance Δr and simultaneous in frame S will be separated by a larger distance in any other frame S'.

13. (*a*) If the kinetic energy K and the momentum p of a particle can be measured, it should be possible to find its mass m and thus identify the particle. Show that

$$m = \frac{(pc)^2 - K^2}{2Kc^2}.$$

(*b*) What does this expression reduce to as $v/c \to 0$, in which v is the speed of the particle? (*c*) Find the mass of a particle whose kinetic energy is 55.0 MeV and whose momentum is 121 MeV/c; express your answer in terms of the mass m of the electron.

14. In a high-energy collision of a primary cosmic-ray particle near the top of the Earth's atmosphere, 120 km above sea level, a pion is created with a total energy of 135 GeV, traveling vertically downward. In its proper frame this pion decays 35.0 ns after its creation. At what altitude above sea level does the decay occur? The rest energy of a pion is 139.6 MeV.

15. A particle of mass m traveling at a relativistic speed makes a completely inelastic collision with an identical particle that is initially at rest. Find (*a*) the speed of the resulting single particle and (*b*) its mass. Express your answers in terms of the Lorentz factor γ of the incident particle.

16. (*a*) Suppose we have a particle accelerated from rest by the action of a force F. Assuming that Newton's second law for a particle, $F = dp/dt$, is valid in relativity, show that the final kinetic energy K can be written, using the work–energy theorem, as $K = \int v\, dp$. (*b*) Using Eq. 20-23 for the relativistic momentum, show that carrying out the integration in (*a*) leads to Eq. 20-27 for the relativistic kinetic energy.

17. (*a*) In experimental high-energy physics, energetic particles are made to circulate in opposite directions in so-called storage rings and permitted to collide head-on. In this arrangement each particle has the same kinetic energy K in the laboratory. The collisions may be viewed as totally inelastic, in that the rest energy of the two colliding particles, plus all available kinetic energy, can be used to generate new particles and to endow them with kinetic energy. Show that the available energy in this arrangement can be written in the form

$$E_{\text{new}} = 2mc^2\left(1 + \frac{K}{mc^2}\right),$$

where m is the mass of the colliding particles. (*b*) How much energy is made available when 100-GeV protons are used in this fashion? (*c*) What proton energy would be required to make 100 GeV available? (Note: Compare your answers with those in Problem 18, which describes another less energy-effective bombarding arrangement.)

18. (*a*) A proton, mass m, accelerated in a proton synchrotron to a kinetic energy K, strikes a second (target) proton at rest in the laboratory. The collision is entirely inelastic in that the rest energy of the two protons, plus all the kinetic energy consistent with the law of conservation of momentum, is available to generate new particles and to endow them with kinetic energy. Show that the energy available for this purpose is given by

$$E_{\text{new}} = 2mc^2\sqrt{1 + \left(\frac{K}{2mc^2}\right)}.$$

(*b*) How much energy is made available when 100-GeV protons are used in this fashion? (*c*) What proton energy would be required to make 100 GeV available? (Note: Compare with Problem 17.)

19. A particle of mass M originally at rest is struck by a particle of mass m moving with speed v_i. After the collision the two particles move in opposite directions with the same speed v_f. Assuming a relativistic, elastic collision, find the ratio of the masses M/m in terms of $\beta = v_i/c$. Show that this reduces to the nonrelativistic value of 3 as $v_i \to 0$. (Note: Although this problem can be solved by hand, it is also a good problem for a computer-aided algebra system such as Maple or Mathematica.)

COMPUTER PROBLEM

1. A spaceship travels to the nearest star (other than the Sun) with a constant acceleration of g as measured by the occupants. The ship spends the first half of the trip accelerating at g and the second half decelerating at g. How long does a one-way trip take as measured by the space travelers? How long does a one-way trip take as measured by people on Earth?

CHAPTER 21

TEMPERATURE

With this chapter we begin our study of thermal physics, *the branch of physics that deals with the changes in the properties of systems that occur when work is done on (or by) them and heat energy is added to (or taken from) them. For systems such as confined gases the properties involved are their pressure, volume, temperature, energy, and—as we will come to learn—entropy, a property that we will introduce in Chapter 24.*

This chapter deals with temperature, *a concept that deeply underlies all of the laws of thermodynamics. We have used this concept in earlier chapters; now we must define it precisely, as we have done for all other physical concepts that we have encountered. We also introduce the concept of an* ideal gas, *which will serve as a convenient system to use in analyzing and illustrating the laws of thermodynamics.*

21-1 TEMPERATURE AND THERMAL EQUILIBRIUM

We all have an ingrained sense of temperature and, indeed, we have used this concept freely in earlier chapters. In this chapter we wish to define temperature in a rigorous way. Just as we went beyond our sense of "push" and "pull" in defining force, we need to go beyond our sense of "hot" and "cold" in defining temperature. Before we can deal directly with temperature, however, we must first establish the concept of *thermal equilibrium,* which is concerned with the question of whether or not the temperatures of two systems are equal.

Figure 21-1*a* shows two systems *A* and *B*, which, among many possibilities, might be blocks of metal or confined gases. They are isolated from one another and from their environment, by which we mean that neither energy nor matter can enter or leave either system. For example, the systems may be surrounded by walls made of thick slabs of Styrofoam, presumed to be both rigid and impermeable. Such walls are said to be *adiabatic,* which you can think of as meaning *thermally insulating.* Changes in the measured properties of either system have no effect on the properties of the other system.

As Fig. 21-1*b* shows, we can replace the adiabatic wall that separates the two systems with one that permits the

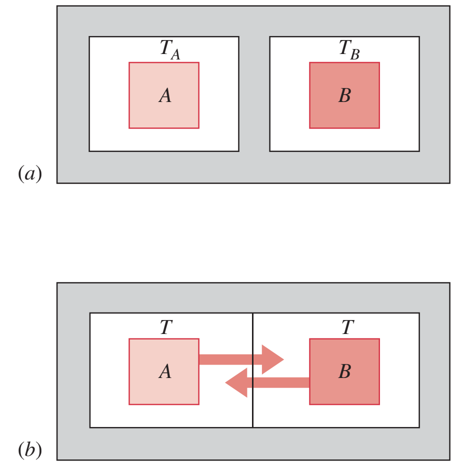


FIGURE 21-1. (*a*) Systems *A* and *B* are separated by an adiabatic wall. The systems have different temperatures T_A and T_B. (*b*) Systems *A* and *B* are separated by a diathermic wall, which permits energy to be exchanged between the systems. The systems will eventually come to thermal equilibrium, upon which they have the same temperature *T*.

flow of energy in a form that we have referred to in Chapter 13 as heat. A thin but rigid sheet of copper might be an example. Such a wall is called *diathermic,* which you can think of as *thermally conducting.*

When the two systems are placed in contact through a diathermic wall, the passage of heat energy through the wall—if it occurs—causes the properties of the two systems to change. If the systems are confined gases, for example, their pressures might change. The changes are relatively rapid at first but become slower as time goes on, until finally all measured properties of each system approach constant values. When this occurs, we say that the two systems are in *thermal equilibrium* with each other. Thus a test of whether or not two systems are in thermal equilibrium is to place them in thermal contact; if their properties do not change, they are in thermal equilibrium; if their properties do change they are not.

It might be inconvenient, or even impossible, to put two systems in thermal contact with each other through a diathermic wall. (The systems might be too bulky to move easily, or they might be very far apart.) We therefore generalize the concept of thermal equilibrium so that systems need not be brought into thermal contact with each other.

One way to test such separated systems is to use a third system C. By placing C in contact with A and then with B, we could discover whether A and B are in thermal equilibrium without ever bringing A and B into direct contact. This is summarized as a postulate called the *zeroth law of thermodynamics,* which is often stated as follows:

> *If systems A and B are each in thermal equilibrium with a third system C, then A and B are in thermal equilibrium with each other.*

This law may seem simple but it is not at all obvious. There are other situations in which a system C may have equivalent interactions with two systems A and B, but A and B do not have a similar interaction with each other. For example, if A and B are unmagnetized iron nails and C is a magnet, then A and C attract each other as do B and C. However, A and B do not.

The zeroth law came to light in the 1930s, long after the first and second laws of thermodynamics had been proposed, accepted, and named. As we discuss later, the zeroth law underlies the concept of temperature, which is fundamental to the first and second laws. The law that establishes the concept of temperature should have a lower number, hence zero.

Temperature

When two systems are in thermal equilibrium, we say that they have the same *temperature.* For example, suppose the systems are two gases that initially have different temperatures, pressures, and volumes. After we place them in contact and wait a sufficiently long time for them to reach thermal equilibrium, their pressures will in general not be equal, nor will their volumes; their temperatures, however, will always be equal in thermal equilibrium. *It is only through this argument based on thermal equilibrium that the notion of temperature can be introduced into physics.*

Although temperature in its everyday use is familiar to all of us, it is necessary to give it a precise meaning if it is to be of value as a scientific measure. Our subjective notion of temperature is not at all reliable. A familiar experience is to touch a metal railing outdoors on a very cold day and then touch a nearby wooden object. The railing well feel colder although in fact both are at the same temperature. What you are testing when you touch a cold object is not only its temperature but also its ability to transfer energy (as heat) away from your (presumably warmer) hand. In such cases your hand is giving a subjective and incorrect measure of temperature. You can also test your subjectivity convincingly by soaking your left hand in cold water and your right hand in warm water. If you then quickly put both hands in water of intermediate temperature, your left hand will sense that the water is warmer than it actually is and your right hand will sense that it is colder.

In practical use of the zeroth law, we identify system C, to which the statement of the law refers, as a *thermometer.* If the thermometer comes separately into thermal equilibrium with systems A and B (which might be widely separated buckets of water) and indicates the same reading, then we may conclude that A and B are in thermal equilibrium and thus indeed have the same temperature. Note that, to test whether two systems have the same temperature, we do not have to establish a temperature scale. If our thermometer (system C) is of the mercury-in-glass type, for example, we do not need to have it marked off in degrees. Simply put the thermometer in contact with system A, mark the mercury level, and then put it in contact with system B, noting whether the mercury reaches the same level.

A statement of the zeroth law in terms of temperature is the following:

> *There exists a scalar quantity called temperature, which is a property of all thermodynamic systems in equilibrium. Two systems are in thermal equilibrium if and only if their temperatures are equal.*

The zeroth law thus defines the concept of temperature and permits us to build and use thermometers.

21-2 TEMPERATURE SCALES

As Table 1-1 shows, temperature (symbol T) is one of the seven base units of the International System of Units (SI). As such we must define it carefully and devise procedures for measuring it that can be reproduced in laboratories around the world. Later in this section we will discuss thermometers based on the familiar Fahrenheit and Celsius scales. These, however, are scales of practical convenience and temperatures measured on them have no deep physical meaning. The scale that is universally adopted as fundamental in physics is the *Kelvin scale.* It is based on the recognition that although there is no apparent limit to how high the temperature of a system can be, there *is* a limit to how low it can be. This *absolute zero of temperature* is defined as zero on the Kelvin scale, which measures temperatures in degrees above this absolute lower limit. Where temperature appears in any equation of fundamental importance in physics, it is certain to refer to this Kelvin (or absolute) scale.

To establish the size of the degree on the Kelvin scale we need to identify a specific calibrating system to which, by international agreement, we assign a specific temperature. We choose for this purpose an arrangement in which ice, liquid water, and water vapor coexist in thermal equilibrium. This point, which is very close to the freezing point of water at atmospheric pressure, is called the *triple point of water.* (The triple point was chosen, rather than the freezing point, because it is more consistently reproducible.) Figure 21-2 shows a triple-point cell of the type used at the National Institute of Standards and Technology (NIST). A thermometer to be calibrated is inserted into the well of the triple-point cell.

The Kelvin temperature at the triple point has been set by international agreement in 1954 to be

$$T_{tr} = 273.16 \text{ K} \qquad \text{(exactly)}, \qquad (21\text{-}1)$$

where K (= kelvin) is the base unit of temperature on the Kelvin scale. The *kelvin,* which is the name we give to the degree on the Kelvin scale, is thus defined as 1/273.16 of the temperature of the triple point of water. In place of Eq. 21-1, the international community could equally well have chosen T_{tr} = 100 K, or any other number, but they did not. The choice they actually made was designed so that the size of the degree on the Kelvin scale (1 kelvin) would equal the size of the degree on the already well-established Celsius scale.

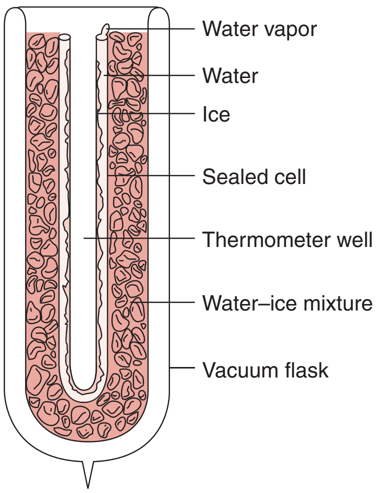


Figure 21-2. The National Institute of Standards and Technology triple-point cell. The U-shaped inner cell contains pure water and is sealed after all the air has been removed. It is immersed in a water–ice bath. The system is at the triple point when ice, water, and water vapor are all present, and in equilibrium, inside the cell. The thermometer to be calibrated is inserted into the central well.

Note that we do not use the degree symbol in reporting a temperature on the Kelvin scale. We might say, for example, that the melting point of lead is 600.7 K, or 600.7 kelvin.

It remains to describe how the Kelvin temperature of a system is actually measured; we shall do so in Section 21-3.

The Celsius and the Fahrenheit Temperature Scales

In nearly all the countries of the world the Celsius scale (formerly called the centigrade scale) is used for all popular and commercial—and some scientific—measurements. Historically, this scale was based on two calibration points: the normal freezing point of water, defined to be 0°C, and the normal boiling point of water, defined to be 100°C. These two points were used to calibrate thermometers and other temperatures were then deduced by interpolation or extrapolation. Note that the degree symbol (°) is used to express temperatures on the Celsius scale.

Today we no longer use these two fixed points to define the Celsius scale; instead, we define a temperature (T_C) on the Celsius scale in terms of the corresponding Kelvin temperature T, by

$$T_C = T - 273.15. \qquad (21\text{-}2)$$

The freezing and boiling points of water (at a pressure of 1 atm) are now measured on the Kelvin scale and then converted to Celsius using Eq. 21-2. The experimental values are, respectively, 0.00°C and 99.975°C, in agreement (for all practical purposes) with the historical basis for defining the Celsius scale. Note also that Eq. 21-2 indicates that the Celsius temperature of the triple point of water is 0.01°C. As we pointed out earlier, this is close to the temperature of the freezing point of water. Also note that, according to Eq. 21-2, the absolute zero of temperature is −273.15°C.

The Fahrenheit scale was also based historically on two fixed points that, after several earlier choices, came to be: (1) the normal freezing point of water, which was defined to be 32°F, and (2) the normal boiling point of water, which was defined to be 212°F. The relationship between the Fahrenheit and the Celsius scales is now taken to be

$$T_F = \tfrac{9}{5}T_C + 32. \qquad (21\text{-}3)$$

As for the Celsius scale, the degree symbol is used in reporting temperatures on the Fahrenheit scale, for example, 98.6°F (normal oral human body temperature).

Transferring between the Celsius and the Fahrenheit scales is easily done by remembering a few corresponding points, such as those shown in Fig. 21-3, which compares the Kelvin, Celsius, and Fahrenheit scales. It is also necessary to make use of the equality between an *interval* of 9

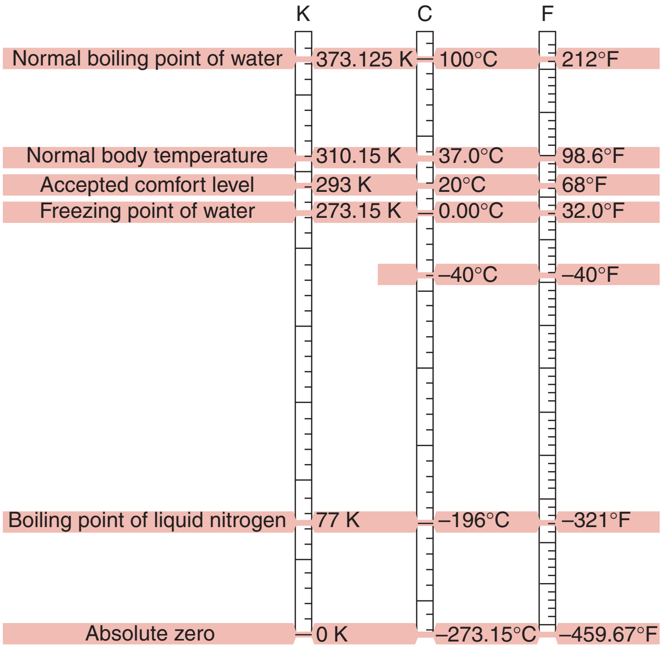


FIGURE 21-3. The Kelvin, Celsius, and Fahrenheit temperature scales compared. Note that the latter two scales coincide at $-40°$.

degrees on the Fahrenheit scale and an *interval* of 5 degrees on the Celsius scale, which we express as

$$9\ F° = 5\ C°. \tag{21-4}$$

Note that these *intervals* are expressed as F° and C°, not as °F or °C. Thus we might write or say: "The temperature here is 90°F. It would be more pleasant if it were 15 F° cooler."

21-3 MEASURING TEMPERATURES

Here we address the problem of measuring the temperatures of a system on the Kelvin scale. Once we have made this measurement, we can easily find the temperature of the system on the Celsius and the Fahrenheit scales, using Eqs. 21-2 and 21-3. To measure a temperature we need a thermometer. What form shall it take?

In principle, any property of a substance that varies with temperature can form the basis for a thermometer. Examples might be the volume of a liquid (as in the common mercury-in-glass thermometer), the pressure of a gas kept at constant volume, the electrical resistance of a wire, the length of a strip of metal, or the color of a lamp filament, all of which vary with temperature and all of which are in common use as thermometers. *The choice of one of these properties leads to a device-sensitive or "private" temperature scale that is defined only for that property and that does not necessarily agree with other choices we might make.* Of course all thermometers will agree, by definition of Eq. 21-1, at the triple point of water. The question is, will they agree at other temperatures, either higher or lower? The answer is that they will not, as Sample Problem 21-1 shows. Even so, a "private" thermometer, when properly calibrated against accepted standards, can be useful as a secondary standard for measuring temperature. Indeed, nearly all temperature measurements are made using such secondary standard thermometers.

Let us assume that our thermometer is based on a system in which we measure the value of an as yet unspecified thermometric property X. The temperature is some function of X. We choose the simplest possible relationship—namely, a linear one

$$T^* = aX \tag{21-5}$$

in which a is a constant. We designate the temperature given by Eq. 21-5 by T^* rather than T because the temperature so measured will be a device-sensitive temperature, not a true Kelvin temperature. We can find the value of a by measuring X at the triple point of water, obtaining the value X_{tr}. We then have, for the temperature as a function of X,

$$T^*(X) = (273.16\ \text{K})\frac{X}{X_{tr}}. \tag{21-6}$$

It remains only to select a suitable temperature-dependent property X and to see whether we can establish a procedure that will yield the true Kelvin temperature rather than T^*.

SAMPLE PROBLEM 21-1. The resistance of a certain coil of platinum wire increases by a factor of 1.392 between the triple point of water and the boiling point of water at atmospheric pressure (that is, the normal boiling point). What temperature for the normal boiling point of water is measured by this thermometer?

Solution The generalized thermometric property X that appears in the defining relation of Eq. 21-6 is, in this case, the resistance R. We are not given R_{tr} but we are told that $R = 1.392\ R_{tr}$. Thus, with R substituted for X, Eq. 21-6 becomes

$$T^*(R) = T_{tr}\frac{R}{R_{tr}} = (273.16\ \text{K})(1.329) = 380.2\ \text{K}.$$

This value gives the "platinum resistance temperature" of boiling water. Other thermometers will give different values. For example, the normal boiling point of water as measured by a thermometer (a *thermocouple*) based on the electric voltage generated by two joined dissimilar wires (copper and constantan) is 412.5 K. The actual Kelvin temperature of the normal boiling point of water (see Fig. 21-3) is 373.125 K. Although such "private scale" thermometers, when properly calibrated, are indispensable for practical use, we cannot rely on them to give consistent measures of temperature on the Kelvin scale.

The Constant-Volume Gas Thermometer

The thermometric property that proves most suitable for measuring temperatures on the Kelvin scale is the pressure p exerted by a fixed volume of gas. The device for realizing

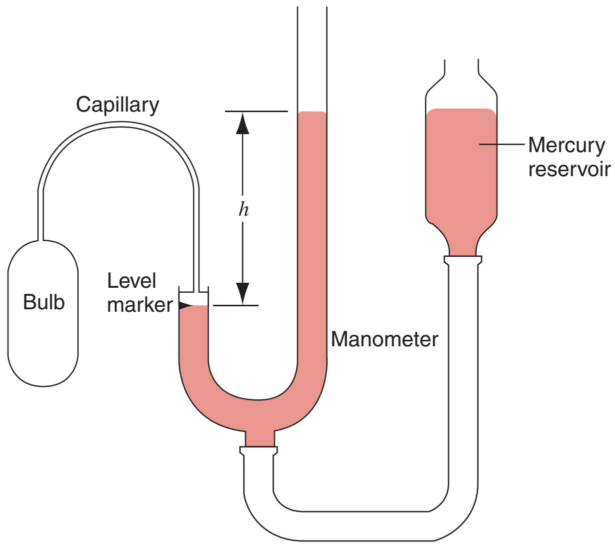


FIGURE 21-4. A constant-volume gas thermometer. The bulb can be immersed in a triple-point cell and then in the bath of a liquid whose temperature we are trying to measure. The difference between the pressure of the gas in the bulb and the atmospheric pressure is found from the height *h* of the mercury column in the manometer. The simplicity of this sketch greatly conceals the complexity of an actual gas thermometer such as may be found, for example, in national standardizing laboratories in many countries.

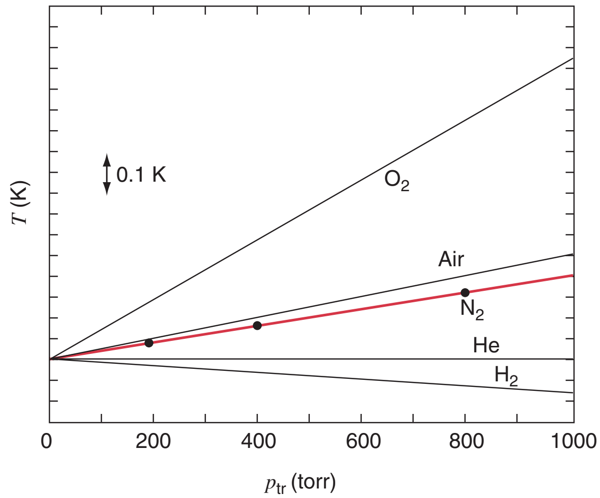


FIGURE 21-5. As the pressure of the nitrogen gas in a constant-volume gas thermometer is reduced from 800 torr to 400 and then to 200, the temperature deduced for the system approaches a limit corresponding to a pressure of zero. Other gases approach the same limit. The full range of the vertical scale is about 1 K for typical conditions.

this procedure in practice is called a *constant-volume gas thermometer.* Figure 21-4 shows a sketch of its essential features. A gas-filled bulb can be alternately immersed in a bath of the liquid whose temperature is to be measured or in a triple-point bath. The volume of the gas in the bulb, which we take to be nitrogen, is maintained constant by raising or lowering the mercury-filled reservoir, so that the level of mercury in the left arm of the manometer always coincides with a fixed marker.

The procedure for measuring a temperature is as follows:

Step 1: (*a*) Immerse the nitrogen-filled bulb in a triple-point bath and read the pressure p_{tr} of the contained gas on the manometer. Let us say that, in a particular case, $p_{tr} =$ 800 torr. (*b*) Immerse the bulb in the bath whose temperature is to be measured and read the new pressure p. Calculate T^* from Eq. 21-6, in which X is replaced with p and X_{tr} with p_{tr}. The result, which we regard as provisional, is plotted as a point at 800 torr in Fig. 21-5.

Step 2: Return the thermometer bulb to the triple-point bath and remove some of the gas, thus decreasing its density. Now p_{tr} has a smaller value—say, 400 torr. Then we return the bulb to the bath whose temperature we are trying to find, measure a new value of p, and calculate a new provisional temperature T^*, also plotted in Fig. 21-5.

We continue this procedure, reducing the amount of gas in the bulb step by step and, at each new lower value of p_{tr} calculating T^*. If we plot the values of T^* against p_{tr}, we can extrapolate the resulting curve to the intersection with the axis at $p_{tr} = 0$. The data points for nitrogen gas and the resultant straight-line extrapolation are shown in Fig. 21-5.

If we repeat this step-wise extrapolation procedure for gases other than nitrogen, we obtain results also shown in Fig. 21-5. We see that, as the triple-point pressure p_{tr} (and thus the gas density) is reduced, the temperature readings of constant-volume gas thermometers approach the same value T, no matter what gas is used. We can regard T as the temperature of the system and we define an *ideal gas temperature scale:*

$$T = (273.16\text{ K}) \lim_{p_{tr}\to 0} \frac{p}{p_{tr}} \qquad (\text{constant } V). \quad (21\text{-}7)$$

In this context, we define an "ideal gas" to be a gas that would read the same temperature T at all pressures, with no need for extrapolation. We will say more about the ideal gas in Section 21-5.

If temperature is to be a truly fundamental physical quantity it is absolutely necessary that its definition be independent of the properties of specific materials. It would not do, for example, to have such a basic quantity as temperature depend on the thermal expansivity of mercury, the electrical resistivity of platinum, or any other such "handbook" property. We choose the gas thermometer as our standard precisely because no such properties are involved in its operation. You can use any gas and you always get the same answer.

The lowest temperature that can be measured with a gas thermometer is about 1 K. To obtain this temperature we must use low-pressure helium, which remains a gas at lower temperatures than any other gas.

TABLE 21-1 Temperatures of Selected Systems

System	*Temperature* (K)
Plasma in fusion test reactor	10^8
Center of Sun	10^7
Surface of Sun	6×10^3
Melting point of tungsten	3.6×10^3
Freezing point of water	2.7×10^2
Normal boiling point of N_2	77
Normal boiling point of ^{4}He	4.2
Mean temperature of universe	2.7
^{3}He–^{4}He dilution refrigerator	5×10^{-3}
Adiabatic demagnetization of paramagnetic salt	10^{-3}
Bose–Einstein condensation experiments	2×10^{-8}

It can be shown that temperatures measured with the constant-volume gas thermometer are true Kelvin temperatures in the range in which the gas thermometer can be used. We must use special methods to measure Kelvin temperatures outside of this range. Table 21-1 lists the Kelvin temperatures of some systems and processes.

The International Temperature Scale

Precise measurement of a temperature with a gas thermometer is a difficult task, requiring many months of painstaking laboratory work and, when completed, has been said to be an international event. In practice therefore, the gas thermometer is used only to establish certain fixed points that can then be used to calibrate other more convenient secondary thermometers.

The International Temperature Scale has been adopted for the calibration of thermometers for scientific or industrial use. This scale consists of a set of procedures for providing in practice the best possible approximations to the Kelvin scale. The adopted scale consists of a set of fixed points, along with specific devices to be used for interpolating between these fixed points and extrapolating beyond the highest fixed point. The International Committee of Weights and Measures reviews and refines the scale about every 20 years. Table 21-2 shows the fixed points of the 1990 version of the International Temperature Scale.

TABLE 21-2 Primary Fixed Points on the 1990 International Temperature Scale[a]

Substance	*State*	*Temperature* (K)
Helium	Boiling point	3–5[c]
Hydrogen	Triple point	13.8033
Hydrogen	Boiling point[b]	17.025–17.045[c]
Hydrogen	Boiling point	20.26–20.28[c]
Neon	Triple point	24.5561
Oxygen	Triple point	54.3584
Argon	Triple point	83.8058
Mercury	Triple point	234.3156
Water	Triple point	273.16
Gallium	Melting point	302.9146
Indium	Freezing point	429.7485
Tin	Freezing point	505.078
Zinc	Freezing point	692.677
Aluminum	Freezing point	933.473
Silver	Freezing point	1234.93
Gold	Freezing point	1337.33
Copper	Freezing point	1357.77

[a] See "The International Temperature Scale of 1990 (ITS-90)," by H. Preston-Thomas, *Metrologia,* 27 (1990), p. 3.
[b] This boiling point is for a pressure of $\frac{1}{3}$ atm. All other boiling points, melting points, or freezing points are for a pressure of 1 atm.
[c] The temperature of the boiling point varies somewhat with the pressure of the gas above the liquid. The temperature scale gives the relationship between T and p that can be used to calculate T for a given p.

21-4 THERMAL EXPANSION

You can often loosen a tight metal jar lid by holding it under a stream of hot water. As its temperature rises, the metal lid expands slightly relative to the glass of the jar. Thermal expansion is not always desirable, as Fig. 21-6 suggests. Roadways of bridges usually include expansion slots to allow for changes in length of the roadway as the temperature changes.

Pipes at refineries often include an expansion loop, so that the pipe will not buckle as the temperature rises. Materials used for dental fillings have expansion properties similar to those of tooth enamel. In aircraft manufacture, rivets and other fasteners are often designed so that they are to be cooled in dry ice before insertion and then allowed to ex-

FIGURE 21-6. Railroad tracks distorted because of thermal expansion on a very hot day. Railroad tracks today come in 1500-ft lengths and, to prevent buckling, are laid at or near the maximum annual temperature of the locality.

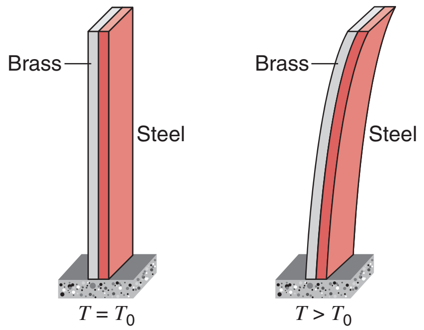


Figure 21-7. A bimetallic strip, consisting of a strip of brass and a strip of steel welded together, at temperature T_0. At temperatures higher than T_0, the strip bends as shown; at lower temperatures it bends the other way. Many thermostats operate on this principle, using the motion of the end of the strip to make or break an electrical contact.

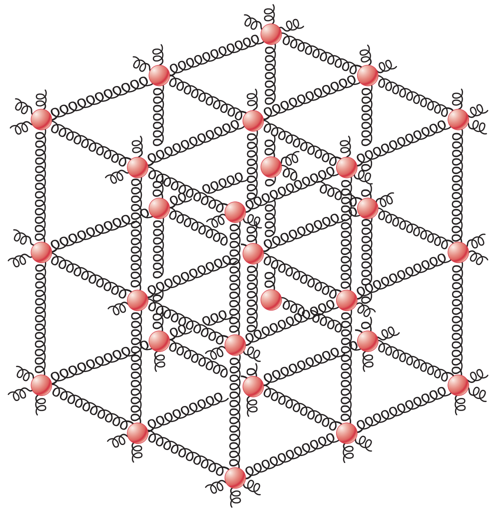


Figure 21-9. A solid behaves in many ways as if it were a collection of atoms joined by elastic forces (here represented by springs).

pand to a tight fit. Thermometers and thermostats may be based on the differences in expansion between the components of a bimetallic strip; see Fig. 21-7. In a thermometer of a familiar type, the bimetallic strip is coiled into a helix that winds and unwinds as the temperature changes; see Fig. 21-8. The familiar liquid-in-glass thermometers are based on the fact that liquids such as mercury or alcohol expand to a different (greater) extent than do their glass containers.

We can understand this expansion by considering a simple model of the structure of a crystalline solid. The atoms are held together in a regular array by electrical forces, which are like those that would be exerted by a set of springs connecting the atoms. We can thus visualize the solid body as a microscopic bedspring (Fig. 21-9). These "springs" are quite stiff and not at all ideal (see Problem 1

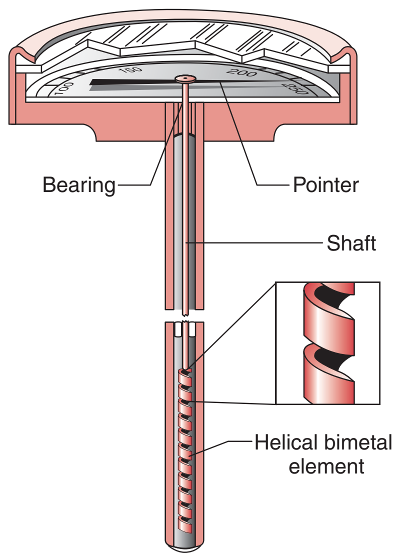


Figure 21-8. A thermometer based on a bimetallic strip. The strip is formed into a helix, which coils or uncoils as the temperature is changed.

of Chapter 17), and there are about 10^{23} of them per cubic centimeter. At any temperature the atoms of the solid are vibrating. The amplitude of vibration is about 10^{-9} cm, about one-tenth of an atomic diameter, and the frequency is about 10^{13} Hz. When the temperature is increased, the atoms vibrate at larger amplitude, and the average distance between atoms increases. (See the discussion of the microscopic basis of thermal expansion at the end of this section.) This leads to an expansion of the whole solid body.

The change in *any* linear dimension of the solid, such as its length, width, or thickness, is called a *linear expansion.* If the length of this linear dimension is L, the change in temperature ΔT causes a change in length ΔL. We find from experiment that, if ΔT is small enough, this change in length ΔL is proportional to the temperature change ΔT and to the original length L. Hence we can write

$$\Delta L = \alpha L \, \Delta T, \tag{21-8}$$

where α, called the *coefficient of linear expansion,* has different values for different materials. Rewriting this formula, we obtain

$$\alpha = \frac{\Delta L / L}{\Delta T}, \tag{21-9}$$

so that α has the meaning of a fractional change in length per degree temperature change.

Strictly speaking, the value of α depends on the actual temperature and the reference temperature chosen to determine L (see Problem 5). However, its variation is usually negligible compared to the accuracy with which measurements need to be made. It is often sufficient to choose an average value that can be treated as a constant over a certain temperature range. In Table 21-3 we list the experimental values for the average coefficient of linear expansion of several common solids. For all the substances listed, the

TABLE 21-3 Some Average Coefficients of Linear Expansion[a]

Substance	$\alpha(10^{-6}$ per C°)
Ice	51
Lead	29
Aluminum	23
Brass	19
Copper	17
Steel	11
Glass (ordinary)	9
Glass (Pyrex)	3.2
Invar alloy	0.7
Quartz (fused)	0.5

[a] Typical average values in the temperature range 0°C to 100°C are shown, except for ice in which the range is − 10°C to 0°C.

change in size consists of an expansion as the temperature rises, because α is positive. The order of magnitude of the expansion is about 1 millimeter per meter length per 100 Celsius degrees. (Note the use of C°, not °C, to express temperature changes here. Note also, that since 1 K is the same as 1 C°, we can use either Kelvin or Celsius temperature differences in Eq. 21-9.)

SAMPLE PROBLEM 21-2. A steel metric scale is to be ruled so that the millimeter intervals are accurate to within about 5×10^{-5} mm at a certain temperature. What is the maximum temperature variation allowable during the ruling?

Solution From Eq. 21-8, we have

$$\Delta T = \frac{\Delta L}{\alpha L} = \frac{5 \times 10^{-5}\ \text{mm}}{(11 \times 10^{-6}/\text{C}°)(1.0\ \text{mm})} = 4.5\ \text{C}°,$$

where we have used the value of α for steel from Table 21-3. The temperature during the ruling must be kept constant to within about 5 C°, and the scale must be used within that same interval of temperature at which it was ruled.

Note that if the alloy invar were used instead of steel, we could achieve the same precision over a temperature interval of about 75 C°; or, equivalently, if we could maintain the same temperature variation (5 C°), we could achieve an accuracy due to temperature changes of about 3×10^{-6} mm.

For many solids, called *isotropic,* the percent change in length for a given temperature change is the same for all lines in the solid. The expansion is quite analogous to a photographic enlargement, except that a solid is three-dimensional. Thus, if you have a flat plate with a hole punched in it, $\Delta L/L$ $(= \alpha\,\Delta T)$ for a given ΔT is the same for the length, thickness, face diagonal, body diagonal, and hole diameter. Every line, whether straight or curved, lengthens in the ratio α per degree temperature rise. If you scratch your name on the plate, the line representing your name has the same fractional change in length as any other line. The analogy to a photographic enlargement is shown in Fig. 21-10.

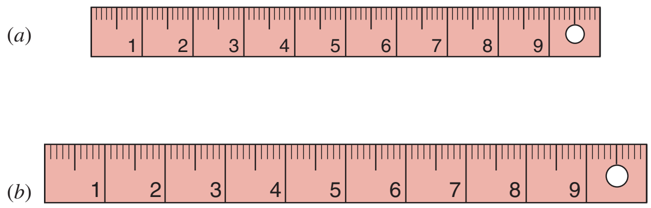


FIGURE 21-10. A steel rule at two different temperatures. The expansion increases in proportion in all dimensions: the scale, the numbers, the hole, and the thickness are all increased by the same factor. (The expansion shown is greatly exaggerated; to obtain such an expansion would require a temperature increase of about 20,000 C°!)

With these ideas in mind, you should be able to show (see Exercises 22 and 23) that to a high degree of accuracy the fractional change in area A per degree temperature change for an isotropic solid is 2α, that is,

$$\Delta A = 2\alpha A\,\Delta T, \qquad (21\text{-}10)$$

and the fractional change in volume V per degree temperature change for an isotropic solid is 3α, that is,

$$\Delta V = 3\alpha V\,\Delta T. \qquad (21\text{-}11)$$

Equations 21-8 to 21-11 cannot be applied to the expansion of fluids, because fluids have no definite shape and so the coefficient of linear expansion is not a meaningful quantity for a fluid. Instead, we define the *coefficient of volume expansion* β of a fluid by analogy with Eq. 21-8 or 21-11:

$$\Delta V = \beta V\,\Delta T. \qquad (21\text{-}12)$$

For liquids, the coefficient of volume expansion is relatively independent of the temperature. Liquids usually expand with increasing temperature (that is, $\beta > 0$). Typical values of β for liquids at room temperature are in the range of 200×10^{-6}/C° to 1000×10^{-6}/C°, more than an order of magnitude larger than the coefficient of volume expansion of most solids (3α from Eq. 21-11). For gases, β is strongly dependent on temperature; in fact, for an ideal gas (discussed in the next section) you can show that $\beta = 1/T$ with T expressed in kelvins (see Exercise 36). For a gas at room temperature and constant pressure, β is about 3300×10^{-6}/C°, as much as an order of magnitude larger than the coefficient of volume expansion for typical liquids.

The most common liquid, water, does not behave like most other liquids. In Fig. 21-11 we show the volume expansion curve for water. Note that above 4°C water expands as the temperature rises, although not linearly. (That is, β is not constant over these large temperature intervals.) As the temperature is lowered from 4°C to 0°C, however, water expands instead of contracting, thus decreasing its density, which is the reason that lakes freeze first at their upper surface. Such an expansion with decreasing temperature is not observed in any other common liquid.

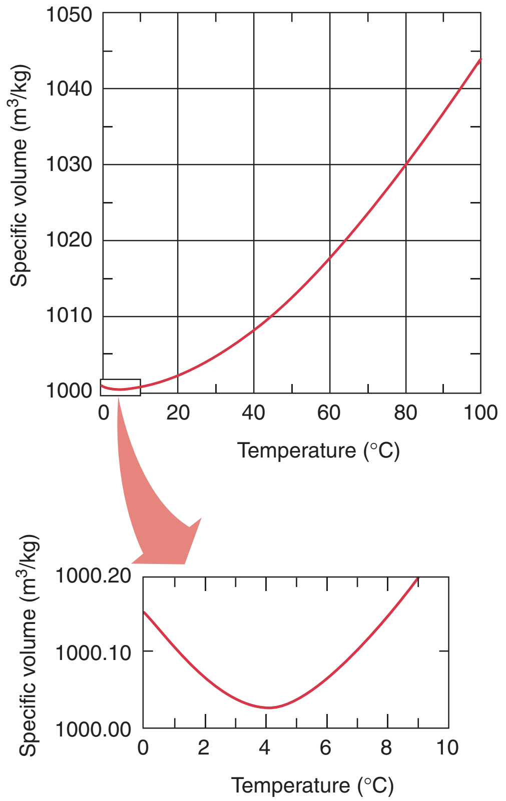


FIGURE 21-11. (*a*) The specific volume (the volume occupied by a particular mass) of water as a function of its temperature. The specific volume is the inverse of the density (the mass per unit volume). (*b*) An enlargement of the region near 4°C, showing a minimum specific volume (or a maximum density).

Microscopic Basis of Thermal Expansion (Optional)

On the microscopic level, thermal expansion of a solid suggests an increase in the average separation between the atoms in the solid. The potential energy curve for two adjacent atoms in a crystalline solid as a function of their internuclear separation is an asymmetric curve like that of Fig. 21-12. As the atoms move close together, their separation decreasing from the equilibrium value r_0, strong repulsive forces come into play, and the potential energy rises steeply ($F = -dU/dr$); as the atoms move farther apart, their separation increasing from the equilibrium value, somewhat weaker attractive forces take over and the potential energy rises more slowly. At a given vibrational energy the separation of the atoms changes periodically from a minimum to a maximum value, the average separation being greater than the equilibrium separation because of the asymmetric nature of the potential energy curve. At still higher vibrational energy the average separation is even greater. The effect is enhanced because, as suggested by Fig. 21-12, the kinetic energy is smaller at larger separations; thus the particles move slower and spend more time at large separations, which then contribute a larger share to the time average.

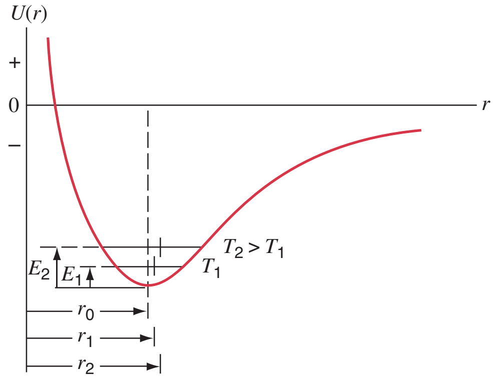


FIGURE 21-12. Potential energy curve for two adjacent atoms in a solid as a function of their internuclear separation distance. The equilibrium separation is r_0. Because the curve is asymmetric, the average separation (r_1, r_2) increases as the temperature (T_1, T_2) and the vibrational energy (E_1, E_2) increase.

Because the vibrational energy increases as the temperature rises, the average separation between atoms increases with temperature, and the entire solid expands.

Note that if the potential energy curve were symmetric about the equilibrium separation, then the average separation would equal the equilibrium separation, no matter how large the amplitude of the vibration. Hence thermal expansion is a direct consequence of the deviation from symmetry of the characteristic potential energy curve of solids.

It should be emphasized that the microscopic models presented here are oversimplifications of a complex phenomenon that can be treated with greater insight using statistical mechanics and quantum theory. ■

21-5 THE IDEAL GAS

Figure 21-5 suggests that real gases such as oxygen, nitrogen, and helium differ from each other as far as the relations among their thermodynamic properties, such as pressure or temperature, are concerned. However, this same figure suggests that, as we examine such real gases at lower and lower densities, their properties seem to converge. That suggests the concept of an *ideal gas*—that is, a gas whose properties represent the limiting behavior of real gases at sufficiently low densities.

The ideal gas is an abstraction, but it is a useful abstraction because (1) real gases—at low enough densities—approximate the behavior of the ideal gas, and (2) the thermodynamic properties of an ideal gas are related to each other in a particularly simple way. Physics is full of useful abstractions and we have met many of them, such as perfectly elastic collisions, massless rods, and unstretchable strings.

Figure 21-13 shows schematically an arrangement with which it is possible to study the properties of real gases and, by extrapolating to sufficiently low densities, to deduce the properties of the ideal gas. An insulated cylinder

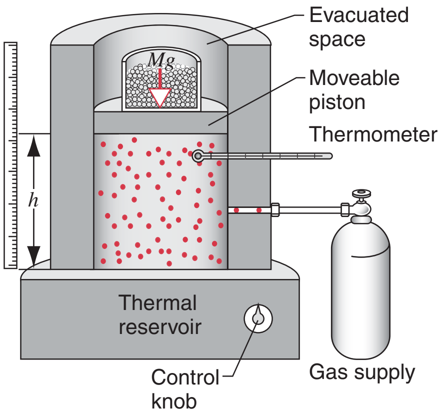


FIGURE 21-13. Gas is confined to a cylinder that is in contact with a thermal reservoir at the (adjustable) temperature T. The piston exerts a total downward force Mg on the gas, which in equilibrium is balanced by the upward force due to the gas pressure. The volume of the gas can be determined from a measurement of the height h of the piston above the bottom of the cylinder, and the temperature of the gas is measured with a suitable thermometer. A gas supply permits additional gas to be added to the cylinder; we assume that a mechanism is also provided for removing gas and for changing the supply to admit different kinds of gas.

that rests on a thermal reservoir (a glorified hot plate) contains a specified quantity of gas, which we can control by adding or removing gas using the gas supply. The temperature of this reservoir—and thus of the gas—can be regulated by turning a control knob. A piston, whose position determines the volume of the gas, can move without friction up and down in the cylinder. Weights, shown here as lead shot, can be added to or removed from the top of the piston, thus determining the pressure exerted by the gas. The variables pressure, volume, temperature, and quantity of gas (number of moles n or number of molecules N) are thus under our control.

From laboratory experiments with real gases, it was found that their pressure p, volume V, and temperature T are related, to a good approximation, by

$$pV = NkT \qquad (21\text{-}13)$$

Here N is the number of molecules contained in the volume V, and k is a constant called the *Boltzmann constant.* Its measured value is, to three significant figures,

$$k = 1.38 \times 10^{-23}\ \text{J/K}. \qquad (21\text{-}14)$$

The temperature T in Eq. 21-13 must *always* be expressed in kelvins.

It is often more useful to write Eq. 21-13 in a slightly different form, expressing the quantity of gas not in terms of the number of molecules N but in terms of the number of moles n. (The mole is one of the seven SI base units; see Section 1-5). Either measures the quantity of gas, and they are related by

$$N = nN_A, \qquad (21\text{-}15)$$

where N_A is the *Avogadro constant*—that is, the number of molecules contained in a mole of any substance. Its value is

$$N_A = 6.02 \times 10^{23}\ \text{molecules/mol}. \qquad (21\text{-}16)$$

In terms of the number of moles, we can write Eq. 21-13 as

$$pV = nRT, \qquad (21\text{-}17)$$

where $R = kN_A$ is a new constant, called the *molar gas constant.* Its value is

$$R = 8.31\ \text{J/mol}\cdot\text{K}. \qquad (21\text{-}18)$$

Equations 21-13 and 21-17 are completely equivalent forms of the *ideal gas law.* This law represents an idealization of the properties of real gases, and it works best as a description of real gases when the pressure and density are low. That is why the lines in Fig. 21-5 representing different gases converged to a single temperature as the pressure (and thus the quantity) of gas was decreased. The ideal gas law also shows why it is critical that the volume of gas in the thermometer of Fig. 21-4 be kept constant, if we want to examine the dependence of pressure on temperature.

In Chapter 22 we explore the ideal gas law by examining the microscopic structure of the gas in terms of the properties of its molecules. It is also possible to "piece together" this law by studying a single relationship between two of the variables in the equation while the others are held constant. Here are three examples of these experiments:

1. The Italian investigator Amadeo Avogadro (1776–1856), for whom the Avogadro constant is named, discovered in 1811 that, under the same conditions of pressure and temperature, equal volumes of different gases contain the same number of molecules ($V \propto N$ for constant p and T). At that time, the very existence of atoms and molecules was much in dispute, and this discovery, known as *Avogadro's law,* was later to provide critical support for the atomic theory.

2. The Anglo-Irish experimenter Robert Boyle (1627–1691) discovered that, if the temperature of a fixed amount of gas is held constant, then the pressure exerted by the gas is inversely proportional to the volume that the gas occupies ($p \propto V^{-1}$ for constant T and N). This observation is known as *Boyle's law.*

3. If the pressure of a fixed quantity of gas is held constant, experiment shows that the volume of the gas is directly proportional to its temperature ($V \propto T$ for constant p and N). These experiments were carried out by the French experimenters Joseph Louis Gay-Lussac (1778–1850) and J.-A.-C. Charles (1746–1823), and this relationship is thus known either as *Gay-Lussac's law* or *Charles' law.*

SAMPLE PROBLEM 21-3. An insulated cylinder fitted with a piston (Fig. 21-13) contains oxygen at a temperature of 20°C and a pressure of 15 atm in a volume of 22 liters. The piston is lowered, decreasing the volume of the gas to 16 liters, and simultaneously the temperature is raised to 25°C. Assuming oxygen to behave like an ideal gas under these conditions, what is the final pressure of the gas?

Solution From Eq. 21-13, since the quantity of gas remains unchanged, we have

$$\frac{p_i V_i}{T_i} = \frac{p_f V_f}{T_f},$$

or

$$p_f = p_i \left(\frac{T_f}{T_i}\right)\left(\frac{V_i}{V_f}\right).$$

Because this is in the form of a ratio, we need not convert p and V into SI units, but *we must express T in absolute (Kelvin) temperature units.* Thus

$$p_f = (15\text{ atm})\left(\frac{273 + 25\text{ K}}{273 + 20\text{ K}}\right)\left(\frac{22\text{ L}}{16\text{ L}}\right) = 21\text{ atm}.$$

MULTIPLE CHOICE

21-1 Temperature and Thermal Equilibrium

1. Consider four objects, A, B, C, and D. It is found that A and B are in thermal equilibrium. It is also found that C and D are in thermal equilibrium. However, A and C are *not* in thermal equilibrium. One can conclude that
(A) B and D are in thermal equilibrium.
(B) B and D could be in thermal equilibrium, but might not be.
(C) B and D cannot be in thermal equilibrium.
(D) the zeroth law of thermodynamics does not apply here, because there are more than three objects.

2. Objects A and B are initially in thermal equilibrium. Objects A and C are originally not in thermal equilibrium, but the two are placed in thermal contact and quickly reach thermal equilibrium. After doing this
(A) B and C will also be in thermal equilibrium.
(B) B and C could be in thermal equilibrium, but might not be.
(C) B and C cannot be in thermal equilibrium.

21-2 Temperature Scales

3. At what temperature do the Fahrenheit and Celsius scales coincide?
(A) $-40°$F (B) 0°F (C) 32°F
(D) 40°F (E) 104°F

4. At what temperature do the Fahrenheit and Kelvin scales coincide?
(A) $-100°$F (B) 273°F (C) 574°F (D) 844°F

21-3 Measuring Temperatures

21-4 Thermal Expansion

5. A large flat slab of metal at temperature T_0 has a hole in it. The metal is warmed to a new temperature $T > T_0$. Upon warming, the area of the hole will
(A) increase. (B) decrease.
(C) remain the same size.
(D) possibly change size, depending on the shape of the hole.

6. Why does a glass sometimes break if you quickly pour boiling water into it?
(A) Hot water expands, pushing the glass out.
(B) The hot water cools when it touches the glass, shrinking and pulling the glass in.
(C) The glass becomes hot and expands, causing the molecules to break.
(D) The inside the glass expands faster than the outside of the glass, causing the glass to break.

7. A mercury-filled glass thermometer is originally at equilibrium in a 20°C water bath. The thermometer is then immersed in a 30°C water bath. The column of mercury in the thermometer will
(A) rise to 30°C and then stop.
(B) first rise above 30°C, then return to 30°C and stop.
(C) first fall below 20°C, then rise to 30°C and stop.
(D) first fall below 20°C, then rise above 30°C, and finally return to 30°C and stop.

8. A strip of copper metal is riveted to a strip of aluminum. The two metals are then heated. What happens?
(A) The strip expands without bending.
(B) The strip expands and bends toward the copper.
(C) The strip expands and bends toward the aluminum.

9. The daily temperature variation of the Golden Gate bridge in San Francisco can be in excess of 20 C°. The bridge is approximately 2 km long and is made of steel (with an asphalt covering on the roadway).
(*a*) What is the approximate change in length of the bridge with this temperature variation?
(A) 4.4 cm (B) 44 cm (C) 4.4 m (D) 44 m
(*b*) If the bridge builders neglected to include expansion joints, then approximately how large of a "bump" would form in the middle of the bridge when it expanded?
(A) 2.1 cm (B) 21 cm (C) 2.1 m (D) 21 m

21-5 The Ideal Gas

10. Which has the higher density (mass per unit volume)—dry air or humid air? Assume that both have the same temperature and pressure.
(A) Dry air (B) Humid air
(C) The densities are the same.

11. Which of the following has the largest particle density (molecules per unit volume)?
(A) 0.8 L of nitrogen gas at 350 K and 100 kPa
(B) 1.0 L of hydrogen gas at 350 K and 150 kPa
(C) 1.5 L of oxygen gas at 300 K and 80 kPa
(D) 2.0 L of helium gas at 300 K and 120 kPa

12. Four different containers each hold 0.5 moles of one of the following gases. Which is at the highest temperature?
(A) 8.0 L of helium gas at 120 kPa
(B) 6.0 L of neon gas at 160 kPa
(C) 4.0 L of argon gas at 250 kPa
(D) 3.0 L of krypton gas at 300 kPa

QUESTIONS

1. Is temperature a microscopic or macroscopic concept?
2. Can we define temperature as a derived quantity, in terms of length, mass, and time? Think of a pendulum, for example.
3. Absolute zero is a minimum temperature. Is there a maximum temperature?
4. Can one object be hotter than another if they are at the same temperature? Explain.
5. Lobster traps are designed so that a lobster can easily get in, but cannot easily get out. Can a diathermic wall be created that allows heat to flow through in one direction only? Explain.
6. Are there physical quantities other than temperature that tend to equalize if two different systems are joined?
7. A piece of ice and a warmer thermometer are suspended in an insulated evacuated enclosure so that they are not in contact. Why does the thermometer reading decrease for a time?
8. What qualities make a particular thermometric property suitable for use in a practical thermometer?
9. What difficulties would arise if you defined temperature in terms of the density of water?
10. Let p_3 be the pressure of the bulb of a constant-volume gas thermometer when the bulb is at the triple-point temperature of 273.16 K and let p be the pressure when the bulb is at room temperature. Given are three constant-volume gas thermometers: for A the gas is oxygen and $p_3 = 20$ cm Hg; for B the gas is also oxygen but $p_3 = 40$ cm Hg; for C the gas is hydrogen and $p_3 = 30$ cm Hg. The measured values of p for the three thermometers are p_A, p_B, and p_C. (*a*) An approximate value of the room temperature T can be obtained with each of the thermometers using

$$T_A = (273.16\text{ K})(p_A/20\text{ cm Hg}),$$

$$T_B = (273.16\text{ K})(p_B/40\text{ cm Hg}),$$

$$T_C = (273.16\text{ K})(p_C/30\text{ cm Hg}).$$

Mark each of the following statements true or false: (1) With the method described, all three thermometers will give the same value of T. (2) The two oxygen thermometers will agree with each other but not with the hydrogen thermometer. (3) Each of the three will give a different value of T. (*b*) In the event that there is a disagreement among the three thermometers, explain how you would change the method of using them to cause all three to give the same value of T.

11. The editor-in-chief of a well-known business magazine, discussing possible warming effects associated with the increasing concentration of carbon dioxide in the Earth's atmosphere (greenhouse effect), wrote: "The polar regions might be three times warmer than now . . ." What do you suppose he meant, and what did he say literally? (See "Warmth and Temperature: A Comedy of Errors," by Albert A. Bartlett, *The Physics Teacher,* November 1984, p. 517.)
12. Although the absolute zero of temperature seems to be experimentally unattainable, temperatures as low as 0.00000002 K have been achieved in the laboratory. Why would physicists strive, as indeed they do, to obtain still lower temperatures? Isn't this low enough for all practical purposes?
13. You put two uncovered pails of water, one containing hot water and one containing cold water, outside in below-freezing weather. The pail with the hot water will usually begin to freeze first. Why? What would happen if you covered the pails?
14. Can a temperature be assigned to a vacuum?
15. Does our "temperature sense" have a built-in sense of direction; that is, does hotter necessarily mean higher temperature, or is this just an arbitrary convention? Celsius, by the way, originally chose the steam point as 0°C and the ice point as 100°C.
16. Many medicine labels inform the user to store below 86°F. Why 86? (Hint: Change to Celsius.) (See *The Science Almanac,* 1985–1986, p. 430.)
17. How would you suggest measuring the temperature of (*a*) the Sun, (*b*) the Earth's upper atmosphere, (*c*) an insect, (*d*) the Moon, (*e*) the ocean floor, and (*f*) liquid helium?
18. Considering the Celsius, Fahrenheit, and Kelvin scales, does any one stand out as "nature's scale"? Discuss.
19. Is one gas any better than another for purposes of a standard constant-volume gas thermometer? What properties are desirable in a gas for such purposes?
20. State some objections to using water-in-glass as a thermometer. Is mercury-in-glass an improvement? If so, explain why.
21. What are the dimensions of α, the coefficient of linear expansion? Does the value of α depend on the unit of length used? When Fahrenheit degrees are used instead of Celsius degrees as the unit of temperature change, does the numerical value of α change? If so, how? If not, prove it.
22. A metal ball can pass through a metal ring. When the ball is heated, however, it gets stuck in the ring. What would happen if the ring, rather than the ball, were heated?
23. A bimetallic strip, consisting of two different metal strips riveted together, is used as a control element in the common thermostat. Explain how it works.
24. Two strips, one of iron and one of zinc, are riveted together side by side to form a straight bar that curves when heated. Why is the iron on the inside of the curve?
25. Explain how the period of a pendulum clock can be kept constant with temperature by attaching vertical tubes of mercury to the bottom of the pendulum.
26. Why should a chimney be freestanding—that is, not part of the structural support of the house?
27. Water expands when it freezes. Can we define a coefficient of volume expansion for the freezing process?
28. Explain why the apparent expansion of a liquid in a glass bulb does not give the true expansion of the liquid.
29. Does the change in volume of an object when its temperature is raised depend on whether the object has cavities inside, other things being equal?
30. Why is it much more difficult to make a precise determination of the coefficient of expansion of a liquid than of a solid?
31. A common model of a solid assumes the atoms to be points executing simple harmonic motion about mean lattice positions. What would be the coefficient of linear expansion of such a lattice?
32. Explain the fact that the temperature of the ocean at great depths is very constant the year round, at a temperature of about 4°C.
33. Explain why lakes freeze first at the surface.

34. What causes water pipes to burst in the winter?

35. What can you conclude about how the melting point of ice depends on pressure from the fact that ice floats on water?

36. Two equal-size rooms communicate through an open doorway. However, the average temperatures in the two rooms are maintained at different values. In which room is there more air?

37. It is found that the weight of an empty, flat, thin plastic bag is not changed when the bag is filled with air. Why not?

38. Why does smoke rise, rather than fall, from a lighted candle?

39. Do the pressure and volume of air in a house change when the furnace raises the temperature significantly? If not, is the ideal gas law violated?

EXERCISES

21-1 Temperature and Thermal Equilibrium

21-2 Temperature Scales

1. The boiling point and the melting point for water on the Fahrenheit scale were chosen so the difference between the two temperatures would be 180 F°, a number that is evenly divisible by 2, 3, 4, 5, 6, and 9. Devise a new temperature scale S so that absolute zero is 0°S and $T_{\text{bp, water}} - T_{\text{mp, water}} =$ 180 S°. (*a*) What is the conversion formula from Celsius to S? (*b*) What are $T_{\text{bp, water}}$ and $T_{\text{mp, water}}$ in S?

2. Absolute zero is -273.15°C. Find absolute zero on the Fahrenheit scale.

3. Repeat Exercise 1, except choose the new temperature scale Q so that absolute zero is 0°Q and $T_{\text{bp, water}} - T_{\text{mp, water}} =$ 100 Q°. (*a*) What is the conversion formula from Celsius to Q? (*b*) What is $T_{\text{bp, water}}$ and $T_{\text{mp, water}}$ in Q? (*c*) This scale actually exists. What is the official name?

4. (*a*) The temperature of the surface of the Sun is about 6000 K. Express this on the Fahrenheit scale. (*b*) Express normal human body temperature, 98.6°F on the Celsius scale. (*c*) In the continental United States, the lowest officially recorded temperature is -70°F at Rogers Pass, Montana. Express this on the Celsius scale. (*d*) Express the normal boiling point of oxygen, -183°C, on the Fahrenheit scale. (*e*) At what Celsius temperature would you find a room to be uncomfortably warm?

5. If your doctor tells you that your temperature is 310 K, should you worry? Explain your answer.

6. At what temperature is the Fahrenheit scale reading equal to (*a*) twice that of the Celsius and (*b*) half that of the Celsius?

21-3 Measuring Temperatures

7. A *resistance thermometer* is a thermometer in which the electrical resistance changes with temperature. We are free to define temperatures measured by such a thermometer in kelvins (K) to be directly proportional to the resistance R, measured in ohms (Ω). A certain resistance thermometer is found to have a resistance R of 90.35 Ω when its bulb is placed in water at the triple-point temperature (273.16 K). What temperature is indicated by the thermometer if the bulb is placed in an environment such that its resistance is 96.28 Ω?

8. A thermocouple is formed from two different metals, joined at two points in such a way that a small voltage is produced when the two junctions are at different temperatures. In a particular iron–constantan thermocouple, with one junction held at 0°C, the output voltage varies linearly from 0 to 28.0 mV as the temperature of the other junction is raised from 0 to 510°C. Find the temperature of the variable junction when the thermocouple output is 10.2 mV.

9. The amplification or *gain* of a transistor amplifier may depend on the temperature. The gain for a certain amplifier at room temperature (20.0°C) is 30.0, whereas at 55.0°C it is 35.2. What would the gain be at 28.0°C if the gain depends linearly on temperature over this limited range?

10. If the gas temperature at the steam point is 373.15 K, what is the limiting value of the ratio of the pressures of a gas at the steam point and at the triple point of water when the gas is kept at constant volume?

11. Two constant-volume gas thermometers are assembled, one using nitrogen as the working gas and the other using helium. Both contain enough gas so that $p_{\text{tr}} = 100$ cm Hg. What is the difference between the pressures in the two thermometers if both are inserted into a water bath at the boiling point? Which pressure is the higher of the two? See Fig. 21-5.

21-4 Thermal Expansion

12. An aluminum flagpole is 33 m high. By how much does its length increase as the temperature increases by 15 C°?

13. The Pyrex glass mirror in the telescope at the Mount Palomar Observatory (the Hale telescope) has a diameter of 200 in. The most extreme temperatures ever recorded on Palomar Mountain are -10°C and 50°C. Determine the maximum change in the diameter of the mirror.

14. A circular hole in an aluminum plate is 2.725 cm in diameter at 12°C. What is its diameter when the temperature of the plate is raised to 140°C?

15. Steel railroad tracks are laid when the temperature is -5.0°C. A standard section of rail is then 12.0 m long. What gap should be left between rail sections so that there is no compression when the temperature gets as high as 42°C?

16. A glass window is 200 cm by 300 cm at 10°C. By how much has its area increased when its temperature is 40°C? Assume that the glass is free to expand.

17. A brass cube has an edge length of 33.2 cm at 20.0°C. Find (*a*) the increase in surface area and (*b*) the increase in volume when it is heated to 75.0°C.

18. What is the volume of a lead ball at -12°C if its volume at 160°C is 530 cm^3?

19. (*a*) From the graph of Fig. 21-11, estimate the coefficient of volume expansion for water at room temperature (20°C). (*b*) What is the coefficient of volume expansion near 4°C?

20. Soon after the Earth formed, heat released by the decay of radioactive elements raised the average internal temperature from 300 to 3000 K, at about which value it remains today. Assuming an average coefficient of volume expansion of 3.2×10^{-5} K^{-1}, by how much has the radius of the Earth increased since its formation?

21. A rod is measured to be 20.05 cm long using a steel ruler at a room temperature of 20°C. Both the rod and the ruler are placed in an oven at 270°C, where the rod now measures 20.11 cm using the same rule. Calculate the coefficient of thermal expansion for the material of which the rod is made.

22. The area A of a rectangular plate is ab. Its coefficient of linear expansion is α. After a temperature rise ΔT, side a is longer by Δa and side b is longer by Δb. Show that if we neglect the small quantity $\Delta a\,\Delta b/ab$ (see Fig. 21-14), then $\Delta A = 2\alpha A\,\Delta T$, verifying Eq. 21-10.

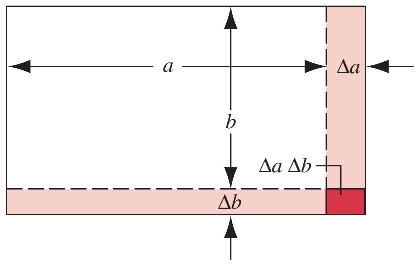


FIGURE 21-14. Exercise 22.

23. Prove that, if we neglect extremely small quantities, the change in volume of a solid upon expansion through a temperature rise ΔT is given by $\Delta V = 3\alpha V \Delta T$, where α is the coefficient of linear expansion. See Eq. 21-11.

24. When the temperature of a copper penny (which is not pure copper) is raised by 100 C°, its diameter increases by 0.18%. Find the percent increase in (*a*) the area of a face, (*b*) the thickness, (*c*) the volume, and (*d*) the mass of the penny. (*e*) Calculate its coefficient of linear expansion.

25. Density is mass divided by volume. If the volume V is temperature dependent, so is the density ρ. Show that the change in density $\Delta\rho$ with change in temperature ΔT is given by

$$\Delta\rho = -\beta\rho\,\Delta T,$$

where β is the coefficient of volume expansion. Explain the minus sign.

26. When the temperature of a metal cylinder is raised from 60 to 100°C, its length increases by 0.092%. (*a*) Find the percent change in density. (*b*) Identify the metal.

27. A steel rod is 3.000 cm in diameter at 25°C. A brass ring has an interior diameter of 2.992 cm at 25°C. At what common temperature will the ring just slide onto the rod?

28. A composite bar of length $L = L_1 + L_2$ is made from a bar of material 1 and length L_1 attached to a bar of material 2 and length L_2 as shown in Fig. 21-15. (*a*) Show that the effective coefficient of linear expansion α for this bar is given by $\alpha = (\alpha_1 L_1 + \alpha_2 L_2)/L$. (*b*) Using steel and brass, design such a composite bar whose length is 52.4 cm and whose effective coefficient of linear expansion is 13×10^{-6}/C°.

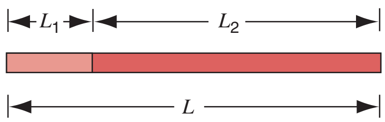


FIGURE 21-15. Exercise 28.

29. At 100°C a glass flask is completely filled by 891 g of mercury. What mass of mercury is needed to fill the flask at −35°C? (The coefficient of linear expansion of glass is 9.0×10^{-6}/C°; the coefficient of volume expansion of mercury is 1.8×10^{-4}/C°.)

30. (*a*) Prove that the change in rotational inertia I with temperature of a solid object is given by $\Delta I = 2\alpha I\,\Delta T$. (*b*) A thin uniform brass rod, spinning freely at 230 rev/s about an axis perpendicular to it at its center, is heated without mechanical contact until its temperature increases by 170 C°. Calculate the change in angular velocity.

31. A cylinder placed in frictionless bearings is set rotating about its axis. The cylinder is then heated, without mechanical contact, until its radius is increased by 0.18%. What is the percent change in the cylinder's (*a*) angular momentum, (*b*) angular velocity, and (*c*) rotational energy?

32. (*a*) Prove that the change in period P of a physical pendulum with temperature is given by $\Delta P = \frac{1}{2}\alpha P \Delta T$. (*b*) A clock pendulum made of invar has a period of 0.500 s and is accurate at 20°C. If the clock is used in a climate where the temperature averages 30°C, what approximate correction to the time given by the clock is necessary at the end of 30 days?

33. A pendulum clock with a pendulum made of brass is designed to keep accurate time at 20°C. How much will the error be, in seconds per hour, if the clock operates at 0°C?

34. An aluminum cup of 110 cm^3 capacity is filled with glycerin at 22°C. How much glycerin, if any, will spill out of the cup if the temperature of the cup and glycerin is raised to 28°C? (The coefficient of volume expansion of glycerin is 5.1×10^{-4}/C°.)

35. A 1.28-m-long vertical glass tube is half-filled with a liquid at 20.0°C. How much will the height of the liquid column change when the tube is heated to 33.0°C? Assume that $\alpha_{glass} = 1.1 \times 10^{-5}$/C° and $\beta_{liquid} = 4.2 \times 10^{-5}$/C°.

21-5 The Ideal Gas

36. (*a*) Using the ideal gas law and the definition of the coefficient of volume expansion (Eq. 21-12), show that $\beta = 1/T$ for an ideal gas at constant pressure. (*b*) In what units must T be expressed? If T is expressed in those units, can you express β in units of $(C°)^{-1}$? (*c*) Estimate the value of β for an ideal gas at room temperature.

37. (*a*) Calculate the volume occupied by 1.00 mol of an ideal gas at standard conditions—that is, pressure of 1.00 atm ($= 1.01 \times 10^5$ Pa) and temperature of 0°C (= 273 K). (*b*) Show that the number of molecules per cubic centimeter (the *Loschmidt number*) at standard conditions is 2.68×10^{19}.

38. The best vacuum that can be attained in the laboratory corresponds to a pressure of about 10^{-18} atm, or 1.01×10^{-13} Pa. How many molecules are there per cubic centimeter in such a vacuum at 22°C?

39. A quantity of ideal gas at 12.0°C and a pressure of 108 kPa occupies a volume of 2.47 m^3. (*a*) How many moles of the gas are present? (*b*) If the pressure is now raised to 316 kPa and the temperature is raised to 31.0°C, how much volume will the gas now occupy? Assume there are no leaks.

40. Oxygen gas having a volume of 1130 cm^3 at 42.0°C and a pressure of 101 kPa expands until its volume is 1530 cm^3 and its pressure is 106 kPa. Find (*a*) the number of moles of oxygen in the system and (*b*) its final temperature.

41. An automobile tire has a volume of 988 in.^{3} and contains air at a gauge pressure of 24.2 lb/in.^{2} where the temperature is -2.60°C. Find the gauge pressure of the air in the tire when its temperature rises to 25.6°C and its volume increases to 1020 in.^{3}. (Hint: It is not necessary to convert from British to SI units. Why? Use $p_{atm} = 14.7$ lb/in.^{2}.)

42. Estimate the mass of the Earth's atmosphere. Express your estimate as a fraction of the mass of the Earth. Recall that atmospheric pressure equals 101 kPa.

43. An air bubble of 19.4 cm^3 volume is at the bottom of a lake 41.5 m deep where the temperature is 3.80°C. The bubble rises to the surface, which is at a temperature of 22.6°C. Take the temperature of the bubble to be the same as that of the surrounding water and find its volume just before it reaches the surface.

44. An open–closed pipe of length $L = 25.0$ m contains air at atmospheric pressure. It is thrust vertically into a freshwater lake until the water rises halfway up in the pipe, as shown in Fig. 21-16. What is the depth h of the lower edge of the pipe? Assume that the temperature is the same everywhere and does not change.

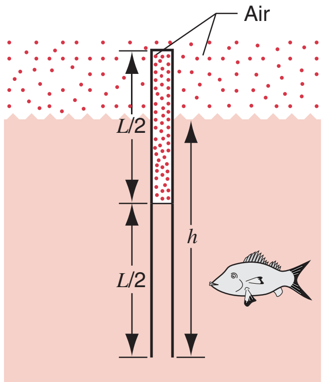


FIGURE 21-16. Exercise 44.

PROBLEMS

1. It is an everyday observation that hot and cold objects cool down or warm up to the temperature of their surroundings. If the temperature difference ΔT between an object and its surroundings ($\Delta T = T_{obj} - T_{sur}$) is not too great, the rate of cooling or warming of the object is proportional, approximately, to this temperature difference; that is,

$$\frac{d\Delta T}{dt} = -A(\Delta T),$$

where A is a constant. The minus sign appears because ΔT decreases with time if ΔT is positive and increases if ΔT is negative. This is known as *Newton's law of cooling.* (*a*) On what factors does A depend? What are its dimensions? (*b*) If at some instant $t = 0$ the temperature difference is ΔT_0, show that it is

$$\Delta T = \Delta T_0 e^{-At}$$

at a time t later.

2. Early in the morning the heater of a house breaks down. The outside temperature is -7.0°C. As a result, the inside temperature drops from 22 to 18°C in 45 min. How much longer will it take for the inside temperature to fall by another 4.0 C°? Assume that the outside temperature does not change and that Newton's law of cooling applies; see Problem 1.

3. Show that when the temperature of a liquid in a barometer changes by ΔT, and the pressure is constant, the height h changes by $\Delta h = \beta h \Delta T$, where β is the coefficient of volume expansion of the liquid. Neglect the expansion of the glass tube.

4. A particular gas thermometer is constructed of two gas-containing bulbs, each of which is put into a water bath, as shown in Fig. 21-17. The pressure difference between the two bulbs is measured by a mercury manometer as shown in the figure. Appropriate reservoirs, not shown in the diagram, maintain constant gas volume in the two bulbs. There is no difference in pressure when both baths are at the triple point of water. The pressure difference is 120 mm Hg when one bath is at the triple point and the other is at the boiling point of water. Finally, the pressure difference is 90.0 mm Hg when one bath is at the triple point and the other is at an unknown temperature to be measured. Find the unknown temperature.

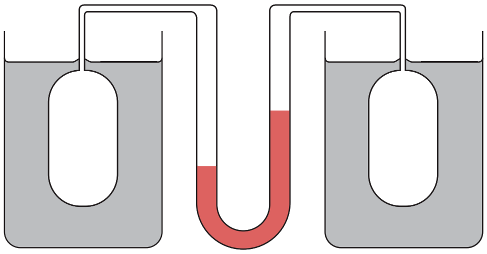


FIGURE 21-17. Problem 4.

5. Show that if α is dependent on the temperature T, then

$$L \approx L_0 \left[1 + \int_{T_0}^{T} \alpha(T)\, dT\right],$$

where L_0 is the length at the reference temperature T_0.

6. In a certain experiment, it was necessary to be able to move a small radioactive source at selected, extremely slow speeds. This was accomplished by fastening the source to one end of an aluminum rod and heating the central section of the rod in a controlled way. If the effective heated section of the rod in Fig. 21-18 is 1.8 cm, at what constant rate must the temperature of the rod be made to change if the source is to move at a constant speed of 96 nm/s?

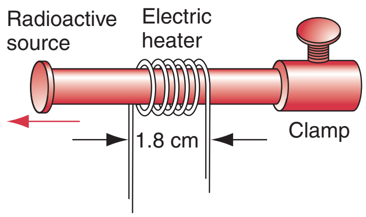


FIGURE 21-18. Problem 6.

7. (*a*) Show that if the lengths of two rods of different solids are inversely proportional to their respective coefficients of linear expansion at the same initial temperature, the difference in length between them will be constant at all temperatures. (*b*) What should be the lengths of a steel and a brass rod at 0°C so that at all temperatures their difference in length is 0.30 m?

8. As a result of a temperature rise of 32 C°, a bar with a crack at its center buckles upward, as shown in Fig. 21-19. If the fixed distance L_0 = 3.77 m and the coefficient linear expansion is 25×10^{-6}/C°, find x, the distance to which the center rises.

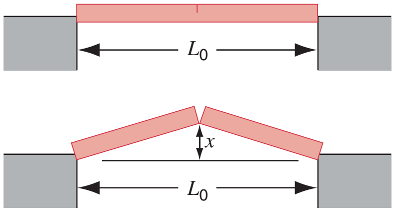


FIGURE 21-19. Problem 8.

9. Figure 21-20 shows the variation of the coefficient of volume expansion of water between 4°C and 20°C. The density of water at 4°C is 1000 kg/m³. Calculate the density of water at 20°C.

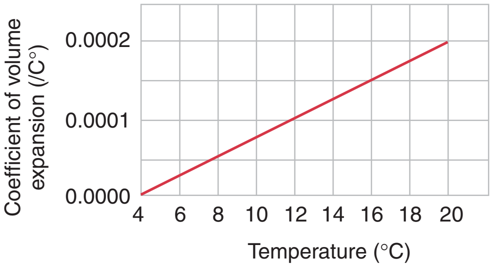


FIGURE 21-20. Problem 9.

10. Consider a mercury-in-glass thermometer. Assume that the cross section of the capillary is constant at A and that V is the volume of the bulb of mercury at 0.00°C. Suppose that the mercury just fills the bulb at 0.00°C. Show that the length L of the mercury column in the capillary at a temperature T, in °C, is

$$L = \frac{V}{A}(\beta - 3\alpha)T,$$

that is, proportional to the temperature, where β is the coefficient of volume expansion of mercury and α is the coefficient of linear expansion of glass.

11. Three equal-length straight rods, of aluminum, invar, and steel, all at 20°C, for an equilateral triangle with hinge pins at the vertices. At what temperature will the angle opposite the invar rod be 59.95°? See Appendix I for needed trigonometric formulas.

12. A glass tube nearly filled with mercury is attached in tandem to the bottom of an iron pendulum rod 100 cm long. How high must the mercury be in the glass tube so that the center of mass of this pendulum will not rise or fall with changes in temperature? (The cross-sectional area of the tube is equal to that of the iron rod. Neglect the mass of the glass. Iron has a density of 7.87×10^3 kg/m³ and a coefficient of linear expansion equal to 12×10^{-6}/C°. The coefficient of volume expansion of mercury is 18×10^{-5}/C°.)

13. An aluminum cube 20 cm on an edge floats on mercury. How much farther will the block sink when the temperature rises from 270 to 320 K? (The coefficient of volume expansion of mercury is 1.8×10^{-4}/C°.)

14. Dumet wire was developed to allow for the expansion of glass in lightbulbs. The wire consists of a core of nickel–steel (invar) surrounded by a sheath of copper. The diameters of the core and of the sheath are chosen so that the wire duplicates the expansion characteristics of glass. (*a*) Show that the ratio of the nickel–steel radius to that of the copper sheath should be

$$\frac{r_{\text{nickel-steel}}}{r_{\text{copper}}} = \sqrt{\frac{\alpha_{\text{copper}} - \alpha_{\text{glass}}}{\alpha_{\text{copper}} - \alpha_{\text{nickel-steel}}}}$$

(*b*) What is a typical value for this ratio?

15. The distance between the towers of the main span of the Golden Gate Bridge near San Francisco is 4200 ft (Fig. 21-21). The sag of the cable halfway between the towers at 50°F is 470 ft. Take $\alpha = 6.5 \times 10^{-6}$/F° for the cable and compute (*a*) the change in length of the cable and (*b*) the change in sag for a temperature change from 10 to 90°F. Assume no bending or separation of the towers and a parabolic shape for the cable.

FIGURE 21-21. Problem 15.

16. A weather balloon is loosely inflated with helium at a pressure of 1.00 atm (= 76.0 cm Hg) and a temperature of 22.0°C. The gas volume is 3.47 m³. At an elevation of 6.50 km, the atmospheric pressure is down to 36.0 cm Hg, and the helium has expanded, being under no restraint from the confining bag. At this elevation the gas temperature is −48.0°C. What is the gas volume now?

17. Two vessels of volumes 1.22 L and 3.18 L contain krypton gas and are connected by a thin tube. Initially, the vessels are at the same temperature, 16.0°C, and the same pressure, 1.44 atm. The larger vessel is then heated to 108°C while the smaller one remains at 16.0°C. Calculate the final pressure. (Hint: There are no leaks.)

18. Container A contains an ideal gas at a pressure of 5.0×10^5 Pa and at a temperature of 300 K. It is connected by a thin tube to container B with four times the volume of A; see Fig. 21-22. B contains the same ideal gas at a pressure of 1.0×10^5 Pa and at a temperature of 400 K. The connecting valve is opened, and equilibrium is achieved at a common pressure while the temperature of each container is kept constant at its initial value. What is the final pressure in the system?

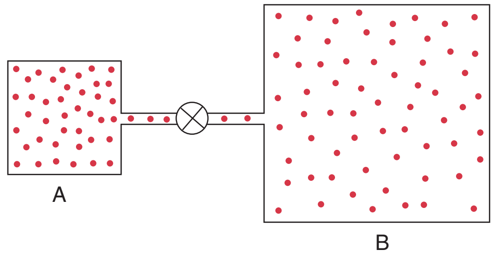


FIGURE 21-22. Problem 18.

19. The variation in pressure in the Earth's atmosphere, assumed to be at a uniform temperature, is given by $p = p_0 e^{-Mgy/RT}$, where M is the molar mass of the air. (See Section 15-3.) Show that $n_V = n_{V,0} e^{-Mgy/RT}$, where n_V is the number of molecules per unit volume.

20. A soap bubble of radius $r_0 = 2.0$ mm floats freely inside a vacuum bell jar. The pressure inside the bell jar is originally $p = 1$ atm. The vacuum pump is turned on and the pressure in the bell jar is slowly decreased to zero while the temperature of the gas inside the bubble remains constant. What is the radius of the soap bubble when the outside pressure drops to zero? The surface tension for a soap bubble is $\gamma = 2.50 \times 10^{-2}$ N/m. (See Computer Problem 1.)

21. A mercury-filled manometer with two unequal-length arms of the same cross-sectional area is sealed off with the same pressure p in the two arms, as in Fig. 21-23. With the temperature constant, an additional $10.0\ \text{cm}^3$ of mercury is admitted through the stopcock at the bottom. The level on the left increases 6.00 cm and that on the right increases 4.00 cm. Find the pressure p.

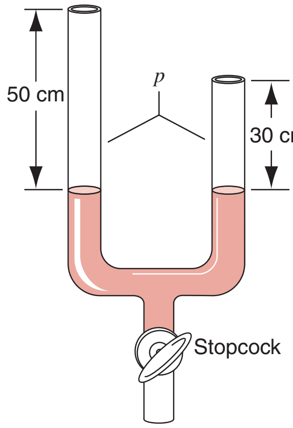


FIGURE 21-23. Problem 21.

22. The "surface tension" of a certain spherical balloon is proportional to the radius of the balloon. Originally the balloon is filled with 10.0 L of an ideal gas at 80°C and 103 kPa. The gas cools to 20°C; find the new volume of the balloon. Assume that the pressure outside the balloon remains at 101 kPa.

COMPUTER PROBLEMS

1. A soap bubble with surface tension $\gamma = 2.50 \times 10^{-2}$ N/m has a radius $r_0 = 2.0$ mm when the pressure outside the bubble is 1.0 atmosphere. (*a*) Numerically calculate the radius of the soap bubble when the pressure outside the bubble drops to 0.5 atm. (*b*) Numerically calculate the radius of the soap bubble if the pressure outside the bubble is raised to 2.0 atm.

2. A small balloon is filled with nitrogen gas (assumed ideal) at the bottom of the Marianas Trench, 35,000 ft beneath the surface of the ocean. The balloon originally has a radius of 1.0 mm, is massless, and is infinitely expandable without any surface tension, but always keeps a spherical shape. Assume the ideal gas inside the balloon is at 4°C throughout this problem. The balloon begins to rise to the surface, as the balloon rises it expands, and as it moves there is a retarding force f proportional to speed v and balloon radius r given by

$$f = 6\pi\eta r v,$$

where $\eta = 1.7 \times 10^{-3}\ \text{N}\cdot\text{s/m}$ is the viscosity of water. (*a*) Calculate the initial buoyant force on the balloon. (*b*) What will be the size of the balloon on the surface? (*c*) Numerically solve this problem to find out how long it takes for the balloon to rise to the surface.

CHAPTER 22

MOLECULAR PROPERTIES OF GASES

In Section 21-5 we introduced the ideal gas law, which is expressed in terms of pressure, volume, and temperature. When we deal with such large-scale, measurable properties of gases, we are taking what we describe as a macroscopic *approach to the subject. The ideal gas law says nothing about the fact that gases—and indeed matter of all kinds—are made up of particles, which may be atoms or molecules.*

In this chapter we take a microscopic *approach and seek to account for the macroscopic properties of a gas in terms of the properties of its molecules. Our plan is to follow the motion of a representative molecule and then average this behavior over all the molecules that make up the system. If the number of molecules is very large—and it usually is—such averages give sharply defined quantities. The formal name for the approach we are taking is* the kinetic theory of gases, *the word "kinetic" suggesting that we are dealing with particles that are in motion.*

22-1 THE ATOMIC NATURE OF MATTER

Today no informed person doubts that all matter is made up of atoms. It may come as a surprise to learn that universal acceptance of the existence of atoms by the scientific community did not occur until the early 1900s. There were many earlier speculations about the atomic nature of matter, dating back to the ancient Greeks, but none were sufficiently firmly supported by experiment to exclude other points of view. Today the hypothesis that atoms exist is so essential to our understanding of the nature of the world around us that the Nobel laureate physicist Richard Feynman could write: "If all scientific knowledge were to be destroyed, I would hope that the knowledge that atoms exist might be spared."

The modern trail to belief in atoms can be said to have started in 1828 when the Scottish botanist Robert Brown observed through his microscope that tiny grains of pollen suspended in water underwent ceaseless random motion. We now call this phenomenon *Brownian motion.* Brown also noted that this same "dancing" motion occurred when particles of finely powdered coal, glass, rocks, and various minerals were suspended in a fluid. The motion seemed to be—and indeed proved to be—a fundamental property of matter.

In 1905, Einstein (unaware of Brown's report of his observations) predicted that the effect should occur and presented it as direct evidence that the fluid in which the particle is suspended is made up of atoms. A particle suspended in a fluid is bombarded on all sides by the atoms of the fluid, which are in constant motion of thermal agitation. Let N be the average number of particle–atom collisions on any one side of the particle during a short time interval Δt. *On average,* the same number of collisions will occur on the other side of the particle. However, because the collisions occur randomly, there will be fluctuations about this average on each side. Thus in any particular interval Δt there will be slightly more collisions on one side of the particle than on the other. These random unbalances occur in three dimensions so the bombarded particle, which typically is many orders of magnitude more massive than the atoms that bombard it, jitters about in the erratic manner that characterizes Brownian motion.

A few years after Einstein's analysis of Brownian motion, the French physical chemist Jean Baptiste Perrin (1870–1942) made quantitative measurements of the effect. Figure 22-1 shows a sample of his data. It displays the Brownian motion of a tiny particle of gum resin suspended in water and viewed through a microscope. Perrin measured the x, y coordinates of the moving particle every 30 s and marked the particle's position with a dot on a graph. (The straight line segments in Fig. 22-1 were drawn simply to connect the dots; the particle does not follow these lines but moves in the same erratic fashion that characterizes the figure as a whole.)

The quantities that can be directly measured from the so-called *random walk* pattern of Fig. 22-1 are Δx and Δy, which are, respectively, the changes in the x and the y coordinates of the particle between successive observations. Because Δx and Δy can be either positive or negative, their average value over many measurements is zero. The significant parameters are the average values of the *squares* of these quantities, $[(\Delta x)^2]_{av}$ and $[(\Delta y)^2]_{av}$, which are inherently positive.

Einstein derived the following expression for $[(\Delta x)^2]_{av}$ if the bombarded particle is a sphere of radius a suspended in a gas:

$$[(\Delta x)^2]_{av} = \frac{RT}{3\pi\eta a N_A}\,\Delta t. \qquad (22\text{-}1)$$

Here η (Greek "eta") is a measure of the viscosity of the gas (see Section 16-6). This quantity enters because, when the suspended particle is given a "kick" because of an unbalance in the atomic bombardment, the particle is slowed down by friction-like viscous forces. R in Eq. 22-1 is the molar gas constant, T is the Kelvin temperature, and N_A is the Avogadro constant.

If N_A were much larger than it actually is, the extent of the Brownian motion would be reduced because the collision rates would be more closely equal on opposite sides of the suspended particle. On the other hand, if N_A were much smaller than it actually is, the Brownian magnitude would be increased. Thus, with $[(\Delta x)^2]_{av}$ measured, Eq. 22-1 can be used to deduce N_A. After collecting much data, of which Fig. 22-1 is a small sample, Perrin found $N_A \approx 6 \times 10^{23}$ molecules/mol, which agreed with results obtained at that time by other methods. For this work, which was so compelling a confirmation of the existence of atoms, Perrin received the 1926 Nobel Prize in physics. In his 1913 book, *Atoms,* Perrin wrote enthusiastically about his Brownian motion observations: "The atomic theory has triumphed. Until recently still numerous, its adversaries, at last overcome, now renounce one after another their misgivings . . . "

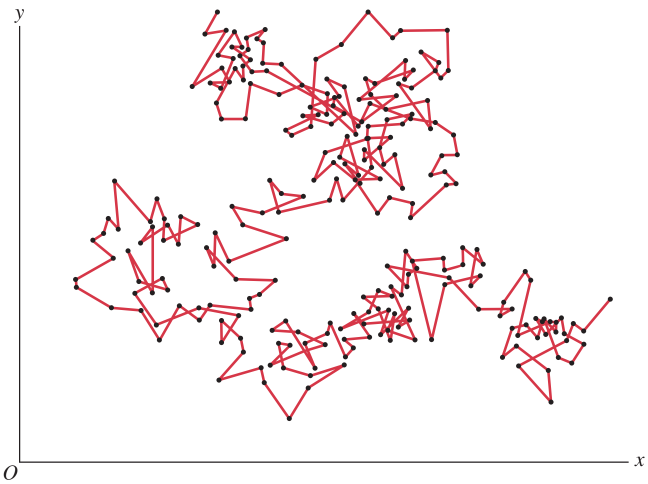


FIGURE 22-1. The Brownian motion of a tiny particle of gum resin of radius about 3μm. The dots, which are connected by straight lines, mark the positions of the particle at 30-s intervals. The path of the particles is an example of a *fractal,* a curve for which any small section resembles the curve as a whole. For example, if we take any short 30-s segment and view it in smaller intervals, perhaps 0.1 s, the plot of the motion in that single 30-s segment would be similar to the entire figure.

Properties of the Ideal Gas

In Section 21-5 we described the macroscopic properties of the ideal gas and showed that they were related by the ideal gas law ($pV = nRT$). Now that we have shown the evidence that matter is really made up of atoms, let us look in a little more detail at the atomic or microscopic properties of the ideal gas. In most of the remaining sections of this chapter we will rely on the ideal gas as our thermodynamic system of choice.

1. *The ideal gas consists of particles, which are in random motion and obey Newton's laws of motion.* These particles may be single atoms or groups of atoms. In either case, we will refer to the particles as "molecules." The molecules move in all directions and with a wide range of speeds.

2. *The total number of molecules is "large."* When a molecule rebounds from the wall of its container, it delivers momentum to it. We assume that the number of molecules is so large that the rate at which momentum is delivered to any area A of the container wall is essentially constant.

3. *The volume occupied by the molecules is a negligibly small fraction of the volume occupied by the gas.* We know that when a gas condenses to liquid form, the volume of the liquid is much smaller than that of the gas. Thus molecules are "small" and our assumption is plausible.

4. *No forces act on a molecule except during a collision, either with the container walls or with another molecule.* If we follow a particular molecule, it moves in a zigzag path consisting of straight-line segments with constant velocity between impulsive encounters.

5. *All collisions are (i) elastic and (ii) of negligible duration.* Part (i) tells us that the total kinetic energy of the molecules is a constant. Part (ii) tells us that the total potential energy of the molecules (which can only come into play during a collision) is negligible.

In the ideal gas model, we take all molecules of a gas of a particular type to be identical and thus to have identical masses. The mass of a molecule is determined by adding the masses of the atoms that make up the molecule. Atomic masses (in units of u), which are often given on a periodic chart of the elements, can be found in Appendix D. For example, the mass of a molecule of sulfur dioxide (SO_2) is given in terms of the atomic masses of sulfur and oxygen as

$$m = m(\text{S}) + 2m(\text{O}) = 32.1\ \text{u} + 2(16.0\ \text{u}) = 64.1\ \text{u}.$$

Instead of the number of molecules N, it is often more convenient to describe the amount of a gas in terms of the number of moles n; the relationship between these two equivalent measures of the quantity of gas was given in Eq. 21-15, $N = nN_A$, where N_A is the Avogadro constant with a value of $N_A = 6.02 \times 10^{23}$ molecules/mol.

The mass of a mole of any substance, called the *molar mass M,* is simply the mass of one molecule times the number of molecules per mole, or

$$M = mN_A. \tag{22-2}$$

The molar mass, measured in grams, is numerically equal to the molecular mass, measured in u. Thus the molar mass of SO_2 is $M = 64.1$ g/mol $= 0.0641$ kg/mol.

In the rest of this chapter we show how the analysis of a gas as a collection of molecules that behave according to Newton's laws gives us a connection between its macroscopic thermodynamic properties and such microscopic properties as the average molecular speed or the average distance a molecule travels between collisions.

22-2 A MOLECULAR VIEW OF PRESSURE

In this section we associate the pressure exerted by a gas on the walls of its container with the constant bombardment of those walls by the molecules of the gas, a point of view perhaps first advanced by the Swiss scientist Daniel Bernoulli (1700–1782) in 1738. We will take the ideal gas as our system and will derive an expression for the pressure it exerts in terms of the properties of the molecules that make it up.

Consider N molecules of an ideal gas confined within a cubical box of edge length L, as in Fig. 22-2. Call the faces at right angles to the x axis A_1 and A_2, each of area L^2. Let us focus our attention on a single molecule of mass m, whose velocity $\vec{v}$ we can resolve into components v_x, v_y, and v_z. When this molecule strikes face A_1, it rebounds with its x component of velocity reversed, because all collisions are assumed to be elastic; that is, $v_x \to -v_x$. There is no effect on v_y or v_z, so that the change in the molecule's momentum has only an x component, given by

$$\text{final momentum} - \text{initial momentum} = -mv_x - (mv_x) = -2mv_x. \tag{22-3}$$

Because the total momentum is conserved in the collision, the momentum imparted to A_1 is $+2mv_x$.

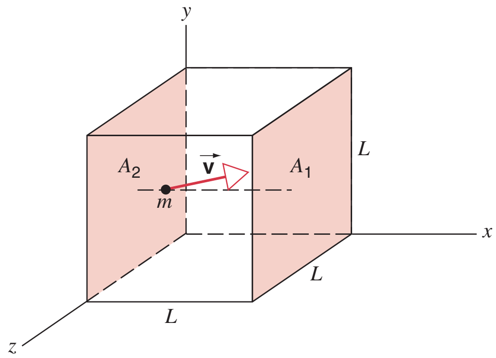


FIGURE 22-2. A cubical box of edge L containing an ideal gas. A molecule of the gas is shown moving with velocity $\vec{v}$ toward side A_1.

Suppose that this molecule reaches A_2 without striking any other molecule on the way. The time required to cross the cube is L/v_x. (If the molecule strikes one of the other faces of the box on the way to A_2, the x component of its velocity does not change, nor does the transit time.) At A_2 it again has its x component of velocity reversed and returns to A_1. Assuming there are no collisions with other molecules, the round trip takes a time $2L/v_x$, which is the time between collisions with A_1. The average impulsive force exerted by this molecule on A_1 is the transferred momentum divided by the time interval between transfers, or

$$F_x = \frac{2mv_x}{2L/v_x} = \frac{mv_x^2}{L}. \tag{22-4}$$

To obtain the *total* force on A_1—that is, the rate at which momentum is imparted to A_1 by *all* the gas molecules—we must sum the quantity mv_x^2/L for all the molecules. Then, to find the pressure, we divide this force by the area of A_1—namely, L^2. The pressure is therefore

$$p = \frac{1}{L^2}\,\frac{mv_{1x}^2 + mv_{2x}^2 + \cdots}{L}$$

$$= \frac{m}{L^3}(v_{1x}^2 + v_{2x}^2 + \cdots), \tag{22-5}$$

where v_{1x} is the x component of the velocity of molecule 1, v_{2x} is that of molecule 2, and so on. If N is the total number of molecules in the container, then Nm is the total mass and Nm/L^3 is the density ρ. Thus $m/L^3 = \rho/N$, and

$$p = \rho\left(\frac{v_{1x}^2 + v_{2x}^2 + \cdots}{N}\right). \tag{22-6}$$

The quantity in parentheses in Eq. 22-6 is the average value of v_x^2 for all the molecules in the container, which we represent by $(v_x^2)_{\text{av}}$. Then

$$p = \rho(v_x^2)_{\text{av}}. \tag{22-7}$$

For any molecule, $v^2 = v_x^2 + v_y^2 + v_z^2$. Because we have many molecules and because they are moving entirely at random, the average values of v_x^2, v_y^2, and v_z^2 are equal, and the value of each is exactly one-third the average value of

v^2. There is no preference among the molecules for motion along any one of the three axes. Hence $(v_x^2)_{av} = \frac{1}{3}(v^2)_{av}$, so that Eq. 22-7 becomes

$$p = \tfrac{1}{3}\rho(v^2)_{av}. \tag{22-8}$$

Although we derived this result by neglecting collisions between molecules, the result is true even when we consider collisions. Because of the exchange of velocities in an elastic collision between identical particles, there will always be a molecule that collides with A_2 with momentum mv_x corresponding to the molecule that left A_1 with this same momentum. Also, the time spent during collisions is negligible compared to the time spent between collisions. Hence our neglect of collisions is merely a convenient device for calculation. Similarly, we could have chosen a container of any shape: the cube merely simplifies the calculation. Although we have calculated the pressure exerted only on the side A_1, it follows from Pascal's law that the pressure is the same on all sides and everywhere in the interior. (This is true only if the density of the gas is uniform. In a large sample of gas, gravitational effects might be significant, and we should take into account the varying density. See Section 15-3 and Problem 19 of Chapter 21.)

The square root of $(v^2)_{av}$ is called the *root-mean-square* speed of the molecules and is a useful measure of average molecular speed. Using Eq. 22-8, we can calculate the root-mean-square speed from measured values of the pressure and density of the gas. Thus

$$v_{rms} \equiv \sqrt{(v^2)_{av}} = \sqrt{\frac{3p}{\rho}}. \tag{22-9}$$

In Eq. 22-8 we relate a macroscopic quantity (the pressure p) to an average value of a microscopic quantity, that is, to $(v^2)_{av}$ or v^2_{rms}.

SAMPLE PROBLEM 22-1. Calculate the root-mean-square speed of hydrogen molecules at 0.00°C and 1.00 atm pressure, assuming hydrogen to be an ideal gas. Under these conditions hydrogen has a density ρ of 8.99×10^{-2} kg/m^3.

Solution Since $p = 1.00$ atm $= 1.01 \times 10^5$ Pa,

$$v_{rms} = \sqrt{\frac{3p}{\rho}} = \sqrt{\frac{3(1.01 \times 10^5 \text{ Pa})}{8.99 \times 10^{-2} \text{ kg/m}^3}} = 1840 \text{ m/s}.$$

This is equal to 4120 mi/h, or just slightly more than a mile per second.

Table 22-1 gives the results of similar calculations for some selected gases at room temperature. The values of v_{rms} in that table refer to the speeds of the molecules between collisions. Because of these collisions, gas molecules are continuously changing direction and do not move very rapidly in any selected direction. This contrast between intercollision speeds and outward diffusion speeds is sometimes said to account for the noticeable time lag between opening a perfume bottle at one end of a room and smelling perfume at the other end. However, the fact that one smells perfume at all can be shown to be due to unavoidable convection currents in the air of the room. If these currents could be eliminated, the time lag would be very much greater indeed. The diffusion speed of one gas into another is very much less than the rms speed of the diffusing molecules.

TABLE 22-1 Some Molecular Speeds at Room Temperature (300 K)

Gas	*Molecular Mass m* (u)	v_{rms} (m/s)
Hydrogen	2.0	1920
Helium	4.0	1370
Water vapor	18.0	645
Nitrogen	28.0	517
Oxygen	32.0	483
Carbon dioxide	44.0	412
Sulfur dioxide	64.1	342

SAMPLE PROBLEM 22-2. The cubical box of Fig. 22-2 is 10 cm on edge and contains oxygen at a pressure of 1.0 atm and a temperature $T = 300$ K. (*a*) How many moles of oxygen are in the box? (*b*) How many molecules? (*c*) At what approximate rate do oxygen molecules strike one face of the box? (Hint: For simplicity, assume that the molecules all move with the same speed v_{rms}, that they do not collide with each other, and that one-third of them move back and forth between each pair of opposing faces of the cube.)

Solution (*a*) Solving the ideal gas equation (Eq. 21-17) for n, the number of moles, we obtain

$$n = \frac{pV}{RT} = \frac{(1.01 \times 10^5 \text{ Pa})(0.10 \text{ m})^3}{(8.31 \text{ J/mol}\cdot\text{K})(300 \text{ K})} = 0.041 \text{ mol}.$$

Here we have replaced V by L^3 and used the fact that, in SI units, 1 atm $= 1.01 \times 10^5$ Pa.

(*b*) The number of molecules follows from Eq. 21-15:

$$\begin{aligned} N = nN_A &= (0.041 \text{ mol})(6.02 \times 10^{23} \text{ molecules/mol}) \\ &= 2.5 \times 10^{22} \text{ molecules.} \end{aligned}$$

(*c*) Consider the back-and forth motion of a single molecule. Its average time between collisions on a particular face is $2L/v_{rms}$ and the rate at which it strikes that face is the inverse of this, or $v_{rms}/2L$. If the box contains N molecules, on our assumption $\frac{1}{3}N$ of them are doing the same thing. So the total rate at which molecules hit the face in question is $(\frac{1}{3}N)(v_{rms}/2L)$. From Table 22-1 we see that v_{rms} for oxygen at 300 K is 483 m/s. Thus

$$\begin{aligned} \text{Rate} \approx \frac{Nv_{rms}}{6L} &= \frac{(2.5 \times 10^{22} \text{ molecules})(483 \text{ m/s})}{(6)(0.1 \text{ m})} \\ &= 2.0 \times 10^{25} \text{ collisions/s.} \end{aligned}$$

A more rigorous analysis, taking into account the varying speeds and directions of the molecules, yields 2.8×10^{25} collisions/s. Thus our approximate answer is not too far removed from the correct answer. In solving problems in physics, we often make grossly simplifying assumptions if we seek only an approximate answer.

SAMPLE PROBLEM 22-3. Natural uranium consists primarily of two isotopes, fissionable ^{235}U (0.7% abundance) and practically nonfissionable ^{238}U (99.3%). (*a*) In UF_6 (uranium hexafluoride) gas containing a natural mixture of these two isotopes at a common temperature T, calculate the ratio of the rms speed of the gas molecules containing ^{235}U to those containing ^{238}U. (*b*) If this gas is passed through a porous barrier, the faster molecules emerge first, and the resulting abundances of the two kinds of gas molecules on the far side of the barrier are proportional to their rms speeds. What will be the relative abundance of gas molecules containing ^{235}U after the passage of the gas through such a barrier? (*c*) How many times must the gas be passed through such a barrier before the abundance of ^{235}U reaches 3%? This abundance is typical of the enrichment of ^{235}U needed for the uranium fuel in fission reactors.

Solution (*a*) Consider two samples of UF_6 gas, identical except that one contains only ^{235}U and the other only ^{238}U. The molecular masses of $^{235}UF_6$ and $^{238}UF_6$ are $m(235) = 235\text{ u} + 6(19\text{ u}) = 349\text{ u}$ and $m(238) = 238\text{ u} + 6(19\text{ u}) = 352\text{ u}$. The ratio of densities—all other factors being equal—is the ratio of the molecular masses so, from Eq. 22-9,

$$\frac{v_{\text{rms}}(235)}{v_{\text{rms}}(238)} = \sqrt{\frac{m(238)}{m(235)}} = \sqrt{\frac{352\text{ u}}{349\text{ u}}} = 1.0043.$$

(*b*) The relative abundance of the two kinds of gas molecules in the mixed gas sample is the same as the relative abundance of the uranium isotopes they contain. On entering the barrier this ratio is $0.007/0.993 = 0.00705$. On our assumption, passage through the barrier increases this ratio by the factor calculated in (*a*), so

$$\text{ratio after 1 pass} = 0.00705 \times 1.0043 = 0.00708.$$

(*c*) The gas entering the first barrier has an isotope ratio, calculated in (*b*), of 0.00705. After passage through a barrier n times, we wish the isotope ratio of the emerging gas to be $0.030/0.97 = 0.03093$. There is an increase in this ratio of 1.0043 at each passage, so

$$(1.0043)^n\,(0.00705) = 0.03093.$$

If we solve this relationship for n (by taking logarithms) we find $n \approx 350$.

22-3 THE MEAN FREE PATH

Suppose that we could follow the zigzag path (Fig. 22-3) of a typical molecule in a gas as it moves around, colliding with other molecules. In particular, let us measure the straight-line distance our chosen molecule travels between collisions and find its average value. We call this quantity the molecule's *mean free path* λ. Because our chosen molecule is not "special," all molecules of the gas have the same mean free path. Of course, we cannot follow a single molecule and make these measurements, but in this section we will calculate the outcome of such measurements.

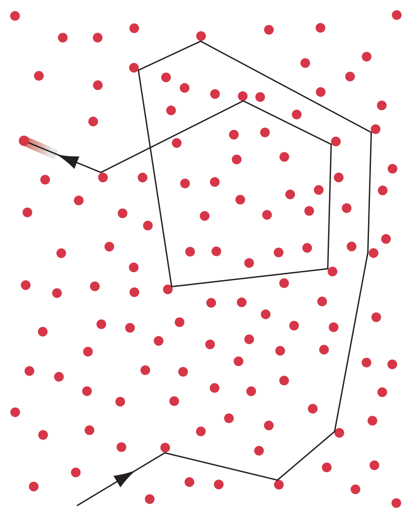


FIGURE 22-3. A molecule traveling through a gas, colliding with other molecules in its path. Of course, the other molecules are themselves moving and experiencing collisions.

Consider the molecules of a gas to be spheres of diameter d. A collision will take place when the centers of two such molecules approach within a distance d of each other. An equivalent description of collisions made by any chosen molecule is to regard that molecule as having a diameter $2d$ and all other molecules as point particles; see Fig. 22-4.

Let us temporarily assume that our molecule of diameter $2d$ exerts no forces on the point molecules among which it moves. In time t our "fat" molecule would sweep out a cylinder of cross-sectional area πd^2, length $L_{\text{cyl}} = vt$ (where v is the speed of the molecule), and volume $V_{\text{cyl}} = \text{area} \times \text{length} = (\pi d^2)(vt)$. Let the volume of the box within which the gas is confined be V and let the box contain N molecules. The number of (point) molecules in the cylinder of Fig. 22-5 is then

$$N_{\text{cyl}} = N\frac{V_{\text{cyl}}}{V} = \frac{N\pi d^2 vt}{V}. \qquad (22\text{-}10)$$

Since our moving molecule and the point molecules *do* exert forces on each other, this number is also the number of collisions experienced by our moving molecule in time t.

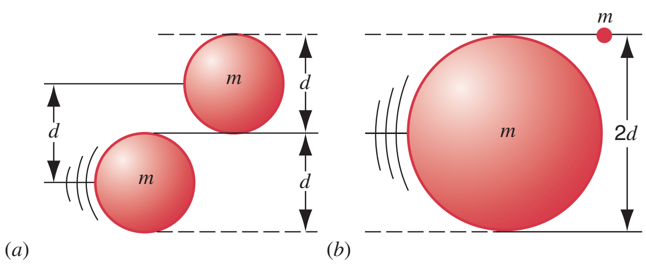


FIGURE 22-4. (*a*) A collision occurs when the centers of two molecules come within a distance d of each other, where d is the molecular diameter. (*b*) An equivalent but more convenient representation is to think of the moving molecule as having a diameter $2d$, all other molecules being points.

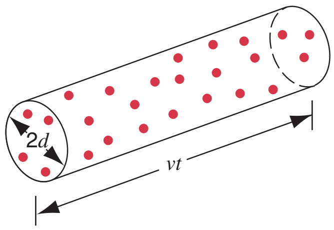


FIGURE 22-5. A molecule with an equivalent diameter $2d$ (as in Fig. 22-4*b*) traveling with speed v sweeps out a cylinder of base area πd^2 and length vt in a time t. The number of collisions suffered by the molecule in this time is equal to the number of molecules (regarded as points) that lie within the cylinder. In actuality, this cylinder would be bent many times as the direction of the molecule's path is changed by collisions; for convenience that path has been straightened.

The cylinder of Fig. 22-5 is, in fact, a broken one, changing direction with every collision.

The mean free path λ is the total distance covered by the moving molecule in time t divided by the number of collisions that it makes in that time, or

$$\lambda = \frac{L_{\text{cyl}}}{N_{\text{cyl}}} = \frac{vtV}{N\pi d^2 vt} = \frac{V}{N\pi d^2}. \qquad (22\text{-}11)$$

As Eq. 21-13 shows, we can write the ideal gas law in the form $pV = NkT$, in which k is the Boltzmann constant. From this equation, $V/N = kT/p$ and Eq. 22-11 becomes

$$\lambda = \frac{kT}{\pi d^2 p}. \qquad (22\text{-}12)$$

Equation 22-12 is based on the assumption of a single moving molecule hitting stationary targets. Actually, the molecule that we are following hits moving targets. When all molecules are moving, the two v's in Eq. 22-11 are not the same and thus do not cancel. The v in the numerator ($= v_{\text{av}}$) is the average molecular speed measured with respect to the box in which the gas is contained. The v in the denominator ($= v_{\text{rel}}$) is the average relative speed with respect to the other molecules. It is this relative speed that determines the collision rate.

We can see qualitatively that $v_{\text{rel}} > v_{\text{av}}$ as follows. Two molecules of speed v moving toward each other have $v_{\text{rel}} = 2v$, which is greater than v. You can easily show that two molecules moving at right angles to each other have $v_{\text{rel}} = \sqrt{2}\, v$, which is also greater than v. Two molecules moving with speed v in the same direction have $v_{\text{rel}} = 0$, which, of course, is less than v. If the angle between the velocities of the colliding molecules (assuming them to have the same speed) is between 0° and 60°, then $0 \leq v_{\text{rel}} \leq v$. If the angle is between 60° and 180° (the latter corresponding to a head-on collision), then $v \leq v_{\text{rel}} \leq 2v$. Because the collisions are random, there is a greater probability that the collision angle will be in the range of 60° to 180° than in the range of 0° to 60°. Thus the relative speed will on the average be greater than v.

A similar conclusion holds on the average if the molecules have a distribution of different speeds. A full calculation, taking into account the actual speed distribution of the molecules, gives $v_{\text{rel}} = \sqrt{2}\, v_{\text{av}}$. As a result, Eq. 22-12 becomes

$$\lambda = \frac{kT}{\sqrt{2}\pi d^2 p} \qquad \text{(mean free path).} \qquad (22\text{-}13)$$

This equation relates two microscopic quantities (λ and d) to two macroscopic quantities (p and T).

For air molecules at sea level, $\lambda \approx 10^{-7}$ m or 0.1 μm. At an altitude of 100 km, the density of air has dropped to such an extent that $\lambda \approx 16$ cm. At 300 km, $\lambda \approx 20$ km. In much scientific and industrial work it is necessary to pump the air out of a sealed container, producing a vacuum. Once the pressure has been reduced to the extent that the mean free path calculated from Eq. 22-13 exceeds the dimensions of the container, the concept of mean free path loses its significance; at that stage molecules collide more often with the container walls than with each other.

The ability of gases to conduct heat, the viscosity of gases, and the rate at which gases diffuse from regions of high concentration to regions of lower concentration are matters of considerable interest, both in science and in industry. All are proportional to the mean free path of the gas molecules. Designers of high-energy particle accelerators, such as those at CERN and Fermilab, go to great lengths to remove as much air as possible from the huge circular rings around which the accelerating particles must circulate thousands of times without colliding with a residual air molecule.

SAMPLE PROBLEM 22-4. What are (*a*) the mean free path and (*b*) the average collision rate for nitrogen at room temperature ($T = 300$ K) and atmospheric pressure ($p = 1.01 \times 10^5$ Pa)? A nitrogen molecule has an effective diameter of $d = 3.15 \times 10^{-10}$ m and, for the conditions stated, an average speed $v_{\text{av}} = 478$ m/s.

Solution (*a*) From Eq. 22-13,

$$\lambda = \frac{kT}{\sqrt{2}\pi d^2 p} = \frac{(1.38 \times 10^{-23}\ \text{J/K})(300\ \text{K})}{(\sqrt{2}\pi)(3.15 \times 10^{-10}\ \text{m})^2(1.01 \times 10^5\ \text{Pa})}$$
$$= 9.3 \times 10^{-8}\ \text{m}.$$

This is about 300 molecular diameters. On average, the distance between molecules in a gas is equal to the cube root of the volume occupied by a single molecule or $(V/N)^{1/3}$. From Eq. 21-13 ($pV = NkT$), we can write this as $(kT/p)^{1/3}$, which proves to be about 3.4×10^{-9} m. This is about 11 molecular diameters. In one mean free path λ a given molecule will pass about 27 other molecules before experiencing a collision.

(*b*) The average collision rate is the average speed divided by the mean free path, or

$$\text{rate} = \frac{v_{\text{av}}}{\lambda} = \frac{478\ \text{m/s}}{9.3 \times 10^{-8}\ \text{m/collision}}$$
$$= 5.1 \times 10^9\ \text{collisions/second}$$

On average, every nitrogen molecule makes more than 5 billion collisions per second!

22-4 THE DISTRIBUTION OF MOLECULAR SPEEDS

We can calculate v_{rms}, the root-mean-square speed of the molecules of an ideal gas, using Eq. 22-9. However, suppose we want to know how the speeds of the molecules are distributed about this average. It is not likely that the molecules would all have this same speed, because collisions between molecules would soon upset this situation. Speeds either close to zero or very much greater than v_{rms} are also relatively unlikely; such speeds would require a sequence of preferential collisions that would be very improbable in a condition of thermal equilibrium.

The Scottish physicist James Clerk Maxwell (1831–1879) first solved the problem of the distribution of speeds in a gas containing a large number of molecules. The *Maxwell speed distribution*—as it is called—for a sample of gas at temperature T containing N molecules, each of mass m, is

$$N(v) = 4\pi N\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT}. \qquad (22\text{-}14)$$

Figure 22-6 shows a plot of this equation for molecules of oxygen at room temperature.

The interpretation of $N(v)$ in Eq. 22-14 is that the (dimensionless) product $N(v)\,dv$ gives the number of molecules having speeds in the range v to $v + dv$. Graphically, this product for $v = 600$ m/s is represented in Fig. 22-6 as the shaded area of the narrow vertical strip located at that speed.

Avoid the temptation to interpret $N(v)$ as "the number of molecules having a speed v." This interpretation is meaningless because, although the number of molecules may be large, it cannot be infinite but the number of available speeds *is* infinite; they cannot be matched up on a one-to-one basis. The probability that a molecule has a precisely stated speed, such as 600.34326759 . . . m/s, is exactly zero. However, the number of molecules whose speeds lie in a narrow range such as 600 m/s to 602 m/s has a definite nonzero value.

If we add up (integrate) the numbers of molecules in each differential speed range dv from $v = 0$ to $v \to \infty$, we must obtain N, the total number in the system. That is, it must be true that

$$N = \int_0^\infty N(v)\,dv. \qquad (22\text{-}15)$$

Note that the integral in Eq. 22-15 can be interpreted as the total area under the speed distribution curve of Fig. 22-6. The number of molecules whose speeds lie between any given values, such as v_1 and v_2, is equal to the area under the speed distribution curve between those limits.

As the temperature increases, the average speed of the molecules increases, so the speed distribution curve must become broader. Because the area under the distribution curve (which is the total number of molecules) remains the same, the distribution curve must also flatten as the temperature rises. Figure 22-7 shows how the speed distribution curve for oxygen molecules at $T = 80$ K is both broadened and flattened as the temperature is increased to 300 K.

The distribution of speeds of molecules in a liquid resembles that of Fig. 22-6. This distribution allows us to understand why water in a saucer will eventually evaporate completely. The speed needed for a molecule of water to

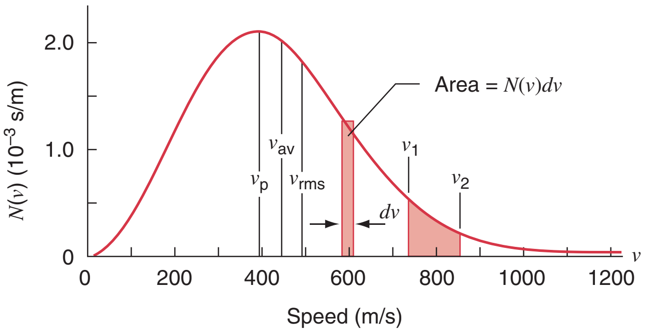


FIGURE 22-6. The Maxwell speed distribution for the molecules of a gas. The plotted curve is characteristic of oxygen molecules at $T = 300$ K. The number of molecules with speeds in any interval dv is $N(v)dv$, indicated by the narrow shaded strip. The number with speeds between any limits v_1 and v_2 is given by the area under the curve between those limits.

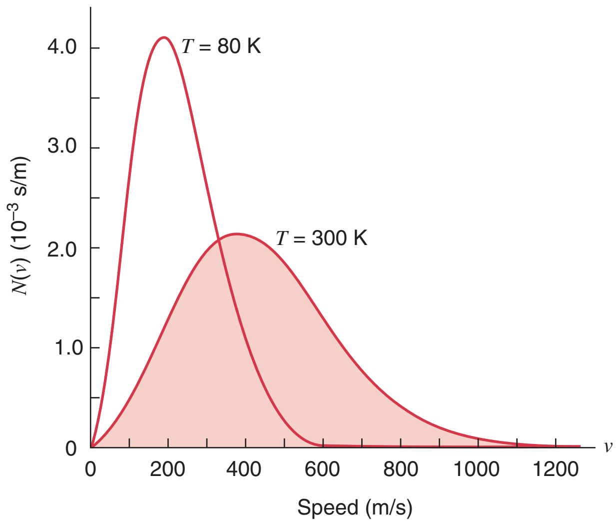


FIGURE 22-7. A comparison of the Maxwell speed distribution for oxygen molecules at two different temperatures. The molecules in general have lower average speeds at the lower temperatures, although both distributions cover the entire range of speeds. The areas of the two distributions are equal, because the total number of molecules is the same in both cases.

escape from the water surface would be very far out indeed on the tail of a distribution curve like that of Fig. 22-6. Only a very small number of molecules would have speeds above this threshold. The escape of these few energetic molecules reduces the average kinetic energy of the remaining molecules, leaving the water at a lower temperature. This explains why evaporation is a cooling process. If the saucer is not thermally isolated from its surroundings, however, energy will flow into the water from these surroundings, maintaining the water in thermal equilibrium with its environment. Energy will flow into the water as heat to compensate for the energy carried away by the escaping "fast" molecules; this process will continue until there is no more water.

Equation 22-14 also shows that the distribution of molecular speeds depends on the mass of the molecule as well as on the temperature. At any given temperature, the smaller the mass of a molecule, the faster it moves. Thus hydrogen is more likely to escape from the Earth's upper atmosphere than oxygen or nitrogen.

Consequences of the Speed Distribution

We can obtain much useful information from Eq. 22-14, the speed distribution equation.

1. *The most probable speed* v_p. This quantity is the speed at which $N(v)$ of Eq. 22-14 has its maximum value. We find it by requiring that $dN/dv = 0$ and solving for v. As you should verify, the result proves to be

$$v_p = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2RT}{M}}. \qquad (22\text{-}16)$$

Here we have made the substitutions $k = R/N_A$ (see Eq. 21-17) and $m = M/N_A$ (see Eq. 22-2).

2. *The average speed* v_{av}. To find the average speed of the molecules, we add up all the individual speeds and divide by the number of molecules. This is most simply done by summing the products of the speed v in each speed interval and the number $N(v)\,dv$ in that interval. Thus

$$v_{av} = \frac{1}{N}\int_0^\infty vN(v)\,dv. \qquad (22\text{-}17)$$

The next step is to substitute for $N(v)$ from Eq. 22-14 and evaluate the integral. The result is

$$v_{av} = \sqrt{\frac{8kT}{\pi m}} = \sqrt{\frac{8RT}{\pi M}}. \qquad (22\text{-}18)$$

3. *The root-mean square speed* v_{rms}. We encountered this quantity earlier, in Eq. 22-9. To find it from the speed distribution equation we proceed as above except that we find the average value of v^2 (rather than the average value of v). This leads, after another integration, to

$$(v^2)_{av} = \frac{1}{N}\int_0^\infty v^2\,N(v)\,dv = \frac{3kT}{m}. \qquad (22\text{-}19)$$

The root-mean-square speed is the square root of this quantity, or

$$v_{rms} = \sqrt{(v^2)_{av}} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3RT}{M}}. \qquad (22\text{-}20)$$

4. *The average translational kinetic energy per molecule* K_{trans}. Note first that, because we assume that our ideal gas is monatomic, translational kinetic energy is the only kind of energy that the molecules can have. An essentially point molecule cannot have energy of rotation, and we assume that there are no changes in the internal energies of the molecules.

To find K_{trans}, we must first find the total translational kinetic energy of the set of N molecules and then divide by N. The total energy K is

$$\begin{aligned} K &= \tfrac{1}{2}m(v_1^2 + v_2^2 + \cdots + v_N^2) \\ &= \tfrac{1}{2}mN\frac{(v_1^2 + v_2^2 + \cdots + v_N^2)}{N} \\ &= \tfrac{1}{2}mNv_{rms}^2. \end{aligned}$$

Replacing v_{rms}^2 from Eq. 22-20 and dividing by N, the total number of molecules, leads to

$$K_{trans} = \tfrac{3}{2}kT. \qquad (22\text{-}21)$$

We will have more to say about this important relation in Chapter 23.

5. *The ideal gas law.* We have derived two equations for v_{rms}, the root-mean-square velocity of the molecules, Eq. 22-9 and Eq. 22-20. Setting these equations equal yields

$$v_{rms}^2 = \frac{3p}{\rho} = \frac{3RT}{M}.$$

The density ρ can be written as nM/V. With this substitution the above equality reduces to $pV = nRT$. Thus we have recovered the ideal gas law from our investigation into the molecular speeds.

Experimental Verification of the Maxwell Speed Distribution

Maxwell derived his speed distribution law (Eq. 22-14) in 1860. At that early date it was not possible to check this law by direct measurement and it was not until about 1920 that the first attempts were made. However, techniques improved rapidly and, in 1955, R. C. Miller and P. Kusch of Columbia University provided a high-precision experimental verification of Maxwell's prediction.

Their apparatus is illustrated in Fig. 22-8. The walls of oven O, containing some thallium metal, were heated, in one set of experiments, to a uniform temperature of 870 ± 4 K. At this temperature thallium vapor, at a pressure of 3.2×10^{-3} torr, fills the oven. Some molecules of thallium vapor escape from slit S into the highly evacuated space outside the oven, falling on the rotating cylinder R.

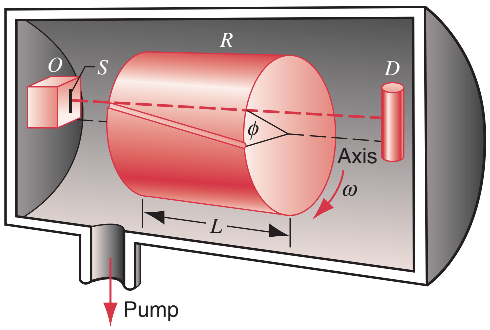


FIGURE 22-8. Apparatus used by Miller and Kusch to verify the Maxwell speed distribution. A beam of thallium molecules leaves the oven O through the slit S, travels through the helical groove in the rotating cylinder R, and strikes the detector D. The angular velocity ω of the cylinder can be varied so that molecules of differing speeds will pass through the cylinder.

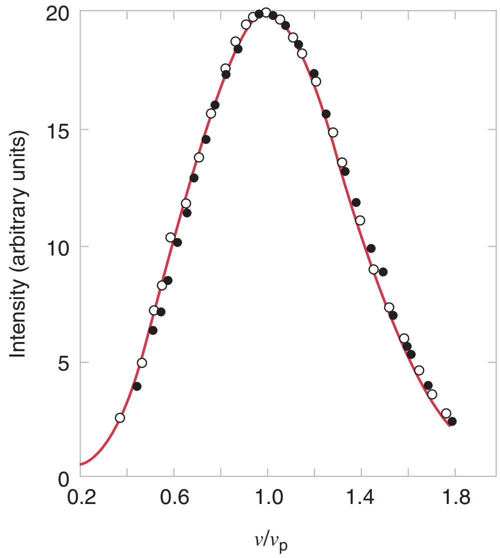


FIGURE 22-9. The results of the experiment to verify the Maxwell speed distribution. The open circles show data taken with the oven temperature at $T = 870$ K, and the filled circles show data at $T = 944$ K. When the distributions are plotted against v/v_p, the two distributions should be identical. The solid curve is the Maxwell distribution. The data agree remarkably well with the curve.

This cylinder, of length L, has about 700 helical grooves cut into it, only one of which is shown in Fig. 22-8. For a given angular speed ω of the cylinder, only molecules of a sharply defined speed v can pass along the grooves without striking the walls. The speed v can be found from

$$\text{time of travel along the groove} = \frac{L}{v} = \frac{\phi}{\omega}$$

or

$$v = \frac{L\omega}{\phi},$$

in which ϕ (see Fig. 22-8) is the angular displacement between the entrance and the exit of a helical groove. Thus the rotating cylinder is a *velocity selector,* in which the speed is selected by the (controllable) angular speed ω. The beam intensity is recorded by detector D as a function of the selected speed v. Figure 22-9 shows the remarkable agreement between theory (the solid line) and experiment (the open and filled circles) for thallium vapor.

The distribution of speeds in the *beam* (as distinguished from the distribution of speeds in the *oven*) is not proportional to $v^2 e^{-mv^2/2kT}$, as in Eq. 22-14, but to $v^3 e^{-mv^2/2kT}$. Consider a group of molecules in the oven whose speeds lie within a certain small range v_1 to $v_1 + \delta v$, where v_1 is less than the most probable speed v_p. We can always find another equal speed interval δv, extending from v_2 to $v_2 + \delta v$, where v_2, which will be greater than v_p, is chosen so that the two speed intervals contain the same number of molecules. However, more molecules in the higher interval than in the lower will escape from slit S to form the beam, because molecules in the higher interval "bombard" the slit with a greater frequency, by precisely the factor v_2/v_1. Thus, other things being equal, fast molecules are favored in escaping from the oven, just in proportion to their speeds, and the molecules in the beam have a v^3 rather than a v^2 distribution. This effect is included in the theoretical curve of Fig. 22-9.

SAMPLE PROBLEM 22-5. The speeds of ten particles in m/s are 0, 1.0, 2.0, 3.0, 3.0, 3.0, 4.0, 4.0, 5.0, and 6.0. Find (*a*) the average speed, (*b*) the root-mean-square speed, and (*c*) the most probable speed of these particles.

Solution (*a*) The average speed is found from

$$v_{av} = \frac{1}{N}\sum_{n=1}^{N} v_n = \frac{1}{10}[0 + 1.0 + 2.0 + 3.0 + 3.0 + 3.0 + 4.0 + 4.0 + 5.0 + 6.0] = 3.1 \text{ m/s}.$$

(*b*) The mean-square speed is the average value of v^2:

$$(v^2)_{av} = \frac{1}{N}\sum_{n=1}^{N} v_n^2 = \frac{1}{10}[0 + (1.0)^2 + (2.0)^2 + (3.0)^2 + (3.0)^2 + (3.0)^2 + (4.0)^2 + (4.0)^2 + (5.0)^2 + (6.0)^2] = 12.5 \text{ m}^2/\text{s}^2,$$

and the root-mean-square speed is

$$v_{rms} = \sqrt{(v^2)_{av}} = \sqrt{12.5 \text{ m}^2/\text{s}^2} = 3.5 \text{ m/s}.$$

(*c*) Of the ten particles, three have speeds of 3.0 m/s, two have speeds of 4.0 m/s, and each of the other five has a different speed. Hence the most probable speed v_p of a particle is

$$v_p = 3.0 \text{ m/s}.$$

SAMPLE PROBLEM 22-6. A container filled with N molecules of oxygen gas is maintained at 300 K. What fraction of the molecules has speeds in the range 599–601 m/s? The molar mass M of oxygen is 0.032 kg/mol.

Solution This speed interval δv (= 2 m/s) is so small that we can treat it as a differential dv. The number of molecules in this interval is $N(v)dv$, and the fraction in that interval is $f = N(v)dv/N$, where $N(v)$ is to be evaluated at $v = 600$ m/s, the midpoint of the range; see the narrow shaded strip in Fig. 22-6. Using Eq. 22-14 with the substitution $m/k = M/R$, we find the fraction

$$f = \frac{N(v)\,dv}{N} = 4\pi\left(\frac{M}{2\pi RT}\right)^{3/2} v^2 e^{-Mv^2/2RT}\,dv.$$

Substituting the given numerical values yields

$$f = 2.6 \times 10^{-3} \text{ or } 0.26\%.$$

At room temperature, 0.26% of the oxygen molecules have speeds that lie in the narrow range between 599 and 601 m/s. If the shaded strip of Fig. 22-6 were drawn to the scale of this problem, it would be a very thin strip indeed.

SAMPLE PROBLEM 22-7. Calculate (*a*) the most probable speed, (*b*) the average speed, and (*c*) the rms speed for oxygen molecules at $T = 300$ K.

Solution (*a*) From Eq. 22-16 we have

$$v_p = \sqrt{\frac{2RT}{M}} = \sqrt{\frac{(2)(8.31\ \text{J/mol}\cdot\text{K})(300\ \text{K})}{0.032\ \text{kg/mol}}} = 395\ \text{m/s}.$$

(*b*) From Eq. 22-18 we have

$$v_{av} = \sqrt{\frac{8RT}{\pi M}} = \sqrt{\frac{(8)(8.31\ \text{J/mol}\cdot\text{K})(300\ \text{K})}{(\pi)(0.032\ \text{kg/mol})}} = 445\ \text{m/s}.$$

(*c*) From Eq. 22-20 we have

$$v_{rms} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{(3)(8.31\ \text{J/mol}\cdot\text{K})(300\ \text{K})}{0.032\ \text{kg/mol}}} = 483\ \text{m/s}.$$

From the equations we have used note that, for any gas at a given temperature,

$$v_p : v_{av} : v_{rms} = 1 : 1.128 : 1.225.$$

22-5 THE DISTRIBUTION OF MOLECULAR ENERGIES

An alternative description of the motion of molecules can be obtained if we look for the distribution in *energy* rather than in speed. That is, we seek the distribution $N(E)$, such that $N(E)dE$ gives the number of molecules with energies between E and $E + dE$.

This problem was first solved by Maxwell. We derive the result, called the *Maxwell–Boltzmann energy distribution,* in the special case that translational kinetic energy is the only form of energy that a molecule can have.

Let us consider again the situation of Sample Problem 22-6, in which we obtained the fraction of oxygen molecules having speeds between 599 and 601 m/s. We found that 0.26% of the molecules in a container at a temperature of 300 K have speeds in that range. An oxygen molecule with a speed of 599 m/s has a kinetic energy of 9.54×10^{-21} J, and one with a speed of 601 m/s has a kinetic energy of 9.60×10^{-21} J. What fraction of the oxygen molecules has kinetic energies in the range of 9.54×10^{-21} to 9.60×10^{-21} J?

A bit of thought should convince you that this fraction must also be 0.26%. It makes no difference whether we count the molecules by their speeds or by their kinetic energies; as long as we set the lower and upper limits of the interval to have corresponding speeds and kinetic energies, we count the same number of molecules between the limits. That is, the number with kinetic energies between E and $E + dE$ is the same as the number with speeds between v and $v + dv$. Mathematically, we express this conclusion as

$$N(E)\,dE = N(v)\,dv, \qquad (22\text{-}22)$$

or

$$N(E) = N(v)\frac{dv}{dE}. \qquad (22\text{-}23)$$

Since the energy is only kinetic, we must have $E = \frac{1}{2}mv^2$ or $v = \sqrt{2E/m}$, and thus

$$\frac{dv}{dE} = \sqrt{\frac{2}{m}}\left(\tfrac{1}{2}E^{-1/2}\right). \qquad (22\text{-}24)$$

Substituting Eqs. 22-14 and 22-24 into Eq. 22-23, we obtain

$$N(E) = \frac{2N}{\sqrt{\pi}}\frac{1}{(kT)^{3/2}}E^{1/2}e^{-E/kT}. \qquad (22\text{-}25)$$

Equation 22-25 is the *Maxwell–Boltzmann energy distribution.* In deriving this result, we have assumed that the gas molecules can have only translational kinetic energy. This distribution therefore applies only to a monatomic gas. In the case of gases with more complex molecules, other factors (such as rotational kinetic energy) will be present in Eq. 22-25. The factor $e^{-E/kT}$, however, is a general feature of the Maxwell–Boltzmann energy distribution that is present no matter what the form of the energy E. This factor which is generally known as the *Boltzmann factor,* is often taken as a rough estimate of the relative probability for a particle to have an energy E in a collection of particles characterized by a temperature T.

Using Eq. 22-25, we can calculate the fraction of the gas molecules having energies between E and $E + dE$, which is given by $N(E)dE/N$. As before, N is the total number of molecules, which is determined from

$$N = \int_0^\infty N(E)\,dE. \qquad (22\text{-}26)$$

One interesting feature of the Maxwell–Boltzmann energy distribution is that it is precisely the same for any gas at a given temperature, no matter what the mass of the molecules (in contrast to the Maxwell speed distribution, Eq. 22-14, in which the mass appears explicitly). Even a "gas" of electrons, to the extent they can be treated as classical particles, has the same energy distribution as a gas of heavy atoms. The effect of increasing the mass m by some factor is to reduce v^2 by the same factor, so that the product mv^2, and thus the kinetic energy, remains the same.

For a simple application of the Boltzmann factor, consider a long, gas-filled vertical container with its bottom resting on the Earth's surface. We'll assume that the gas in the container is in thermal equilibrium at a uniform temperature T. A molecule at a height y above the bottom has energy $E_0 + mgy$, where E_0 is the energy of a similar molecule at the bottom of the container. Using the Boltzmann factor $e^{-E/kT}$, we can deduce that the number of molecules at height y is, compared with the number at $y = 0$,

$$\frac{N(y)}{N(0)} = \frac{e^{-(E_0+mgy)/kT}}{e^{-E_0/kT}} \qquad (22\text{-}27)$$

or

$$N(y) = N_0 e^{-mgy/kT} \qquad (22\text{-}28)$$

where $N_0 = N(0)$. With $kT = pV/N$ from the ideal gas law, the factor m/kT in the exponent can be written as $mN/pV = \rho/p$, where ρ is the density of the gas. Because we have assumed the gas to be at a uniform temperature, we must have $\rho/p = \rho_0/p_0$, with ρ_0 and p_0 being the values of the density and pressure at the Earth's surface. Furthermore, because the number of molecules in a small volume element at any height is proportional to the density at that height, which is in turn proportional to the pressure, we can write Eq. 22-28 as

$$p(y) = p_0 e^{-mgy/kT} = p_0 e^{-gy\rho_0/p_0}. \qquad (22\text{-}29)$$

Equation 22-29 is identical with Eq. 15-12 for the pressure in the atmosphere as a function of the height above the Earth's surface. We also derived Eq. 15-12 under the assumption of a uniform temperature for the atmosphere, and it is comforting that the dynamic approach used in Chapter 15 and the present statistical approach give the same result.

SAMPLE PROBLEM 22-8. Find (*a*) the average energy and (*b*) the most probable energy of a gas in thermal equilibrium at temperature T.

(*a*) The average energy E_{av} can be written, in analogy with Eq. 22-17, as

$$E_{av} = \frac{1}{N}\int_0^\infty E\,N(E)\,dE.$$

Substituting Eq. 22-25, we obtain

$$E_{av} = \frac{2}{\sqrt{\pi}(kT)^{3/2}}\int_0^\infty E^{3/2}e^{-E/kT}\,dE. \qquad (22\text{-}30)$$

To evaluate this integral, make the substitution $x^2 = E/kT$ and convert it to a standard form for a definite integral given in Appendix I. The result, which you should verify, is

$$E_{av} = \tfrac{3}{2}kT \qquad (22\text{-}31)$$

a result that agrees precisely with Eq. 22-21 for this case in which we have assumed kinetic energy to be the only type of energy that the gas molecules may have.

(*b*) To find the most probable energy, we take the derivative of Eq. 22-25, set the result equal to zero, and solve for the energy. The result, which you should derive, is

$$E_p = \tfrac{1}{2}kT.$$

Note that this is *not* equal to $\frac{1}{2}mv_p^2$, which gives an energy of kT. Can you explain why the energy corresponding to the most probable speed is not the most probable energy?

22-6 EQUATIONS OF STATE FOR REAL GASES

The equation of state for an ideal gas holds well enough for real gases at sufficiently low densities. However, it does not hold *exactly* for real gases at *any* density and departs more and more the greater the density. There is much interest in finding an equation of state that describes real gases over a range of densities. We discuss two of the many approaches.

The Virial Expansion

Our first approach to an equation of state for a real gas is to write

$$pV = nRT\left[1 + B_1\frac{n}{V} + B_2\left(\frac{n}{V}\right)^2 + \cdots\right], \qquad (22\text{-}32)$$

in which $B_1, B_2, \ldots$, called *virial coefficients,* are functions of temperature and grow successively smaller as the series progresses. It is clear that, at small molar densities ($n/V \to 0$), this equation of state reduces to the ideal gas law. This must be the case for all equations of state for gases because the ideal gas law holds in the limit of low densities. The virial coefficients must be found empirically, by fitting Eq. 22-32 to experimental data.

The van der Waals Equation of State

This equation, proposed in 1873 by the Dutch physicist Johannes Diderik van der Waals (1837–1923), is

$$\left(p + \frac{an^2}{V^2}\right)(V - nb) = nRT, \qquad (22\text{-}33)$$

in which a and b are constants whose values must be obtained by experiment. Comparison of Eq. 22-33 with the ideal gas law ($pV = nRT$) suggests that van der Waals (who received the 1910 Nobel Prize for his work) arrived at his equation by correcting perceived points of failure in the ideal gas law. That is indeed the case. Note that if the constants a and b are put equal to zero (or if we allow the molar density n/V to become very small), Eq. 22-33 reduces to the ideal gas law. We now investigate the line of reasoning that led to the terms containing these constants.

The Volume Correction. In Section 22-1 we assumed (property 3) that the volume occupied by the molecules of an ideal gas is negligible. This is not quite true for real gases. Let us regard each molecule of a real gas as a hard sphere of diameter d. Two such molecules cannot approach each other so close that the distance between their centers would be less than d (Fig. 22-10). The "free volume" per mole available for one molecule is therefore decreased by the volume of a hemisphere of *radius* d centered on the other molecule. If we estimate d as 2.5×10^{-10} m (a typical molecular diameter), then we can find an approximate value of b of

$$b = \tfrac{1}{2}N_A(\tfrac{4}{3}\pi d^3) = 2 \times 10^{-5}\ \text{m}^3/\text{mol}.$$

The factor of $\frac{1}{2}$ comes about because, as two molecules approach each other, the volume within which they interact is not a full sphere but the hemisphere facing the direction of approach. Under standard conditions of temperature and pressure the molar density n/V of an ideal gas is 45 mol/m^3. Thus $bn/V \approx 0.0009$ or about 0.1%; under these conditions, the volume correction b is relatively small.

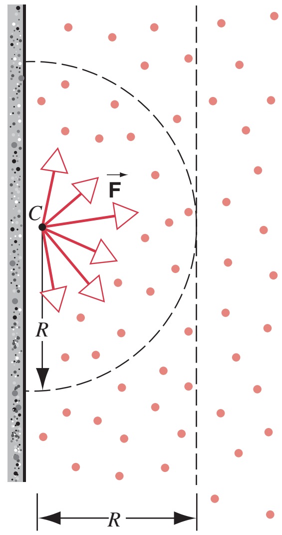


FIGURE 22-11. A gas molecule C (here considered to be a point) near the wall of the container experiences a net force away from the wall due to the attraction of the surrounding molecules within the range R of the force between molecules. The net pressure on the walls of the container is reduced by all such molecules within a distance R of the walls.

The Pressure Correction. In Section 22-1 (property 4) we assumed that the molecules of an ideal gas exert forces on each other only during collisions. That is also not quite true for real gases. A molecule in the body of the gas would experience no net force due to the forces exerted on it by the surrounding molecules; that is, these forces would balance out to zero. However, that is not true for a molecule located near the wall of the containing vessel, as in Fig. 22-11. Such a molecule would experience a net force of attraction away from the wall because of its interaction with the nearby molecules that are within the range of the attractive force that it exerts. Thus the pressure measured at the wall is somewhat less than what we may call the true pressure that exists in the body of the gas.

The reduction in pressure owing to the collisions of molecule C with the wall is proportional to the number of molecules in the hemisphere within the range R of its attractive force and thus to the number of molecules per unit volume or, alternatively, to n/V. The net effect due to *all* the molecules that strike the wall (C is a typical member of this group) is also proportional to the number of molecules per unit volume or to n/V. The total reduction in pressure is proportional to the product of these two quantities, or $(n/V)^2$.

That is, if we triple the number of molecules in a given container, molecule C will experience three times the unbalanced force. In the entire gas there will be three times as many molecules like C. The overall pressure reduction thus increases ninefold. If p in Eq. 22-33 is to be the measured pressure, we must increase it by a term proportional to $(n/V)^2$—that is, by an^2/V^2—to obtain the "true" pressure.

Figure 22-12 compares a pV plot of an ideal gas at various temperatures with a plot of Eq. 22-33 for carbon dioxide gas. Note that the deviation from ideal behavior occurs primarily at high pressures and low temperatures. For CO_2 at 264 K, the graph contains a region of positive slope, indicating that as we decrease the volume in this region the pressure also decreases. Since this behavior is contrary to expectations for a gas, it suggests that some of the CO_2 is condensing to a liquid, leaving less of it in the gaseous state. The van der Waals equation thus suggests the existence of mixtures of different phases, which the ideal gas model cannot do. If we were to compress a sample of CO_2, we would find that the actual $T = 264$ K graph would not follow the curve shown in Fig. 22-12b, but instead would follow the dashed horizontal segment AB in that figure.

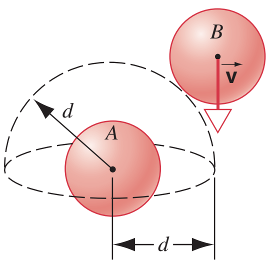


FIGURE 22-10. If molecules of a gas are considered to behave like hard spheres, then the center of molecule B is not permitted to move within the hemisphere of radius d centered on molecule A. Here d is the diameter of a molecule. The free volume available for molecule B is reduced by the volume of such a hemisphere centered on each molecule of the gas.

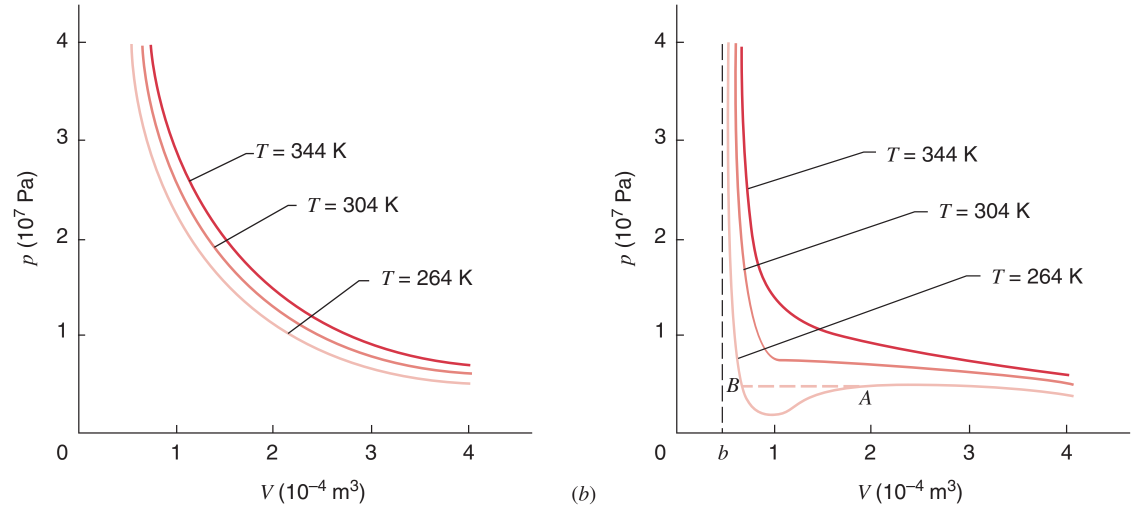


Figure 22-12. pV graphs for one mole of (*a*) an ideal gas and (*b*) CO_2 determined from the van der Waals equation. Note that at large volume, the ideal and van der Waals graphs behave similarly. As the temperature is raised, the van der Waals graphs behave more like those of the ideal gas. Note also that, as the pressure becomes very large, the volume approaches the value of b, as Eq. 22-33 requires, rather than the value of zero, as the ideal gas equation of state would predict. The dashed line AB shows a more realistic representation of the behavior at $T = 264$ K. As the gas is compressed from A, some of the gas condenses into a liquid, and the pressure remains constant.

Sample Problem 22-9. For oxygen the van der Waals coefficients have been measured to be $a = 0.138\ \text{J}\cdot\text{m}^3/\text{mol}^2$ and $b = 3.18 \times 10^{-5}\ \text{m}^3/\text{mol}$. Assume that 1.00 mol of oxygen at $T = 50$ K is confined to a box of volume $0.0224\ \text{m}^3$. What pressure does the gas exert according to (*a*) the ideal gas law and (*b*) the van der Waals equation?

Solution (*a*) The ideal gas law yields

$$p = \frac{nRT}{V} = \frac{(1.00\ \text{mol})(8.31\ \text{J/mol}\cdot\text{K})(50\ \text{K})}{0.0224\ \text{m}^3}$$

$$= 1.85 \times 10^4\ \text{Pa} = 0.184\ \text{atm}.$$

(*b*) The pressure and the volume correction terms in the van der Waals equation (Eq. 22-33) are

$$\frac{an^2}{V^2} = \frac{(0.138\ \text{J}\cdot\text{m}^3/\text{mol}^2)(1.00\ \text{mol})^2}{(0.0224\ \text{m}^3)^2} = 275\ \text{Pa}$$

and

$$b = 3.18 \times 10^{-5}\ \text{m}^3/\text{mol}.$$

Substituting these quantities into the van der Waals equation and solving that equation for p yields

$$p = 1.83 \times 10^4\ \text{Pa} = 0.181\ \text{atm}.$$

For oxygen under these conditions the ideal gas law and the van der Waals equation yield results that are within 2% of each other. Note that the pressure correction term is only (275 Pa)/(1.83×10^4 Pa) or about 1.5%. The volume correction bn/V is only ($3.18 \times 10^{-5}\ \text{m}^3/\text{mol}$)(1.00 mol)/(0.0224 m^3) or about 0.14%. For lower temperatures, as the gas moves in the direction of liquifaction, the van der Waals equation will better agree with experiment than will the ideal gas law.

22-7 THE INTERMOLECULAR FORCES (Optional)

Forces between molecules are of electromagnetic origin. All molecules contain electric charges in motion. These molecules are electrically neutral in the sense that the negative charge of the electrons is equal and opposite to the positive charge of the nuclei. This does not mean, however, that molecules do not interact electrically. For example, when two molecules approach each other, the charges on each are disturbed and depart slightly from their usual positions in such a way that the average distance between opposite charges in the two molecules is a little smaller than that between like charges. Hence an attractive intermolecular force results. This internal rearrangement takes place only when molecules are fairly close together, so that these forces act only over short distances; they are short-range forces. If the molecules come very close together, so that their outer charges begin to overlap, the intermolecular force becomes repulsive. The molecules repel each other because there is no way for a molecule to rearrange itself internally to prevent repulsion of the adjacent external electrons. It is this repulsion on contact that accounts for the billiard-ball character of molecular collisions in gases. If it were not for this repulsion, molecules would move right through each other instead of rebounding on collision.

Let us assume that molecules are approximately spherically symmetrical. Then we can describe intermolecular forces graphically by plotting the mutual potential energy of two molecules, U, as a function of distance r between their centers. The force F acting on each molecule is related to the potential energy U by $F = -dU/dr$. In Fig. 22-13*a* we

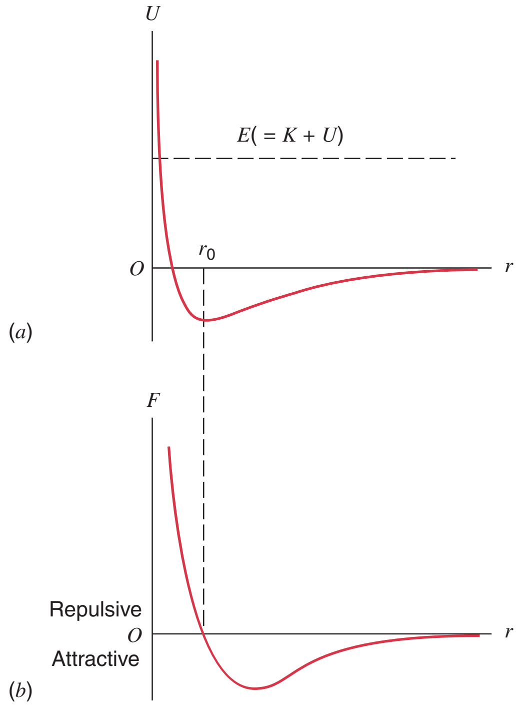


FIGURE 22-13. (*a*) The mutual potential energy U of two molecules as a function of their separation distance r. The mechanical energy E is indicated by the horizontal line. (*b*) The radial force between the molecules, given by $-dU/dr$, corresponding to this potential energy. The potential energy is a minimum at the equilibrium separation r_0, at which point the force is zero.

plot a typical $U(r)$. Here we can imagine one molecule to be fixed at O. Then the other molecule is repelled from O when the slope of U is negative and is attracted to O when the slope is positive. At r_0 no force acts between the molecules; the slope is zero there. In Fig. 22-13*b* we plot the mutual force $F(r)$ corresponding to this potential energy function. The line E in Fig. 22-13*a* represents the mechanical energy of the colliding molecules. The intersection of $U(r)$ with this line is a "turning point" of the motion (see Section 12-5). The separation of the centers of two molecules at the turning point is the distance of closest approach. The separation distance at which the mutual potential energy is zero may be taken as the approximate distance of closest approach in a low-energy collision and hence as the diameter of the molecule. For simple molecules the diameter is about 2.5×10^{-10} m. The distance r_0 at which the potential is a minimum (the equilibrium point) is about 3.5×10^{-10} m for simple molecules, and the force and potential energy approach zero as r increases to about 10^{-9} m, or about 4 diameters. The molecular force thus has a very short range. Of course, different molecules have different sizes and internal arrangement of charges so that intermolecular forces vary from one molecule to another. However, they always show the qualitative behavior indicated in Fig. 22-13.

In a solid, molecules vibrate about the equilibrium position r_0. Their total energy E is negative—that is, lying below the horizontal axis in Fig. 22-13*a*. The molecules do not have enough energy to escape from the potential valley (that is, from the attractive binding force). The centers of vibration O are more or less fixed in a solid. In a liquid the molecules have greater vibrational energy about centers that are free to move but that remain about the same distance from one another. Molecules have their greatest kinetic energy in the gaseous state. In a gas the average distance between the molecules is considerably greater than the effective range of intermolecular forces, and the molecules move in straight lines between collisions. Maxwell discusses the relation between the kinetic theory model of a gas and the intermolecular forces as follows: "Instead of saying that the particles are hard, spherical, and elastic, we may if we please say that the particles are centers of force, of which the action is insensible except at a certain small distance, when it suddenly appears as a repulsive force of very great intensity. It is evident that either assumption will lead to the same results."

It is interesting to compare the measured intermolecular forces with the gravitational force of attraction between molecules. If we choose a separation distance of 4×10^{-10} m, for example, the force between two helium atoms is about 6×10^{-13} N. The gravitational force at that separation is about 7×10^{-42} N, smaller than the intermolecular force by a factor of 10^{29}! This is a typical result and shows that gravitation is negligible in intermolecular forces.

Although the intermolecular forces appear to be small by ordinary standards, we must remember that the mass of a molecule is so small (about 10^{-26} kg) that these forces can impart instantaneous accelerations of the order of 10^{15} m/s^2 (10^{14} *g*). These accelerations may last for only a very short time, of course, because one molecule can very quickly move out of the range of influence of the other. ■

MULTIPLE CHOICE

22-1 The Atomic Nature of Matter

1. Which *two* of the following cases do *not* correspond to the behavior of an ideal gas?
   (A) A molecule loses kinetic energy when it collides elastically with another molecule.
   (B) There is a potential energy associated with the interaction between molecules.
   (C) Collisions can change the internal energy of molecules.
   (D) The speed of a molecule is unchanged after a collision with the walls of the container.

2. The gas in a closed container consists of a mixture of helium and krypton. This mixture can be treated as an ideal gas if it is assumed that the helium and krypton atoms have the same average
   (A) mass. (B) speed.
   (C) momentum. (D) kinetic energy.

22-2 A Molecular View of Pressure

3. Where does the factor of "3" come from in Eq. 22-9?
(A) It is an approximation for π.
(B) It is found from comparing the units of pressure and density.
(C) It is related to the number of spatial dimensions.
(D) It arises from integrating v^2 to find the average.

22-3 The Mean Free Path

4. (*a*) At approximately what density, in molecules/m^3, does the mean free path of nitrogen molecules equal the size of a room ($\approx$ 3 m)?
(A) 10^{23} molecules/m^3 (B) 10^{20} molecules/m^3
(C) 10^{18} molecules/m^3 (D) 10^{9} molecules/m^3
(*b*) Assuming that room temperature is 300 K, what is the approximate pressure?
(A) 10^{-1} atm (B) 10^{-2} atm
(C) 10^{-5} atm (D) 10^{-7} atm

5. The density of gas in a bell jar is kept constant while varying the temperature. If the temperature is doubled, then the mean free path will
(A) double. (B) remain the same.
(C) decrease by half.

6. In a fixed amount of gas, how would the mean free path be affected if
(*a*) the density of the gas is doubled?
(*b*) the mean molecular speed is doubled?
(*c*) both the density and mean molecular speed are doubled?
(A) The mean free path will also double.
(B) The mean free path will remain the same.
(C) The mean free path will decrease by one-half.
(D) The mean free path will decrease to one-fourth its original value.

22-4 The Distribution of Molecular Speeds

7. Rank v_p, v_{av}, and v_{rms} from highest to lowest at $T = 350$ K for hydrogen molecules.
(A) $v_{rms} > v_p > v_{av}$ (B) $v_{rms} > v_{av} > v_p$
(C) $v_{av} > v_{rms} > v_p$ (D) $v_p > v_{av} > v_{rms}$

8. The root-mean-square speed of molecules in still air at room temperature is closest to
(A) walking speed (2 m/s).
(B) the speed of a fast car (30 m/s).
(C) the speed of a supersonic airplane (500 m/s).
(D) escape speed from Earth (1.1×10^4 m/s).
(E) the speed of light (3×10^8 m/s).

9. Which of the following speeds divides the molecules in a gas in thermal equilibrium so that half have speeds faster, and half have speeds slower?
(A) v_p (B) v_{av} (C) v_{rms}
(D) None of the above.

10. Which of the following speeds corresponds to a molecule with the average kinetic energy?
(A) v_p (B) v_{av} (C) v_{rms}
(D) None of the above.

11. Consider the distribution of speeds shown in Fig. 22-14. Which is the correct ordering for the speeds?
(A) $v_{rms} < v_{av} < v_p$ (B) $v_{rms} < v_p < v_{av}$
(C) $v_{av} < v_{rms} < v_p$ (D) $v_{av} < v_p < v_{rms}$
(E) $v_p < v_{av} < v_{rms}$

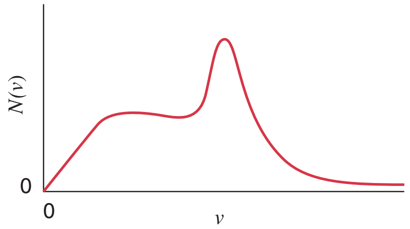


FIGURE 22-14. Multiple-choice question 11.

22-5 The Distribution of Molecular Energies

22-6 Equations of State for Real Gases

12. A certain (fictitious!) gas is found to obey the van der Waals equation exactly. The volume of the gas is changed from $V_i = 1000nb$ to $V_f = 2000nb$. Assume that a is small compared to pV^2/n^2, but not negligible. If the change in volume occurred at constant pressure, then
(A) T_f is slightly more than $2T_i$.
(B) T_f is slightly less than $2T_i$.
(C) T_f is exactly equal to $2T_i$.
(D) The answer depends on the value of an^2/pV^2.

13. A plasma is a gas consisting of charged particles. If all the particles have the same charge, then the forces between the particles will be repulsive at all distances. What would be the sign of a in the van der Waals equation for this type of gas?
(A) Positive (B) Negative
(C) The sign would depend on the sign of the charges in the plasma.
(D) There would be no sign, since the van der Waals equation applies only for attractive forces.

22-7 Intermolecular Forces

QUESTIONS

1. In kinetic theory we assume that the number of molecules in a gas is large. Real gases behave like an ideal gas at low densities. Are these statements contradictory? If not, what conclusion can you draw from them?

2. We have assumed that the walls of the container are elastic for molecular collisions. Actually, the walls may be inelastic. Why does this make no difference as long as the walls are at the same temperature as the gas?

3. We have assumed that the force exerted by molecules on the wall of a container is steady in time. How is this justified?

4. We know that a stone will fall to the ground if we release it. We put no constraint on molecules of the air, yet they do not all fall to the ground. Why not?

5. How is the speed of sound related to the gas variables in the kinetic theory model?

6. Why doesn't the Earth's atmosphere leak away? At the top of the atmosphere atoms will occasionally be headed out with a speed exceeding the escape speed. Isn't it just a matter of time?

7. Titan, one of Saturn's many moons, has an atmosphere, but our own Moon does not. What is the explanation?

8. How, if at all, would you expect the composition of the air to change with altitude?

9. Would a gas whose molecules were true geometric points obey the ideal gas law?

10. Why do molecules not travel in perfectly straight lines between collisions and what effect, easily observable in the laboratory, occurs as a result?

11. Suppose we want to obtain ^{238}U instead of ^{235}U as the end product of a diffusion process. Would we use the same process? If not, explain how the separation process would have to be modified.

12. Considering the diffusion of gases into each other, can you draw an analogy to a large jostling crowd with many "collisions" on a large inclined plane with a slope of a few degrees?

13. Would you expect real molecules to be spherically symmetrical? If not, how would the potential energy function of Fig. 22-13 change?

14. Although real gases can be liquefied, an ideal gas cannot be. Explain.

15. Show that as the volume per mole of a gas increases, the van der Waals equation tends to the equation of state of an ideal gas.

16. Consider the case in which the mean free path is greater than the longest straight line in a vessel. Is this a perfect vacuum for a molecule in this vessel?

17. List effective ways of increasing the number of molecular collisions per unit time in a gas.

18. Give a qualitative explanation of the connection between the mean free path of ammonia molecules in air and the time it takes to smell the ammonia when a bottle is opened across the room.

19. If molecules are not spherical, what meaning can we give to d in Eq. 22-13 for the mean free path? In which gases would the molecules act most nearly like rigid spheres?

20. In what sense is the mean free path a macroscopic property of a gas rather than a microscopic one?

21. Since the actual force between molecules depends on the distance between them, forces can cause deflections even when molecules are far from "contact" with one another. Furthermore, the deflection caused should depend on how long a time these forces act and hence on the relative speed of the molecules. (*a*) Would you then expect the measured mean free path to depend on temperature, even though the density remains constant? (*b*) If so, would you expect λ to increase or decrease with temperature? (*c*) How does this dependence enter into Eq. 22-13?

22. When a can of mixed nuts is shaken, why does the largest nut generally end up on the surface, even if it is denser than the others?

23. Justify qualitatively the statement that, in a mixture of molecules of different kinds in complete equilibrium, each kind of molecule has the same Maxwellian distribution in speed that it would have if the other kinds were not present.

24. A gas consists of N particles. Explain why $v_{rms} \geq v_{av}$ regardless of the distribution of speeds.

25. What observation is good evidence that not all molecules of a body are moving with the same speed at a given temperature?

26. The fraction of molecules within a given range δv of the rms speed decreases as the temperature of a gas rises. Explain.

27. Figure 22-15 shows the distribution of the x component of the velocities of the molecules in a container at a fixed temperature. (*a*) The distribution is symmetrical about $v_x = 0$; make this plausible. (*b*) What does the total area under the curve represent? (*c*) How would the distribution change with an increase in temperature? (*d*) What is the most probable value of v_x? (*e*) Is the most probable speed equal to zero? Explain.

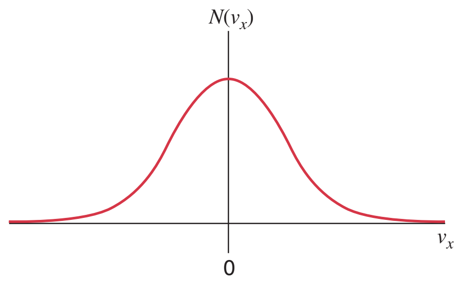


FIGURE 22-15. Question 27.

28. The slit system in Fig. 22-8 selects only those molecules moving in the $+x$ direction. Does this destroy the validity of the experiment as a measure of the distribution of speeds of molecules moving in all directions?

29. List examples of the Brownian motion in physical phenomena.

30. A golf ball is suspended from the ceiling by a long thread. Explain in detail why its Brownian motion is not readily apparent.

31. Let ρ_n be the number of molecules per unit volume in a gas. If we define ρ_n for a very small volume in a gas—say, one equal to 10 times the volume of an atom—then ρ_n fluctuates with time through the range of values zero to some maximum value. How then can we justify a statement that ρ_n has a definite value at every point in a gas?

EXERCISES

22-1 The Atomic Nature of Matter

1. Gold has a molar (atomic) mass of 197 g/mol. Consider a 2.56-g sample of pure gold vapor. (*a*) Calculate the number of moles of gold present. (*b*) How many atoms of gold are present?

2. (*a*) Find the number of molecules in 1.00 m^3 of air at 20.0°C and at a pressure of 1.00 atm. (*b*) What is the mass of this volume of air? Assume that 75% of the molecules are nitrogen (N_2) and 25% are oxygen (O_2).

3. A steel tank contains 315 g of ammonia gas (NH_3) at an absolute pressure of 1.35×10^6 Pa and temperature 77.0°C. (*a*) What is the volume of the tank? (*b*) The tank is checked later when the temperature has dropped to 22.0°C and the absolute pressure has fallen to 8.68×10^5 Pa. How many grams of gas leaked out of the tank?

4. (*a*) Consider 1.00 mol of an ideal gas at 285 K and 1.00 atm pressure. Imagine that the molecules are for the most part evenly spaced at the centers of identical cubes. Using Avogadro's constant and taking the diameter of a molecule to be 3.00×10^{-8} cm, find the length of an edge of such a cube and calculate the ratio of this length to the diameter of a molecule. The edge length is an estimate of the distance between molecules in the gas. (*b*) Now consider a mole of water having a volume of 18 cm^3. Again imagine the molecules to be evenly spaced at the centers of identical cubes and repeat the calculation in (*a*).

5. Consider a sample of argon gas at 35.0°C and 1.22 atm pressure. Suppose that the radius of a (spherical) argon atom is 0.710×10^{-10} m. Calculate the fraction of the container volume actually occupied by atoms.

22-2 A Molecular View of Pressure

6. The mass of the H_2 molecule is 3.3×10^{-24} g. If 1.6×10^{23} hydrogen molecules per second strike 2.0 cm^2 of wall at an angle of 55° with the normal when moving with a speed of 1.0×10^5 cm/s, what pressure do they exert on the wall?

7. At 44.0°C and 1.23×10^{-2} atm the density of a gas is 1.32×10^{-5} g/cm^3. (*a*) Find v_{rms} for the gas molecules. (*b*) Using the ideal gas law, find the number of moles per unit volume (molar density) of the gas. (*c*) By combining the results of (*a*) and (*b*), find the molar mass of the gas and identify it.

8. A cylindrical container of length 56.0 cm and diameter 12.5 cm holds 0.350 moles of nitrogen gas at a pressure of 2.05 atm. Find the rms speed of the nitrogen molecules.

22-3 Mean Free Path

9. At standard temperature and pressure (0°C and 1.00 atm) the mean free path in helium gas is 285 nm. Determine (*a*) the number of molecules per cubic meter and (*b*) the effective diameter of the helium atoms.

10. At 2500 km above the Earth's surface the density is about 1.0 molecule/cm^3. (*a*) What mean free path is predicted by Eq. 22-13 and (*b*) what is its significance under these conditions? Assume a molecular diameter of 2.0×10^{-8} cm.

11. At what frequency would the wavelength of sound be on the order of the mean free path in nitrogen at 1.02 atm pressure and 18.0°C? Take the diameter of the nitrogen molecule to be 315 pm.

12. In a certain particle accelerator the protons travel around a circular path of diameter 23.5 m in a chamber of 1.10×10^{-6} mm Hg pressure and 295 K temperature. (*a*) Calculate the number of gas molecules per cubic meter at this pressure. (*b*) What is the mean free path of the gas molecules under these conditions if the molecular diameter is 2.20×10^{-8} cm?

13. In Sample Problem 22-4, at what temperature is the average rate of collision equal to 6.0×10^9 s^{-1}? The pressure remains unchanged.

22-4 The Distribution of Molecular Speeds

14. The speeds of a group of ten molecules are 2.0, 3.0, 4.0, . . . , 11 km/s. (*a*) Find the average speed of the group. (*b*) Calculate the root-mean-square speed of the group.

15. (*a*) Ten particles are moving with the following speeds: four at 200 m/s, two at 500 m/s, and four at 600 m/s. Calculate the average and root-mean-square speeds. Is $v_{rms} > v_{av}$? (*b*) Make up your own speed distribution for the ten particles and show that $v_{rms} \geq v_{av}$ for your distribution. (*c*) Under what condition (if any) does $v_{rms} = v_{av}$?

16. Calculate the root-mean-square speed of ammonia (NH_3) molecules at 56.0°C. An atom of nitrogen has a mass of 2.33×10^{-26} kg and an atom of hydrogen has a mass of 1.67×10^{-27} kg.

17. The temperature in interstellar space is 2.7 K. Find the root-mean-square speed of hydrogen molecules at this temperature. (See Table 22-1.)

18. Verify Eq. 22-16 by evaluating $dN(v)/dv = 0$ and solving for v.

19. Evaluate the integral in Eq. 22-17 to verify Eq. 22-18.

20. Evaluate the integral in Eq. 22-19 to verify that $(v^2)_{av} = 3kT/m$.

21. Calculate the root-mean-square speed of smoke particles of mass 5.2×10^{-14} g in air at 14°C and 1.07 atm pressure.

22. At what temperature do the atoms of helium gas have the same rms speed as the molecules of hydrogen gas at 26.0°C?

23. (*a*) Compute the temperatures at which the rms speed is equal to the speed of escape from the surface of the Earth for molecular hydrogen and for molecular oxygen. (*b*) Do the same for the Moon, assuming the gravitational acceleration on its surface to be 0.16*g*. (*c*) The temperature high in the Earth's upper atmosphere is about 1000 K. Would you expect to find much hydrogen there? Much oxygen?

24. You are given the following group of particles (N_n represents the number of particles that have a speed v_n):

N_n	v_n (km/s)
2	1.0
4	2.0
6	3.0
8	4.0
2	5.0

(*a*) Compute the average speed v_{av}. (*b*) Compute the root-mean-square speed v_{rms}. (*c*) Among the five speeds shown, which is the most probable speed v_p for the entire group?

25. In the apparatus of Miller and Kusch (see Fig. 22-8) the length L of the rotating cylinder is 20.4 cm and the angle ϕ is 0.0841 rad. What rotational speed corresponds to a selected speed v of 212 m/s?

26. It is found that the most probable speed of molecules in a gas at temperature T_2 is the same as the rms speed of the molecules in this gas when its temperature is T_1. Calculate T_2/T_1.

27. Show that, for atoms of mass m emerging as a beam from a small opening in an oven of temperature T, the most probable speed is $v_p = \sqrt{3kT/m}$.

28. An atom of germanium (diameter = 246 pm) escapes from a furnace (T = 4220 K) with the root-mean-square speed into a chamber containing atoms of cold argon (diameter = 300 pm) at a density of 4.13×10^{19} atoms/cm^3. (*a*) What is the speed of the germanium atom? (*b*) If the germanium atom and an argon atom collide, what is the closest distance between their centers, considering each as spherical? (*c*) Find the initial collision frequency experienced by the germanium atom.

22-5 The Distribution of Molecular Energies

29. Calculate the fraction of particles in a gas moving with translational kinetic energy between $0.01kT$ and $0.03kT$. (*Hint:* For $E \ll kT$, the term $e^{-E/kT}$ in Eq. 22-25 can be replaced with $1 - E/kT$. Why?)

30. Find the fraction of particles in a gas having translational kinetic energies within a range $0.02kT$ centered on the most probable energy E_p. (*Hint:* In this region, $N(E) \approx$ constant. Why?)

22-6 Equations of State for Real Gases

31. Estimate the van der Waals constant b for H_2O knowing that one kilogram of water has a volume of 0.001 m^3. The molar mass of water is 18 g/mol.

32. The value of the van der Waals constant b for oxygen is 32 cm^3/mol. Compute the diameter of an O_2 molecule.

33. Show that the constant a in the van der Waals equation can be written in units of

$$\frac{\text{energy per particle}}{\text{particle density}}.$$

22-7 Intermolecular Forces

PROBLEMS

1. At 0°C and 1.000 atm pressure the densities of air, oxygen, and nitrogen are, respectively, 1.293 kg/m^3, 1.429 kg/m^3, and 1.250 kg/m^3. Calculate the fraction by mass of nitrogen in the air from these data, assuming only these two gases to be present.

2. *Dalton's law* states that when mixtures of gases having no chemical interaction are present together in a vessel, the pressure exerted by each constituent at a given temperature is the same as it would exert if it alone filled the whole vessel, and that the total pressure is equal to the sum of the partial pressures of each gas. Derive this law from kinetic theory, using Eq. 22-8.

3. A container encloses two ideal gases. Two moles of the first gas are present, with molar mass M_1. Molecules of the second gas have a molar mass $M_2 = 3M_1$, and 0.5 mol of this gas is present. What fraction of the total pressure on the container wall is attributable to the second gas? (Hint: See Problem 2.)

4. Calculate the mean free path for 35 spherical jelly beans in a jar that is vigorously shaken. The volume of the jar is 1.0 L and the diameter of a jelly bean is 1.0 cm.

5. The mean free path A of the molecules of a gas may be determined from measurements (for example, from measurement of the viscosity of the gas). At 20.0°C and 75.0 cm Hg pressure such measurements yield values of $\lambda(\text{argon}) = 9.90 \times 10^{-6}$ cm and $\lambda(\text{nitrogen}) = 27.5 \times 10^{-6}$ cm. (*a*) Find the ratio of the effective cross-section diameters of argon to nitrogen. (*b*) What would be the value of the mean free path of argon at 20.0°C and 15.0 cm Hg? (*c*) What would be the value of the mean free path of argon at −40.0°C and 75.0 cm Hg?

6. The probability that a gas molecule will travel a distance between r and $r + dr$ before colliding with another molecule is given by $Ae^{-cr}dr$, where A and c are constants. By setting the average distance of travel to be equal to the mean free path, find A and c in terms of the number of molecules N and the mean free path λ.

7. Two containers are at the same temperature. The first contains gas at pressure p_1 whose molecules have mass m_1 with a root-mean-square speed $v_{\text{rms},1}$. The second contains molecules of mass m_2 at pressure $2p_1$ that have an average speed $v_{\text{av},2} = 2v_{\text{rms},1}$. Find the ratio $m_1\!:\!m_2$ of the masses of their molecules.

8. A gas, not necessarily in thermal equilibrium, consists of N particles. The speed distribution is not necessarily Maxwellian. (*a*) Show that $v_{\text{rms}} \geq v_{\text{av}}$ regardless of the distribution of speeds. (*b*) When would the equality hold?

9. Figure 22-16 shows a hypothetical speed distribution of N gas molecules with $N(v) = Cv^2$ for $0 < v < v_0$ and $N(v) = 0$ for $v > v_0$. Find (*a*) an expression for C in terms of N and v_0, (*b*) the average speed of the particles, and (*c*) the rms speed of the particles.

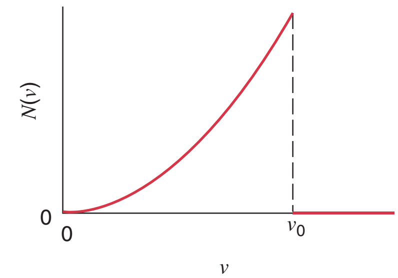


FIGURE 22-16. Problem 9.

10. A gas of N particles has the hypothetical speed distribution shown in Fig. 22-17 [$N(v) = 0$ for $v > 2v_0$]. (*a*) Express a in terms of N and v_0. (*b*) How many of the particles have speeds between $1.50v_0$ and $2.00v_0$? (*c*) Express the average speed of the particles in terms of v_0. (*d*) Find v_{rms}.

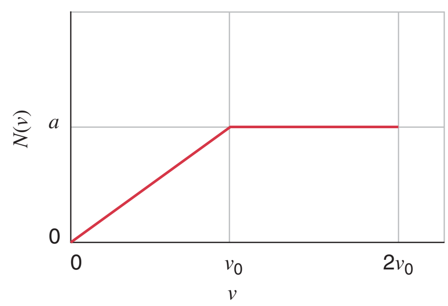


FIGURE 22-17. Problem 10.

11. For a gas in which all molecules travel with the same speed v_{av}, show that $v_{\text{rel}} = \frac{4}{3}v_{\text{av}}$ rather than $\sqrt{2}v_{\text{av}}$ (which is the result obtained when we consider the actual distribution of molecular speeds). See Eq. 22-13.

12. The Sun is a huge ball of hot ideal gas. The glow surrounding the Sun in the ultraviolet image shown in Fig. 22-18 is the corona—the atmosphere of the Sun. Its temperature and pressure are 2.0×10^6 K and 0.030 Pa. Calculate the rms speed of free electrons in the corona.

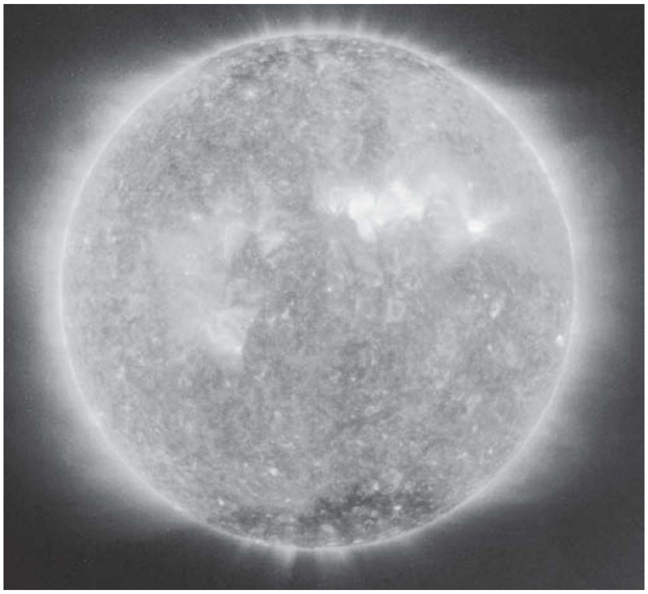


FIGURE 22-18. Problem 12.

13. Consider a gas at temperature T occupying a volume V to consist of a mixture of atoms—namely, N_a atoms of mass m_a each having an rms speed v_a and N_b atoms of mass m_b each having an rms speed v_b. (*a*) Give an expression for the total pressure exerted by the gas. (*b*) Suppose now that $N_a = N_b$ and that the different atoms combine at constant volume to form molecules of mass $m_a + m_b$. Once the temperature returns to its original value, what would be the ratio of the pressure after combination to the pressure before?

14. Find *all* of the virial coefficients for a gas that obeys the van der Waals equation of state.

15. The envelope and basket of a hot-air balloon have a combined mass of 249 kg, and the envelope has a capacity of 2180 m^3. When fully inflated, what should be the temperature of the enclosed air to give the balloon a lifting capacity of 272 kg (in addition to its own mass)? Assume that the surrounding air, at 18.0°C, has a density of 1.22 kg/m^3.

16. Very small solid particles, called grains, exist in interstellar space. They are continually bombarded by hydrogen atoms of the surrounding interstellar gas. As a result of these collisions, the grains execute Brownian movement in both translation and rotation. Assume that the grains are uniform spheres of diameter 4.0×10^{-6} cm and density 1.0 g/cm^3, and that the temperature of the gas is 100 K. Find (*a*) The root-mean-square speed of the grains between collisions and (*b*) the approximate rate (rev/s) at which the grains are spinning. (Assume that the average translational kinetic energy and average rotational kinetic energy are equal.)

17. As Fig. 22-11 suggests, if the intermolecular forces are large enough, the measured pressure p of a gas that obeys the van der Waals equation of state could be zero. (*a*) For what value of the volume per mole would this occur? (Hint: There are two solutions; find them both and interpret them.) (*b*) Show that there is a maximum temperature for zero pressure to occur, and find this maximum temperature in terms of the a and b parameters in the van der Waals equation. (*c*) Assuming that oxygen obeys the van der Waals equation with $a = 0.138$ J·m^3/mol^2 and $b = 3.18 \times 10^{-5}$ m^3/mol, find the maximum temperature for which $p = 0$ for oxygen and compare this value with the normal boiling point of oxygen.

COMPUTER PROBLEMS

1. Write a program to simulate the random walk of a particle. The particle starts at the origin, and can then take a step with Δx and Δy increments assigned randomly between -1 and 1. (*a*) Allow the particle to "walk" through 200 steps, and graph the motion as was done in Fig. 22-1. Choose the scale of the graph to just fit the data. (*b*) Allow the particle to walk through 2000 steps, but this time plot the position of the particle only at the end of each 10 steps. Again, choose the scale of the graph to just fit the data. (*c*) Repeat, but now allow the particle to walk through 20,000 steps, and only plot the position at the end of each 100 steps. Compare the three graphs. How does the size of the graph grow with the number of steps? Do the graphs look similar? If the graphs were shuffled, would you be able to tell which was which?

2. Consider a van der Waals gas with $a = 0.10$ J·m^3/mol^2 and $b = 1.0 \times 10^{-4}$ m^3/mol. (*a*) Find the temperature T_{cr}, pressure p_{cr}, and volume V_{cr} where $\partial p/\partial V = 0$ and $\partial^2 p/\partial V^2 = 0$. (*b*) Graph the pressure along isotherms as a function of volume for $0.80T_{cr}$, $0.85T_{cr}$, $0.90T_{cr}$, $0.95T_{cr}$, $1.00T_{cr}$, $1.05T_{cr}$, and $1.10T_{cr}$. The graphs should extend from $V = 0$ to $V = 5V_{cr}$. (*c*) What is physically significant about the T_{cr} isotherm?

CHAPTER 23

THE FIRST LAW OF THERMODYNAMICS

In earlier chapters we used the concept of heat, without defining it carefully. In this chapter we explore the nature of heat in more detail. With the concepts of work, heat, and internal energy now in hand, we return to the first law of thermodynamics—first discussed in Chapter 13—for a deeper analysis. We conclude by applying this law to a number of thermodynamic processes, once more choosing the ideal gas as our system of interest.

23-1 HEAT: ENERGY IN TRANSIT

It is a common observation that if you place a cup of hot coffee or a glass of ice water on a table at room temperature, the coffee will get colder and the ice water will get warmer, the temperature of each approaching that of the room. In each case the object will tend toward thermal equilibrium with its environment.

We have mentioned earlier that such approaches to thermal equilibrium must involve some sort of exchange of energy between the system and the environment. In Section 13-7 (which you should review) we defined the heat Q to be the energy that is transferred, such as from the coffee to the room or from the room to the ice water. Specifically:

> *Heat is energy that flows between a system and its environment because of a temperature difference between them.*

Figure 23-1 summarizes this view. If the temperature T_S of a system is less than the temperature T_E of the system's environment, energy flows *into* the system as heat. We choose our sign convention so that Q is positive in this case, which tends to increase the internal energy E_{int} of the system. Conversely, when $T_S > T_E$, energy flows out of the system (thereby decreasing E_{int}), and we take Q for this case to be negative.

Like other forms of energy, heat can be expressed in the SI unit of joules (J). In Section 13-7 we listed the relationship of the joule to other units in which heat energy is sometimes measured.

Misconceptions about Heat

Heat is similar to work in that both represent ways of transferring energy. Neither heat nor work is an intrinsic property of a system; that is, we cannot say that a system

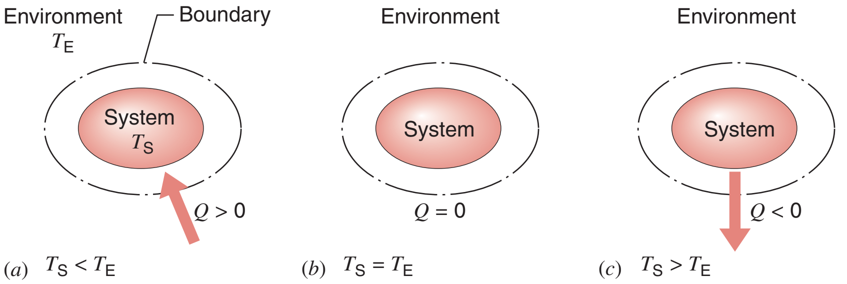


FIGURE 23-1. (*a*) If the temperature T_S of a system is less than the temperature T_E of its environment, heat is transferred into the system until thermal equilibrium is established, as in (*b*). (*c*) If the temperature of a system is greater than that of its environment, heat is transferred out of the system.

"contains" a certain amount of heat or work. Unlike properties such as pressure, temperature, and internal energy, heat and work are not properties of the state of the system; they are not *state functions.* Instead, we say that a certain amount of energy can be transferred, either into or out of the system, as heat or as work. Both heat and work are thus associated with a *thermodynamic process*—that is, with the interaction between the system and its environment as the system changes from one equilibrium state to another.

As we indicated in Section 13-7, in common usage, heat is often confused with temperature or internal energy. When cooking instructions say, "heat at 300 degrees," it is temperature (on the Fahrenheit scale!) that is being discussed. We also may hear someone refer to the "heat generated" in the brake linings of a car as it is braked to a halt. In this case, both the temperature and the internal energy of the brake linings have increased because of frictional work done on them. The rise in temperature of the brake linings did not occur because heat was transferred to the brake linings from some external object at a higher temperature. There is no such object. The only transfer of heat in this case is *from* the high-temperature brake linings *to* their immediate surroundings.

23-2 THE TRANSFER OF HEAT

We know that heat is transferred between a system and its environment when their temperatures differ. However, we have not yet described the mechanisms by which this transfer takes place. There are three of them: thermal conduction, convection, and radiation. We will discuss each in turn.

Thermal Conduction

If you leave a poker in a fire for a long enough time, its handle will become hot. Energy is transferred from the fire to the handle by *thermal conduction* along the metal shaft. In metals—as we shall learn in Chapter 49—some of the atomic electrons are free to move about within the confines of the metallic object and thus are able to pass along increases in their kinetic energy from regions of higher temperature to regions of lower temperature. In this way a region of rising temperature passes along the shaft to your hand.

Consider a thin slab of a homogeneous material of thickness Δx and area A (Fig. 23-2). One face is held at a constant temperature T and the other at a somewhat higher constant temperature $T + \Delta T$, both temperatures being uniform over their respective surfaces. Consider the rate H $(= Q/\Delta t)$ at which heat is transferred through the slab. (The SI unit for H is the joule/second, which is the watt.) Experiment shows that H is (1) directly proportional to A—the more area available, the more heat can be transferred per unit time; (2) inversely proportional to Δx—the thicker

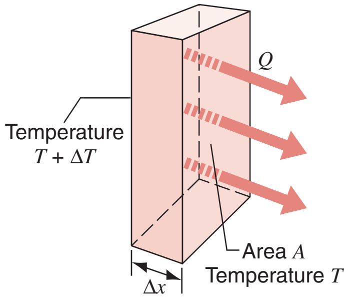


FIGURE 23-2. Heat Q flows through a rectangular slab of material of thickness Δx and area A.

the slab, the less heat will be transferred per unit time; and (3) directly proportional to ΔT—the larger the temperature difference is, the more heat will be transferred.

We can summarize these experimental findings as

$$H = kA\frac{\Delta T}{\Delta x}, \qquad (23\text{-}1)$$

in which the proportionality constant k is called the *thermal conductivity* of the material. The SI unit of k is the watt per meter kelvin (W/m · K).

Table 23-1 shows some values of k for selected substances. A substance with a large value of k is a good thermal conductor; one with a small value of k is a poor thermal conductor or, equivalently, a good thermal insulator. Figure 23-3 shows a patio in which concrete slabs are separated by fir strips. As Table 23-1 shows, the thermal conductivity of concrete is more than five times that of fir; heat conduction from the (warmer) ground through the concrete

TABLE 23-1 Some Thermal Conductivities and *R*-Values[a]

Material	*Conductivity, k* (W/m · K)	*R-Value* ($ft^2 \cdot F° \cdot h/Btu$)
Metals		
Stainless steel	14	0.010
Lead	35	0.0041
Aluminum	235	0.00061
Copper	401	0.00036
Silver	428	0.00034
Gases		
Air (dry)	0.026	5.5
Helium	0.15	0.96
Hydrogen	0.18	0.80
Building materials		
Polyurethane foam	0.024	5.9
Rock wool	0.043	3.3
Fiberglass	0.048	3.0
Fir	0.14	1.0
Concrete	0.80	0.18
Window glass	1.0	0.14

[a] Values are for room temperature. Note that values of k are given in SI units and those of R in the customary British units. The R-values are for a 1-in. slab.

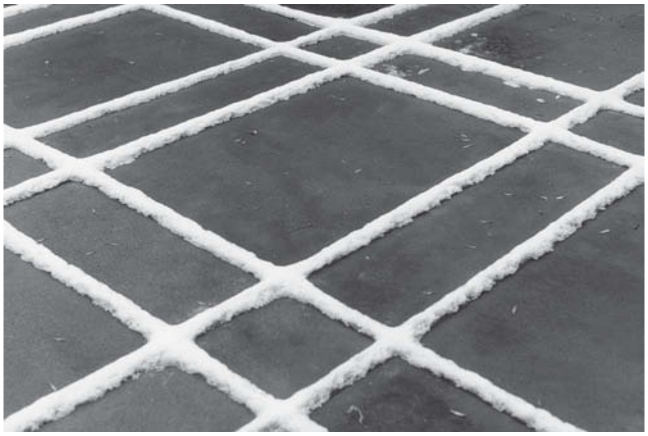


FIGURE 23-3. Snow melts on the concrete, but not on the fir strips between the concrete sections, because concrete is a better thermal conductor than wood.

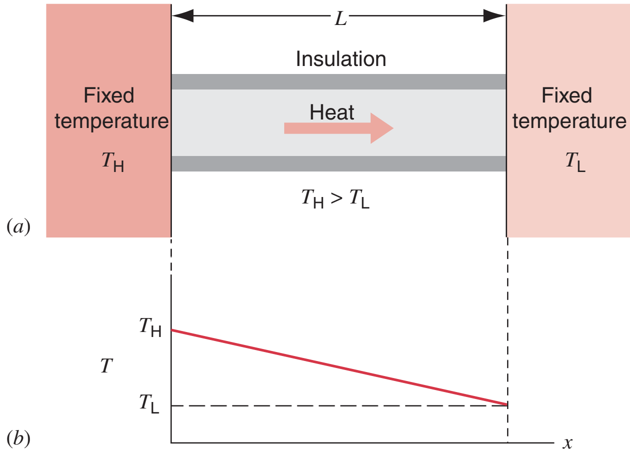


FIGURE 23-4. (*a*) Conduction of heat through an insulated conducting rod. (*b*) The variation of temperature along the rod.

and the fir to the (cooler) air causes the snow above the concrete to melt first.

In choosing building materials, one often finds them rated in terms of the *thermal resistance* or *R*-value, defined by

$$R = \frac{L}{k}, \tag{23-2}$$

where L is the thickness of the material through which the heat is transfered. Thus the lower the conductivity is, the higher is the R-value: good insulators have high R-values. Numerically, the R-value is evaluated according to Eq. 23-2 expressed in the British units of $\text{ft}^2 \cdot \text{F}° \cdot \text{h/Btu}$. The R-value is determined for a certain thickness of material. For example, a 1-in. thickness of fiberglass has $R = 3$, whereas a 1-in. thickness of wood has $R = 1$ (and therefore conducts heat at three times the rate of fiberglass). One inch of air has $R = 5$, but air is a poor thermal insulator because it can transfer more heat by convection, and the thermal conductivity is thus not a good measure of the insulating value of air. Table 23-1 shows the R-values of one-inch slabs of some materials.

Now let us consider two applications of Eq. 23-1. We first take the case of a long rod of length L and uniform cross section A (Fig. 23-4*a*), in which one end is maintained at the high temperature T_H and the other end at the low temperature T_L.* We call this a *steady state* situation: the temperatures and the rate of heat transfer are constant in time. In this situation, every increment of energy that enters the rod at the hot end leaves it at the cold end. Put another way, through any cross section along the length of the rod, we would measure the same rate of heat transfer.

For this case, we can write Eq. 23-1 as

$$H = kA\frac{T_H - T_L}{L}. \tag{23-3}$$

Here L is the thickness of the material in the direction of heat transfer. The rate of heat flow H is a constant, and the temperature decreases in linear fashion between the ends of the rod (Fig. 23-4*b*).

We now consider the case in which the slab has infinitesimal thickness dx and temperature difference dT between its faces. In this limit, we obtain

$$H = -kA\frac{dT}{dx}. \tag{23-4}$$

The derivative dT/dx is called the *temperature gradient.* We choose the positive direction of the variable x in Eq. 23-4 to be the direction in which heat is transferred. Because heat flows in the direction of *decreasing* temperature, the gradient dT/dx is inherently negative. We introduce a minus sign into Eq. 23-4 to ensure that H, the rate of heat transfer, will be a positive quantity.

Equation 23-4 is particularly applicable in cases where the cross section of the material through which heat is being transferred is not uniform. Sample Problem 23-2 is an illustrative example.

SAMPLE PROBLEM 23-1. Consider a compound slab consisting of two materials having different thicknesses, L_1 and L_2, and different thermal conductivities, k_1 and k_2. If the temperatures of the outer surfaces are T_1 and T_2 (with $T_2 > T_1$), find the rate of heat transfer through the compound slab (Fig. 23-5) in a steady state.

* The ends of the rod can be considered to be immersed in *thermal reservoirs,* which can supply or absorb an unlimited amount of heat while maintaining a constant temperature. A thermal reservoir might be a material of such large quantity or ability to absorb heat that the heat flowing to or from the rod makes a negligible difference in its temperature. Or it might be a mixture of steam and water maintained at the boiling point or ice and water at the melting point, so that the heat absorbed causes a change in phase but no change in temperature. Other possibilities for thermal reservoirs include furnaces or refrigerators in which the heat is ultimately converted to or from mechanical work while keeping the temperature fixed.

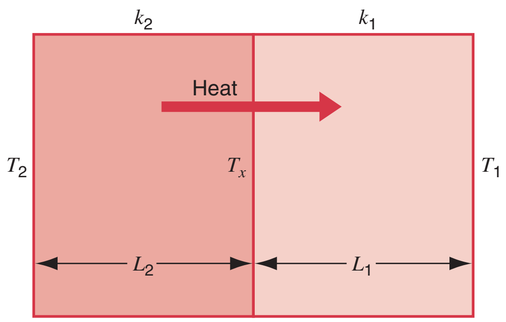


FIGURE 23-5. Sample Problem 23-1. Conduction of heat through two layers of matter with different thermal conductivities.

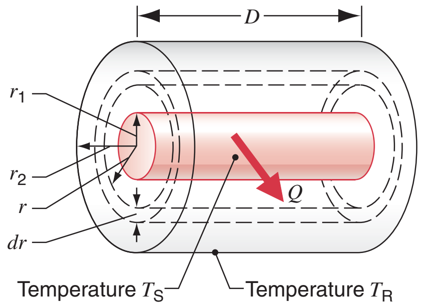


FIGURE 23-6. Sample Problem 23-2. The inner surface (radius r_1) of the insulation on a cylindrical pipe is at the temperature T_S and the outer surface (radius r_2) is at T_R. The same heat Q flows through every cylindrical shell of insulation, such as the intermediate one of thickness dr and radius r shown by the dashed lines.

Solution Let T_x be the temperature at the interface between the two materials. Then the rate of heat transfer through slab 2 is

$$H_2 = \frac{k_2 A(T_2 - T_x)}{L_2}$$

and that through slab 1 is

$$H_1 = \frac{k_1 A(T_x - T_1)}{L_1}.$$

In a steady state $H_2 = H_1$, so that

$$\frac{k_2 A(T_2 - T_x)}{L_2} = \frac{k_1 A(T_x - T_1)}{L_1}.$$

Let H be the rate of heat transfer (the same for all sections). Then, solving for T_x and substituting into either of the equations for H_1 or H_2, we obtain

$$H = \frac{A(T_2 - T_1)}{(L_1/k_1) + (L_2/k_2)} = \frac{A(T_2 - T_1)}{R_1 + R_2}.$$

The extension to any number of sections in series is

$$H = \frac{A(T_2 - T_1)}{\Sigma\,(L_n/k_n)} = \frac{A(T_2 - T_1)}{\Sigma\, R_n}. \qquad (23\text{-}5)$$

SAMPLE PROBLEM 23-2. A thin, cylindrical metal pipe is carrying steam at a temperature of $T_S = 100°\mathrm{C}$. The pipe has a diameter of 5.4 cm and is wrapped with a thickness of 5.2 cm of fiberglass insulation. A length $D = 6.2$ m of the pipe passes through a room in which the temperature is $T_R = 11°\mathrm{C}$. (*a*) At what rate does heat energy pass through the insulation? (*b*) How much additional insulation must be added to reduce the heat transfer rate by half?

Solution (*a*) Figure 23-6 illustrates the geometry appropriate to the calculation. In the steady state, the rate of heat transfer H will be constant and will be the same for every thin cylindrical shell, such as the one indicated by the dashed lines in Fig. 23-6. We can regard this shell as a slab of material, having a thickness dr and an area of $2\pi rD$. Applying Eq. 23-4 to this geometry, we have

$$H = -kA\frac{dT}{dr} = -k(2\pi rD)\frac{dT}{dr}$$

or

$$H\frac{dr}{r} = -2\pi kD\, dT.$$

We assume that the thin metal pipe is at the temperature of the steam, so it does not enter into the calculation. We integrate from the inner radius r_1 of the insulation at temperature T_S to the outer radius r_2 at temperature T_R:

$$\int_{r_1}^{r_2} H\frac{dr}{r} = -2\pi kD\int_{T_S}^{T_R} dT.$$

Removing the constant H from the integral on the left and carrying out the integrations, we obtain

$$H \ln\frac{r_2}{r_1} = -2\pi kD(T_R - T_S) = 2\pi kD(T_S - T_R).$$

Solving for H and inserting the numerical values, we find

$$H = \frac{2\pi kD(T_S - T_R)}{\ln(r_2/r_1)}$$

$$= \frac{2\pi(0.048\ \mathrm{W/m\cdot K})(6.2\ \mathrm{m})(89\ \mathrm{K})}{\ln(7.9\ \mathrm{cm}/2.7\ \mathrm{cm})} = 155\ \mathrm{W}.$$

Note that, if we had not inserted a minus sign into Eq. 23-4, the algebraic sign of H would not have been positive.

(*b*) To reduce the heat transfer rate by half, we must increase r_2 to the value r_2' such that the denominator in the above expression for H becomes twice as large; that is,

$$\frac{\ln(r_2'/r_1)}{\ln(r_2/r_1)} = 2.$$

Solving for r_2', we find

$$r_2' = \frac{r_2^2}{r_1} = \frac{(7.9\ \mathrm{cm})^2}{2.7\ \mathrm{cm}} = 23\ \mathrm{cm}.$$

Thus we need nearly four times the thickness of insulation to reduce the heat transfer by half! This effect is due to the increase in the area, and therefore in the mass, contained in each thin slab as we increase the radius in the cylindrical geometry. There is more material available to conduct heat at the outer radii, and we must therefore supply an increasing amount of insulation as r grows larger. This differs from the linear geometry, in which the heat transferred decreases linearly as the insulation thickness increases. In the spherical geometry (which might be appropriate to calculating the heat energy transferred from the Earth's core to its surface), the calculation is still different; see Problem 3.

Convection

If you look at the flame of a candle or a match, you are watching heat energy being transported upward by *convection.* Heat transfer by convection occurs when a fluid, such as air or water, is in contact with an object whose temperature is higher than that of its surroundings. The temperature of the fluid that is in contact with the hot object increases, and (in most cases) the fluid expands. The warm fluid is less dense than the surrounding cooler fluid, so it rises because of buoyant forces; see Fig. 23-7. The surrounding cooler fluid falls to take the place of the rising warmer fluid, and a convective circulation is set up.

Atmospheric convection plays a fundamental role in determining the global climate patterns and in our daily weather variations. Glider pilots and condors alike seek the convective thermals that, rising from the warmer Earth beneath, keep them aloft. Huge energy transfers take place within the oceans by the same process. The outer region of the Sun, called the *photosphere,* contains a vast array of convection cells that transport energy to the solar surface and give the surface a granulated appearance. Finally, there are thought to be huge convective cells within the mantle of the Earth, their outermost surfaces being the tectonic plates whose motions move the continents.

We have so far been describing *free* or *natural* convection. Convection can also be forced, as when a furnace blower causes air circulation to heat the rooms of a house.

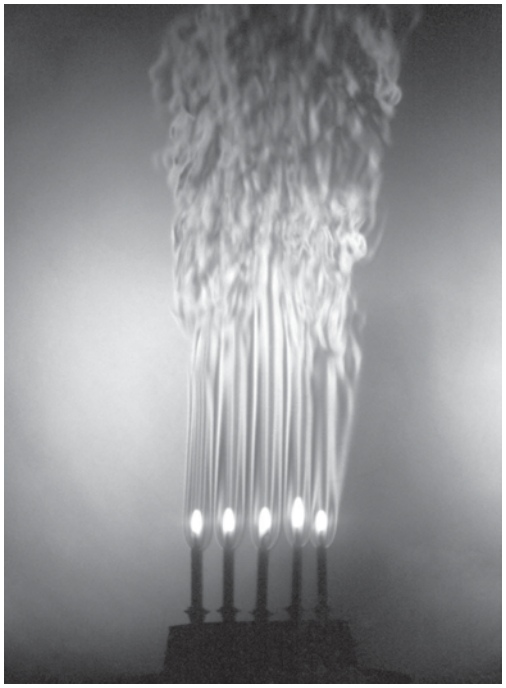


FIGURE 23-7. Air rises by convection around a heated cylinder. The dark areas represent regions of uniform temperature.

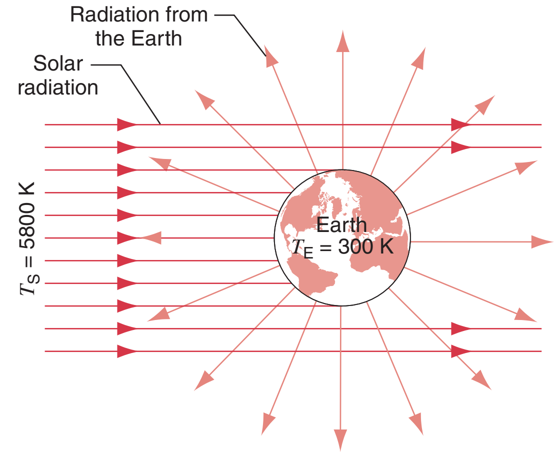


FIGURE 23-8. Solar radiation is intercepted by the Earth and is (mostly) absorbed. The temperature T_E of the Earth adjusts itself to a value at which the Earth's heat loss by radiation is just equal to the solar heat that it absorbs.

Radiation

Energy is carried from the Sun to us by electromagnetic waves that travel freely through the near vacuum of the intervening space. If you stand near a bonfire or an open fireplace, you are warmed by the same process. All objects emit such electromagnetic radiation because of their temperature and also absorb some of the radiation that falls on them from other objects. The higher the temperature of an object is, the more it radiates. We shall see in Chapter 45 of this text that the energy radiated by an object is proportional to the fourth power of its (Kelvin) temperature. The average temperature of our Earth, for example, levels off at about 300 K because at that temperature the Earth radiates energy into space at the same rate that it receives it from the Sun; see Fig. 23-8.

23-3 THE FIRST LAW OF THERMODYNAMICS

In Chapter 13 we discussed the fundamental concept of conservation of energy in a system of particles. As we did in the case of conservation of momentum in Chapter 7, we concentrated our attention on a particular collection of particles or objects that we defined as our *system.* We drew an imaginary boundary that separated the system from its *environment,* and then we carefully accounted for all interactions between the system and its environment. Sometimes, as in the case of momentum conservation, we characterize those interactions in terms of forces. Other times it is more convenient to characterize those interactions in terms of energy transfer.

We are free to define our system in any convenient way, as long as we are consistent and can account for all energy transfers to or from the system. For example, we might define the system to be a block of metal that is at a lower temperature than its environment, so that the interaction involves a transfer of heat from the environment to the block. Or we might define a system to be water and ice that are mixed together in an insulated container. In this case there is an exchange of energy *within* the system but no interaction with the environment.

For a thermodynamic system, in which internal energy is the only type of energy the system may have, the law of conservation of energy may be expressed as

$$Q + W = \Delta E_{\text{int}}. \tag{23-6}$$

In this section we examine this equation, which is a statement of the *first law of thermodynamics.* In this equation:

Q is the energy transferred (as heat) between the system and its environment because of a temperature difference between them. A heat transfer that occurs entirely within the system boundary is not included in Q.

W is the work done on (or by) the system by forces that act through the system boundary. Work done by forces that act entirely within the system boundary is not included in W.

ΔE_{int} is the change in the internal energy of the system that occurs when energy is transferred into or out of the system as heat or work.

By convention we have chosen Q to be positive when heat is transferred *into* the system and W to be positive when work is done *on* the system. With these choices, positive values of Q and W each serve to *increase* the internal energy of the system.*

Equation 23-6 is a restricted form of the general law of conservation of energy. For example, the system as a whole may be in motion in our frame of reference. That is, there may be kinetic energy associated with the motion of the center of mass of the system. If that were the case, we would have to add a term ΔK_{cm} to the right side of Eq. 23-6. However, in the systems we discuss the center of mass of the system will always be at rest in our reference frame so that no such term is needed.

Figure 23-9 suggests how Eq. 23-6 is to be applied. The system starts in an *initial equilibrium state* i in Fig. 23-9*a*, in which the properties of the system, such as its internal energy E_{int}, have definite constant values. We then permit the system to undergo a *thermodynamic process*—that is, to interact with its environment as in Fig. 23-9*b*—during which work may be done and/or heat energy exchanged. When the process is concluded, the system ends up in a *final equilibrium state* f, in which the properties of the system will, in general, have different constant values.

There are many processes by which we can take a system from a specified initial state to a specified final state. In general, the values of Q and W will differ, depending on the process we choose. However, experiment shows that, although Q and W may differ individually, their sum $Q + W$ is the same for all processes that connect the given initial and final states. As Eq. 23-6 suggests, this is the experimental basis for regarding the internal energy E_{int} as a true *state function*—that is, as just as much an inherent property of a system as pressure, temperature, and volume. To stress this point of view, we can express the first law of thermodynamics formally in these words:

In any thermodynamic process between equilibrium states i *and* f, *the quantity* $Q + W$ *has the same value for any path between* i *and* f. *This quantity is equal to the change in value of a state function called the internal energy* E_{int}.

The first law of thermodynamics is a general result that is thought to apply to every process in nature that proceeds between equilibrium states. It is not necessary that every stage of the process be an equilibrium state, only the initial and the final states. For example, we can apply the first law to the explosion of a firecracker in an insulated steel drum. We can account for the energy balance before the explosion and after the system has returned to equilibrium, and for this calculation we need not worry that the intermediate condition is turbulent and that pressure and temperature are not well defined.

Because of its generality, the first law is somewhat incomplete as a description of nature. It tells us that energy must be conserved in every process, but it does not tell us whether any particular process that conserves energy can

* Some authors define work done *by* the system to be positive, in which case the first law would be written $Q - W = \Delta E_{\text{int}}$. We have chosen to define work done *on* the system to be positive, so that thermodynamic work will have the same sign convention that we used in earlier chapters for mechanical work.

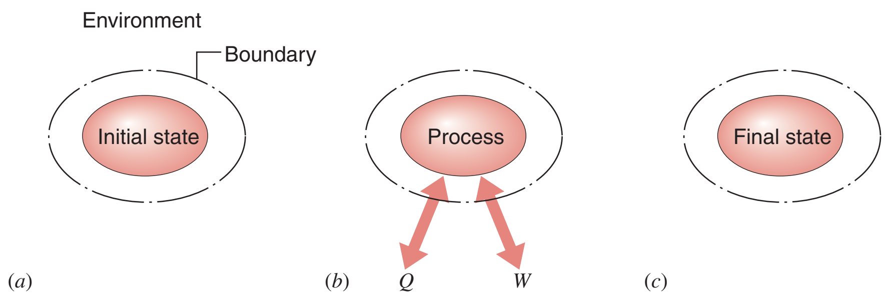


FIGURE 23-9. (*a*) A system in an initial state in equilibrium with its surroundings. (*b*) A thermodynamic process during which the system may exchange heat Q or work W with its environment. (*c*) A final equilibrium state reached as a result of the process.

actually occur. The explosion of a firecracker, for example, releases chemical energy stored in the gunpowder that eventually raises the temperature of the gas in the drum. We can imagine the hot gas giving its thermal energy back to the combustion products, turning them once again into gunpowder and reassembling the firecracker, but this never happens. Conservation of energy works either way but nature seems to go in a preferred direction. The *second* law of thermodynamics, which we discuss in Chapter 24, accounts for this distinction.

In thermal physics as in mechanics, you must be quite clear as to the system to which you are applying fundamental laws such as Eq. 23-6. Figure 23-10, for example, shows a heating coil immersed in water contained in an insulated bucket. The current through the coil is supplied by an (ideal) generator that is driven by a weight that falls with constant speed. Let us see what values of Q, W, and ΔE_{int} result for different arbitrary choices of what we choose to call our system:

1. *System = water alone.* Heat is delivered to the water from the coil so that $Q > 0$. No work is done because the water does not move under the influence of any external force that acts on it. Thus $W = 0$. From the first law, then (Eq. 23-6) $\Delta E_{\text{int}} > 0$. The heat delivered to the water causes its internal energy, and thus its temperature, to rise.

2. *System = coil + weight.* As long as the weight is falling at a constant rate the coil maintains a constant temperature. Thus the system is in a steady state, with no energy transfers occurring within the system boundary. Thus $\Delta E_{\text{int}} = 0$. Heat energy is transferred from coil to water out of this system, so that $Q < 0$. Work is done by the (external) gravitational force so that $W > 0$. The system acts as a conduit for energy, the work done by the gravitational force being delivered as heat energy to the water.

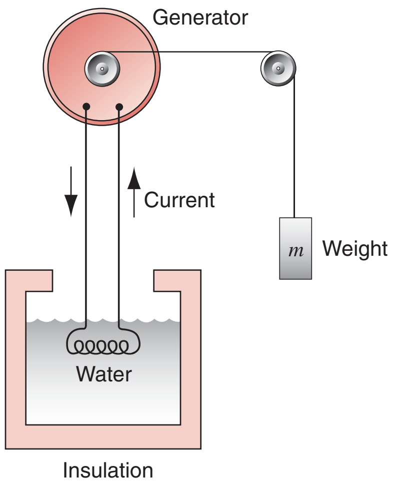


FIGURE 23-10. A heating coil is immersed in water, the electric current through the coil being provided by an (ideal) generator that is driven by a falling weight. Values of Q, W, and ΔE_{int} in Eq. 23-6 depend critically on what parts of this arrangement we choose to define as the "system."

3. *System = coil + weight + water.* Here the gravitational force does work on this system so that $W > 0$. The insulation of the bucket prevents heat transfer to the environment so that $Q = 0$. From Eq. 23-6 then, $\Delta E_{\text{int}} > 0$. Again, work done by an external force produces an increase in the internal energy, and thus the temperature, of the system.

4. *System = coil + weight + water + Earth.* In this case the gravitational force is internal to the system so that $W = 0$. Also, $Q = 0$, as for system choice 3 above. From Eq. 23-6 then, we must have $\Delta E_{\text{int}} = 0$. The internal energy of part of the system rises because of the rise in temperature of the water. However, the internal energy of another part of the system falls—and by the same amount—because the falling weight and the Earth move closer together, thus reducing their potential energy.

The lesson to learn from this analysis is to define your system carefully and stay with that definition throughout your analysis.

23-4 HEAT CAPACITY AND SPECIFIC HEAT

We can change the state of a body by transferring energy to or from it in the form of heat, or by doing work on the body. One property of a body that may change in such a process is its temperature T. The change in temperature ΔT that corresponds to the transfer of a particular quantity of heat energy Q will depend on the circumstances under which the heat was transferred. For example, in the case of a gas confined to a cylinder with a movable piston, we can add heat and keep the piston fixed (thus keeping the volume constant), or we can add heat and allow the piston to move but keep the force on the piston constant (thus keeping the gas under constant pressure). We can even change the temperature by doing work on a system, such as by rubbing together two objects that exert frictional forces on one another; in this case, no heat transfer need occur.

It is convenient to define the *heat capacity* C of a body as the ratio of the amount of heat energy Q transferred to a body in any process to its corresponding temperature change ΔT; that is,

$$C = \frac{Q}{\Delta T}. \tag{23-7}$$

The word "capacity" may be misleading because it suggests the essentially meaningless statement "the amount of heat a body can hold," whereas what is meant is simply the energy per degree of temperature change that is transferred as heat when the temperature of the body changes.

The heat capacity per unit mass of a body, called *specific heat capacity* or usually just *specific heat,* is characteristic of the material of which the body is composed:

$$c = \frac{C}{m} = \frac{Q}{m\,\Delta T}. \tag{23-8}$$

The heat capacity is characteristic of a particular object, but the specific heat characterizes a substance. Thus we speak, on one hand, of the heat capacity of a copper pot but, on the other, of the specific heat of copper.

Neither the heat capacity of a body nor the specific heat of a material is constant; both depend on the temperature (and possibly on other variables as well, such as the pressure). The previous equations give only average values for these quantities in the temperature range of ΔT. In the limit, as $\Delta T \to 0$, we can speak of the specific heat at a particular temperature T.

We can find the heat that must be given to a body of mass m, whose material has a specific heat c, to increase its temperature from initial temperature T_i to final temperature T_f by dividing the temperature change into N small intervals ΔT_n, assuming that c_n is constant in each small interval, and summing the contributions to the total heat transfer from all intervals $n = 1, 2, \ldots, N$. This gives

$$Q = \sum_{n=1}^{N} mc_n \, \Delta T_n. \tag{23-9}$$

In the differential limit this becomes

$$Q = m \int_{T_i}^{T_f} c \, dT, \tag{23-10}$$

where c may be a function of the temperature. At ordinary temperatures and over ordinary temperature intervals, specific heats can be considered to be constants. For example, the specific heat of water varies by less than 1% over the interval from 0°C to 100°C. We can therefore write Eq. 23-10 in the more generally useful form

$$Q = mc(T_f - T_i). \tag{23-11}$$

TABLE 23-2 Heat Capacities of Some Substances[a]

Substance	*Specific Heat Capacity* (J/kg·K)	*Molar Heat Capacity* (J/mol·K)
Elemental solids		
Lead	129	26.7
Tungsten	135	24.8
Silver	236	25.5
Copper	387	24.6
Carbon	502	6.02
Aluminum	900	24.3
Other solids		
Brass	380	
Granite	790	
Glass	840	
Ice (−10°C)	2220	
Liquids		
Mercury	139	
Ethyl alcohol	2430	
Seawater	3900	
Water	4190	

[a] Measured at room temperature and atmospheric pressure, except where noted.

Equation 23-8 does not define specific heat uniquely. We must also specify the conditions under which the heat Q is added to the material. One common condition is that the specimen remain at normal (constant) atmospheric pressure while we add the heat, but there are many other possibilities, each leading, in general, to a different value for c. To obtain a unique value for c we must indicate the conditions, such as specific heat at constant pressure c_p, specific heat at constant volume c_V, and so on.

Table 23-2 shows values for the specific heat capacities of a number of common substances, measured under conditions of constant pressure. Although the units are expressed in terms of K, we can also work with temperatures in °C, because a temperature *difference* in C° is equal to the same temperature difference in K.

SAMPLE PROBLEM 23-3. A cube of copper of mass $m_c = 75$ g is placed in an oven at a temperature of $T_0 = 312$°C until it comes to thermal equilibrium. The cube is then dropped quickly into an insulated beaker containing a quantity of water of mass $m_w = 220$ g. The heat capacity of the beaker alone is $C_b = 190$ J/K. Initially the water and the beaker are at a temperature of $T_i = 12.0$°C. What is the final equilibrium temperature T_f of the system consisting of the copper + water + beaker?

Solution Once the copper cube has been dropped into the beaker, no energy enters or leaves the system copper + water + beaker, either as heat or as work, so that there is no change in the internal energy of this system. However, there are changes in the internal energies of the three objects—which we now regard as subsystems—that make up the system. These three internal energy changes must add up to zero, or

$$\Delta E_{int,c} + \Delta E_{int,w} + \Delta E_{int,b} = 0.$$

However, $W = 0$ for each object (because no work is done on any object) so that, from Eq. 23-6, we must have

$$Q_c + Q_w + Q_b = 0. \tag{23-12}$$

From Eqs. 23-7 and 23-11, the heat transfers for these subsystems are:

$$\text{Copper:} \quad Q_c = m_c c_c (T_f - T_0)$$
$$\text{Water:} \quad Q_w = m_w c_w (T_f - T_i)$$
$$\text{Beaker:} \quad Q_b = C_b (T_f - T_i)$$

Note that we have written the temperature differences as the final temperature minus the initial temperature, so that Q_w and Q_b are positive (indicating that heat energy is transferred *into* the water and beaker subsystems, thus increasing their internal energies) and Q_c is negative (indicating that heat energy is transferred *from* this subsystem, corresponding to a decrease in its internal energy). Substituting these heat transfers into Eq. 23-12 above, we obtain

$$m_w c_w (T_f - T_i) + C_b (T_f - T_i) + m_c c_c (T_f - T_0) = 0.$$

Solving for T_f and substituting, we have

$$T_f = \frac{m_w c_w T_i + C_b T_i + m_c c_c T_0}{m_w c_w + C_b + m_c c_c}$$

$$= \frac{(0.220\text{ kg})(4190\text{ J/kg}\cdot\text{K})(12°\text{C}) + (190\text{ J/K})(12°\text{C}) + (0.075\text{ kg})(387\text{ J/kg}\cdot\text{K})(312°\text{C})}{(0.220\text{ kg})(4190\text{ J/kg}\cdot\text{K}) + 190\text{ J/K} + (0.075\text{ kg})(387\text{ J/kg}\cdot\text{K})}$$

$$= 19.6°\text{C}.$$

Note that, because all temperatures were part of temperature *differences,* we can use °C in this expression. In most thermodynamic expressions, however, only Kelvin temperatures can be used.

From the given data you can show that

$$Q_w = 7010\ \text{J}, \quad Q_b = 1440\ \text{J}, \quad \text{and} \quad Q_c = -8450\ \text{J}.$$

The algebraic sum of these three heat transfers is indeed zero, as Eq. 23-12 requires.

Heats of Transformation

When heat enters a solid or a liquid, the temperature of the sample does not necessarily rise. Instead, the sample may change from one *phase* or *state* (that is, solid, liquid, or gas) to another. Thus ice melts and water boils, absorbing heat in each case without a temperature change. In the reverse processes (water freezes, steam condenses), heat is released by the sample, again at a constant temperature.

The amount of heat per unit mass that must be transferred to produce a phase change is called the *heat of transformation* or *latent heat* (symbol L) for the process. The total heat transferred in a phase change is then

$$Q = Lm, \tag{23-13}$$

where m is the mass of the sample that changes phase. The heat transferred during melting or freezing is called the *heat of fusion* (symbol L_f), and the heat transferred during boiling or condensing is called the *heat of vaporization* (symbol L_v). Table 23-3 shows the heats of transformation of some substances.

Knowledge of heat capacities and heats of transformation is important because we can measure a heat transfer by determining either the temperature change of a material of known heat capacity or the amount of a substance of known heat of transformation converted from one phase to another. For example, in low-temperature systems involving liquid helium at 4 K, the rate at which helium gas boils from the liquid gives a measure of the rate at which heat enters the system.

TABLE 23-3 Some Heats of Transformation

Substance[a]	*Melting Point* (K)	*Heat of Fusion* (kJ/kg)	*Boiling Point* (K)	*Heat of Vaporization* (kJ/kg)
Hydrogen	14.0	58.6	20.3	452
Oxygen	54.8	13.8	90.2	213
Mercury	234	11.3	630	296
Water	273	333	373	2256
Lead	601	24.7	2013	858
Silver	1235	105	2485	2336
Copper	1356	205	2840	4730

[a] Substances are listed in order of increasing melting points.

Heat Capacities of Solids

Recall that the *specific heat capacity* of any material (SI unit: J/kg · K) is the heat capacity per unit mass. In Table 23-2 we see that the values of this quantity vary widely from one solid material to another. If we multiply the specific heat capacity by the molar mass M we obtain the *molar heat capacity* (SI unit: J/mol · K) or the heat capacity per mole. Table 23-2 shows that, with few exceptions (see carbon) the molar heat capacities of all solids have values close to 25 J/mol · K. This remarkable experimental observation was first pointed out in 1819 by the French scientists P. L. Dulong (1785–1838) and A. T. Petit (1791–1820).

In comparing molar heat capacities, we are, in effect, comparing samples that contain the same number of moles rather than samples that have the same mass. Samples with the same number of moles have the same number of atoms, and we conclude that the heat energy required *per atom* to raise the temperature of a solid by a given amount seems—with a few exceptions—to be about the same for all solids. This is striking evidence for the atomic theory of matter.

Actually, molar heat capacities vary with temperature, approaching zero as $T \rightarrow 0$ and approaching the so-called Dulong–Petit value only at relatively high temperatures. Figure 23-11 shows the variation for lead, aluminum, and carbon. The low value of the molar heat capacity for carbon listed in Table 23-2 occurs because, at room temperature, carbon has not yet achieved its limiting value.*

* The data plotted in Fig. 23-11 are the molar heat capacities at constant volume. It is almost impossible to keep a solid from expanding as you increase its temperature so the direct measurements of molar heat capacity are made under conditions of constant pressure. The constant-volume values plotted in the figure are found by making a small theory-based correction to the measured constant-pressure values.

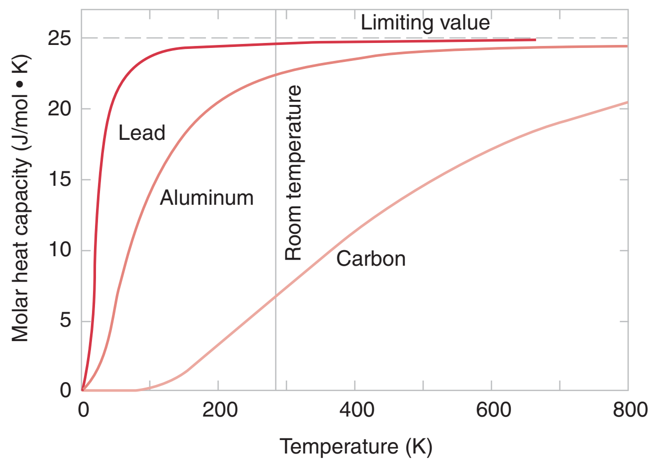


FIGURE 23-11. The molar heat capacity of three elements as a function of temperature. At high temperatures, the heat capacities of all solids approach the same limiting value. For lead and aluminum, that value is nearly reached at room temperature; for carbon it is not.

We will learn in Section 23-6 that the Dulong–Petit high-temperature limit for the molar heat capacity can be understood on the basis of classical physics. However, to understand the complete temperature variation of the molar heat capacity requires an analysis based on quantum physics. Einstein was quick to realize that measurements of the molar heat capacity provide a sensitive probe of the manner in which atoms absorb energy—a matter of deep significance. Understanding the temperature variation of the molar heat capacities of solids was the first problem to which Einstein turned his attention after the introduction of quantum theory, and he provided a preliminary but insightful solution in 1906.*

The data plotted in Fig. 23-11 vary smoothly and characterize materials that do not change their state in that temperature range. That is, they do not melt or change from one crystal structure to another. Measurements of heat capacity are useful in studying such changes. For example, Fig. 23-12*a* shows the variation of the molar heat capacity of tantalum for temperatures in the range 3–5.5 K. It seems likely that something is happening to tantalum at $T = 4.4$ K and indeed it is. Above that temperature, tantalum conducts electricity in the same way that copper and other familiar electrical conductors do. Below that temperature, however, the electrical resistance of tantalum completely disappears; it becomes a so-called *superconductor.*

* Details of Einstein's calculation can be found in *Modern Physics,* by Kenneth S. Krane (Wiley, 1996), Chapter 10.

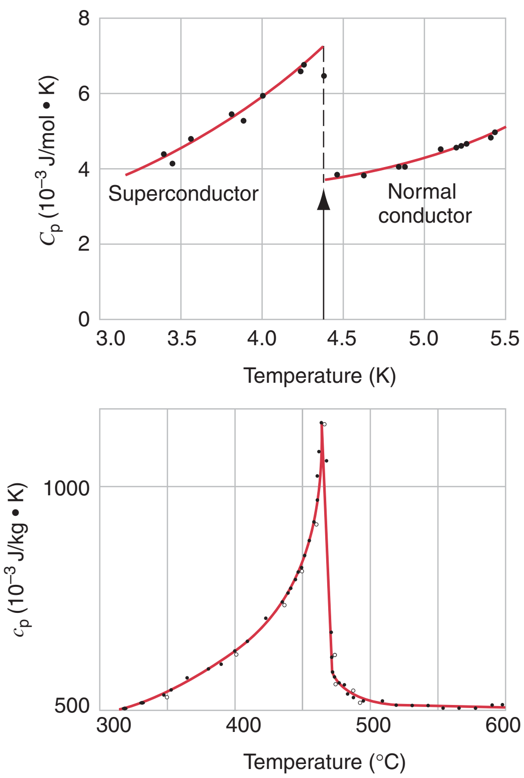


FIGURE 23-12. (*a*) The molar heat capacity of tantalum near its superconducting transition temperature. (*b*) The specific heat capacity of brass.

For another example, Fig. 23-12*b* shows the specific heat capacity of brass in the range 300–600°C. X-ray analysis shows that a change in the crystal structure of brass occurs at about 460°C, from a very ordered structure below that temperature to a rather disordered structure above it.

23-5 WORK DONE ON OR BY AN IDEAL GAS

So far in this chapter we have explored energy transfers as heat in relation to the first law of thermodynamics. In this section we explore energy transfers as work and—as we have done before—we choose the ideal gas as our thermodynamic system of interest. The stylized apparatus of Fig. 21-13 suggests how work might be done either on an ideal gas or by it under various conditions.

If we increase the temperature of the gas in the cylinder of Fig. 21-13, the gas expands and raises the piston against gravity; the gas does (positive) work on the piston. The upward force exerted on the piston by the gas due to its pressure p is given by pA, where A is the area of the piston. By Newton's third law, the force exerted *on* the gas *by* the piston is equal and opposite to the force exerted *on* the piston *by* the gas. Using Eq. 11-14, we can therefore write the work W done on the gas as

$$W = \int F_x \, dx = \int (-pA)dx. \qquad (23\text{-}14)$$

Here dx represents the displacement of the piston, and the minus sign enters because the force exerted on the gas by the piston is in a direction opposite to the displacement of the piston. If we *reduce* the temperature of the gas, it contracts instead of expanding; the work done on the gas in that case is positive. We assume that the process described by Eq. 23-14 is carried out slowly, so that the gas can be considered to be in thermal equilibrium at all intermediate stages. Otherwise, the pressure would not be clearly defined during the process, and the integral in Eq. 23-14 could not easily be evaluated.

We can write Eq. 23-14 in a more general form that turns out to be very useful. If the piston moves through a distance dx, then the volume of the gas changes by an amount $dV = A\,dx$. Thus the work done on the gas is

$$W = -\int p \, dV. \qquad (23\text{-}15)$$

The integral is carried out between the initial volume V_i and the final volume V_f.

Equation 23-15 is the most general result for the work done on a gas. It makes no reference to the outside agent that does the work; it states simply that the work done on the gas can be calculated from the pressure and the change in volume of the gas itself. Note that the algebraic sign of the work is implicitly contained in Eq. 23-15; if the gas expands, dV is

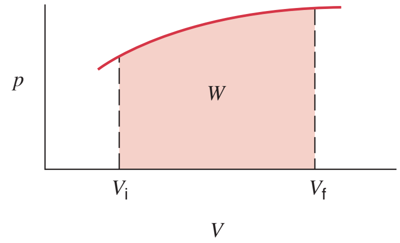


FIGURE 23-13. The magnitude of the work W done on a gas by a process of arbitrarily varying pressure is equal to the area under the pressure curve on a pV diagram between the initial volume V_i and the final volume V_f.

positive and W is negative, p being a scalar quantity having only positive values. Conversely, if the gas is compressed, dV is negative and the work done on the gas is positive.

Equation 23-15 is analogous to the general result for the work done on a system by a variable force F. You will recall from Fig. 11-12 that if we plot F against x, the work done by F is simply the area under the curve between x_i and x_f. Figure 23-13 shows the similar situation for the work done on the gas. A graph in the form of Fig. 23-13 is called a *pV diagram,* with p plotted on the vertical axis (like F) and V plotted on the horizontal axis (like x). *The magnitude of the work done on the gas is equal to the area under the curve representing the process on a pV diagram.* The sign of W is determined according to whether $V_f > V_i$ (in which case W is negative, as in Fig. 23-13), or $V_f < V_i$ (in which case W is positive). Once again, the work done *on* the gas is negative if the process increases the volume of the gas and positive if the process reduces the volume of the gas.

The pressure force is clearly nonconservative, as Fig. 23-14 demonstrates. Let us suppose we wish to take our ideal gas from the initial conditions V_i and p_i (point A) to the final conditions V_f and p_f (point D). There are many different paths we can take between A and D, of which two are shown in Fig. 23-14. Along path 1 (ABD), we first increase the pressure from p_i to p_f at constant volume. (We might accomplish this by turning up the control knob on the thermal reservoir, increasing the temperature of the gas, while we simultaneously add just the right amount of additional weight to the piston to keep it from moving.) We then follow path BD by increasing the temperature but adding no additional weight to the piston, so that the pressure remains constant at the value p_f while the volume increases from V_i to V_f. The work done in this entire procedure is the area of the rectangle $BDFE$ (the area below the line BD).

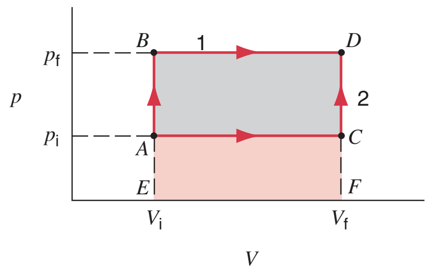


FIGURE 23-14. A gas is taken from the pressure and volume at point A to the pressure and volume at point D along two different paths, ABD and ACD. Along path 1 (ABD) the work is equal to the area of the rectangle $BDFE$, whereas along path 2 (ACD) the work is equal to the area of the rectangle $ACFE$.

We can find W_1, the work done on the gas along path 1, by considering the work done along the two segments AB and BD:

$$W_1 = W_{AB} + W_{BD}.$$

Because the volume is constant along AB, it follows from Eq. 23-15 that $W_{AB} = 0$. Along BD, the pressure is constant (at the value p_f) and comes out of the integral. The result is

$$\begin{aligned} W_1 &= W_{AB} + W_{BD} \\ &= 0 - \int p\,dV = -p_f \int_{V_i}^{V_f} dV = -p_f(V_f - V_i). \end{aligned}$$

To follow path 2 (ACD), we first increase the temperature while holding the pressure constant at p_i (that is, adding no additional weight to the piston), so that the volume increases from V_i to V_f. We then increase the pressure from p_i to p_f at the constant volume V_f by increasing the temperature and adding weight to the piston to keep it from moving. The work done in this case is the area under the line AC or the rectangle $ACFE$. We can compute this as

$$\begin{aligned} W_2 &= W_{AC} + W_{CD} \\ &= -\int p\,dV + 0 = -p_i \int_{V_i}^{V_f} dV = -p_i(V_f - V_i). \end{aligned}$$

Clearly $W_1 \neq W_2$, and the work depends on the path.

We can perform a variety of operations on the gas and evaluate the work done in each case.

Work Done at Constant Volume

The work is zero for any process in which the volume remains constant (as in segments AB and CD in Fig. 23-14):

$$W = 0 \qquad \text{(constant } V\text{)}. \tag{23-16}$$

We deduce directly from Eq. 23-15 that $W = 0$ if V is constant. Note that it is not sufficient that the process start and end with the same volume; the volume must be constant throughout the process for the work to vanish. For example, consider process $ACDB$ in Fig. 23-14. The volume starts and ends at V_i, but the work is certainly not zero. The work is zero only for vertical paths such as AB, representing a process at constant volume.

Work Done at Constant Pressure

Here we can easily apply Eq. 23-15, because the constant p comes out of the integral:

$$\begin{aligned} W &= -p \int dV \\ &= -p(V_f - V_i) \qquad \text{(constant } p\text{)}. \end{aligned} \tag{23-17}$$

Examples are the segments AC and BD in Fig. 23-14. Note that the work done on the gas is negative for both of these segments, because the volume increases in both processes.

Work Done at Constant Temperature

In the gas expands or contracts at constant temperature, the relationship between p and V, given by the ideal gas law $(pV = nRT)$, is

$$pV = \text{constant}.$$

On a pV diagram, the plot of the equation pV = constant is exactly like a plot of the equation xy = constant on an xy coordinate system: it is a hyperbola, as shown in Fig. 23-15.

A process done at constant temperature is called an *isothermal* process, and the corresponding hyperbolic curve on the pV diagram is called an *isotherm.* To find the work done on a gas during an isothermal process, we use Eq. 23-15, but we must find a way of carrying out the integral when p varies. To do this we use the ideal gas equation of state to write $p = nRT/V$, and thus

$$W = -\int_{V_i}^{V_f} p\, dV = -\int_{V_i}^{V_f} \frac{nRT}{V}\, dV = -nRT \int_{V_i}^{V_f} \frac{dV}{V},$$

where the last step can be made because we are taking T to be a constant. Carrying out the integral, we find

$$W = -nRT \ln \frac{V_f}{V_i} \qquad (\text{constant } T). \qquad (23\text{-}18)$$

Note that this is also negative whenever $V_f > V_i$ ($\ln x$ is positive for $x > 1$) and positive whenever $V_f < V_i$.

Work Done in Thermal Isolation

Let us remove the gas cylinder in Fig. 21-13 from contact with the thermal reservoir and rest it on a slab of insulating material. The gas will then be in complete thermal isolation from its surroundings; if we do work on it, its temperature will change, in contrast to its behavior when it was in contact with the thermal reservoir. A process carried out in thermal isolation is called an *adiabatic* process.

If we allow the gas to change its volume with no other constraints, we state—and we will derive it in Section 23-8—that the path it will follow is represented on a pV diagram by the parabola-like curve

$$pV^{\gamma} = \text{constant}, \qquad (23\text{-}19)$$

as shown in Fig. 23-16. The dimensionless parameter γ, called the *ratio of specific heats,* must be determined by experiment for any particular gas. Its values are typically in the range 1.1–1.8. Because γ is greater than 1, the curve pV^{γ} = constant is a bit steeper than the curve pV = constant at any point at which they intersect. As Fig. 23-16 shows, this means that the work done by the gas in expanding adiabatically from V_i to V_f will be somewhat smaller in magnitude than the work done in expanding isothermally between these same two volumes.

We can find the "constant" in Eq. 23-19 if we know γ and also the pressure and volume at any particular point on the curve. If we choose the initial point p_i, V_i in Fig. 23-16, the "constant" has the value $p_i V_i^{\gamma}$ and we can write Eq. 23-19 as

$$pV^{\gamma} = p_i V_i^{\gamma}$$

or

$$p = \frac{p_i V_i^{\gamma}}{V^{\gamma}}. \qquad (23\text{-}20)$$

We can now find the adiabatic work:

$$\begin{aligned} W &= -\int_{V_i}^{V_f} p\, dV \\ &= -\int_{V_i}^{V_f} \frac{p_i V_i^{\gamma}}{V^{\gamma}}\, dV = -p_i V_i^{\gamma} \int_{V_i}^{V_f} \frac{dV}{V^{\gamma}} \\ &= -\frac{p_i V_i^{\gamma}}{\gamma - 1}\left(V_i^{1-\gamma} - V_f^{1-\gamma}\right). \end{aligned}$$

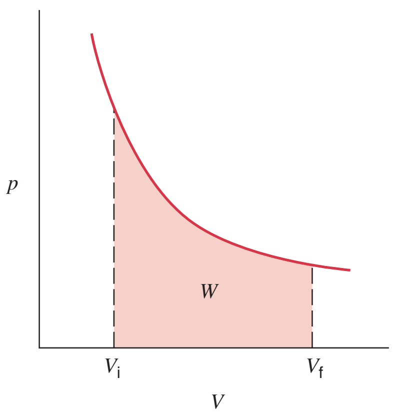


FIGURE 23-15. A process done at constant temperature (isothermal process) is represented by a hyperbola on a pV diagram. The work done in changing the volume is equal to the area under the curve between V_i and V_f.

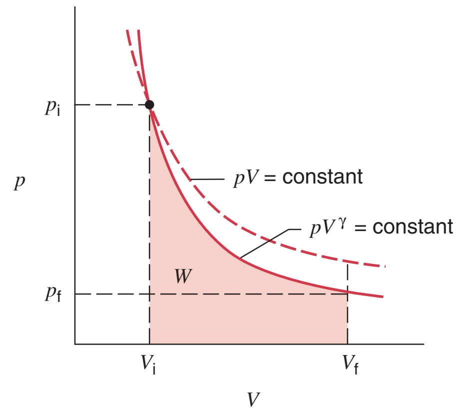


FIGURE 23-16. An adiabatic process is represented on a pV diagram by the hyperbola-like curve pV^{γ} = constant. The work done in changing the volume is equal to the area under the curve between V_i and V_f. Because $\gamma > 1$, the adiabatic curve has a steeper negative slope than the isothermal curve pV = constant.

By bringing a factor of $V_i^{\gamma-1}$ inside the parentheses, we can write the adiabatic work in the form

$$W = \frac{p_i V_i}{\gamma - 1}\left[\left(\frac{V_i}{V_f}\right)^{\gamma-1} - 1\right]. \qquad (23\text{-}21)$$

If the gas expands, then $V_i/V_f < 1$, and since a number less than 1 raised to any positive power remains less than one, the work is again shown to be negative. By further using $p_i V_i^{\gamma} = p_f V_f^{\gamma}$, we can also write the adiabatic work in equivalent form as

$$W = \frac{1}{\gamma - 1}(p_f V_f - p_i V_i) \quad \text{(adiabatic)}. \qquad (23\text{-}22)$$

SAMPLE PROBLEM 23-4. A sample of gas consisting of 0.11 mol is compressed from a volume of 4.0 m³ to 1.0 m³ while its pressure increases from 10 to 40 Pa. Compare the work done along the three different paths shown in Fig. 23-17.

Solution Path 1 consists of two processes, one at constant pressure followed by another at constant volume. The work done at constant pressure is found from Eq. 23-17,

$$W = -p(V_f - V_i) = -(10 \text{ Pa})(1.0 \text{ m}^3 - 4.0 \text{ m}^3) = 30 \text{ J}.$$

The work done at constant volume is zero (see Eq. 23-16), so the total work for path 1 is

$$W_1 = 30 \text{ J} + 0 = 30 \text{ J}.$$

Path 2 represents an isothermal process, along which T = constant. Thus $p_i V_i = p_f V_f = nRT$. The work done during the isothermal process can be found using Eq. 23-18, substituting $p_i V_i$ for nRT, which gives

$$W_2 = -p_i V_i \ln\frac{V_f}{V_i} = -(10 \text{ Pa})(4.0 \text{ m}^3)\ln\frac{1.0 \text{ m}^3}{4.0 \text{ m}^3} = 55 \text{ J}.$$

Path 3 consists of a process at constant volume, for which the work is again zero, followed by a process at constant pressure, and so the total work for path 3 is

$$W_3 = 0 - p_f(V_f - V_i) = -(40 \text{ Pa})(1.0 \text{ m}^3 - 4.0 \text{ m}^3) = 120 \text{ J}.$$

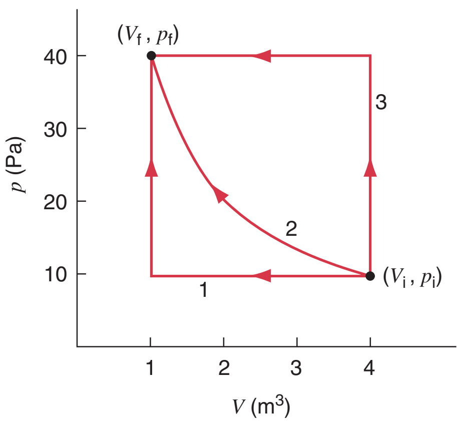


FIGURE 23-17. Sample Problem 23-4. A gas is taken from initial point i to final point f along three different paths. Path 2 is an isotherm.

Note that the work is positive for all three processes, and that the magnitudes increase according to the area under each path on the pV diagram.

SAMPLE PROBLEM 23-5. (*a*) Find the bulk modulus B for an adiabatic process involving an ideal gas. (*b*) Use the adiabatic bulk modulus to find the speed of sound in the gas as a function of temperature. Evaluate for air at room temperature (20°C).

Solution (*a*) In the differential limit, the bulk modulus (see Eq. 15-5) can be written

$$B = -V\frac{dp}{dV}.$$

For an adiabatic process, Eq. 23-19 (pV^{γ} = constant) gives, taking the derivative with respect to V,

$$\frac{d(pV^{\gamma})}{dV} = \left(\frac{dp}{dV}\right)V^{\gamma} + p(\gamma V^{\gamma-1}) = 0,$$

or

$$V\frac{dp}{dV} = -\gamma p.$$

Thus

$$B = \gamma p$$

for an adiabatic process involving an ideal gas.

(*b*) In Section 19-3, we determined that the speed of sound in a gas can be written

$$v = \sqrt{B/\rho},$$

where B is the bulk modulus and ρ is the density of the gas. Using the result of part (*a*) and the ideal gas equation of state ($pV = nRT$), we obtain

$$v = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma(nRT/V)}{\rho}} = \sqrt{\frac{\gamma nRT}{\rho V}}.$$

The quantity ρV is the total mass of the gas, which can also be written nM, where n is the number of moles and M is the molar mass. Making this substitution, we have

$$v = \sqrt{\frac{\gamma RT}{M}}.$$

Thus the speed of sound in a gas depends on the square root of the temperature. For air, the average molar mass is about 0.0290 kg/mol, and the parameter γ is about 1.4. Thus for T = 20°C = 293 K,

$$v = \sqrt{\frac{(1.4)(8.31 \text{ J/mol}\cdot\text{K})(293 \text{ K})}{0.0290 \text{ kg/mol}}} = 343 \text{ m/s}.$$

23-6 THE INTERNAL ENERGY OF AN IDEAL GAS

In Section 22-4 we showed that the average translational kinetic energy per molecule of an ideal monatomic gas is

$$K_{\text{trans}} = \tfrac{3}{2}kT. \qquad (23\text{-}23)$$

For such a gas this is the entire store of internal energy because there is no other form the internal energy can take. The molecules of an ideal monatomic gas have no potential energy; they cannot vibrate, nor is any energy associated with their rotation.

The total internal energy of n moles of an ideal monatomic gas is then the number of molecules ($= nN_A$) times the average energy per molecule:

$$E_{int} = (nN_A)(K_{trans}) = (nN_A)(\tfrac{3}{2}kT)$$

or

$$E_{int} = \tfrac{3}{2}nRT. \qquad (23\text{-}24)$$

Here (see Eq. 21-17) we have replaced $N_A k$ with its equal, the molar gas constant R.

Equation 23-24 shows that, if we change the internal energy of the gas—by doing work on it or transferring heat to it—its temperature will change, so that

$$\Delta E_{int} = \tfrac{3}{2}nR\,\Delta T. \qquad (23\text{-}25)$$

Let us now consider, not the monatomic or point molecule that has been the focus of our attention so far, but a molecule consisting of two point particles separated by a given distance. This model gives a better description of diatomic gases, such as O_2, N_2, or CO (carbon monoxide). Such a molecule can acquire kinetic energy by rotating about its center of mass, and we need to consider contributions to the internal energy of the gas from the rotational motions of its molecules as well as from their translational motions.

The rotational kinetic energy of a diatomic molecule, illustrated in Fig. 23-18, can be written

$$K_{rot} = \tfrac{1}{2}I_x\,\omega_x^2 + \tfrac{1}{2}I_y\,\omega_y^2,$$

where I is the rotational inertia of the molecule for rotation about a particular axis. For point masses, no kinetic energy is associated with rotation about the z axis because $I_z = 0$. The total kinetic energy of a diatomic molecule is the sum of its translational and rotational terms, or

$$K = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}mv_y^2 + \tfrac{1}{2}mv_z^2 + \tfrac{1}{2}I_x\,\omega_x^2 + \tfrac{1}{2}I_y\,\omega_y^2. \qquad (23\text{-}26)$$

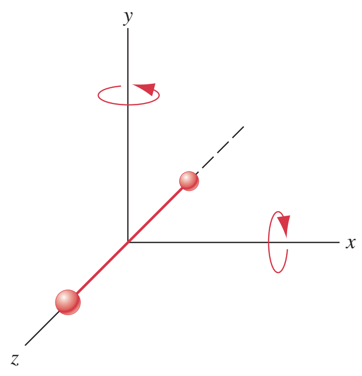


FIGURE 23-18. A diatomic molecule, consisting of two atoms considered to be point particles, is shown with its axis along the z axis of a coordinate system. In this orientation, the rotational inertia for rotations about the z axis is zero, and thus there is no term in the kinetic energy corresponding to such rotations. The rotational inertias for rotations about the x and y axes are not zero, and thus there are kinetic energy terms for such rotations.

To find the total internal energy of the gas, we must find the average energy of a single molecule and then multiply by the number of molecules.

The five terms in Eq. 23-26 represent independent ways in which a molecule can absorb energy and are called *degrees of freedom.* A monatomic gas has three degrees of freedom, since it has only translational kinetic energy ($K = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}mv_y^2 + \tfrac{1}{2}mv_z^2$.)

As Eq. 23-26 shows, a diatomic molecule has five degrees of freedom, three translational degrees and two rotational degrees. If we increase the internal energy of such a gas by an amount ΔE_{int}, it is clear (because all directions in the gas are equivalent) that the three translational degrees will absorb the same amount of energy. Similarly, the two rotational degrees will absorb the same amount of energy but there seems to be no reason why these two amounts should be the same.

However, James Clerk Maxwell derived a theorem called the *equipartition of energy theorem,* which asserts that the energy of a molecule is shared equally, on average, among *all* independent ways in which the molecule can absorb energy. Specifically,

> *When the number of molecules is large, the average energy per molecule is* $\tfrac{1}{2}kT$ *for each independent degree of freedom.*

We have already encountered an equipartition of energy situation in our studies of the one-dimensional simple harmonic oscillator. In this case energy can be stored in either kinetic or potential form and, as Fig. 17-8 suggests, on average the available energy is shared equally between these two forms.

Let us use the equipartition of energy theorem to write an expression for the internal energy of a monatomic ideal gas. The average energy per molecule is $\tfrac{3}{2}kT$ (3 degrees of freedom $\times\ \tfrac{1}{2}kT$ for each degree of freedom). The total energy for N molecules is

$$E_{int} = N(\tfrac{3}{2}kT) = \tfrac{3}{2}nRT \qquad \text{(monatomic gas)}. \quad (23\text{-}27)$$

Equation 23-27 is identical with Eq. 23-24. For a diatomic gas, with five degrees of freedom, the result is

$$E_{int} = N(\tfrac{5}{2}kT) = \tfrac{5}{2}nRT \qquad \text{(diatomic gas)}. \quad (23\text{-}28)$$

A polyatomic gas (more than two atoms per molecule) generally has three possible axes of rotation (unless the three atoms lie in a straight line, as for CO_2). The kinetic energy of a single molecule would then have a sixth term, $\tfrac{1}{2}I_z\omega_z^2$. For six degrees of freedom, the internal energy is

$$E_{int} = N(\tfrac{6}{2}kT) = 3nRT \qquad \text{(polyatomic gas)}. \quad (23\text{-}29)$$

Equations 23-27, 23-28, and 23-29 show us a fact that is inherent in the equipartition of energy theorem—namely, that no matter what the nature of its molecules, *The internal energy of an ideal gas depends only on its temperature.*

It does not depend on its pressure or its volume.

So far we have considered only the contributions of the translational or rotational kinetic energy to the internal energy of the gas. Other kinds of energy may also contribute. For example, a diatomic molecule that is free to vibrate (imagine two point atoms connected by a spring) has two additional contributions to the energy: the potential energy of the spring and the kinetic energy of the oscillating atoms. Thus a diatomic molecule free to translate, rotate, and vibrate would have $7(= 3 + 2 + 2)$ degrees of freedom. For polyatomic molecules, the number of vibrational terms can be greater than two. The vibrational modes in the internal energy are usually apparent only at high gas temperatures, where the more violent collisions can cause the molecule to vibrate.

Molar Heat Capacities of Solids

We can also apply the equipartition of energy theorem to the molar heat capacities of solids, a topic that we discussed in Section 23-4. As Fig. 21-9 suggests, an atom in a solid is fixed in a lattice. The atom can oscillate back and forth about its equilibrium position in three independent directions, thus displaying three degrees of freedom associated with its kinetic energy. The atom also has potential energy, associated with the forces between it and its neighboring atoms, again in three independent directions. This gives rise to three more degrees of freedom for a total of six. The average energy per atom is then $6 \times \frac{1}{2}kT = 3kT$. For a sample containing N atoms, the total internal energy is then

$$E_{\text{int}} = N(3kT) = 3nN_{\text{A}}kT = 3nRT,$$

in which n is the number of moles.

Suppose that energy Q is added to the solid sample as heat, raising its temperature by ΔT. Because no work is done in this process ($W = 0$), the first law of thermodynamics ($Q + W = \Delta E_{\text{int}}$) yields

$$Q = \Delta E_{\text{int}} = 3nR\,\Delta T.$$

The molar heat capacity is then

$$C = \frac{Q}{n\,\Delta T} = \frac{3nR\,\Delta T}{n\,\Delta T} = 3R$$

$$= (3)(8.31\text{ J/mol}\cdot\text{K}) \approx 25\text{ J/mol}\cdot\text{K}.$$

As Fig. 23-11 shows, this is simply the experimentally observed high-temperature limit for the molar heat capacities of solids. Note that, although the (classical) equipartition of energy theorem gives the correct value for the molar heat capacity in the limit of sufficiently high temperatures, it fails at lower temperatures. In this region only a treatment based on quantum physics proves to agree with experiment.

23-7 HEAT CAPACITIES OF AN IDEAL GAS

We have used the equipartition of energy theorem to calculate the molar heat capacity of a solid. Let us now use it to calculate the molar heat capacities of an ideal gas. The measured heat capacity of a substance depends on the manner in which the heat is added to it. In the case of a gas, for example, is the volume held constant during the process? Is the pressure held constant? We explore both possibilities.

Molar Heat Capacity at Constant Volume

Let us introduce n moles of a gas into a cylinder fitted with a piston. We fix the position of the piston so that there can be no volume change and thus no work done, and then we add an amount of energy Q to the gas as heat. From the first law of thermodynamics (Eq. 23-6) we have, because $W = 0$,

$$Q = \Delta E_{\text{int}}. \qquad (23\text{-}30)$$

We let C_{V} represent the molar heat capacity at constant volume, so that

$$C_{\text{V}} = \frac{Q}{n\,\Delta T} = \frac{\Delta E_{\text{int}}}{n\,\Delta T}. \qquad (23\text{-}31)$$

From Eq. 23-27, for a monatomic ideal gas $\Delta E_{\text{int}} = \frac{3}{2}nR\,\Delta T$, and so

$$C_{\text{V}} = \tfrac{3}{2}R = 12.5\text{ J/mol}\cdot\text{K} \quad \text{(monatomic gas).} \qquad (23\text{-}32)$$

Repeating this derivation using Eqs. 23-28 and 23-29 for diatomic and polyatomic gases, we find

$$C_{\text{V}} = \tfrac{5}{2}R = 20.8\text{ J/mol}\cdot\text{K} \quad \text{(diatomic gas),} \qquad (23\text{-}33)$$

$$C_{\text{V}} = 3R = 24.9\text{ J/mol}\cdot\text{K} \quad \text{(polyatomic gas).} \qquad (23\text{-}34)$$

Molar Heat Capacity at Constant Pressure

Figure 23-19 shows two ideal gas isotherms differing in temperature by ΔT. Path AB is the constant-volume process considered previously. Path AC is a constant-pressure

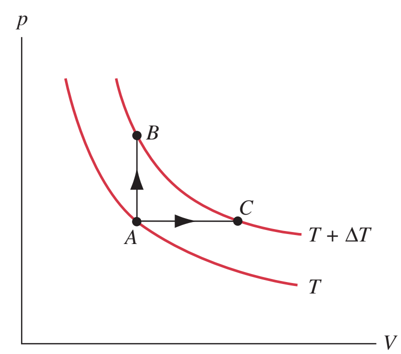


FIGURE 23-19. Two ideal-gas isotherms differing in temperature by ΔT are connected by the constant-volume process AB and the constant-pressure process AC.

process that connects the same two isotherms. In Section 23-6 we established that *the internal energy of an ideal gas depends only on the temperature.* For all paths connecting the two isotherms of Fig. 23-19, the change in internal energy has the same value, because all paths correspond to the same change in temperature. In particular, the change in internal energy is the same for paths AB and AC.

$$\Delta E_{\text{int},AB} = \Delta E_{\text{int},AC}. \tag{23-35}$$

There are two contributions to the change in internal energy along path AC—the heat Q transferred to the gas and the work W done on the gas:

$$\Delta E_{\text{int},AC} = Q + W. \tag{23-36}$$

Note the sign conventions that are implicit in Eq. 23-36. Heat transferred *from* the environment is considered to be positive and tends to increase the internal energy. If the volume decreases, the work done on the gas by the environment is positive, which tends to increase the internal energy. If the volume increases ($W < 0$), we regard the gas as doing work on the environment, which tends to decrease the supply of internal energy of the gas.

The heat transferred in a constant-pressure process can be written

$$Q = nC_p\,\Delta T, \tag{23-37}$$

where C_p is the *molar heat capacity at constant pressure.* Equation 23-15 gives the work along path AC as $W = -p\,\Delta V$, which can be written for this constant-pressure process using the ideal gas law as

$$W = -p\,\Delta V = -nR\,\Delta T. \tag{23-38}$$

Using Eq. 23-31 to obtain the change in internal energy along path AB, we can substitute into Eq. 23-36 to find

$$nC_V\,\Delta T = nC_p\,\Delta T - nR\,\Delta T$$

or

$$C_p = C_V + R. \tag{23-39}$$

From Eqs. 23-32 to 23-34 we then find the molar heat capacities at constant pressure:

$$C_p = \tfrac{5}{2}R = 20.8\ \text{J/mol}\cdot\text{K} \qquad \text{(monatomic gas)}, \tag{23-40}$$

$$C_p = \tfrac{7}{2}R = 29.1\ \text{J/mol}\cdot\text{K} \qquad \text{(diatomic gas)}, \tag{23-41}$$

$$C_p = 4R = 33.3\ \text{J/mol}\cdot\text{K} \qquad \text{(polyatomic gas)}. \tag{23-42}$$

Another parameter of interest, which can be directly measured independently of the values of C_p and C_V, is the *ratio of molar heat capacities* γ, defined as

$$\gamma = \frac{C_p}{C_V}. \tag{23-43}$$

Because the specific heat capacity is related to the molar heat capacity by $c = C/M$, where M is the molar mass of the substance, we can also express γ as c_p/c_V. For this reason γ is often called the *ratio of specific heats* or *specific heat ratio.* We used γ previously in the expression for the speed of sound in a gas (Sample Problem 23-5) and in the relationship between pressure and volume in an adiabatic process (Eq. 23-19).

TABLE 23-4 Molar Heat Capacities of Gases

Gas	C_p (J/mol · K)	C_V (J/mol · K)	$C_p - C_V$ (J/mol · K)	γ
Monatomic				
Ideal	20.8	12.5	8.3	1.67
He	20.8	12.5	8.3	1.66
Ar	20.8	12.5	8.3	1.67
Diatomic				
Ideal	29.1	20.8	8.3	1.40
H_2	28.8	20.4	8.4	1.41
N_2	29.1	20.8	8.3	1.40
O_2	29.4	21.1	8.3	1.40
Polyatomic				
Ideal	33.3	24.9	8.3	1.33
CO_2	37.0	28.5	8.5	1.30
NH_3	36.8	27.8	9.0	1.31

Using Eqs. 23-40 to 23-42 for C_p and Eqs. 23-32 to 23-34 for C_V, we obtain

$$\gamma = \tfrac{5}{3} = 1.67 \qquad \text{(monatomic gas)}, \tag{23-44}$$

$$\gamma = \tfrac{7}{5} = 1.40 \qquad \text{(diatomic gas)}, \tag{23-45}$$

$$\gamma = \tfrac{4}{3} = 1.33 \qquad \text{(polyatomic gas)}. \tag{23-46}$$

Table 23-4 shows a comparison of observed values with the predictions of the ideal gas model. The agreement is excellent.

SAMPLE PROBLEM 23-6. A family enters a winter vacation cabin that has been unheated for such a long time that the interior temperature is the same as the outside temperature (0°C). The cabin consists of a single room of floor area 6 m by 4 m and height 3 m. The room contains one 2-kW electric heater. Assuming that the room is perfectly airtight and that all the heat from the electric heater is absorbed by the air, none escaping through the walls or being absorbed by the furnishings, how long after the heater is turned on will the air temperature reach the comfort level of 21°C (= 70°F)?

Solution Let us assume that the air in the room (which is mostly nitrogen and oxygen) behaves like an ideal diatomic gas, so that (according to Table 23-4) $C_V = 20.8$ J/mol · K. The volume of the room is

$$V = (6\ \text{m})(4\ \text{m})(3\ \text{m}) = 72\ \text{m}^3 = 72{,}000\ \text{L}.$$

Since 1 mole of an ideal gas occupies 22.4 L at 0°C and 1 atm, the number of moles is

$$n = (72{,}000\ \text{L})/(22.4\ \text{L/mol}) = 3.2 \times 10^3\ \text{mol}.$$

If the room is airtight (see the discussion below), we can regard the absorption of heat to take place at constant volume, for which

$$Q = nC_V\,\Delta T = (3.2 \times 10^3\ \text{mol})(20.8\ \text{J/mol}\cdot\text{K})(21\ \text{K})$$
$$= 1.4 \times 10^6\ \text{J}.$$

The heater delivers a power P of 2 kW and can provide this energy in a time of

$$t = \frac{Q}{P} = \frac{1.4 \times 10^6 \text{ J}}{2 \times 10^3 \text{ W}} = 700 \text{ s},$$

or about 12 min.

This problem contained some very unphysical assumptions about the absorption of heat in this room. Try to estimate the heat capacity of some pieces of furniture to see whether neglecting their effect on the heat absorption (and thus on the time to bring the room to comfort level) was reasonable. The heat loss through the walls of the room, which we considered in Section 23-2, also will have a considerable effect on this problem.

Is the assumption about the room being airtight reasonable? If the air in the cabin were originally at a pressure of 1 atm when the temperature was 0°C, what will be the interior pressure at 21°C? What will be the resulting outward force on the roof and walls? A more reasonable assumption might be that the room is not quite airtight, but that as the temperature rises some air will escape, thereby keeping the pressure constant. See Problem 16 for a calculation based on this assumption.

SAMPLE PROBLEM 23-7. Consider once again the situation of Sample Problem 23-4, in which 0.11 mole of an ideal gas begins at the initial point with volume $V_i = 4.0 \text{ m}^3$ and pressure $p_i = 10$ Pa. Let the cylinder be removed from the thermal reservoir, and let us compress the gas adiabatically until its volume is $V_f = 1.0 \text{ m}^3$. Find the change in internal energy of the gas, assuming it to be helium (a monatomic gas with $\gamma = 1.66$).

Solution To find the change in internal energy, we can use Eq. 23-27 if we know the change in temperature. We can find the initial temperature using the ideal gas law (since p_i and V_i are known), and we can find the final temperature if we know the pressure and volume of the final point. The final pressure can be found using the adiabatic relationship of Eq. 23-19:

$$p_f = \frac{p_i V_i^\gamma}{V_f^\gamma} = \frac{(10 \text{ Pa})(4.0 \text{ m}^3)^{1.66}}{(1.0 \text{ m}^3)^{1.66}} = 100 \text{ Pa}.$$

On the pV diagram of Fig. 23-17, the final point reached in the adiabatic process lies vertically far above the final point reached in the isothermal process (40 Pa). This is consistent with the adiabatic curves being steeper than the isothermal curves, as shown in Fig. 23-16.

We can now proceed to find the initial and final temperatures and then the change in internal energy:

$$T_i = \frac{p_i V_i}{nR} = \frac{(10 \text{ Pa})(4.0 \text{ m}^3)}{(0.11 \text{ mol})(8.31 \text{ J/mol}\cdot\text{K})} = 44 \text{ K}.$$

$$T_f = \frac{p_f V_f}{nR} = \frac{(100 \text{ Pa})(1.0 \text{ m}^3)}{(0.11 \text{ mol})(8.31 \text{ J/mol}\cdot\text{K})} = 109 \text{ K}.$$

$$\Delta E_{int} = \tfrac{3}{2} nR\Delta T$$
$$= \tfrac{3}{2}(0.11 \text{ mol})(8.31 \text{ J/mol}\cdot\text{K})(109 \text{ K} - 44 \text{ K}) = 89 \text{ J}.$$

The change in internal energy is positive. This is consistent with the first law of thermodynamics ($Q + W = \Delta E_{int}$) because $Q = 0$ for this adiabatic process and the work done in compressing the gas is positive.

23-8 APPLICATIONS OF THE FIRST LAW OF THERMODYNAMICS

Now that we have examined many of the properties of the ideal gas, including its internal energy (Section 23-6) and its heat capacities (Section 23-7), we are ready to study the various processes that a system consisting of an ideal gas can undergo.

Adiabatic Processes

In an adiabatic process the system is well insulated so that no heat enters or leaves, in which case $Q = 0$. The first law becomes, in this case,

$$\Delta E_{int} = W \qquad \text{(adiabatic process).} \qquad (23\text{-}47)$$

Let us derive the relationship between p and V for an adiabatic process carried out on an ideal gas, which we used in Section 23-5. We assume the process to be carried out slowly, so that the pressure is always well defined. For an ideal gas, we can write Eq. 23-31 as

$$dE_{int} = nC_V\, dT.$$

Thus

$$p\, dV = -dW = -dE_{int} = -nC_V\, dT. \qquad (23\text{-}48)$$

The equation of state of the gas can be written in differential form as

$$d(pV) = d(nRT)$$

$$p\, dV + V\, dp = nR\, dT. \qquad (23\text{-}49)$$

However, $p\, dV$ is simply $-dW$, which is equal to $-dE_{int}$ (since Eq. 23-47 can be written in differential form as $dE_{int} = dW$). Solving Eq. 23-49 for $V\, dp$ and substituting Eq. 23-48, we have

$$V\, dp = nC_V\, dT + nR\, dT = nC_p\, dT, \qquad (23\text{-}50)$$

where the last result has been obtained using Eq. 23-39, $C_p = C_V + R$. We now take the ratio between Eqs. 23-50 and 23-48, which gives

$$\frac{V\, dp}{p\, dV} = \frac{nC_p\, dT}{-nC_V\, dT} = -\frac{C_p}{C_V} = -\gamma,$$

using Eq. 23-43 for the ratio of molar heat capacities γ. Rewriting, we find

$$\frac{dp}{p} = -\gamma \frac{dV}{V},$$

which we can integrate between initial state i and final state f

$$\int_{p_i}^{p_f} \frac{dp}{p} = -\gamma \int_{V_i}^{V_f} \frac{dV}{V}$$

$$\ln \frac{p_f}{p_i} = -\gamma \ln \frac{V_f}{V_i},$$

which can be written

$$p_i V_i^\gamma = p_f V_f^\gamma. \qquad (23\text{-}51)$$

Since i and f are arbitrary points, we can write this equation as

$$pV^{\gamma} = \text{constant}. \tag{23-52}$$

Equations 23-51 and 23-52 give the relationship between the pressure and volume of an ideal gas that undergoes an adiabatic process. Given the values of the pressure and volume at the initial point, the adiabatic process will proceed through final points whose pressure and volume can be calculated from Eq. 23-51. Equivalently, Eq. 23-52 defines a family of curves on a pV diagram. Every adiabatic process can be represented by a segment of one of these curves (Fig. 23-20).

We can rewrite these results in terms of temperature, using the ideal gas equation of state:

$$(pV)V^{\gamma-1} = \text{constant}$$

$$TV^{\gamma-1} = \text{constant}. \tag{23-53}$$

The constant in Eq. 23-53 is not the same as that in Eq. 23-52. Equivalently, we can write Eq. 23-53 as

$$T_i V_i^{\gamma-1} = T_f V_f^{\gamma-1}$$

$$T_f = T_i \left(\frac{V_i}{V_f}\right)^{\gamma-1} \tag{23-54}$$

Suppose we compress a gas in an adiabatic process. Then $V_i > V_f$, and Eq. 23-54 requires that $T_f > T_i$. The temperature of the gas rises as it is compressed, as we frequently observe from the warming of a bicycle pump. Conversely, the temperature falls when a gas expands, which is often used as a means to achieve low temperatures in the laboratory (see Fig. 23-20).

Sound waves in air can be represented in terms of adiabatic processes. At audio frequencies, air is a poor conductor of heat. There is an increase in temperature in the compression zones of a sound wave, but due to the poor conduction there is no appreciable heat transfer to the neighboring cooler rarefactions; the process is thus adiabatic. The compressions and expansions of steam in a steam engine, or of the hot gases in the cylinders of an internal combustion engine, are also essentially adiabatic, because there is insufficient time for heat to be transferred.

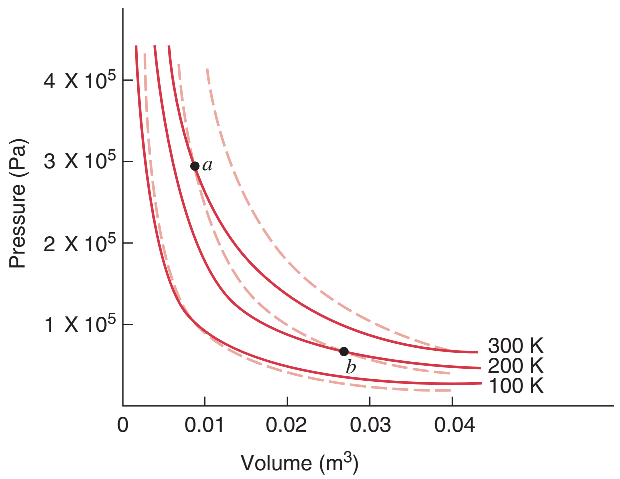


FIGURE 23-20. Isothermal processes (solid lines) and adiabatic processes (dashed lines) carried out on 1 mole of a diatomic ideal gas. Note that an adiabatic *increase* in volume (for example, the segment *ab*) is always accompanied by a *decrease* in temperature.

Isothermal Processes

In an isothermal process, the temperature remains constant. If the system is an ideal gas, then the internal energy must therefore also remain constant. With $\Delta E_{int} = 0$, the first law gives

$$Q + W = 0 \qquad \text{(isothermal process; ideal gas)}. \tag{23-55}$$

If an amount of (positive) work W is done on the gas, an equivalent amount of heat $Q = -W$ is released by the gas to the environment. None of the work done on the gas remains with the gas as stored internal energy.

Figure 23-20 compares isothermal and adiabatic processes for 1 mole of a monatomic ideal gas.

Constant-Volume Processes

If the volume of a gas remains constant, it can do no work. Thus $W = 0$, and the first law gives

$$\Delta E_{int} = Q \qquad \text{(constant-volume process)}. \tag{23-56}$$

In this case all the heat that enters the gas ($Q > 0$) is stored as internal energy ($\Delta E_{int} > 0$).

Cyclical Processes

In a cyclical process we carry out a sequence of operations that eventually restores the system to its initial state, as, for example, the three-step process illustrated in Fig. 23-21. Because the process starts and finishes at the point A, the internal energy change for the cycle is zero. Thus, according to the first law,

$$Q + W = 0 \qquad \text{(cyclical process)}, \tag{23-57}$$

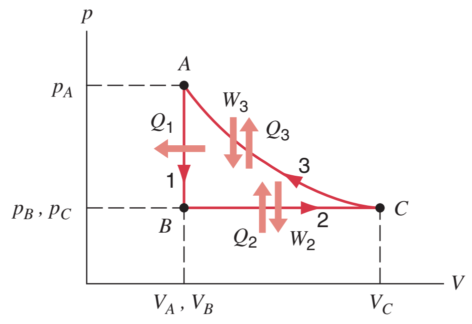


FIGURE 23-21. A gas undergoes a cyclical process starting at point A and consisting of (1) a constant-volume process AB, (2) a constant-pressure process BC, and (3) an isothermal process CA.

TABLE 23-5 Applications of the First Law

Process	*Restriction*	*First Law*	*Other Results*
All	None	$\Delta E_{int} = Q + W$	$\underline{\Delta E_{int} = nC_V\,\Delta T}$, $W = -\int p\,dV$
Adiabatic	$Q = 0$	$\Delta E_{int} = W$	$\underline{W = (p_f V_f - p_i V_i)/(\gamma - 1)}$
Constant volume	$W = 0$	$\Delta E_{int} = Q$	$Q = nC_V\,\Delta T$
Constant pressure	$\Delta p = 0$	$\Delta E_{int} = Q + W$	$W = -p\Delta V$, $Q = nC_p\,\Delta T$
Isothermal	$\underline{\Delta E_{int} = 0}$	$\underline{Q = -W}$	$\underline{W = -nRT\ln(V_f/V_i)}$
Cycle	$\Delta E_{int} = 0$	$Q = -W$	
Free expansion	$Q = W = 0$	$\Delta E_{int} = 0$	$\underline{\Delta T = 0}$

Items underlined apply only to ideal gases; all other items apply in general.

where Q and W represent the totals for the cycle. In Fig. 23-21, the total work is positive, because there is more positive area under the curve representing step 3 than there is negative area under the line representing step 2. Thus $W > 0$ and it follows from Eq. 23-57 that $Q < 0$. In fact, for any cycle that is done in a counterclockwise direction, we must have $W > 0$ (and thus $Q < 0$), whereas cycles performed in the clockwise direction have $W < 0$ and $Q > 0$.

Free Expansion

Figure 23-22 represents the process known as *free expansion.* The gas is initially in one side of the container, and when the stopcock is opened, the gas expands into the previously evacuated half. No weights can be raised in this process, so no work is done. The container is insulated, so the process is adiabatic. Hence, with $W = 0$ and $Q = 0$, the first law gives

$$\Delta E_{int} = 0 \qquad \text{(free expansion).} \qquad (23\text{-}58)$$

Thus the internal energy of an ideal gas undergoing a free expansion remains constant, and because the internal energy of an ideal gas depends only on the temperature, its temperature must similarly remain constant.

The free expansion is a good example of a *nonequilibrium* process. If a gas has a well-defined pressure and volume (and therefore temperature), we can show the state of the gas as a point on a pV diagram. The assignment of a temperature to the gas means that it must be in thermal equilibrium; each point on a pV diagram therefore represents a system in equilibrium. In the case of a free expansion, the initial state (all gas on one side) is an equilibrium state, as is the final state; but at intermediate times, as the gas rushes from one side to the other, the temperature and the pressure do not have unique values, and we cannot plot this process on a pV diagram. Only the initial and final points appear on the graph. Nevertheless, we can still use the first law to analyze this process, because the change in internal energy depends only on the initial and final points.

Table 23-5 summarizes the processes we have considered and their energy transfers.

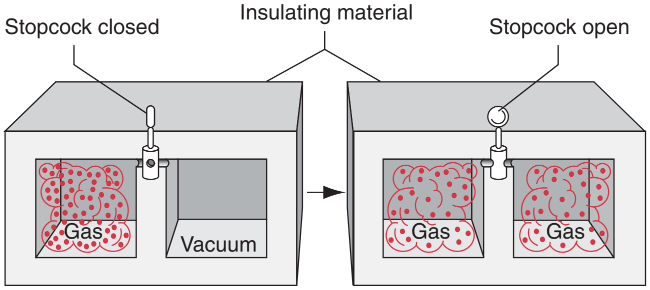


FIGURE 23-22. Free expansion. Opening the stopcock allows gas to flow from one side of the insulated container to the other. No work is done, and no heat is transferred to the environment.

SAMPLE PROBLEM 23-8. Let 1.00 kg of liquid water be converted to steam by boiling at standard atmospheric pressure; see Fig. 23-23. The volume changes from an initial value of 1.00×10^{-3} m^3 as a liquid to 1.671 m^3 as steam. For this process, find (*a*) the work done on the system, (*b*) the heat added to the system, and (*c*) the change in the internal energy of the system.

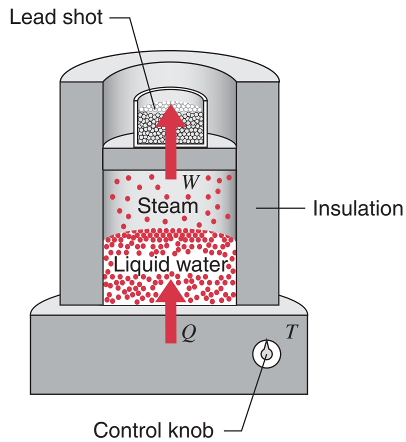


FIGURE 23-23. Sample Problem 23-8. Water is boiling at constant pressure. Heat flows from the reservoir until the water has changed completely into steam. Work is done by the expanding gas as it lifts the piston.

Solution (*a*) The work done on the gas during this constant-pressure process is given by Eq. 23-17:

$$\begin{aligned} W &= -p(V_f - V_i) \\ &= -(1.01 \times 10^5 \text{ Pa})(1.671 \text{ m}^3 - 1.00 \times 10^{-3} \text{ m}^3) \\ &= -1.69 \times 10^5 \text{ J} = -169 \text{ kJ}. \end{aligned}$$

The work done on the system is negative; equivalently, positive work is done *by* the system on its environment in lifting the weighted piston of Fig. 23-23.
(*b*) From Eq. 23-13 we have

$$Q = Lm = (2256 \text{ kJ/kg})(1.00 \text{ kg}) = 2260 \text{ kJ}.$$

This quantity is positive, as is appropriate for a process in which heat is transferred to the system.
(*c*) We find the change in internal energy from the first law:

$$\Delta E_{\text{int}} = Q + W = 2260 \text{ kJ} + (-169 \text{ kJ}) = 2090 \text{ kJ}.$$

This quantity is positive, indicating that the internal energy of the system has increased during the boiling process. This energy represents the internal work done in overcoming the strong attraction that the H_2O molecules have for each other in the liquid state.

We see that, when water boils, about 7.5% (169 kJ/2260 kJ = 0.075) of the added heat goes into external work in pushing back the atmosphere. The rest goes into internal energy that is added to the system.

SAMPLE PROBLEM 23-9. The cycle shown in Fig. 23-21 consists of three processes, starting at point A: a reduction in pressure at constant volume from point A to point B; an increase in volume at constant pressure from point B to point C; an isothermal compression (decrease in volume) from point C back to point A. Let the cycle be carried out on 0.75 mol of a diatomic ideal gas, with $p_A = 3.2 \times 10^3$ Pa, $V_A = 0.21$ m^3, and $p_B = 1.2 \times 10^3$ Pa. For each of the three processes and for the cycle, find Q, W, and ΔE_{int}.

Solution The first step is to find the values of p, V, and T at each point. At point A, we are given p_A and V_A, and we can solve for T_A from the ideal gas law:

$$T_A = \frac{p_A V_A}{nR} = \frac{(3.2 \times 10^3 \text{ Pa})(0.21 \text{ m}^3)}{(0.75 \text{ mol})(8.31 \text{ J/mol}\cdot\text{K})} = 108 \text{ K}.$$

At point B, we are given p_B and V_B $(= V_A)$, and we can similarly find T_B:

$$T_B = \frac{p_B V_B}{nR} = \frac{(1.2 \times 10^3 \text{ Pa})(0.21 \text{ m}^3)}{(0.75 \text{ mol})(8.31 \text{ J/mol}\cdot\text{K})} = 40 \text{ K}.$$

At point C, we know p_C $(= p_B)$ and T_C $(= T_A$, because process CA is an isotherm). We can then find V_C:

$$V_C = \frac{nRT_C}{p_C} = \frac{(0.75 \text{ mol})(8.31 \text{ J/mol}\cdot\text{K})(108 \text{ K})}{1.2 \times 10^3 \text{ Pa}} = 0.56 \text{ m}^3.$$

With this information, we can now calculate the heat transfer, work done, and change in internal energy for each process. For process 1 (AB), we have

$$\begin{aligned} Q_1 &= nC_V(T_B - T_A) \\ &= (0.75 \text{ mol})(20.8 \text{ J/mol}\cdot\text{K})(40 \text{ K} - 108 \text{ K}) = -1060 \text{ J}, \\ W_1 &= 0 \qquad \text{(constant-volume process)}, \\ \Delta E_{\text{int},1} &= Q_1 + W_1 = -1060 \text{ J} + 0 = -1060 \text{ J}. \end{aligned}$$

The system transfers energy to the environment as heat during process 1, and its temperature falls, corresponding to a negative change in internal energy.

For the constant-pressure process 2 (BC), we obtain

$$\begin{aligned} Q_2 &= nC_p(T_C - T_B) \\ &= (0.75 \text{ mol})(29.1 \text{ J/mol}\cdot\text{K})(108 \text{ K} - 40 \text{ K}) = 1480 \text{ J}, \\ W_2 &= -p(V_C - V_B) \\ &= -(1.2 \times 10^3 \text{ Pa})(0.56 \text{ m}^3 - 0.21 \text{ m}^3) = -420 \text{ J}, \\ \Delta E_{\text{int},2} &= Q_2 + W_2 = 1480 \text{ J} + (-420 \text{ J}) = 1060 \text{ J}. \end{aligned}$$

Energy is transferred to the gas as heat during process 2, and in expanding the gas does work on its environment (the environment does negative work on the gas.)

Along the isotherm (CA), the work is given by Eq. 23-18;

$$\begin{aligned} W_3 &= -nRT_C \ln\frac{V_A}{V_C} \\ &= -(0.75 \text{ mol})(8.31 \text{ J/mol}\cdot\text{K})(108 \text{ K}) \ln\frac{0.21 \text{ m}^3}{0.56 \text{ m}^3} \\ &= 660 \text{ J}, \\ \Delta E_{\text{int},3} &= 0 \text{ (isothermal process)}, \\ Q_3 &= \Delta E_{\text{int},3} - W_3 = 0 - 660 \text{ J} = -660 \text{ J}. \end{aligned}$$

For the cycle, we have

$$\begin{aligned} Q &= Q_1 + Q_2 + Q_3 = -1060 \text{ J} + 1480 \text{ J} + (-660 \text{ J}) \\ &= -240 \text{ J}, \\ W &= W_1 + W_2 + W_3 = 0 + (-420 \text{ J}) + 660 \text{ J} = 240 \text{ J}, \\ \Delta E_{\text{int}} &= \Delta E_{\text{int},1} + \Delta E_{\text{int},2} + \Delta E_{\text{int},3} = -1060 \text{ J} + 1060 \text{ J} + 0 = 0. \end{aligned}$$

Note that, as expected for the cycle, $\Delta E_{\text{int}} = 0$ and $Q = -W$. The total work for the cycle is positive, as we expect for a cycle that is done in the counterclockwise direction.

In solving problems of this type, we can use expressions that give directly the heat transfer in adiabatic ($Q = 0$), constant-pressure, and constant-volume processes. For other processes, such as the isothermal step in this problem, we can find Q only by first finding ΔE_{int} and W and then using the first law.

MULTIPLE CHOICE

23-1 Heat: Energy in Transit

23-2 The Transfer of Heat

1. Two identical long, thin, solid cylinders are used to conduct heat from a reservoir at temperature T_{hot} to a reservoir at temperature T_{cold}. Originally the cylinders are connected in series as shown in Fig. 23-24*a*, and the rate of heat transfer is H_0. If the cylinders are connected in parallel instead as shown in Fig. 23-24*b*, then what would be the rate of heat transfer?

(A) $16H_0$ (B) $4H_0$ (C) $2H_0$ (D) $H_0/2$

(E) The answer depends on the thermal conductivity, k, of the cylinders.

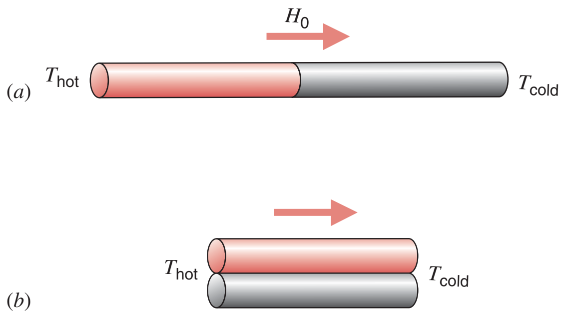


FIGURE 23-24. Multiple-choice question 1.

2. Two long, thin, solid cylinders are identical in size, but they are made of different substances with two different thermal conductivities. The two cylinders are connected in series between a reservoir at temperature T_{hot} and a reservoir at temperature T_{cold}. The temperature at the boundary between the two cylinders is T_b. One can conclude that

(A) T_b is exactly half way between T_{hot} and T_{cold}.
(B) T_b is closer to T_{hot} than it is to T_{cold}.
(C) T_b is closer to T_{cold} than it is to T_{hot}.
(D) T_b is closer to the temperature of the reservoir that is in contact with the cylinder with the lower thermal conductivity.
(E) T_b is closer to the temperature of the reservoir that is in contact with the cylinder with the higher thermal conductivity.

3. A spherical constant temperature heat source of radius r_1 is at the center of a uniform solid sphere of radius r_2. The rate at which heat is transferred through the surface of the sphere is proportional to

(A) $r_2^2 - r_1^2$. (B) $r_2 - r_1$.
(C) $\ln r_1 - \ln r_2$. (D) $1/r_2 - 1/r_1$.
(E) $(1/r_2 - 1/r_1)^{-1}$.

23-3 The First Law of Thermodynamics

4. Which of the following processes must violate the first law of thermodynamics? (There may be more than one answer!)

(A) $W > 0$, $Q < 0$, and $\Delta E_{int} = 0$
(B) $W > 0$, $Q < 0$, and $\Delta E_{int} > 0$
(C) $W > 0$, $Q < 0$, and $\Delta E_{int} < 0$
(D) $W < 0$, $Q > 0$, and $\Delta E_{int} < 0$
(E) $W > 0$, $Q > 0$, and $\Delta E_{int} < 0$

23-4 Heat Capacity and Specific Heat

5. A 100-g cube of aluminum originally at 120°C is placed into an insulated container of water originally at 18°C. After some time the system reaches equilibrium, and the final temperature of the water is 22°C. What is the final temperature of the aluminum cube?

(A) It is greater than 22°C.
(B) It is equal to 22°C.
(C) It is less than 22°C.
(D) It could be more or less than 22°C, depending on the mass of water present.

6. Block A is a 50-g aluminum block originally at 90°C. Block B is a 100-g aluminum block originally at 45°C. The blocks are placed in two separate 1.0 liter containers of water that were originally at 20°C. When the systems reach thermal equilibrium, which aluminum block will have the higher final temperature?

(A) Block A (B) Block B
(C) The blocks will have the same final temperature.
(D) The answer depends on the specific heat of water.

7. A 1-kg block of ice at 0°C is placed into a perfectly insulated, sealed container that has 2 kg of water also at 0°C. The water and ice completely fill the container, but the container is flexible. After some time one can expect that

(A) the water will freeze so that the mass of the ice will increase.
(B) the ice will melt so that the mass of the ice will decrease.
(C) both the amount of water and the amount of ice will remain constant.
(D) both the amount of water and the amount of ice will decrease.

23-5 Work Done on or by an Ideal Gas

8. In which of the paths between initial state i and final state f in Fig. 23-25 is the work done on the gas the greatest?

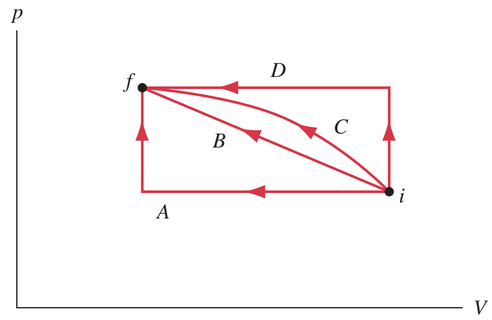


FIGURE 23-25. Multiple-choice question 8.

9. Which of the following is *not* a necessary condition for a process involving an ideal gas to do work? (There may be more than one correct answer!)

(A) $\Delta T \neq 0$ (B) $\Delta p \neq 0$
(C) $\Delta V \neq 0$ (D) $Q \neq 0$

23-6 The Internal Energy of an Ideal Gas

10. Consider the following processes that can be done on an ideal gas: constant volume, $\Delta V = 0$; constant pressure, $\Delta p = 0$; and constant temperature, $\Delta T = 0$. (*a*) For which process does $W = 0$? (*b*) For which process does $Q = 0$? (*c*) For which of these processes does $W + Q = 0$? (*d*) For which of these processes does $\Delta E_{int} = Q$? (*e*) For which of these processes does $\Delta E_{int} = W$?

(A) $\Delta V = 0$ (B) $\Delta p = 0$ (C) $\Delta T = 0$
(D) None of these

23-7 Heat Capacities of an Ideal Gas

11. Which type of ideal gas will have the largest value for $C_p - C_V$?

(A) Monatomic (B) Diatomic (C) Polyatomic
(D) The value will be the same for all.

12. What would be the most likely value for C_T, the molar heat capacity at constant temperature?

(A) 0 (B) $0 < C_T < C_V$
(C) $C_V < C_T < C_p$ (D) $C_T = \infty$

23-8 Applications of the First Law of Thermodynamics

13. Which of the following processes is forbidden by the first law of thermodynamics? (There may be more than one correct answer!)
    (A) An ice cube is placed in hot coffee; the ice gets colder and the coffee gets hotter.
    (B) Solid wax is placed in a hot metal pan; the wax melts and the metal pan cools.
    (C) Cold water is placed in a cold glass; the glass gets colder and the water gets colder.
    (D) A student builds an automobile engine that converts into work the heat energy released when water changes to ice.
    (E) Dry ice can be made by allowing carbon dioxide gas to expand in a bag.

QUESTIONS

1. Temperature and heat are often confused, as in "the heat is really severe today." By example, distinguish between these two concepts as carefully as you can.
2. Give an example of a process in which no heat is transferred to or from the system but the temperature of the system changes.
3. Can heat be considered a form of stored (or potential) energy? Would such an interpretation contradict the concept of heat as energy in the process of transfer because of a temperature difference?
4. Can heat be added to a substance without causing the temperature of the substance to rise? If so, does this contradict the concept of heat as energy in the process of transfer because of a temperature difference?
5. Why must heat energy be supplied to melt ice? After all, the temperature does not change.
6. Explain the fact that the presence of a large body of water nearby, such as a sea or ocean, tends to moderate the temperature extremes of the climate on adjacent land.
7. As ice is heated it melts, forming a liquid, and then it boils. However, as solid carbon dioxide is heated it goes directly to the vapor state—we say it sublimes—without passing through a liquid state. How could liquid carbon dioxide be produced?
8. Pails of hot and cold water are set out in freezing weather. Explain how (*a*) if the pails have lids, the cold water will freeze first, but (*b*) if the pails do not have lids, it is possible for the hot water to freeze first.
9. Why does the boiling temperature of a liquid increase with pressure?
10. A block of wood and a block of metal are at the same temperature. When the blocks feel cold, the metal feels colder than the wood; when the blocks feel hot, the metal feels hotter than the wood. Explain. At what temperature will the blocks feel equally hot or cold?
11. How can you best use a spoon to cool a cup of coffee? Stirring—which involves doing work—would seem to heat the coffee rather than cool it.
12. How does a layer of snow protect plants during cold weather? During freezing spells, citrus growers in Florida often spray their fruit with water, hoping it will freeze. How does that help?
13. Explain the wind-chill effect.
14. You put your hand in a hot oven to remove a casserole and burn your fingers on the hot dish. However, the air in the oven is at the same temperature as the casserole dish but it does not burn your fingers. Why not?
15. Metal workers have observed that they can dip a hand very briefly into hot molten metal without ill effects. Explain.
16. Why is thicker insulation used in an attic than in the walls of a house?
17. Is ice always at 0°C? Can it be colder? Can it be warmer? What about an ice–water mixture?
18. (*a*) Can ice be heated to a temperature above 0°C without its melting? Explain. (*b*) Can water be cooled to a temperature below 0°C without its freezing? Explain. (See "The Undercooling of Liquids," by David Turnbull, *Scientific American,* January 1965, p. 38.)
19. Explain why your finger sticks to a metal ice tray just taken from your refrigerator.
20. It is difficult to "boil" eggs in water at the top of a high mountain because water boils there at a relatively low temperature. What is a simple, practical way of overcoming this difficulty?
21. Will a 3-minute egg cook any faster if the water is boiling furiously than if it is simmering quietly?
22. Water is a much better coolant than most liquids. Why? Would there be instances in which another liquid might be preferred?
23. Explain why the latent heat of vaporization of a substance might be expected to be considerably greater than its latent heat of fusion.
24. Explain why the specific heat at constant pressure is greater than the specific heat at constant volume.
25. Why is the difference between C_p and C_V often neglected for solids?
26. Can C_p ever be less than C_V? If so, give an example.
27. Real gases always cool when making a free expansion, whereas an ideal gas does not. Explain.
28. Discuss the similarities and especially the distinctions between heat, work, and internal energy.
29. Discuss the process of the freezing of water from the point of view of the first law of thermodynamics. Remember that ice occupies a greater volume than an equal mass of water.
30. A thermos bottle contains coffee. The thermos bottle is vigorously shaken. Consider the coffee as the system. (*a*) Does its temperature rise? (*b*) Has heat been added to it? (*c*) Has work been done on it? (*d*) Has its internal energy changed?

31. Is the temperature of an isolated system (no interaction with the environment) conserved? Explain.

32. Is heat the same as internal energy? If not, give an example in which a system's internal energy changes without a flow of heat across the system's boundary.

33. Can you tell whether the internal energy of a body was acquired by heat transfer or by the performance of work?

34. If the pressure and volume of a system are given, is the temperature always uniquely determined?

35. Keeping in mind that the internal energy of a body consists of kinetic energy and potential energy of its particles, how would you distinguish between the internal energy of a body and its temperature?

36. Explain how we might keep a gas at a constant temperature during a thermodynamic process.

37. On a winter day the temperature on the inside surface of a wall is much lower than room temperature and that of the outside surface is much higher than the outdoor temperature. Explain.

38. Can heat energy be transferred through matter by radiation? If so, give an example. If not, explain why.

39. Why does stainless steel cookware often have a layer of copper or aluminum on the bottom?

40. Consider that heat can be transferred by convection and radiation, as well as by conduction, and explain why a thermos bottle is doubled-walled, evacuated, and silvered.

41. A lake freezes first at its upper surface. Is convection involved? What about conduction and radiation?

42. Explain why the temperature of a gas drops in an adiabatic expansion.

43. Comment on this statement: "There are two ways to carry out an adiabatic process. One is to do it quickly and the other is to do it in an insulated box."

EXERCISES

23-1 Heat: Energy in Transit

23-2 The Transfer of Heat

1. The average rate at which heat flows out through the surface of the Earth in North America is 54 mW/m^2 and the average thermal conductivity of the near surface rocks is 2.5 W/m·K. Assuming a surface temperature of 10°C, what should be the temperature at a depth of 33 km (near the base of the crust)? Ignore the heat generated by radioactive elements; the curvature of the Earth can also be ignored.

2. Calculate the rate at which heat would be lost on a very cold winter day through a 6.2 m × 3.8 m brick wall 32 cm thick. The inside temperature is 26°C and the outside temperature is −18°C; assume that the thermal conductivity of the brick is 0.74 W/m·K.

3. Consider the slab shown in Fig. 23-2. Suppose that $\Delta x = 24.9$ cm, $A = 1.80$ m^2, and the material is copper. If $T = -12.0$°C, $\Delta T = 136$ C°, and a steady state is reached, find (*a*) the temperature gradient, (*b*) the rate of heat transfer, and (*c*) the temperature at a point in the rod 11.0 cm from the high-temperature end.

4. (*a*) Calculate the rate at which body heat flows out through the clothing of a skier, given the following data: the body surface area is 1.8 m^2 and the clothing is 1.2 cm thick; skin surface temperature is 33°C, whereas the outer surface of the clothing is at 1.0°C; the thermal conductivity of the clothing is 0.040 W/m·K. (*b*) How would the answer change if, after a fall, the skier's clothes become soaked with water? Assume that the thermal conductivity of water is 0.60 W/m·K.

5. Four square pieces of insulation of two different materials, all with the same thickness and area A, are available to cover an opening of area $2A$. This can be done in either of the two ways shown in Fig. 23-26. Which arrangement, (*a*) or (*b*), would give the lower heat flow if $k_2 \neq k_1$?

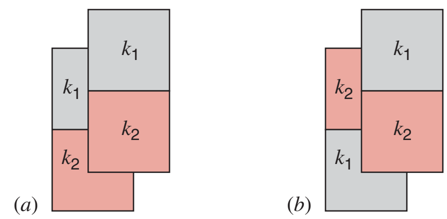


FIGURE 23-26. Exercise 5.

6. Show that the temperature T_x at the interface of a compound slab (see Sample Problem 23-1) is given by

$$T_x = \frac{R_1 T_1 + R_2 T_2}{R_1 + R_2}.$$

7. Ice has formed on a shallow pond and a steady state has been reached with the air above the ice at −5.20°C and the bottom of the pond at 3.98°C. If the total depth of ice + water is 1.42 m, how thick is the ice? (Assume that the thermal conductivities of ice and water are 1.67 and 0.502 W/m·K, respectively.)

8. Two identical rectangular rods of metal are welded end to end as shown in Fig. 23-27*a*, and 10 J of heat flows through the rods in 2.0 min. How long would it take for 30 J to flow through the rods if they are welded as shown in Fig. 23-27*b*?

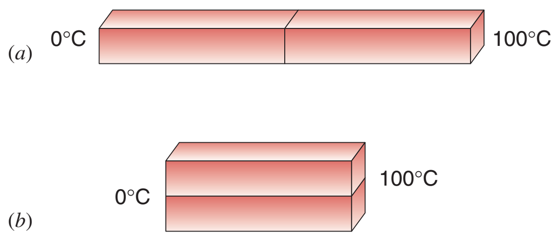


FIGURE 23-27. Exercise 8.

9. An idealized representation of the air temperature as a function of distance from a single-pane window on a calm, winter day is shown in Fig. 23-28. The window dimensions are 60 cm × 60 cm × 0.50 cm. (*a*) At what rate does heat flow out through the window? (Hint: The temperature drop across the glass is very small.) (*b*) Estimate the difference in temperature between the inner and outer glass surfaces.

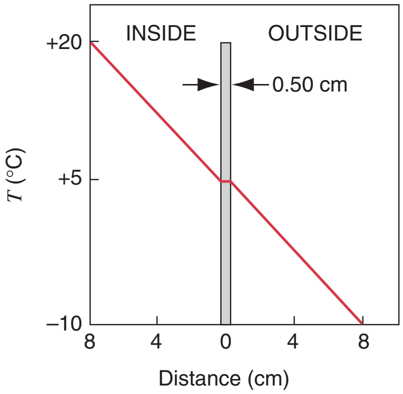


FIGURE 23-28. Exercise 9.

23-3 The First Law of Thermodynamics

10. Consider that 214 J of work are done on a system, and 293 J of heat are extracted from the system. In the sense of the first law of thermodynamics, what are the values (including algebraic signs) of (*a*) W, (*b*) Q, and (*c*) ΔE_{int}?

11. When a system is taken from state i to state f along the path iaf in Fig. 23-29, it is found that $Q = 50$ J and $W = -20$ J. Along the path ibf, $Q = 36$ J. (*a*) What is W along the path ibf? (*b*) If $W = +13$ J for the curved return path fi, what is Q for this path? (*c*) Take $E_{\text{int},i} = 10$ J. What is $E_{\text{int},f}$? (*d*) If $E_{\text{int},b} = 22$ J, find Q for process ib and process bf.

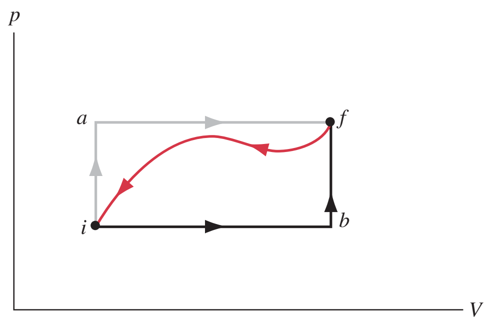


FIGURE 23-29. Exercise 11.

23-4 Heat Capacity and Specific Heat

12. Icebergs in the North Atlantic present hazards to shipping (see Fig. 23-30), causing the length of shipping routes to increase by about 30% during the iceberg season. Strategies for destroying icebergs include planting explosives, bombing, torpedoing, shelling, ramming, and painting with lampblack. Suppose that direct melting of the iceberg, by placing heat sources in the ice, is tried. How much heat is required to melt 10% of a 210,000-metric-ton iceberg? (One metric ton = 1000 kg.)

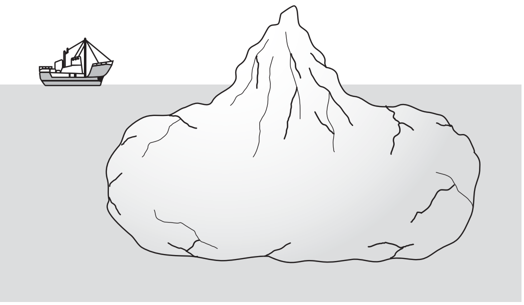


FIGURE 23-30. Exercise 12.

13. In a certain solar house, energy from the Sun is stored in barrels filled with water. In a particular winter stretch of five cloudy days, 5.22 GJ are needed to maintain the inside of the house at 22.0°C. Assuming that the water in the barrels is at 50.0°C, what volume of water is required?

14. A small electric immersion heater is used to boil 136 g of water for a cup of instant coffee. The heater is labeled 220 watts. Calculate the time required to bring this water from 23.5°C to the boiling point, ignoring any heat losses.

15. How much water remains unfrozen after 50.4 kJ of heat have been extracted from 258 g of liquid water initially at 0°C?

16. (*a*) Compute the possible increase in temperature for water going over Niagara Falls, 49.4 m high. (*b*) What factors would tend to prevent this possible rise?

17. A 146-g copper bowl contains 223 g of water; both bowl and water are at 21.0°C. A very hot 314-g copper cylinder is dropped into the water. This causes the water to boil, with 4.70 g being converted to steam, and the final temperature of the entire system is 100°C. (*a*) How much heat was transferred to the water? (*b*) How much to the bowl? (*c*) What was the original temperature of the cylinder?

18. Calculate the minimum amount of heat required to completely melt 130 g of silver initially at 16.0°C. Assume that the specific heat does not change with temperature. See Tables 23-2 and 23-3.

19. An aluminum electric kettle of mass 0.560 kg contains a 2.40-kW heating element. It is filled with 0.640 L of water at 12.0°C. How long will it take (*a*) for boiling to begin and (*b*) for the kettle to boil dry? (Assume that the temperature of the kettle does not exceed 100°C at any time.)

20. What mass of steam at 100°C must be mixed with 150 g of ice at 0°C, in a thermally insulated container, to produce liquid water at 50°C?

21. A 21.6-g copper ring has a diameter of 2.54000 cm at its temperature of 0°C. An aluminum sphere has a diameter of 2.54533 cm at its temperature of 100°C. The sphere is placed on top of the ring (Fig. 23-31), and the two are allowed to come to thermal equilibrium, no heat being lost to the surroundings. The sphere just passes through the ring at the equilibrium temperature. Find the mass of the sphere.

22. (*a*) Two 50-g ice cubes are dropped into 200 g of water in a glass. If the water were initially at a temperature of 25°C, and if the ice came directly from a freezer at −15°C, what is the final temperature of the drink? (*b*) If only one ice cube had been used in (*a*), what would be the final temperature of the drink? Neglect the heat capacity of the glass.

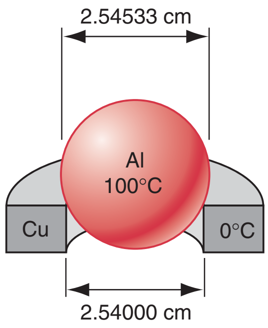


FIGURE 23-31. Exercise 21.

23. A certain substance has a molar mass of 51.4 g/mol. When 320 J of heat are added to a 37.1-g sample of this material, its temperature rises from 26.1 to 42.0°C. (*a*) Find the specific heat of the substance. (*b*) How many moles of the substance are present? (*c*) Calculate the molar heat capacity of the substance.

23-5 The Work Done on or by an Ideal Gas

24. A sample of gas expands from 1.0 to 5.0 m^3 while its pressure decreases from 15 to 5.0 Pa. How much work is done on the gas if its pressure changes with volume according to each of the three processes shown in the pV diagram in Fig. 23-32?

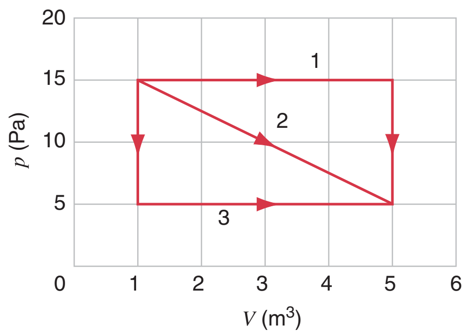


FIGURE 23-32. Exercise 24.

25. Suppose that a sample of gas expands from 2.0 to 8.0 m^3 along the diagonal path in the pV diagram shown in Fig. 23-33. It is then compressed back to 2.0 m^3 along either path 1 or path 2. Compute the net work done on the gas for the complete cycle in each case.

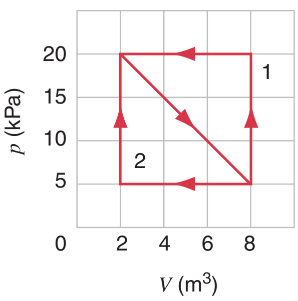


FIGURE 23-33. Exercise 25.

26. Air that occupies 0.142 m^3 at 103 kPa gauge pressure is expanded isothermally to zero gauge pressure and then cooled at constant pressure until it reaches its initial volume. Compute the work done on the gas.

27. Calculate the work done by an external agent in compressing 1.12 mol of oxygen from a volume of 22.4 L and 1.32 atm pressure to 15.3 L at the same temperature.

28. (*a*) One liter of gas with $\gamma = 1.32$ is at 273 K and 1.00 atm pressure. It is suddenly (adiabatically) compressed to half its original volume. Find its final pressure and temperature. (*b*) The gas is now cooled back to 273 K at constant pressure. Find the final volume. (*c*) Find the total work done on the gas.

29. Gas occupies a volume of 4.33 L at a pressure of 1.17 atm and a temperature of 310 K. It is compressed adiabatically to a volume of 1.06 L. Determine (*a*) the final pressure and (*b*) the final temperature, assuming the gas to be an ideal gas for which $\gamma = 1.40$. (*c*) How much work was done on the gas?

30. An air compressor takes air at 18.0°C and 1.00 atm pressure and delivers compressed air at 2.30 atm pressure. The compressor operates at 230 W of useful power. Assume that the compressor operates adiabatically. (*a*) Find the temperature of the compressed air. (*b*) How much compressed air, in liters, is delivered each second?

23-6 The Internal Energy of an Ideal Gas

31. Calculate the total rotational kinetic energy of all the molecules in 1 mole of air at 25.0°C.

32. Calculate the internal energy of 1 mole of an ideal gas at 250°C.

33. An ideal gas experiences an adiabatic compression from $p = 122$ kPa, $V = 10.7$ m^3, $T = -23.0$°C to $p = 1450$ kPa, $V = 1.36$ m^3. (*a*) Calculate the value of γ. (*b*) Find the final temperature. (*c*) How many moles of gas are present? (*d*) What is the total translational kinetic energy per mole before and after the compression? (*e*) Calculate the ratio of the rms speed before to that after the compression.

34. A cosmic-ray particle with energy 1.34 TeV is stopped in a detecting tube that contains 0.120 mol of neon gas. Once this energy is distributed among all the atoms, by how much is the temperature of the neon increased?

23-7 Heat Capacities of an Ideal Gas

35. In an experiment, 1.35 mol of oxygen (O_2) are heated at constant pressure starting at 11.0°C. How much heat must be added to the gas to double its volume?

36. Twelve grams of nitrogen (N_2) in a steel tank are heated from 25.0 to 125°C. (*a*) How many moles of nitrogen are present? (*b*) How much heat is transferred to the nitrogen?

37. A 4.34-mol sample of an ideal diatomic gas experiences a temperature increase of 62.4 K under constant-pressure conditions. (*a*) How much heat was added to the gas? (*b*) By how much did the internal energy of the gas increase? (*c*) By how much did the internal translational kinetic energy of the gas increase?

38. The mass of a helium atom is 6.66×10^{-27} kg. Compute the specific heat at constant volume for helium gas (in J/kg·K) from the molar heat capacity at constant volume.

39. A container holds a mixture of three nonreacting gases: n_1 moles of the first gas with molar specific heat at constant volume C_1, and so on. Find the molar specific heat at constant volume of the mixture, in terms of the molar specific heats and quantities of the three separate gases.

23-8 Applications of the First Law of Thermodynamics

40. Gas within a chamber passes through the cycle shown in Fig. 23-34. Determine the net heat added to the gas during process CA if $Q_{AB} = 20$ J, $Q_{BC} = 0$, and $W_{BCA} = -15$ J.

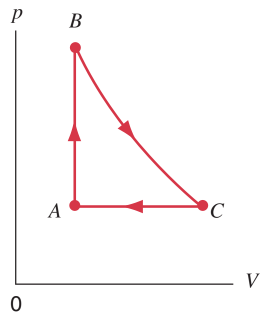


FIGURE 23-34. Exercise 40.

41. A sample of n moles of an ideal gas undergoes an isothermal expansion. Find the heat flow into the gas in terms of the initial and final volumes and the temperature.

42. A quantity of ideal gas occupies an initial volume V_0 at a pressure p_0 and a temperature T_0. It expands to volume V_1 (*a*) at constant pressure, (*b*) at constant temperature, and (*c*) adiabatically. Graph each case on a pV diagram. In which case is Q greatest? Least? In which case is W greatest? Least? In which case is ΔE_{int} greatest? Least?

43. (*a*) A monatomic ideal gas initially at 19.0°C is suddenly compressed to one-tenth its original volume. What is its temperature after compression? (*b*) Make the same calculation for a diatomic gas.

44. In Fig. 23-35, assume the following values:

$$p_{\text{i}} = 2.20 \times 10^5 \text{ Pa}, \quad V_{\text{i}} = 0.0120 \text{ m}^3,$$

$$p_{\text{f}} = 1.60 \times 10^5 \text{ Pa}, \quad V_{\text{f}} = 0.0270 \text{ m}^3.$$

For each of the three paths shown, find the value of Q, W, and $Q + W$. (Hint: Find P, V, T at points A, B, C. Assume an ideal monatomic gas.)

45. A quantity of ideal monatomic gas consists of n moles initially at temperature T_1. The pressure and volume are then slowly doubled in such a manner as to trace out a straight line on the pV diagram. In terms of n, R, and T_1, find (*a*) W, (*b*) ΔE_{int}, and (*c*) Q. (*d*) If one were to define an equivalent specific heat for this process, what would be its value?

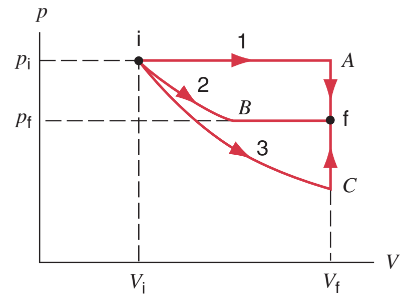


FIGURE 23-35. Exercise 44.

46. Gas within a chamber undergoes the processes shown in the pV diagram of Fig. 23-36. Calculate the net heat added to the system during one complete cycle.

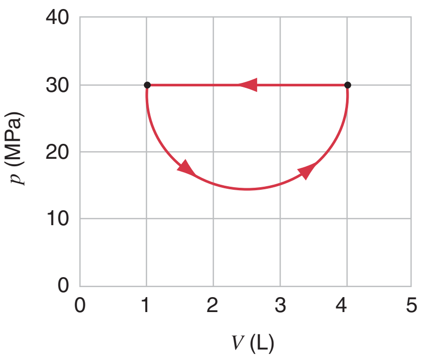


FIGURE 23-36. Exercise 46.

47. Let 20.9 J of heat be added to a particular ideal gas. As a result, its volume changes from 63.0 to 113 cm^3 while the pressure remains constant at 1.00 atm. (*a*) By how much does the internal energy of the gas change? (*b*) If the quantity of gas present is 2.00×10^{-3} mol, find the molar heat capacity at constant pressure. (*c*) Find the molar heat capacity at constant volume.

48. The temperature of 3.15 mol of an ideal polyatomic gas is raised 52.0 K by each of three different processes: at constant volume, at constant pressure, and by an adiabatic compression. Complete a table, showing for each process the heat added, the work done on the gas, the change in internal energy of the gas, and the change in total translational kinetic energy of the gas molecules.

PROBLEMS

1. (*a*) Calculate the rate of heat loss through a glass window of area 1.4 m^2 and thickness 3.0 mm if the outside temperature is −20°F and the inside temperature is +72°F. (*b*) A storm window is installed having the same thickness of glass but with an air gap of 7.5 cm between the two windows. What will be the corresponding rate of heat loss presuming that conduction is the only important heat-loss mechanism?

2. A cylindrical silver rod of length 1.17 m and cross-sectional area 4.76 cm^2 is insulated to prevent heat loss through its surface. The ends are maintained at a temperature difference of 100 C° by having one end in a water–ice mixture and the other in boiling water and steam. (*a*) Find the rate at which heat is transferred along the rod. (*b*) Calculate the rate at which ice melts at the cold end.

3. Assuming k is constant, show that the radial rate of flow of heat in a substance between two concentric spheres is given by

$$H = \frac{(T_1 - T_2)4\pi k r_1 r_2}{r_2 - r_1},$$

where the inner sphere has a radius r_1 and temperature T_1, and the outer sphere has a radius r_2 and temperature T_2.

4. (*a*) Use the data in Exercise 1 to calculate the rate at which heat flows out through the surface of the Earth. (*b*) Suppose that this heat flux is due to the presence of a hot core in the Earth and that this core has a radius of 3470 km. Assume also that the material lying between the core and the surface of the Earth contains no sources of heat and has an average thermal conductivity of 4.2 W/m · K. Use the result of Problem 3 to calculate the temperature of the core. (Assume that the Earth's surface is at 0°C.) The answer obtained is too high by a factor of about 10. Why?

5. At low temperatures (below about 50 K), the thermal conductivity of a metal is proportional to the absolute temperature; that is, $k = aT$, where a is a constant with a numerical value that depends on the particular metal. Show that the rate of heat flow through a rod of length L and cross-sectional area A whose ends are at temperatures T_1 and T_2 is given by

$$H = \frac{aA}{2L}(T_1^2 - T_2^2).$$

(Ignore heat loss from the surface.)

6. A container of water has been outdoors in cold weather until a 5.0-cm-thick slab of ice has formed on its surface (Fig. 23-37). The air above the ice is at −10°C. Calculate the rate of formation of ice (in centimeters per hour) on the bottom surface of the ice slab. Take the thermal conductivity and density of ice to be 1.7 W/m · K and 0.92 g/cm³. Assume that no heat flows through the walls of the container.

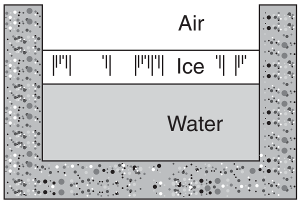


FIGURE 23-37. Problem 6.

7. A person makes a quantity of iced tea by mixing 520 g of the hot tea (essentially water) with an equal mass of ice at 0°C. What are the final temperature and mass of ice remaining if the initial hot tea is at a temperature of (*a*) 90.0°C and (*b*) 70.0°C?

8. A *flow calorimeter* is used to measure the specific heat of a liquid. Heat is added at a known rate to a stream of the liquid as it passes through the calorimeter at a known rate. Then a measurement of the resulting temperature difference between the inflow and the outflow points of the liquid stream enables us to compute the specific heat of the liquid. A liquid of density 0.85 g/cm³ flows through a calorimeter at the rate of 8.2 cm³/s. Heat is added by means of a 250-W electric heating coil, and a temperature difference of 15 C° is established in steady-state conditions between the inflow and the outflow points. Find the specific heat of the liquid.

9. Water standing in the open at 32°C evaporates because of the escape of some of the surface molecules. The heat of vaporization is approximately equal to ϵn, where ϵ is the average energy of the escaping molecules and n is the number of molecules per kilogram. (*a*) Find ϵ. (*b*) What is the ratio of ϵ to the average kinetic energy of H_2O molecules, assuming that the kinetic energy is related to temperature in the same way as it is for gases?

10. A thermometer of mass 0.055 kg and heat capacity 46.1 J/K reads 15.0°C. It is then completely immersed in 0.300 kg of water and it comes to the same final temperature as the water. If the thermometer reads 44.4°C, what was the temperature of the water before insertion of the thermometer, neglecting other heat losses?

11. From Fig. 23-11, estimate the amount of heat needed to raise the temperature of 0.45 mol of carbon from 200 to 500 K. (Hint: Approximate the actual curve in this region with a straight-line segment).

12. The molar heat capacity of silver, measured at atmospheric pressure, is found to vary with temperature between 50 and 100 K by the empirical equation

$$C = 0.318T - 0.00109T^2 - 0.628,$$

where C is in J/mol · K and T is in K. Calculate the quantity of heat required to raise 316 g of silver from 50.0 to 90.0 K. The molar mass of silver is 107.87 g/mol.

13. The gas in a cloud chamber at a temperature of 292 K undergoes a rapid expansion. Assuming the process is adiabatic, calculate the final temperature if $\gamma = 1.40$ and the volume expansion ratio is 1.28.

14. Calculate the work done on n moles of a van der Waals gas in an isothermal expansion from volume V_i to V_f.

15. A thin tube, sealed at both ends, is 1.00 m long. It lies horizontally, the middle 10.0 cm containing mercury and the two equal ends containing air at standard atmospheric pressure. If the tube is now turned to a vertical position, by what amount will the mercury be displaced? Assume that the process is (*a*) isothermal and (*b*) adiabatic. (For air, $\gamma = 1.40$.) Which assumption is more reasonable?

16. A room of volume V is filled with diatomic ideal gas (air) at temperature T_1 and pressure p_0. The air is heated to a higher temperature T_2, the pressure remaining constant at p_0 because the walls of the room are not airtight. Show that the internal energy content of the air remaining in the room is the same at T_1 and T_2 and that the energy supplied by the furnace to heat the air has all gone to heat the air outside the room. If we add no energy to the air, why bother to light the furnace? (Ignore the furnace energy used to raise the temperature of the walls, and consider only the energy used to raise the air temperature.)

17. The molar atomic mass of iodine is 127 g. A standing wave in a tube filled with iodine gas at 400 K has nodes that are 6.77 cm apart when the frequency is 1000 Hz. Determine from these data whether iodine gas is monatomic or diatomic.

18. Figure 23-38*a* shows a cylinder containing gas and closed by a movable piston. The cylinder is submerged in an ice–water mixture. The piston is quickly pushed down from position 1 to position 2. The piston is held at position 2 until the gas is again at 0°C and then is slowly raised back to position 1. Figure 23-38*b* is a *pV* diagram for the process. If 122 g of ice are melted during the cycle, how much work has been done on the gas?

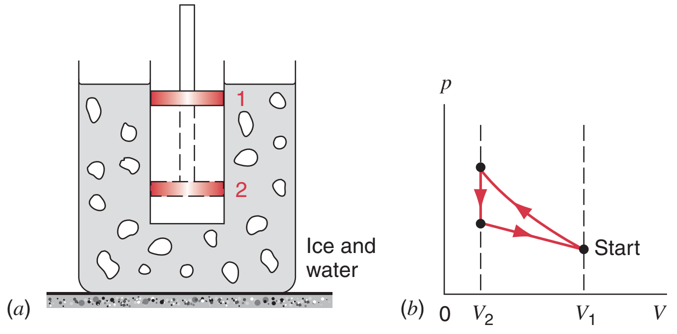


FIGURE 23-38. Problem 18.

19. An engine carries 1.00 mol of an ideal monatomic gas around the cycle shown in Fig. 23-39. Process *AB* takes place at constant volume, process *BC* is adiabatic, and process *CA* takes place at a constant pressure. (*a*) Compute the heat Q, the change in internal energy E_{int}, and the work W for each of the three processes and for the cycle as a whole. (*b*) If the initial pressure at point *A* is 1.00 atm, find the pressure and the volume at points *B* and *C*. Use 1 atm $= 1.013 \times 10^5$ Pa and R $=$ 8.314 J/mol · K.

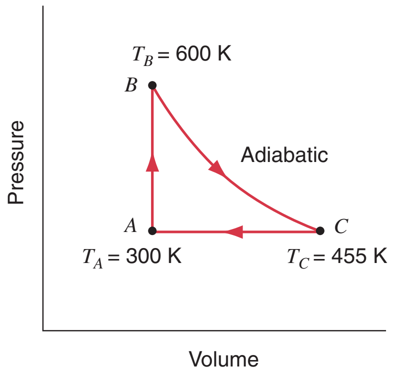


FIGURE 23-39. Problem 19.

20. A cylinder has a well-fitted, 2.0-kg metal piston whose cross-sectional area is 2.0 cm³ (Fig. 23-40). The cylinder contains water and steam at constant temperature. The piston is observed to fall slowly at a rate of 0.30 cm/s because heat flows out of the cylinder through the cylinder walls. As this happens, some steam condenses in the chamber. The density of the steam inside the chamber is 6.0×10^{-4} g/cm³ and the atmospheric pressure is 1.0 atm. (*a*) Calculate the rate of condensation of steam. (*b*) At what rate is heat leaving the chamber? (*c*) What is the rate of change of internal energy of the steam and water inside the chamber?

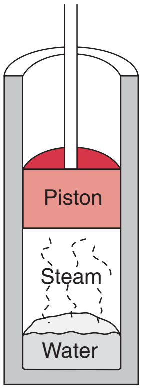


FIGURE 23-40. Problem 20.

21. In a motorcycle engine, after combustion occurs in the top of the cylinder, the piston is forced down as the mixture of gaseous products undergoes an adiabatic expansion. Find the average power involved in this expansion when the engine is running at 4000 rpm, assuming that the gauge pressure immediately after combustion is 15.0 atm, the initial volume is 50.0 cm³, and the volume of the mixture at the bottom of the stroke is 250 cm³. Assume that the gases are diatomic and that the time involved in the expansion is one-half that of the total cycle.

COMPUTER PROBLEMS

1. The theoretical specific heat capacity of a solid at temperature T is given by the Debye formula

$$c_V = 9\left[4\left(\frac{T}{\Theta}\right)^3 \int_0^{\Theta} \frac{x}{T^2}\,\frac{dx}{e^{x/T}-1} - \frac{\Theta/T}{e^{\Theta/T}-1}\right],$$

where Θ is a constant, called the Debye temperature, that depends on the substance. (*a*) Numerically integrate this expression to find the specific heat capacity of aluminum at room temperature, using $\Theta_{\text{aluminum}} = 420$ K. Compare your result to the measured value. (*b*) Generate a graph of the specific heat capacity of aluminum for the range $T = 0$ to $T = 500$ K.

2. The specific heat capacity of aluminum at low temperatures is given by

$$c_V = \frac{12\pi^4}{5} R \left(\frac{T}{420\text{ K}}\right)^3.$$

A 1.0-kg block of aluminum originally at 20 K is placed into a device (left in Roswell, New Mexico by aliens) that can extract 1000 J of heat energy from the aluminum every minute. (*a*) How long before the temperature of the aluminum is 1 K? (*b*) What is the temperature of the aluminum after 12 hours? (*c*) Can the aluminum ever be cooled to absolute zero with this device?

CHAPTER 24

ENTROPY AND THE SECOND LAW OF THERMODYNAMICS

We can imagine many processes that never happen, even though they do not violate the law of conservation of energy. For instance, hot coffee resting in a mug might give up some internal thermal energy and spontaneously begin to rotate. A glass of cool water might spontaneously change into an ice cube in a glass of warmer water. Even though such things never happen, we commonly see them happening in the reverse direction. The second law of thermodynamics, the subject of this chapter, deals with the directions in which processes occur. It is often said that the second law gives a preferred direction to the "arrow of time," telling us that systems naturally evolve with time in one direction but not in the other.

We have seen that the zeroth law of thermodynamics leads to the concept of temperature. Similarly, the first law of thermodynamics leads to the concept of internal energy. The second law establishes still another concept; entropy, *a quantity in terms of which the second law of thermodynamics is expressed. We will examine entropy from both a macroscopic and a microscopic point of view.*

24-1 ONE-WAY PROCESSES

There is a property of things that happen naturally in the world around us that is strange beyond belief. Yet we are so used to it that we hardly ever think about it. It is this:

> *All naturally occurring processes proceed in one direction only. They never, of their own accord, proceed in the opposite direction.*

Consider the following examples:

> *Example 1:* If you drop a stone, it falls to the ground. A stone resting on the ground never, of its own accord, leaps up into the air.
>
> *Example 2:* A cup of hot coffee left on your desk gradually cools down. It never gets hotter all by itself.
>
> *Example 3:* If you put a drop of ink in a glass of water, the molecules of ink eventually spread uniformly throughout the volume of the water. They never, of their own accord, regroup into a drop-shaped clump.

If you saw any of these processes happen in reverse, you would probably suspect that you had been tricked.

Such spontaneous one-way processes are *irreversible,* which means that once they have started they keep on going. More precisely, you cannot make them go backward by making any small change in their environment. Essentially, all naturally occurring processes are irreversible.

Although the "wrong-way" events we have described above do not occur, none of them would violate the law of conservation of energy. Consider these examples again:

> *Example 1:* The ground could spontaneously cool a little, giving up some of its internal thermal energy to the resting stone as kinetic energy, allowing it to leap up. But it does not happen.
>
> *Example 2:* Here we are dealing only with the direction of energy transfer, not with changes in its amount. Energy might flow from the surrounding air into the coffee, instead of the other way around. But it does not.

Example 3: Here no energy transfers are involved. All that is needed is for the ink molecules, each of which is free to move throughout the water, all to return simultaneously to somewhere near their original locations. That will never happen.

It is not the *energy* of the system that controls the direction of irreversible processes; it is another property that we introduce in this chapter—the *entropy* (symbol S) of the system. Although we have not discussed entropy up to this point, it is just as much a property of the state of a system as are temperature, pressure, volume, and internal energy. We shall define entropy in the following section but, to see where we are going, let us at once state its central property, which we can call the *entropy principle:*

If an irreversible process occurs in a closed system, the entropy of that system always increases; it never decreases.

Entropy is different from energy in that it does *not* obey a conservation law. No matter what changes occur within a closed system, the energy of that system remains constant. Its entropy, however, always increases for irreversible processes.

In this chapter we are concerned with *changes* in entropy—that is, with ΔS rather than S. If a process occurs irreversibly in a closed system, the entropy principle tells us that $\Delta S > 0$. The "backward" processes that we have described—if they occurred—would have $\Delta S < 0$ and would violate the entropy principle.

There are two equivalent ways to define the change in the entropy of a system: (1) A macroscopic approach, involving heat transfer and the temperature at which the transfer occurs, and (2) a microscopic approach, involving counting the ways in which the atoms or molecules that make up the system can be arranged. We use the first approach in Section 24-2 and the second in Section 24-9.

24-2 DEFINING ENTROPY CHANGE

In this section we define the entropy change ΔS that occurs when a closed system changes from a well-defined initial state to an equally well-defined final state by a process that we can describe as *reversible.* In a reversible process, we make a small change in a system and its environment; by reversing that change, the system and its environment will return to their original conditions. For example, when we place a hot block of metal and a cool block of metal into contact, heat is transferred from the hotter block to the cooler one. That is an irreversible process; we cannot reverse any step in the procedure that would cause the heat flow to reverse direction and restore the blocks to their original temperatures. On the other hand, consider a piece of metal on a hot plate at a temperature T. If we increase the temperature of the hot plate by a small step dT, a small amount of heat dQ is transferred from the hot plate to the block. If we then *decrease* the temperature of the hot plate by dT, an equal amount of heat dQ is transferred from the block to the hot plate. The block and the hot plate are restored to their original conditions; the heat transferred in this way is done by a reversible process.

For another example, consider the gas in the cylinder shown in Fig. 21-13. If we remove a small amount of lead shot from the container on the piston, a small quantity of heat dQ will be transferred to the gas from the thermal reservoir; if we replace that amount of lead shot, the same quantity of heat dQ flows back to the reservoir, and the system and its environment are restored to their original conditions in this reversible process.

In a truly reversible process, there would be no losses of energy due to turbulence, friction, or other dissipative effects. Clearly the reversible process is an abstraction, because all natural processes will result in these types of energy losses and hence be irreversible. For example, if there is friction in the piston of Fig. 21-13, the system will not return to its original configuration when we return the lead shot to the container. However, by improving the apparatus and making other experimental refinements, we can approach arbitrarily close to reversibility. More importantly, the strictly reversible process is a simple and useful abstraction that helps us to analyze and understand more complex processes, just as the ideal gas concept is an abstraction that helps us to understand the behavior of real gases.

We begin our discussion of entropy by simply stating the definition of entropy change for a reversible process and then examining its consequences. The definition is

$$\Delta S = \int_{\mathrm{i}}^{\mathrm{f}} \frac{dQ}{T} \qquad \text{(reversible).} \qquad (24\text{-}1)$$

Here dQ is the increment of heat energy that is transferred into or out of the (closed) system at (Kelvin) temperature T, and the integral is evaluated from the initial state i of the system to its final state f. Both the heat transferred and the temperature at which the transfer takes place are equally important in defining the entropy change.

If the process is isothermal, so that the heat transfer takes place at a constant temperature T, then Eq. 24-1 reduces to

$$\Delta S = \frac{Q}{T} \qquad \text{(reversible, isothermal).} \qquad (24\text{-}2)$$

Because the (Kelvin) temperature T is always positive, it follows from Eqs. 24-1 and 24-2 that the entropy change has the same algebraic sign as the heat Q. That is, if heat energy is *added* (reversibly) to a closed system ($Q > 0$), the entropy of that system *increases* ($\Delta S > 0$) and conversely. The unit of entropy that follows from its defining equation is the joule/kelvin.

Entropy as a State Property

If entropy, like pressure, internal energy, and temperature, were not a true property of a given equilibrium state of a system, we would not find it a useful quantity. Let us now prove specifically that entropy is such a *state property* for the important case of an ideal gas.

We write the first law of thermodynamics in differential form as

$$dQ + dW = dE_{\text{int}}.$$

We then replace dW with $-p\,dV$, and we use Eq. 23-31 to replace dE_{int} with $nC_V\,dT$. Solving for dQ then leads to

$$dQ = p\,dV + nC_V\,dT.$$

Using the ideal gas law, we replace p in this equation with nRT/V and we then divide each term in the resulting equation by T. This gives us

$$\frac{dQ}{T} = nR\frac{dV}{V} + nC_V\frac{dT}{T}.$$

Now let us integrate each term of this equation between an arbitrary initial state i and an arbitrary final state f. The quantity on the left is then the entropy change defined by Eq. 24-1, so we get

$$\Delta S = nR\ln\frac{V_f}{V_i} + nC_V\ln\frac{T_f}{T_i}.$$

We did not specify a path in carrying out the integration, so the above result must hold for all (reversible) paths. Thus the change in entropy between the initial and final states of an ideal gas depends only on properties of the initial state (T_i and V_i) and of the final state (T_f and V_f). It is totally independent of the process by which the ideal gas moves from its initial to its final state. Thus entropy is indeed a state property, characteristic of the particular state of a system and *not* dependent on how the system arrived at that state.

Sample Problem 24-1. An insulating vessel containing 1.8 kg of water is placed on a hot plate, both the water and hot plate being initially at 20°C. The temperature of the hot plate is raised very slowly to 100°C, at which point the water begins to boil. What is the entropy change of the water during this process?

Solution The water and the hot plate are essentially in thermal equilibrium at all times so that the process is reversible. That is, by lowering the temperature of the hot plate slightly at any stage of the process, we could cause the temperature of the water to stop rising and to begin to fall. We choose the water alone as our system and, because the process is reversible, we can use Eq. 24-1 to calculate the entropy change.

The heat energy required to raise the temperature of the water by an amount dT is

$$dQ = mc\,dT$$

in which m is the mass of the water and c is the specific heat of water. Equation 24-1 then becomes

$$\begin{aligned}\Delta S &= \int_{T_i}^{T_f}\frac{mc\,dT}{T}\\ &= mc\int_{T_i}^{T_f}\frac{dT}{T} = mc\ln\frac{T_f}{T_i}\\ &= (1.8\text{ kg})(4190\text{ J/kg}\cdot\text{K})\ln\frac{(273+100)\text{ K}}{(273+20)\text{ K}}\\ &= 1820\text{ J/K}.\end{aligned}$$

Note that, as part of the definition of ΔS, we had to change the initial and final temperatures from the Celsius to the Kelvin scale. Because heat is transferred to the system to increase its temperature, the change in the entropy of the system is positive.

24-3 ENTROPY CHANGE FOR IRREVERSIBLE PROCESSES

We can use Eq. 24-1 to calculate the entropy change for a process only if that process is reversible. However, a reversible process—like an ideal gas—is an idealization. All processes that we encounter in the real world involve friction or turbulence, or have some other aspect that makes them essentially irreversible. How, then, do we calculate the entropy change for an irreversible process?

We rely on the fact that entropy is a state property. That is, when a closed system proceeds from an initial state i to a final state f, the entropy change depends only on the properties of these two states. It does not depend at all on the process that connects the states or even whether that process is reversible or irreversible. Thus, we can find the entropy change ΔS for a system that proceeds from state i to state f by an irreversible process using the following procedure:

1. Find a reversible process that connects these same two states. Any of the many such processes will do. It makes sense to choose the simplest.

2. Use Eq. 24-1 to calculate ΔS for this chosen *equivalent reversible process*. The result will hold for the original irreversible process as well.

Let us explore this prescription for what must be one of the most familiar of irreversible processes, the falling stone of Example 1 in Section 24-1. Figure 24-1*a* shows the initial state of the system. For convenience, we allow the stone to fall into a thermally insulated bucket of water. Figure 24-1*b* shows the final state. The stone now rests in the bucket and the temperature of the water and the resting stone has risen from an initial value T_i to a final value T_f.

We take as our system *stone + water.* No heat energy is transferred through the boundary of this system so that $Q = 0$. Work, in amount $m_s gh$, where m_s is the mass of the stone, is done on the system by the gravitational force that acts on the stone. Thus $W = +m_s gh$. From the first law of thermodynamics, we must then have

$$\Delta E_{\text{int}} = Q + W = 0 + m_s gh = +m_s gh.$$

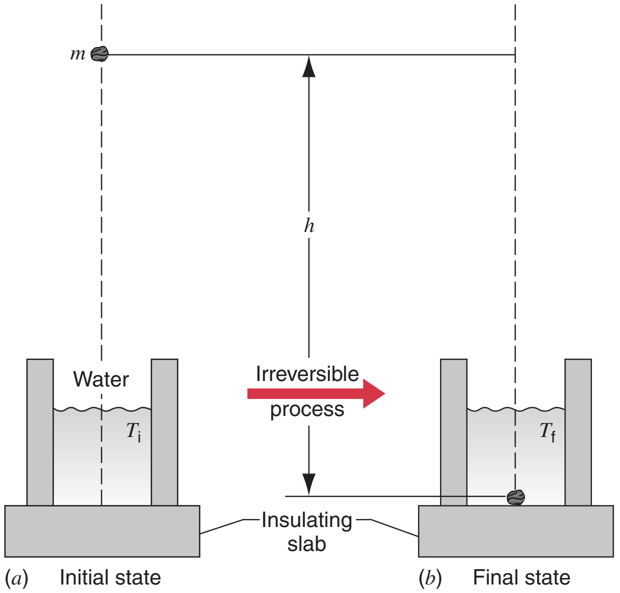


FIGURE 24-1. An irreversible process between two equilibrium states. A stone of mass m is dropped from a height h into water contained in a thermally insulated bucket. The temperature of the water (and the stone) rises from an initial value T_i to a higher value T_f.

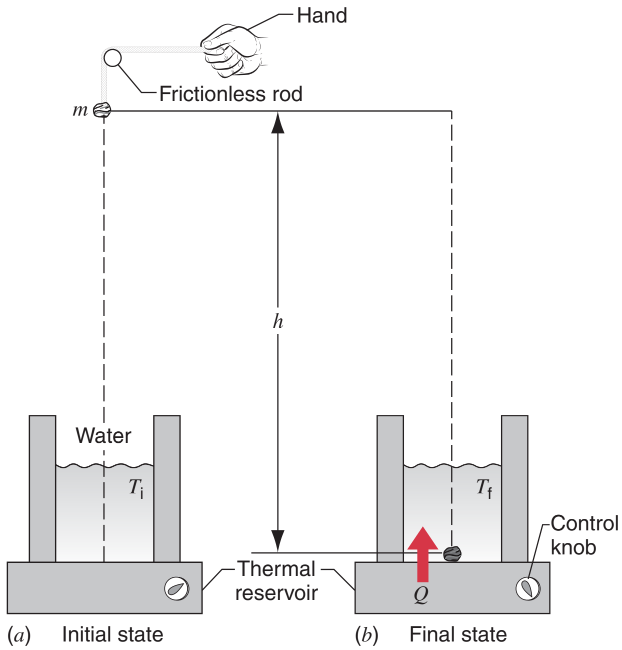


FIGURE 24-2. A reversible process connecting the same initial and final states shown in Fig. 24-1. The water bucket now rests on a *thermal reservoir* whose temperature can be adjusted by means of a control knob. First the stone is lowered slowly at the end of a string. Then the temperature of the water (and the stone) is increased slowly from T_i to T_f by adjusting the temperature control knob. During this process, heat energy Q is transferred from the reservoir to the water.

This increase in internal energy shows up as a small increase in the temperature of the water–stone system.

Now, following step 1 of our procedure, let us find an equivalent reversible process that connects the two states of Fig. 24-1. In Fig. 24-2*a*, we place the water bucket, not on an insulating slab, but on a thermal reservoir whose controllable temperature we set to T_i. We do not allow the stone to fall freely but we attach it to a string and lower it very slowly. After the stone is in the water, we increase the temperature of the reservoir slowly to T_f, using the temperature control knob. The initial and final states in Fig. 24-2 are exactly the same as those in Fig. 24-1.

The process of Fig. 24-2 is truly reversible. We could change the direction of the process at any stage by making small adjustments in the environment of the system—that is, by raising the stone instead of lowering it and by extracting heat energy instead of adding it.

Now let us examine the energy transfers that occur in the equivalent reversible process of Fig. 24-2. The net force acting on the stone is now zero, the force of gravity being balanced by the upward-directed tension in the string. Thus $W = 0$. Because the initial and final states of Fig. 24-2 are the same as those for Fig. 24-1 (and the internal energy is a state property) we must have $\Delta E_{\text{int}} = +m_s gh$ in each case. From the first law of thermodynamics we then have

$$Q = \Delta E_{\text{int}} - W = m_s gh - 0 = +m_s gh. \tag{24-3}$$

Heat in this amount must enter the system from the thermal reservoir if we are to increase the system temperature from T_i to T_f. Knowing Q, we can then calculate the entropy change for the equivalent reversible process, using Eq. 24-1; see Sample Problem 24-2. Because Q is positive (heat *enters* the system) the entropy change will also be positive. *This same (positive) entropy change also holds for the irreversible process of Fig. 24-1.*

In the following three sample problems, we will examine three irreversible processes that occur in closed systems and will show that, in accord with the entropy principle, the entropy always increases.

SAMPLE PROBLEM 24-2. A stone of mass $m_s = 1.5$ kg falls through a vertical height $h = 2.5$ m into a bucket containing a mass $m_w = 4.5$ kg of water, as in Fig. 24-1. The initial temperatures of the water and the stone are 300 K. (*a*) What is the temperature rise ΔT of the system *water + stone*? (*b*) What is the entropy change ΔS of this system? (*c*) What would be the entropy change for the reverse process—that is, for the system to cool down, transferring its energy to the stone in kinetic form, allowing it to leap 2.5 m into the air? (It will never happen!) The specific heat of water is $c_w = 4190$ J/kg·K and that of the stone material is $c_s = 790$ J/kg·K.

Solution (*a*) Figure 24-2 shows an equivalent reversible process that we can use to calculate the entropy change of the falling stone. In terms of the temperature change ΔT of both the water and the stone, the heat transfer Q in the equivalent reversible process of Fig. 24-2*b* is

$$Q = m_w c_w \Delta T + m_s c_s \Delta T, \qquad (24\text{-}4)$$

a positive quantity. We have seen from Eq. 24-3 that Q is also given by

$$Q = m_s gh = (4.5 \text{ kg})(9.8 \text{ m/s}^2)(2.5 \text{ m}) = +110 \text{ J}.$$

Substituting this value for Q into Eq. 24-4 and solving for ΔT yields

$$\Delta T = \frac{Q}{m_w c_w + m_s c_s} = \frac{+110 \text{ J}}{(4.5 \text{ kg})(4190 \text{ J/kg}\cdot\text{K}) + (1.5 \text{ kg})(790 \text{ J/kg}\cdot\text{K})} = +5.5 \times 10^{-3} \text{ K} = +5.5 \text{ mK}.$$

Because temperature is a state property, this calculated temperature rise holds both for the equivalent reversible process of Fig. 24-2 and for the original irreversible process of Fig. 24-1.

(*b*) Now let us calculate the entropy change for the equivalent reversible process of Fig. 24-2. The temperature change (5.5 mK) is so small that we can say that heat Q is transferred from the reservoir to the system at essentially a constant temperature of 300 K. Thus we can calculate ΔS from Eq. 24-2. From that equation, then

$$\Delta S = \frac{Q}{T} = \frac{+110 \text{ J}}{300 \text{ K}} = +0.37 \text{ J/K}.$$

Note that heat Q is transferred *into* the system from the reservoir and is thus positive. Therefore ΔS is also positive, signifying an *increase* in entropy.

Although we have calculated ΔS for the reversible process of Fig. 24-2 it applies equally well to the irreversible process of Fig. 24-1. When a stone falls to earth, the entropy of the system increases, just as the entropy principle requires.

(*c*) In the reverse process, heat energy in the amount $Q = -110$ J would have to be transferred *from* the system of Fig. 24-1*b*, causing its temperature to *fall* by 5.5 mK. Having acquired this energy in kinetic form, the stone would then leap 2.5 m into the air, restoring the system to that of Fig. 24-1*a*. The entropy change calculation proceeds just as in (*b*) except that, because heat is extracted from the system, Q is now negative and so must be ΔS. If this backward process happened, it would have $\Delta S = -0.37$ J/K, in violation of the entropy principle.

You may be inclined to say, "This 'backward' process does not happen because it violates the entropy principle." A better statement is, "Because we have observed that this backward process—and countless others like it—never happen, scientists have been led to correlate all these observations by formulating the entropy principle."

SAMPLE PROBLEM 24-3. Figure 24-3*a* shows a paper cup containing a mass $m = 0.57$ kg of hot water and a similar cup containing an equal mass of cold water. The initial temperature of the hot water is $T_{iH} = 90°\text{C} = 363$ K; that of the cold water is $T_{iC} = 10°\text{C} = 283$ K. When the insulating shutter separating the two enclosures is removed, as in Fig. 24-3*b*, the hot water and the cold water eventually come to thermal equilibrium at a temperature of $T_f = 50°\text{C} = 323$ K. What is the entropy change of the system for this irreversible process? The specific heat of water is $c = 4190$ J/kg · K; the heat capacity of the paper cups is negligible.

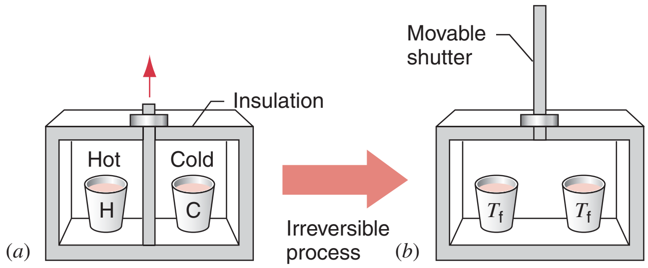


FIGURE 24-3. Sample Problem 24-3. (*a*) In the initial state, two cups of water H and C, identical except for their temperatures, are in an insulating box and are separated by an insulating shutter. (*b*) When the shutter is removed, the cups exchange heat and come to a final state, both with the same temperature T_f. The process is irreversible.

Solution As always, when seeking to find the entropy change for an irreversible process, we must start by finding an equivalent reversible process that connects the same initial and final states. Figure 24-4 shows an arrangement that we can use to carry out such a process.

Step 1: Having first adjusted the temperature of the thermal reservoir to $T_{iH} = 363$ K we place the cup of hot water (cup H) on it and surround it with a thermally insulating container. We then lower the temperature of the reservoir slowly and reversibly to $T_f = 323$ K. For each temperature change by an amount dT during this process, an amount of heat given by $dQ = mc\,dT$ is transferred *from* the hot water. From Eq. 24-1, the entropy change of the hot water is

$$\Delta S_H = \int \frac{dQ}{T} = \int_{T_{iH}}^{T_f} \frac{mc\,dT}{T} = mc \ln \frac{T_f}{T_{iH}} = (0.57 \text{ kg})(4190 \text{ J/kg}\cdot\text{K}) \ln \frac{323 \text{ K}}{363 \text{ K}} = -279 \text{ J/K}.$$

Because heat is transferred from the hot water, we expect the entropy change to be negative, as indeed it turns out to be.

Step 2: Now put the cup of cold water (cup C) in a similar thermally insulating container, having first adjusted the tempera-

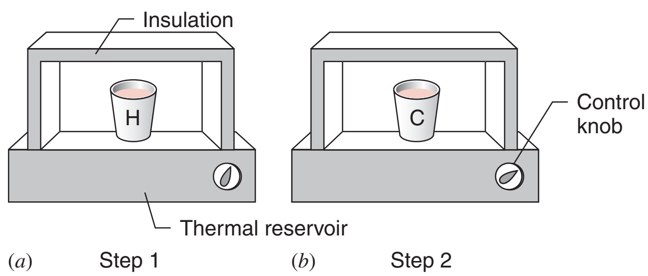


FIGURE 24-4. The cups of Fig. 24-3 can proceed from their initial state to their final state in a reversible way if we use a reservoir with a controllable temperature (*a*) to extract heat reversibly from cup H and (*b*) to add heat reversibly to cup C.

ture of the thermal reservoir to $T_{iC} = 283$ K. Then increase the temperature of the reservoir, slowly and reversibly, to T_f, the final equilibrium temperature of the system. During this process, for every increment of temperature dT an amount of heat $dQ = mc\,dT$ is transferred *to* the system. The entropy change for the cold water can be calculated as above, the result being

$$\Delta S_C = (0.57\text{ kg})(4190\text{ J/kg}\cdot\text{K})\ln\frac{323\text{ K}}{283\text{ K}} = +316\text{ J/K}.$$

Step 3: The net entropy change for the entire system is

$$\Delta S = \Delta S_H + \Delta S_C = -279\text{ J/K} + 316\text{ J/K} = +37\text{ J/K}.$$

This is also the entropy change for the irreversible process of Fig. 24-3. Once again we see that the entropy of a closed system *increases* during an irreversible process.

SAMPLE PROBLEM 24-4. Let $n = 0.55$ mol of an ideal gas at room temperature ($T = 293$ K) be confined in the left thermally insulated chamber of the apparatus of Fig. 24-5. The right chamber is evacuated and the two chambers—which are of equal volume—are connected by a tube containing a stopcock. If you open the stopcock, the gas will rush to fill the evacuated chamber and will eventually settle down into a state of thermal equilibrium, filling both chambers. What is the entropy change of the gas for this irreversible process?

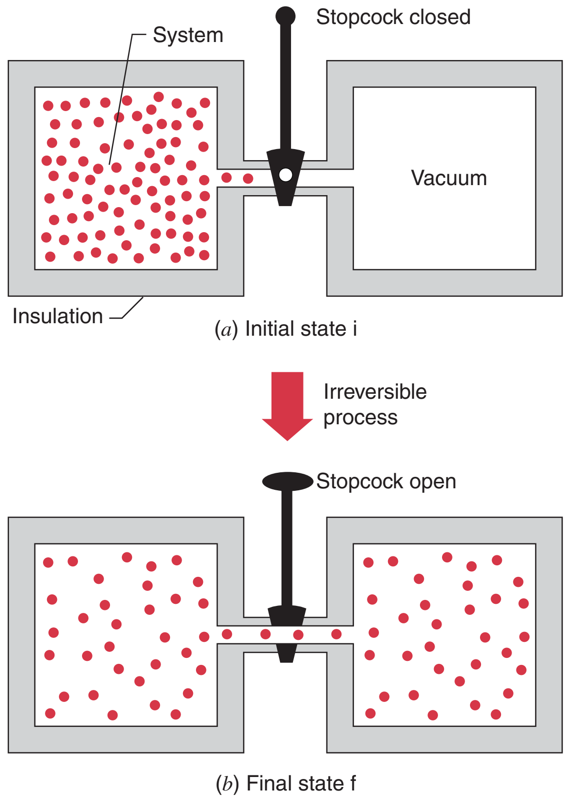


FIGURE 24-5. Sample Problem 24-4. The free expansion of an ideal gas. (*a*) The gas is confined to the left half of an insulated container by a closed stopcock. (*b*) When the stopcock is opened, the gas rushes to fill the entire container. This process is irreversible; that is, it does not occur in reverse, with the gas spontaneously collecting itself in the left half of the container.

Solution The process of Fig. 24-5 is a *free expansion,* a process that we examined in Section 23-8. We learned there that, if the gas is ideal—which we assume—the temperature of the final state is the same as that of the initial state. The free expansion is clearly *not* reversible; we cannot return the system to its previous state by making a small change in its environment.

As in the two previous sample problems, to calculate the entropy change, we must first find an equivalent reversible process that takes the system from the initial state of Fig. 24-5 to its final state. Figure 24-6 shows how such a process can be carried out; it is the reversible isothermal expansion of an ideal gas.

We confine 0.55 mol of the gas to an insulated cylinder that rests on a thermal reservoir set to $T = 293$ K. We place enough lead shot on top of the piston so that the pressure and volume of the gas are those of the initial state of Fig. 24-5*a*. We then remove the shot very slowly until the pressure and volume of the gas are those of the final state of Fig. 24-5*b*. This slow process *is* reversible: at each step, we could return a small amount of lead shot to its container, and a corresponding small amount of heat would be transferred *to* the reservoir *from* the gas. During this expansion process, a total heat energy Q is transferred *from* the reservoir *to* the gas to maintain the temperature constant as the gas expands.

We can find Q from the first law of thermodynamics, which we write in differential form as

$$dQ + dW = dE_{int}.$$

The internal energy of an ideal gas depends only on its temperature (see Eq. 23-24) and, because the temperature does not change ($dT = 0$) we must also have $dE_{int} = 0$. Replacing dW by $-p\,dV$ and substituting nRT/V for p, we have

$$dQ = -dW = p\,dV = nRT\frac{dV}{V}.$$

Integrating between the initial volume and the final volume yields

$$Q = \int dQ = nRT\int_{V_i}^{V_f}\frac{dV}{V} = nRT\ln\frac{V_f}{V_i}.$$

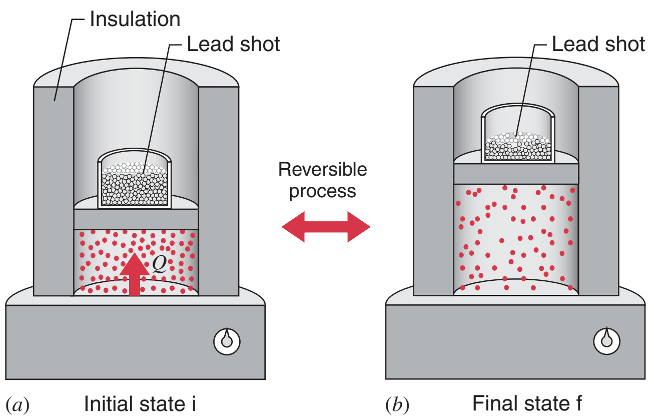


FIGURE 24-6. The isothermal expansion of an ideal gas, done in a reversible way. The gas has the same initial state i and same final state f as in the irreversible process of Fig. 24-5.

We can calculate the entropy change of the isothermal expansion process of Fig. 24-6 from Eq. 24-2, because the temperature is constant throughout. Thus, bearing in mind that $V_f/V_i = 2$,

$$\Delta S = \frac{Q}{T} = nR \ln \frac{V_f}{V_i}$$

$$= (0.55 \text{ mol})(8.31 \text{ J/mol}\cdot\text{K})(\ln 2) = +3.17 \text{ J/K}.$$

This is also the entropy change for the irreversible process of Fig. 24-5. As we expect from the entropy principle, it is positive. We have now provided three examples to demonstrate the entropy increases in irreversible processes that occur in closed systems.

24-4 THE SECOND LAW OF THERMODYNAMICS

Before we express the second law of thermodynamics in terms of entropy change, we must solve a little puzzle. We saw in Sample Problem 24-4 that if we cause the gas in Fig. 24-6 to undergo a reversible expansion from (*a*) to (*b*) in that figure, the change in entropy of the gas—which we take as our system—is positive. However, because the process is reversible, we can just as easily cause the gas to undergo a reversible compression, making it go from (*b*) to (*a*), simply by slowly adding lead shot to the piston of Fig. 24-6*b* until the original volume of the gas is restored. In this reverse process, heat must be transferred *from* the gas to keep its temperature from rising. Hence Q is negative and, from Eq. 24-2, so is the entropy change of the gas.

Doesn't this decrease in entropy of the gas violate our expectation that entropy should always increase? No, because the expectation that entropy should always increase holds only for *irreversible* processes occurring in *closed* systems. First, the procedure suggested by Fig. 24-6 is not irreversible. Second, because energy is transferred as heat from the gas to the reservoir, the system (that is, the gas) is not closed.

We can always close a system by enlarging it to include those parts of its environment with which it interacts. In Fig. 24-6, for example, we can choose as our system the *gas + reservoir*, rather than the gas alone. If the process in that figure then goes from (*b*) to (*a*), heat Q moves from the gas to the reservoir—that is, from one part of our larger system to the other. We can calculate the entropy changes of the gas and the reservoir separately with Eq. 24-2, which applies to an isothermal process like that of Fig. 24-6. We get

$$\Delta S_{gas} = -\frac{|Q|}{T} \quad \text{and} \quad \Delta S_{res} = +\frac{|Q|}{T},$$

in which $|Q|$ is the absolute value of the heat transfer, a positive quantity. The entropy change of the closed system *gas + reservoir* is the sum of these two quantities, which is zero. Thus, although the entropy of the gas decreases, that of the reservoir increases, and by the same amount.

With this as background, we can now extend the statement we made in Section 24-1 about entropy changes to include both reversible and irreversible processes. The extended statement, which we call the *second law of thermodynamics,* is:

When changes occur within a closed system its entropy either increases (for irreversible processes) or remains constant (for reversible processes). It never decreases.

In equation form this statement becomes

$$\Delta S \geq 0. \tag{24-5}$$

The "greater than" sign applies to irreversible processes and the "equals" sign to reversible processes. No exceptions to the second law of thermodynamics have ever been found.

Although entropy may decrease in part of a closed system, there will always be an equal (or larger) entropy increase in another part of that system so that the entropy of the system as a whole never decreases.

24-5 ENTROPY AND THE PERFORMANCE OF ENGINES

A heat engine, or more simply, an *engine,* is a device that extracts energy from its environment in the form of heat and does useful work. At the heart of every engine is a *working substance.* In an automobile engine, for example, the working substance is a gas–air mixture. For an engine to do work on a sustained basis, the working substance must operate in a *cycle.* That is, it must pass through a closed series of thermodynamic processes, returning again and again to any arbitrarily selected state. Let us see what the laws of thermodynamics can tell us about the operation of engines.

A Carnot Engine

We have seen that we can learn much about real gases by analyzing an ideal gas, which obeys the simple law $pV = nRT$. This is a useful plan because, although an ideal gas does not exist, any real gas approaches ideal behavior as closely as you wish if its density is low enough. In much the same spirit we choose to study (real) engines by analyzing the behavior of an ideal engine.

Figure 24-7 shows schematically the operation of our ideal engine. We call it a *Carnot engine,* after the French scientist and engineer N. L. Sadi Carnot (pronounced "carno"), who first proposed the concept in 1824. It is amazing that Carnot was able to analyze the performance of this engine some 25 years before the first law of thermodynamics was discovered and the concept of entropy was established.

During each cycle of the engine of Fig. 24-7, the working substance absorbs heat $|Q_H|$ from a reservoir at constant temperature T_H and discharges heat $|Q_L|$ to a second

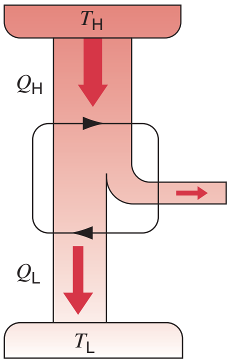


FIGURE 24-7. The elements of a Carnot engine. The two black arrowheads on the central loop suggest the working substance operating in a cycle. Heat Q_H is transferred from the high-temperature reservoir at temperature T_H to the working substance. Heat Q_L is transferred from the working substance to the low-temperature reservoir at temperature T_L. Work W is done by the engine (actually by the working substance) on something in the environment.

reservoir at a constant lower temperature T_L. We assume that all thermodynamic processes involved in the operation of the engine are *reversible,* which means that no dissipative processes such as turbulence and friction and no irreversible heat transfers can be present. Although the Carnot engine is a hypothetical engine, we can learn much about real engines by analyzing its performance.

The Carnot Cycle

Figure 24-8 shows a pressure–volume (or pV) plot of the cycle followed by the working substance of the Carnot engine of Fig. 24-7. As indicated by the arrows, the cycle is traversed in the clockwise direction. To carry out the Carnot cycle physically, imagine the working substance of the

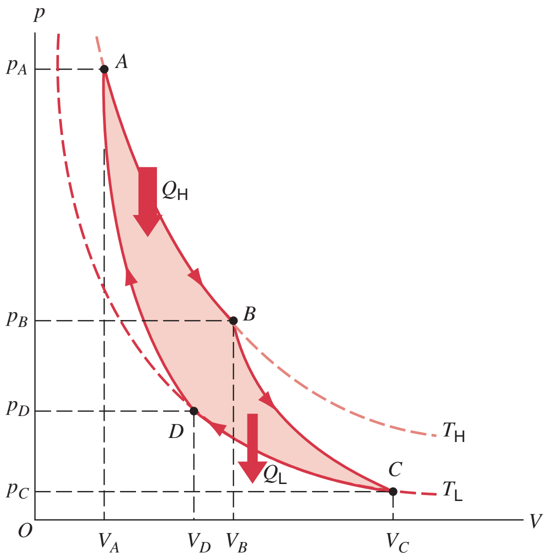


FIGURE 24-8. The Carnot cycle plotted on a pV diagram for an ideal gas as the working substance.

Carnot engine to be a gas, confined to an insulating cylinder with a weighted moveable piston. The cylinder may be placed at will on an insulating slab or on either of two thermal reservoirs, one at the high temperature T_H and the other at the low temperature T_L. Figure 24-8 shows that, if we place the cylinder in contact with the high-temperature reservoir T_H, heat Q_H is transferred *to* the working substance from this reservoir as the gas undergoes an isothermal *expansion* from volume V_A to volume V_B. Similarly, with the working substance in contact with the low-temperature reservoir T_L, heat Q_L is transferred *from* the working substance to this reservoir, as the gas undergoes an isothermal *compression* from volume V_C to volume V_D.

In preparing Fig. 24-7, we have assumed that heat transfers to or from the working substance can take place *only* during the isothermal processes *AB* and *CD* of Fig. 24-8. Therefore, processes *BC* and *DA* in that figure, which connect the two isotherms, T_H and T_L, must be (reversible) adiabatic processes; that is, they must be processes in which no heat is transferred. To ensure this, during processes *BC* and *DA* the cylinder is placed on an insulating slab as the volume of the working substance is changed.

As we have defined it (see Section 23-5), work done *on* a gas is negative when the gas expands (its volume increases). In considering the performance of engines, however, we are more interested in the work done *by* the gas on its environment. When the gas expands, it does *positive* work on the environment—for example, it can lift a weight. We continue to define the thermodynamic work W as we did in Chapter 23 as the work done *on* the gas, but in our discussion of engines we will use $|W|$ to represent the work done *by* an expanding gas on its environment. This notation reminds us that negative work done *on* the gas corresponds to positive work done *by* the gas.

During the consecutive processes *AB* and *BC* of Fig. 24-8, the working substance is expanding and thus doing positive work as it raises the weighted piston. This work is represented in Fig. 24-8 by the area under the curve *ABC*. During the consecutive processes *CD* and *DA*, the working substance is being compressed, which means that it is doing negative work on its environment or, equivalently, that its environment is doing positive work on it as the loaded piston descends. This work is represented by the area under the curve *CDA*. The *net work per cycle,* which is represented by W in Fig. 24-7, is the difference between these two areas, a negative quantity equal to the area enclosed by the cycle *ABCDA* and shown shaded in Fig. 24-8. This work W is performed on some outside object, perhaps being used to lift a weight.

It is also instructive to plot the Carnot cycle on a temperature–entropy (or *T-S*) diagram, as in Fig. 24-9. Note that, in this plot, the isotherms are horizontal lines. Verify that the points *A*, *B*, *C*, and *D* in Fig. 24-9 correspond to the points so labeled in the pressure–volume diagram of Fig. 24-8. Figure 24-9 shows that, during process *AB*, the entropy of the working substance increases. From Eq. 24-2, this increase is $|Q_H|/T_H$ because heat Q_H is transferred to

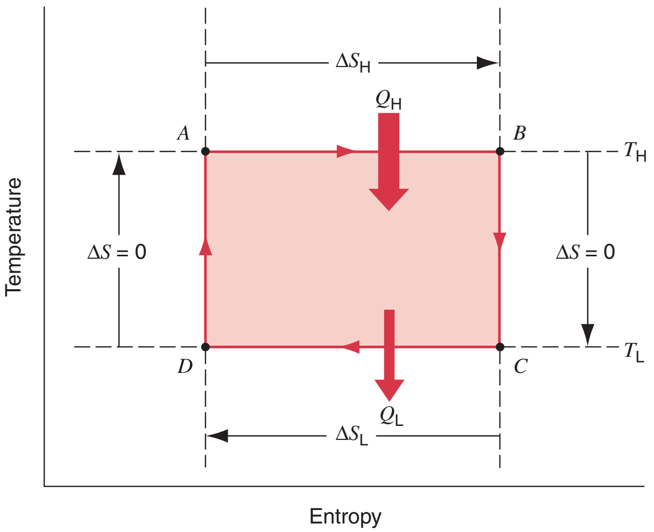


Figure 24-9. The Carnot cycle shown on a temperature–entropy plot. Entropy changes occur during processes *AB* and *CD* but not during processes *BC* and *DA*. The plot will have this rectangular shape regardless of the nature of the working substance.

the working substance, reversibly and at constant temperature T_H. Similarly, during the process *CD* in Fig. 24-9, heat Q_L is being transferred (reversibly, and at constant temperature T_L) from the working substance and, as a result, its entropy decreases. Processes *BC* and *DA* in Fig. 24-9 are adiabatic; that is, there is no reversible transfer of heat so that, again from Eq. 24-2, the entropy remains constant. Figure 24-9 shows clearly that the Carnot cycle consists of two isothermal processes (in which the temperature remains constant) and two so-called *isentropic* processes (during which the entropy remains constant).

Because the engine operates in a cycle, the working substance must return again and again to any arbitrarily selected state in that cycle. If X represents any state property of the working substance such as pressure, temperature, volume, internal energy, or entropy, we must have $\Delta X = 0$ for every cycle. In particular, we must have

$$\Delta E_{\text{int}} = 0 \quad \text{and} \quad \Delta S = 0 \tag{24-6}$$

for every cycle of the working substance. We will use these conclusions later.

Efficiency of a Carnot Engine

The purpose of an engine is to transform as much of the extracted heat Q_H into work as possible. We measure its success in doing so by its *thermal efficiency* ϵ, defined as the work the engine does per cycle ("what you get") divided by the heat energy it absorbs per cycle ("what you pay for"), or

$$\epsilon = \frac{\text{energy you get}}{\text{energy you pay for}} = \frac{|W|}{|Q_H|}. \tag{24-7}$$

Let us apply the first law of thermodynamics ($\Delta E_{\text{int}} = Q + W$) to the working substance as it undergoes one cycle of operation. Q is then the *net* heat transfer per cycle, W is the *net* work, and (from Eq. 24-6) $\Delta E_{\text{int}} = 0$. That law then becomes

$$|W| = |Q_H| - |Q_L|. \tag{24-8}$$

Combining Eqs. 24-7 and 24-8 yields

$$\epsilon = \frac{|Q_H| - |Q_L|}{|Q_H|} = 1 - \frac{|Q_L|}{|Q_H|}. \tag{24-9}$$

Now let us see what entropy considerations have to say about the operation of a Carnot engine. In this engine, there are two reversible heat transfers and thus two changes in entropy, one (ΔS_H) at temperature T_H and one (ΔS_L) at T_L. As Eq. 24-6 reminds us, the net entropy change per cycle must be zero, so that, consistent with Fig. 24-9,

$$\Delta S_H = -\Delta S_L, \tag{24-10}$$

which, because ΔS_L is negative, we can write as

$$\frac{|Q_H|}{T_H} = \frac{|Q_L|}{T_L}. \tag{24-11}$$

The quantity on the left is the magnitude of the entropy change at the high-temperature reservoir and that on the right is the magnitude of the entropy change at the low-temperature reservoir. We also see from Eq. 24-11 that, because $T_H > T_L$, we must have $|Q_H| > |Q_L|$. That is, more energy is extracted as heat from the high-temperature reservoir than is delivered to the low-temperature reservoir.

Combining Eqs. 24-9 and 24-11, we obtain the efficiency of a Carnot engine:

$$\epsilon = 1 - \frac{T_L}{T_H} \quad \text{(Carnot efficiency).} \tag{24-12}$$

Because $T_L < T_H$, the Carnot engine necessarily has a thermal efficiency less than unity—that is, less than 100%. We can see this from Fig. 24-7, which shows that only part of the heat energy extracted from the high-temperature reservoir is available to do work, the rest being delivered to the low-temperature reservoir. We will show in Section 24-7 that *no real engine can have a thermal efficiency greater than that calculated from Eq. 24-12.*

Note that, in deriving Eq. 24-12, we did not need to specify the nature of the working substance, nor did we do so. We conclude:

> *Equation 24-12 gives the efficiency of all Carnot engines working between the same two fixed temperatures, regardless of the nature of their working substance.*

Search for a "Perfect" Engine

Inventors continually try to improve engine efficiency by reducing the energy $|Q_L|$ that is "thrown away" during each cycle. The inventor's dream is to produce the *perfect engine,* diagrammed in Fig. 24-10, in which Q_L is reduced to zero and Q_H is converted completely into work. Such an engine on an ocean liner, for example, could extract heat from

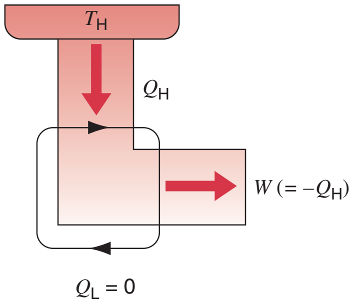


FIGURE 24-10. The elements of a perfect engine—that is, one that converts heat Q_H from a high-temperature reservoir directly to work W with 100% efficiency.

the ocean and use it to drive the propellers, with no fuel cost. An automobile, fitted with such an engine, could extract heat energy from the surrounding air and use it to drive the car, again with no fuel cost. Alas, a perfect engine is only a dream! Inspection of Eq. 24-12 shows that we can achieve 100% engine efficiency (that is, $\epsilon = 1$) only if $T_L = 0$ or $T_H \to \infty$, requirements that are impossible to meet. Instead, decades of practical engineering experience have led to the following alternative version of the second law of thermodynamics:

> *No series of processes is possible whose sole result is the absorption of heat from a thermal reservoir and the complete conversion of this energy to work.*

In short, *there are no perfect engines.*

To summarize: the thermal efficiency given by Eq. 24-12 applies only to Carnot engines. Real engines, in which the processes that form the engine cycle are not reversible, have lower efficiencies. If your car were powered by a Carnot engine, it would have an efficiency of about 55% according to Eq. 24-12; its actual efficiency is probably about 25%. A nuclear power plant, taken in its entirety, is an engine. Its high-temperature reservoir is the reactor core and its low-temperature reservoir is a nearby river. Work is done on a steam turbine, which drives an alternator, generating electrical power. If a nuclear power plant operated as a Carnot engine, its efficiency would be about 40%; its actual efficiency is about 30%. In designing engines of any type, there is simply no way to overcome the efficiency limitation imposed by Eq. 24-12.

Other Reversible Engines

Equation 24-12 does not apply to all reversible engines, but only to engines that have two (and only two) thermal reservoirs, as in Fig. 24-7. In short, it applies only to Carnot engines. For example, Fig. 24-11 shows the operating cycle of an ideal (that is, reversible) *Stirling engine.* Comparison with the Carnot cycle of Fig. 24-8 shows that each engine has isothermal heat transfers at temperatures T_H and T_L. However, the two isotherms of the Stirling engine cycle of Fig. 24-11 are connected, not by adiabatic processes as for

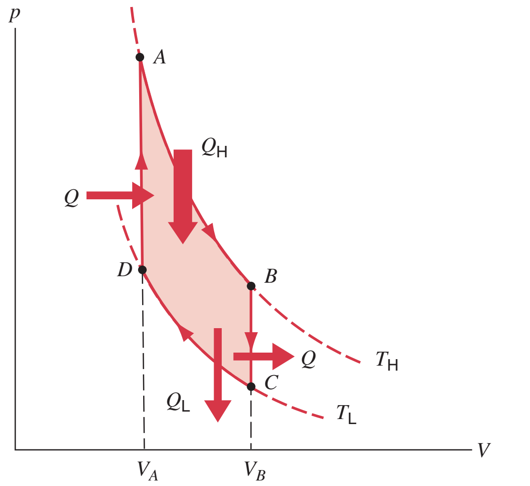


FIGURE 24-11. A pV plot for the working substance of an ideal (that is, reversible) Stirling engine, assumed for convenience to be an ideal gas. Compare the Carnot engine cycle of Fig. 24-8. Each cycle has two isothermal processes, but in the Stirling engine these processes are connected by two constant-volume (not constant-entropy) processes, along which heat transfers also occur.

the Carnot engine, but by constant-volume processes. To increase the temperature of a gas at constant volume reversibly from T_L to T_H (as in process DA of Fig. 24-11) requires a heat transfer to the working substance from a thermal reservoir whose temperature can be varied smoothly between those limits. Although reversible heat transfers (and corresponding entropy changes) occur in only two of the processes that form the cycle of a Carnot engine, they occur in all four of the processes that form the cycle of a Stirling engine. All four heat exchanges must be taken into account in deriving the thermal efficiency of a Stirling engine. Its efficiency will be lower than that of a Carnot engine operating between the same two temperatures. Real Stirling engines, in contrast to ideal Stirling engines, will have still lower efficiencies.

SAMPLE PROBLEM 24-5. The turbine in a steam power plant takes steam from a boiler at 520°C and exhausts it to a condenser at 100°C. What is the maximum possible efficiency of the turbine?

Solution The maximum efficiency is that of a Carnot engine operating between the same two temperatures or, from Eq. 24-12,

$$\epsilon_{max} = 1 - \frac{T_L}{T_H} = 1 - \frac{(273 + 100)\text{ K}}{(273 + 520)\text{ K}}$$
$$= 0.53 \quad \text{or} \quad 53\%$$

Note that the temperatures in Eq. 24-12 *must* be expressed in kelvins. Because of friction, turbulence, and unwanted heat transfers, actual efficiencies of about 40% may be realized for a turbine of this type. Note that the theoretical maximum efficiency depends *only* on the two temperatures involved, not on the pressures or other factors.

24-6 ENTROPY AND THE PERFORMANCE OF REFRIGERATORS

A *refrigerator* is a device that uses work to transfer thermal energy from a low-temperature reservoir to a high-temperature reservoir as it continuously repeats a set series of thermodynamic processes. In a household refrigerator, for example, work is done by an electrical compressor to transfer thermal energy from the food storage compartment (a low-temperature reservoir) to the room (a high-temperature reservoir).

An *air conditioner* is a refrigerator. Its low-temperature reservoir is the room that is to be cooled, and its high-temperature reservoir is the (presumably warmer) outdoors. A *heat pump,* which is also a refrigerator, is an air conditioner that can be operated in reverse to heat a room. The room is now the high-temperature reservoir and heat is transferred to it from the (presumably cooler) outdoors.

Figure 24-12 shows the basic elements of a refrigerator. If we assume that the processes involved in the operation of the refrigerator are reversible, then we have an *ideal refrigerator.* Comparison of Fig. 24-12 with Fig. 24-7 shows that an ideal refrigerator is simply a Carnot engine running backward, with the directions of all energy transfers, either as heat or work, reversed. Thus we call the ideal refrigerator of Fig. 24-12 a *Carnot refrigerator.*

The designer of a refrigerator would like to extract as much heat $|Q_L|$ as possible from the low-temperature reservoir ("what you want") for the least amount of work ("what you pay for"). We take, as a measure of the efficiency of a refrigerator, the ratio

$$K = \frac{\text{what you want}}{\text{what you pay for}} = \frac{|Q_L|}{|W|}. \quad (24\text{-}13)$$

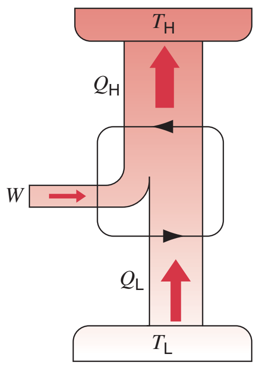


FIGURE 24-12. The elements of a Carnot refrigerator. The two black arrowheads on the central loop suggest the working substance operating in a cycle, as if on a pV plot. Heat Q_L is transferred to the working substance from the low-temperature reservoir. Heat Q_H is transferred to the high-temperature reservoir from the working substance. Work W is done on the refrigerator (on the working substance) by something in the environment.

The larger the value of K, called the *coefficient of performance,* the more efficient is the refrigerator.

The first law of thermodynamics, applied to the working substance of the refrigerator, yields

$$|W| = |Q_H| - |Q_L|$$

so that Eq. 24-13 becomes

$$K = \frac{|Q_L|}{|Q_H| - |Q_L|}. \quad (24\text{-}14)$$

Because a Carnot refrigerator is simply a Carnot engine working backward, Eq. 24-11 holds for it. If we combine that equation with Eq. 24-14 we find, after a little algebra,

$$K = \frac{T_L}{T_H - T_L} \quad \text{(Carnot refrigerator).} \quad (24\text{-}15)$$

For a typical room air conditioner, $K \approx 2.5$, which means (see Eq. 24-13) that the unit removes 2.5 J of thermal energy from the room for every joule of electrical energy supplied to it. As Eq. 24-15 shows, the value of K becomes larger as the temperatures of the two reservoirs become more nearly the same. That is why heat pumps are more effective in temperate climates than in climates where the outside temperature fluctuates between wide limits.

Search for a "Perfect" Refrigerator

It would be nice to own a refrigerator that did not require some input of work—that is, one that would run without being plugged in. Figure 24-13 shows another "inventor's dream," a *perfect refrigerator,* for which $W = 0$, so that (see Eq. 24-13) $K \to \infty$. However, as we shall now see, such a device would violate the second law of thermodynamics.

Let us take as our system the working substance and also the two heat reservoirs. Only by including the reservoirs can we ensure that the system is closed. Because the unit operates in a cycle, the entropy of the working substance does not change during one cycle. The entropies of

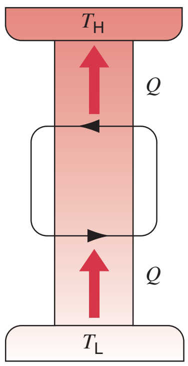


FIGURE 24-13. The elements of a perfect refrigerator—that is, one that transfers heat from a low-temperature reservoir to a high-temperature reservoir without any input of work.

the two reservoirs, however, do change, the net entropy change for the entire closed system (see Fig. 24-13) being

$$\Delta S = -\frac{|Q|}{T_L} + \frac{|Q|}{T_H},$$

in which heat Q leaves the low-temperature reservoir and, in the same amount, enters the high-temperature reservoir. Because $T_H > T_L$, the net change in entropy per cycle for a perfect refrigerator would be negative, a violation of the second law of thermodynamics. If you want your refrigerator to work, you must plug it in!

This result leads to a third equivalent formulation of the second law of thermodynamics, often called the Clausius version, after Rudolph Clausius (1822–1888) who first introduced the concept of entropy:

> *No process is possible whose sole result is the transfer of heat from a reservoir at one temperature to another reservoir at a higher temperature.*

In short, *there are no perfect refrigerators.*

SAMPLE PROBLEM 24-6. A household refrigerator, whose coefficient of performance K is 4.7, extracts heat from the food chamber at the rate of 250 J/cycle. (*a*) How much work per cycle is required to operate the refrigerator? (*b*) How much heat per cycle is discharged to the room?

Solution (*a*) The basic definition of coefficient of performance, Eq. 24-13, relates K to the work done and the heat Q_L removed from the low-temperature reservoir. Solving that equation for W yields

$$|W| = \frac{|Q_L|}{K} = \frac{250 \text{ J/cycle}}{4.7} = 53 \text{ J/cycle}.$$

(*b*) Applying the first law of thermodynamics to the working substance of the refrigerator yields

$$-|Q_H| + |Q_L| + |W| = \Delta E_{int}.$$

Here $\Delta E_{int} = 0$ because the working substance operates in a cycle. Solving the above equation for $|Q_H|$ and inserting the known data yields

$$|Q_H| = |W| + |Q_L| = 53 \text{ J/cycle} + 250 \text{ J/cycle} = 303 \text{ J/cycle}.$$

We see that the refrigerator is an efficient room heater. By paying for 53 J of energy (to run the compressor) we get 303 J of heat energy delivered to the room. If we heated the room with an electric heater, we would get only 53 J of heat energy for every 53 J of work we paid for.

SAMPLE PROBLEM 24-7. A heat pump is a device that—acting as a refrigerator—can heat a house by transferring heat energy from the outside to the inside of the house; the process is driven by work done on the device. The outside temperature is −10°C, and the interior is to be kept at 22°C. To maintain the temperature by making up for normal heat losses it is necessary to deliver heat to the interior at the rate of 16 kW. At what minimum rate must energy be supplied to the heat pump?

Solution The low-temperature reservoir is the great outdoors, at $T_L = (273 - 10) = 263$ K, and the high-temperature reservoir is the house interior at $T_H = (273 + 22) = 295$ K. The maximum coefficient of performance of a heat pump is given by Eq. 24-15, or

$$K = \frac{T_L}{T_H - T_L} = \frac{263 \text{ K}}{295 \text{ K} - 263 \text{ K}} = 8.22.$$

Applying the first law of thermodynamics to Eq. 24-13 yields

$$K = \frac{|Q_L|}{|W|} = \frac{|Q_H| - |W|}{|W|}.$$

Solving for $|W|$ and dividing by Δt, the duration of a cycle, to express the result in terms of power yields

$$|W/\Delta t| = \frac{|Q_H/\Delta t|}{K+1} = \frac{16 \text{ kW}}{8.22 + 1} = 1.7 \text{ kW}.$$

Herein lies the "magic" of the heat pump. By using the heat pump as a refrigerator to heat a house by cooling the great outdoors, you can deliver 16 kW to the interior of the house as heat but you need pay for only the 1.7 kW it takes to run the pump.

Actually, the 1.7 kW is a theoretical minimum because it is based on the assumption that the heat pump is a Carnot refrigerator. In practice, a greater power input would be required but there would still be a very considerable saving over, say, heating the house directly with electric heaters.

24-7 THE EFFICIENCIES OF REAL ENGINES

In this section we wish to show that no real engine can have an efficiency greater than that of a Carnot engine operating between the same two temperatures. That is, no real engine can have an efficiency greater than that given by Eq. 24-12.

Let us assume that an inventor has constructed an engine, engine X, whose efficiency ϵ_X—it is claimed—is greater than ϵ_C, the efficiency of a Carnot engine. That is,

$$\epsilon_X > \epsilon_C \qquad \text{(a claim).} \qquad (24\text{-}16)$$

Let us couple engine X to a Carnot refrigerator, as in Fig. 24-14*a*. We adjust the strokes of the Carnot refrigerator so that the work it requires per cycle is just equal to that provided by engine X. Thus no (external) work is performed on or by the combination *engine + refrigerator* of Fig. 24-14*a*, which we take as our system.

If Eq. 24-16 is true then, from the definition of efficiency (see Eq. 24-7), we must have

$$\frac{|W|}{|Q_{HX}|} > \frac{|W|}{|Q_{HC}|}.$$

Here Q_{HX} is the heat extracted from the high-temperature reservoir by engine X and Q_{HC} is that same quantity for the Carnot refrigerator when it operates as an engine. This inequality requires that

$$|Q_{HC}| > |Q_{HX}|. \qquad (24\text{-}17)$$

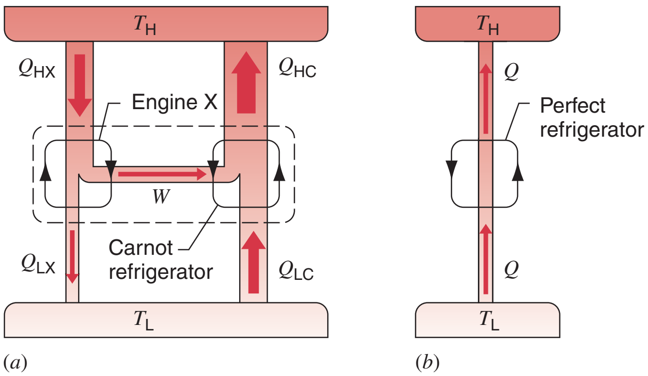


FIGURE 24-14. (*a*) Engine X drives a Carnot refrigerator. If engine X were more efficient than a Carnot engine, then the combination would be equivalent to the perfect refrigerator shown in (*b*).

Now let us apply the first law of thermodynamics separately to the working substances of the Carnot refrigerator and also that of engine X. Because we have chosen the work done by engine X to be equal to the work done on the Carnot refrigerator, we have

$$|W| = |Q_{HC}| - |Q_{LC}| = |Q_{HX}| - |Q_{LX}|,$$

which we can write as

$$|Q_{HC}| - |Q_{HX}| = |Q_{LC}| - |Q_{LX}| = Q. \quad (24\text{-}18)$$

Because of Eq. 24-17, the quantity Q in Eq. 24-18 must be positive.

Comparison of Eq. 24-18 with Fig. 24-14 shows that the net effect of engine X and the Carnot refrigerator, working in combination, is to transfer heat energy Q from a low-temperature reservoir to a high-temperature reservoir without the requirement of work; see Fig. 24-14*b*. Thus the combination acts like the perfect refrigerator of Fig. 24-13, whose existence is a violation of the second law of thermodynamics.

Something must be wrong. We conclude that the claim made in Eq. 24-16 cannot be correct and

> *No real engine can have an efficiency greater than that of a Carnot engine working between the same two temperatures.*

At most, a real engine can have an efficiency equal to the Carnot engine efficiency, given by Eq. 24-12. In that case, of course, engine X *is itself* a Carnot engine. Because real engines are irreversible, their efficiencies will always be less than the limit set by Eq. 24-12.

SAMPLE PROBLEM 24-8. The inventor of engine X claims that it has a work output $W = 120$ J per cycle and operates between the boiling and freezing points of water with an efficiency of $\epsilon_X = 75\%$ (*a*) How does this claimed efficiency compare with the efficiency of a Carnot engine operating between the same two temperatures? (*b*) If engine X actually existed, how much heat energy Q_H would it extract from the high-temperature reservoir per cycle? (*c*) If engine X actually existed, how much heat energy Q_L would it discharge to the low-temperature reservoir per cycle? (*d*) Once more, assuming that engine X actually exists, what would be the entropy change per cycle for the entire engine, including the working substance and both reservoirs?

Solution (*a*) From Eq. 24-12, which applies only to a Carnot engine, we have

$$\epsilon_C = 1 - \frac{T_L}{T_H}$$
$$= 1 - \frac{(273 + 0)\text{ K}}{(273 + 100)\text{ K}} = 0.268 \approx 27\%.$$

As we have shown in this section, no real engine can have an efficiency greater than that of a Carnot engine operating between the same two temperatures. Regardless of the inventor's claim, engine X must have an efficiency less than 27%. Something is wrong.

(*b*) From Eq. 24-7 we have

$$|Q_H| = \frac{|W|}{\epsilon_X} = \frac{120\text{ J}}{0.75} = 160\text{ J}.$$

(*c*) Applying the first law of thermodynamics to the working substance of engine X yields

$$|W| = |Q_H| - |Q_L|.$$

(Because the working substance operates in a cycle, $\Delta E_{int} = 0$.) Solving the above equation for Q_L and substituting numerical data yields

$$|Q_L| = |Q_H| - |W| = 160\text{ J} - 120\text{ J} = 40\text{ J}.$$

(*d*) The system we have chosen is closed so that we can apply the second law of thermodynamics in the form of Eq. 24-5. Bearing in mind that, because the engine operates in a cycle, the entropy change per cycle of its working substance is zero, we have

$$\Delta S_X = \Delta S_H + \Delta S_L + \Delta S_{WS}$$
$$= -\frac{|Q_H|}{T_H} + \frac{|Q_L|}{T_L} + 0$$
$$= -\frac{160\text{ J}}{(273 + 100)\text{ K}} + \frac{40\text{ J}}{(273 + 0)\text{ K}} + 0$$
$$= -0.429\text{ J/K} + 0.147\text{ J/K} + 0 = -0.28\text{ J/K}.$$

The terms on the right are, respectively, the entropy changes of the high-temperature reservoir (a negative quantity), of the low-temperature reservoir (a positive quantity), and of the working substance.

Note that, as we expect, the entropy of engine X *decreases* steadily ($\Delta S_X < 0$) as the cycles progress. This is a clear violation of the second law, which states that the entropy of a closed system can never decrease. Once again, something is wrong.

24-8 THE SECOND LAW REVISITED

So far we have presented three statements of the second law of thermodynamics as it applies to closed systems, namely:

1. The entropy of such systems never decreases. That is, as Eq. 24-5 shows, $\Delta S \geq 0$.

2. You cannot change heat energy into work with 100% efficiency. That is, there are no perfect engines.

3. You cannot transfer heat energy from a low-temperature reservoir to a higher temperature reservoir without doing work. That is, there are no perfect refrigerators.

At first glance these statements seem quite different. However, they are all completely equivalent. If you accept any one of these statements, you must also accept the other two. If any one of them is false, the other two are also false.

In Section 24-5 we showed that statement 2 follows from statement 1 and in Section 24-6 we showed that statement 3 also follows from statement 1. Here we will show that statements 2 and 3 are completely equivalent.

Consider what could happen if statement 2 were false and we could actually build a perfect engine, converting heat Q_H entirely into work W. Let us use this work output to drive a Carnot refrigerator, as shown in Fig. 24-15*a*. This refrigerator transfers heat $|Q_{HC}| = |Q_{LC}| + |W|$ into the high-temperature reservoir.

Let us regard the combination of the perfect engine and the Carnot refrigerator as a single device, as shown by the dashed boundary lines in Fig. 24-15*a*. The work W is an internal feature of this device and does not represent an exchange of energy between the device and its environment. The overall effect of the combined device is to take heat $|Q_{LC}|$ from the low-temperature reservoir and deliver to the high-temperature reservoir a net amount of heat equal to $|Q_{HC}| - |Q_H|$. However, $|Q_H| = |W|$, so, applying the first law of thermodynamics to the Carnot refrigerator,

$$|Q_{HC}| - |Q_H| = |Q_{HC}| - |W| = |Q_{LC}|.$$

Thus, as Fig. 24-15*b* shows, our combined device acts like a perfect refrigerator, taking heat $|Q_{LC}|$ from the low-temperature reservoir and transferring it to the high-temperature reservoir, with no external work performed.

Thus, if you can build a perfect engine, you can also build a perfect refrigerator. By a similar argument, you can show that, if you can build a perfect refrigerator, you can also build a perfect engine. Thus, a violation of statement 2 of the second law above implies a violation of statement 3, and conversely. The two statements are logically equivalent.

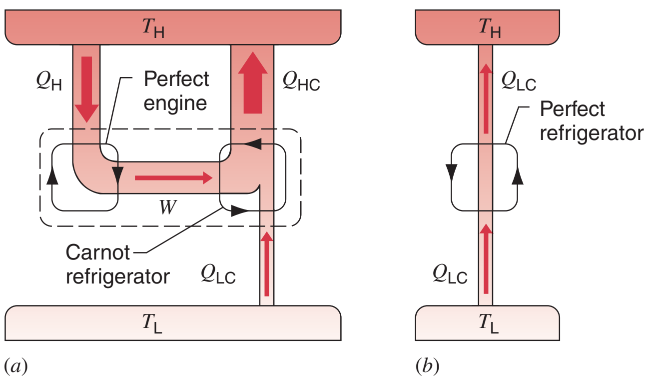


FIGURE 24-15. (*a*) A Carnot refrigerator, driven by a perfect engine, is equivalent to (*b*) a perfect refrigerator.

24-9 A STATISTICAL VIEW OF ENTROPY

In our discussion of entropy so far we have said nothing about the fact that matter is made up of atoms. In this section we take that fact as central and we will see that we can approach the entropy concept from that direction. We start with the simple problem of counting the number of ways that we can divide a small number of atoms (for generality, we will call them molecules) between the two halves of a box. This is a problem in the general field of *statistical mechanics.*

Imagine that we distribute—by hand—eight molecules between the two halves of a box. The molecules are indistinguishable so we can pick the first one to put in the box in eight different ways. We then have seven choices for the second molecule, six choices for the third, and so on. The total number of ways in which we can put the eight molecules in the box is the product of these independent choices, or

$$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40{,}320.$$

We write this in mathematical shorthand as

$$8! = 40{,}320,$$

in which 8! is pronounced "eight factorial." Your calculator can probably calculate factorials for you. By definition, we take $0! = 1$, a fact that we will use later.

However, not all of these 40,320 ways are independent. We have overcounted and that number is too large. Consider the *configuration* of Fig. 24-16*a*, for example, in which there happen to be five molecules in one half of the box and three in the other. Because the molecules are identical, there is no way that, simply by looking at the five

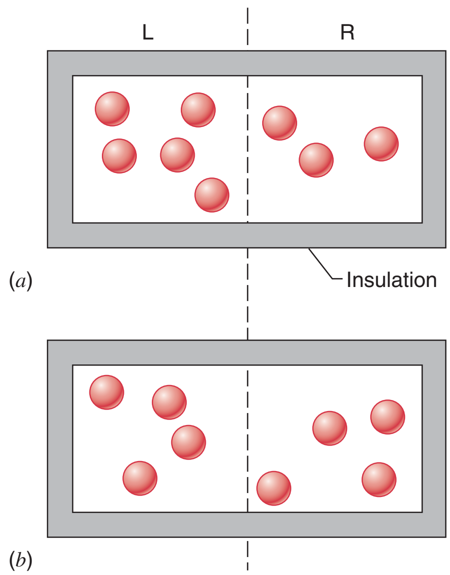


FIGURE 24-16. An insulated box contains eight gas molecules. Each molecule has the same probability of being in the left half (L) of the box as in the right half (*R*). The arrangement in (*a*) corresponds to configuration IV in Table 24-1, and that in (*b*) corresponds to configuration V.

molecules, we can deduce the order in which we put them there. We can, in fact, put these five molecules in place in $5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$ different ways, all leading to the same configuration. Similarly, we can put the three molecules in the other half of the box in $3 \times 2 \times 1 = 3! = 6$ different ways.

To get the number of truly independent ways of arriving at the configuration of Fig. 24-16*a*, we need to divide 40,320 by 120 and also by 6, obtaining 56. We call each of these 56 ways a *microstate* of the configuration and we call the number of microstates that lead to a given configuration the *multiplicity* w of that configuration. Thus the configuration of Fig. 24-16*a* has a multiplicity of 56, which means that it contains 56 microstates.

Extending our consideration from eight molecules to N molecules, the multiplicity of the configuration in which N_1 molecules are in one half of the box and N_2 in the other is given by

$$w = \frac{N!}{N_1!\, N_2!}. \tag{24-19}$$

We can use this general relation to verify that, for eight molecules ($N = 8$), the multiplicity of the configuration of Fig. 24-16*a* ($N_1 = 5$, $N_2 = 3$) is indeed 56.

Table 24-1 shows that, for eight molecules, there are nine configurations, which we have labeled with Roman numerals. The multiplicities, calculated from Eq. 24-19, are also shown. We see that Fig. 24-16*a* shows configuration IV and Fig. 24-16*b* shows configuration V. The total number of microstates of the eight-molecule system is 256 ($= 2^8$).

The basic assumption of our statistical approach to thermodynamics is perhaps a surprising one; namely,

All microstates of a system are equally probable.

Thus, as the eight molecules in the system of Fig. 24-16 jostle around in their random fashion, the system will spend, on average, the same amount of time in each of the 256 microstates listed in Table 24-1. Note that all *configurations* are *not* equally probable. In fact, the system will spend 70 times longer in configuration V than in configuration I, because configuration V includes 70 times as many microstates. Configuration V is clearly the most favored configuration, with the greatest probability of occurrence. This begins to be familiar, reminding us that in thermal equilibrium we are likely to find the molecules of a gas uniformly distributed throughout the volume of their container.

Eight molecules in a box ($N = 8$) are not very many on which to base a conclusion about the real world. Let us increase the number N to 50 (still a small number!) and again compare the length of time in which 25 molecules are in each half of the box to that in which all the molecules are in a given half of the box. The ratio is not 70 to 1 (as it is for $N = 8$ in Table 24-1) but about 1.2×10^{14} to 1. If you could count the microstates for the configuration $N_1 = N_2 = 25$ at the rate of one per second, it would take you about four million years to complete the task! Imagine how many microstates there are for the much more reasonable case of $N \approx 10^{22}$, which is about the number of air molecules in a child's balloon. The probability is then completely overwhelming for an even distribution of molecules between the two halves of the box.

For large values of N, nearly all the microstates correspond to an essentially equal division of the molecules between the two halves of the box, as Fig. 24-17 indicates. Even though the measured temperature and pressure of the gas remain constant, at the molecular level the gas is churning away endlessly, its molecules "visiting" all the possible microstates with equal probabilities. However, so few microstates lie outside the narrow central peak of Fig. 24-17 that the gas molecules can be considered always to be evenly divided between the two halves of the box. There is a chance that all of the molecules will find themselves in one half of the box but, for large numbers of molecules, it is a vanishingly small one.

TABLE 24-1 Eight Molecules in a Box

Configuration	N_1	N_2	*Multiplicity*[a] w	*Entropy* (10^{-23} J/K)[b]
I	8	0	1	0
II	7	1	8	2.87
III	6	2	28	4.60
IV	5	3	56	5.56
V	4	4	70	5.86
VI	3	5	56	5.56
VII	2	6	28	4.60
VIII	1	7	8	2.87
IX	0	8	1	0
Total number of microstates			256	

[a] Calculated from Eq. 24-19 [b] Calculated from Eq. 24-20

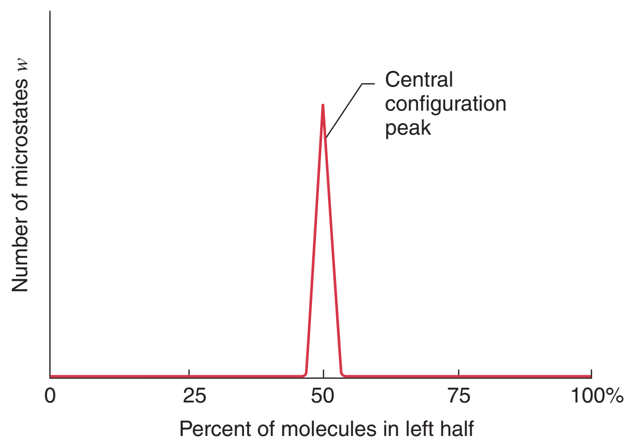


FIGURE 24-17. For a *large* number of molecules in a box, a plot of the number of microstates that require various percentages of the molecules to be in the left half of the box. Nearly all the microstates correspond to an approximately equal sharing of the molecules between the two halves of the box; those microstates form the central *configuration peak* on the plot. For $N \approx 10^{22}$, the central configuration peak would be much too narrow to be drawn on this plot.

We see that, left to themselves, systems tend toward configurations with the highest value of w—that is, toward configurations with the highest probability of occurrence. We have also seen that systems tend toward configurations with the greatest entropy. There must be a relationship between probability and entropy. Any such relationship must take into account these two facts:

1. The probability of occurrence of two subsystems is the *product* of their individual probabilities.

2. The entropy of a system consisting of two subsystems is the *sum* of their individual entropies.

That is, probabilities (as for coin tossing) are multiplicative and entropies (as for energies or volumes) are additive.

This suggests that the relationship between entropy and probability must involve a logarithm because that is a simple way (and, indeed, it is the only way) in which quantities that are multiplied are related to quantities that are added. Thus,

$$\ln (a \times b) = \ln a + \ln b.$$

The Austrian physicist Ludwig Boltzmann first pointed out the relationship between entropy and probability in 1877, by advancing the logarithmic relation, now called *Boltzmann's entropy equation,*

$$S = k \ln w, \tag{24-20}$$

in which k ($= 1.38 \times 10^{-23}$ J/K) is the Boltzmann constant that we first encountered in Section 21-5 and w is the multiplicity associated with the configuration whose entropy S we wish to calculate. We have used Eq. 24-20 to calculate the entropy of the nine configurations of Table 24-1.

In using Eq. 24-19 to calculate w, a problem may arise in that your calculator will flash an overflow signal if you try to find the factorial of a number much greater than a few hundred—still a very small number where macroscopic systems are concerned. Fortunately, there is a very good approximation (known as *Stirling's approximation*) not for $N!$ but for $\ln N!$. As Eqs. 24-19 and 24-20 show, $\ln N!$ is just what we need to calculate the entropy. Stirling's approximation is

$$\ln N! \approx N \ln N - N. \tag{24-21}$$

(Incidentally, the Stirling of the approximation and the Stirling whose engine we described in Fig. 24-11 are not the same person.)

Entropy and Disorder

Boltzmann's entropy equation (Eq. 24-20) can be used to calculate the entropy of thermodynamic systems much more complex than the simple one we have examined—namely, the distribution of molecules between the two halves of a box. Let us apply it qualitatively—for example, to the observation that a stirred cup of coffee, swirling in its cup, will eventually stop swirling and come to rest. We will focus our attention, not on the positions of the molecules (as we did for the molecules-in-a-box problem) but on their velocities.

The final resting state, in which the velocities of the molecules are randomly directed, contains many more microstates than does the initial state, in which the velocities of most molecules are pointed in or near the direction of swirl. Put another way, there are many more ways that you could assign velocities to the molecules to produce a resting cup of coffee than to produce a swirling cup of coffee. That is,

$$w_{\text{resting}} > w_{\text{swirling}}.$$

From Eq. 24-20 it then follows that

$$S_{\text{resting}} > S_{\text{swirling}}.$$

According to the second law of thermodynamics, the coffee—left to itself—will change spontaneously in the direction in which its entropy increases. It will never change in the opposite direction. That is, the "natural" behavior is from swirling to resting. A resting cup of coffee will never start to swirl all by itself.

Entropy is often associated with *disorder* and the second law of thermodynamics is sometimes cast as a statement that the disorder of a closed system always increases. This seems clear enough for our swirling cup of coffee. The final state, with the randomly directed motions of its molecules, is more disordered than the initial state, with the directed motions of a substantially large number of its molecules. In general, however, the association of entropy with disorder requires a careful definition of disorder, appropriate to the process at hand. Responding to the order evident in much of our life experience—including life itself—the physicist and science writer Timothy Ferris, perhaps pushing the concept to a limit, has written

> "Entropy can decrease locally even while it increases on the cosmic scale. One might go so far as to say that the excitement generated by life, art, science, and the spectacle of a bustling city with its libraries and theaters is at root the excitement of seeing the law of entropy being defeated—in one place at least, for a while."

SAMPLE PROBLEM 24-9. (*a*) In how many independent ways can 200 molecules be divided evenly between the two halves of a box? (*b*) How many microstates are there that correspond to 150 molecules in one half of the box and 50 in the other?

Solution (*a*) In this problem we have $N = 200$ and $N_1 = N_2 = 100$. From Eq. 24-19,

$$\begin{aligned} w &= \frac{N!}{N_1!\,N_2!} \\ &= \frac{200!}{100!\,100!} = \frac{2.22 \times 10^{373}}{(3.72 \times 10^{156})(3.72 \times 10^{156})} \\ &= 1.60 \times 10^{60}. \end{aligned}$$

Note how large the factorials are. You can calculate them on a hand calculator using Stirling's approximation (Eq. 24-21).

(*b*) In this case, $N = 200$, $N_1 = 150$, and $N_2 = 50$. Again using Eq. 24-19 we have

$$w = \frac{N!}{N_1!\,N_2!}$$

$$= \frac{200!}{(150!)\,(50!)} = \frac{2.22 \times 10^{373}}{(1.86 \times 10^{261})(1.71 \times 10^{63})}$$

$$= 6.97 \times 10^{48}.$$

By dividing these two multiplicities, you can learn that the 100/100 split is about 200 billion times more likely than the 150/50 split. As N increases, the $N_1 = N_2$ split comes to dominate, as Fig. 24-17 shows.

MULTIPLE CHOICE

24-1 One-Way Processes

24-2 Defining Entropy Change

1. For which of the following processes is the entropy change zero?
(A) Isobaric (B) Isothermal
(C) Adiabatic (D) Constant volume
(E) None of these, since $\Delta S > 0$ for all processes.

2. One mole of an ideal gas is originally at p_0, V_0, and T_0. The gas is heated at constant volume to $2T_0$, then allowed to expand at constant temperature to $2V_0$, and finally it is allowed to cool at constant pressure to T_0. The net entropy change for this ideal gas is
(A) $\Delta S = (5R/2)\ln 2$. (B) $\Delta S = 5R/2$.
(C) $\Delta S = R\ln 2$. (D) $\Delta S = 3R/2$.
(E) $\Delta S = 0$.

24-3 Entropy Change for Irreversible Processes

3. A block of aluminum originally at 80°C is placed into an insulated container of water originally at 25°C. After a while the system reaches an equilibrium temperature of 31°C.
(*a*) During this process
(A) $\Delta S_{\text{aluminum}} > 0$.
(B) $\Delta S_{\text{aluminum}} = 0$.
(C) $\Delta S_{\text{aluminum}} < 0$.
(*b*) During this process
(A) $\Delta S_{\text{water}} > 0$.
(B) $\Delta S_{\text{water}} = 0$.
(C) $\Delta S_{\text{water}} < 0$.
(*c*) During this process
(A) $|\Delta S_{\text{water}}| > |\Delta S_{\text{aluminum}}|$.
(B) $|\Delta S_{\text{water}}| = |\Delta S_{\text{aluminum}}|$.
(C) $|\Delta S_{\text{water}}| < |\Delta S_{\text{aluminum}}|$.

24-4 The Second Law of Thermodynamics

4. Which of the following is a consequence of the second law of thermodynamics?
(A) Heat can flow only from high temperature to low temperature.
(B) Objects in contact will tend toward having the same temperature.
(C) Any system that produces order from disorder must have an external influence.

24-5 Entropy and the Performance of Engines

5. A Carnot engine discharges 3 J of heat into the low-temperature reservoir for every 2 J of work output.
(*a*) What is the efficiency of this Carnot engine?
(A) 1/3 (B) 2/5 (C) 3/5 (D) 2/3
(*b*) For this engine $T_L = 27$°C. What can be concluded about T_H?
(A) $T_H = 627$°C. (B) $T_H = 227$°C.
(C) $T_H > 627$°C. (D) $T_H < 227$°C.
(E) 227°C $< T_H <$ 627°C.

24-6 Entropy and the Performance of Refrigerators

6. Consider an ideal heat pump and a perfect electric heater. The electric heater converts 100% of the electrical energy into heat energy; the heat pump converts 100% of the electrical energy into work, which then powers a Carnot refrigerator. Which is the more "efficient" way to heat a home? (Ignore maintenance or start-up costs.)
(A) The electric heater is always more efficient.
(B) The heat pump is always more efficient.
(C) The heat pump is more efficient if the outside temperature is not too warm.
(D) The heat pump is more efficient if the outside temperature is not too cold.

24-7 The Efficiencies of Real Engines

7. A real engine has an efficiency of 33%. The engine has a work output of 24 J per cycle.
(*a*) How much heat energy is extracted from the high-temperature reservoir per cycle?
(A) 8 J (B) 16 J (C) 48 J (D) 72 J
(E) The question can be answered only if the engine is a Carnot engine.
(*b*) How much heat energy is discharged into the low-temperature reservoir per cycle?
(A) 8 J (B) 16 J (C) 48 J (D) 72 J
(E) The question can be answered only if the engine is a Carnot engine.
(*c*) For this engine $T_L = 27$°C. What can be concluded about T_H?
(A) $T_H = 450$°C. (B) $T_H = 177$°C.
(C) $T_H > 177$°C. (D) $T_H < 177$°C.
(E) 177°C $< T_H <$ 450°C.

8. A real engine operates at 75% of the efficiency of a Carnot engine operating between the same two temperatures. This real engine has a power output of 100 W and discharges heat into the 27°C low-temperature reservoir at a rate of 300 J/s. What is the temperature of the high-temperature reservoir?
(A) 27°C (B) 77°C (C) 127°C (D) 177°C

24-8 The Second Law Revisited

9. An inventor claims to have invented four engines, each of which operates between heat reservoirs at 400 and 300 K.

Data on each engine, per cycle of operation, are as follows:

	Q_{in}	Q_{out}	$\lvert W \rvert$
Engine 1	200 J	−175 J	40 J
Engine 2	500 J	−200 J	400 J
Engine 3	600 J	−200 J	400 J
Engine 4	100 J	− 90 J	10 J

(*a*) Which of these engines violate the first law of thermodynamics? (There may be more than one correct answer!)

(A) 1 (B) 2 (C) 3 (D) 4

(*b*) Which of these engines violate the second law of thermodynamics? (There may be more than one correct answer!)

(A) 1 (B) 2 (C) 3 (D) 4

24-9 A Statistical View of Entropy

10. Ten identical particles are to be divided up into two containers.

(*a*) How many microstates belong to the configuration of three particles in one container and seven in the other?

(A) 120 (B) 30240
(C) 3628800 (D) 6.3×10^9

(*b*) How many different configurations are possible?

(A) 1 (B) 11
(C) 120 (D) 1024
(E) 3628800

(*c*) What is the total number of microstates for the ten-particle system?

(A) 1 (B) 11
(C) 120 (D) 1024
(E) 3628800

(*d*) Which configuration has the largest number of microstates?

(A) 0, 10 (B) 3, 7 (C) 4, 6 (D) 5, 5

11. Six identical molecules are in one box, and two are in another box. The two boxes are brought together and the molecules mix together so that four molecules are in each box. What is the change in entropy when this happens?

(A) $k/2$ (B) $k \ln (5/2)$
(C) $k \ln (4/3)$ (D) $k \ln 20$

QUESTIONS

1. Are any of the following phenomena reversible: (*a*) breaking an empty soda bottle; (*b*) mixing a cocktail; (*c*) winding a watch; (*d*) melting an ice cube in a glass of iced tea; (*e*) burning a log of firewood; (*f*) puncturing an automobile tire; (*g*) heating electrically an insulated block of metal; (*h*) isothermally expanding a nonideal gas against a piston; (*i*) finishing the “Unfinished Symphony”; (*j*) writing this book?

2. Give some examples of irreversible processes in nature.

3. Are there any natural processes that are reversible?

4. Give a qualitative explanation of how frictional forces between moving surfaces produce internal energy. Why does the reverse process (internal energy producing relative motion of those surfaces) not occur?

5. Is a human being a heat engine? Explain.

6. Could we not just as well define the efficiency of an engine as $\epsilon = \lvert W \rvert / \lvert Q_{out} \rvert$ rather than as $\epsilon = \lvert W \rvert / \lvert Q_{in} \rvert$? Why don’t we?

7. The efficiencies of nuclear power plants are less than those of fossil-fuel plants. Why?

8. Can a given amount of mechanical energy be converted completely into heat energy? If so, give an example.

9. An inventor suggested that a house might be heated in the following manner: A system resembling a refrigerator draws heat from the Earth and rejects heat to the house. The inventor claimed that the heat supplied to the house can exceed the work done by the engine of the system. What is your comment?

10. Comment on the statement: “A heat engine converts disordered mechanical motion into organized mechanical motion.”

11. Is a heat engine operating between the warm surface water of a tropical ocean and the cooler water beneath the surface a possible concept? Is the idea practical? (See “Solar Sea Power,” by Clarence Zener, *Physics Today,* January 1973, p. 48.)

12. Can we calculate the work done during an irreversible process in terms of an area on a pV diagram? Is any work done?

13. If a Carnot engine is independent of the working substance, then perhaps real engines should be similarly independent, to a certain extent. Why then, for real engines, are we so concerned to find suitable fuels such as coal, gasoline, or fissionable material? Why not use stones as fuel?

14. Under what conditions would an ideal heat engine be 100% efficient?

15. What factors reduce the efficiency of a heat engine from its ideal value?

16. You wish to increase the efficiency of a Carnot engine as much as possible. You can do this by increasing T_H a certain amount, keeping T_L constant, or by decreasing T_L the same amount, keeping T_H constant. Which would you do?

17. Explain why a room can be warmed by leaving open the door of an oven but cannot be cooled by leaving open the door of a kitchen refrigerator.

18. Why do you get poorer gasoline mileage from your car in winter than in summer?

19. From time to time inventors will claim to have perfected a device that does useful work but consumes no (or very little) fuel. What do you think is most likely true in such cases: (*a*) the claimants are right, (*b*) the claimants are mistaken in their measurements, or (*c*) the claimants are swindlers? Do you think that such a claim should be examined closely by a panel of scientists and engineers? In your opinion, would the time and effort be justified?

20. We have seen that real engines always discard substantial amounts of heat to their low-temperature reservoirs. It seems a shame to throw this heat energy away. Why not use this heat to run a second engine, the low-temperature reservoir of the first engine serving as the high-temperature reservoir of the second?

21. Give examples in which the entropy of a system decreases and explain why the second law of thermodynamics is not violated.

22. Do living things violate the second law of thermodynamics? As a chicken grows from an egg, for example, it becomes more and more ordered and organized. Increasing entropy, however, calls for disorder and decay. Is the entropy of a chicken actually decreasing as it grows?

23. Two containers of gases at different temperatures are isolated from the surroundings and separated from each other by a partition that allows heat exchange. What would have to happen if the entropy were to decrease? To increase? What is likely to happen?

24. Is there a change in entropy in purely mechanical motions?

25. Show that the total entropy increases when work is converted into heat by friction between sliding surfaces. Describe the increase in disorder.

26. Heat energy flows from the Sun to the Earth. Show that the entropy of the Earth–Sun system increases during this process.

27. Is it true that the heat energy of the universe is steadily growing less available? If so, why?

28. Consider a box containing a very small number of molecules—say, five. It must sometimes happen by chance that all these molecules find themselves in the left half of the box, the right half being completely empty. This is simply the reverse of free expansion, a process that we have declared to be *irreversible.* What is your explanation?

29. A rubber band feels warmer than its surroundings immediately after it is quickly stretched; it becomes noticeably cooler when it is allowed to contract suddenly. Also, a rubber band supporting a load contracts on being heated. Explain these observations using the fact that the molecules of rubber consist of intertwined and cross-linked long chains of atoms in roughly random orientation.

30. What entropy change occurs, if any, when a pack of 52 cards is shuffled into one particular arrangement? Is the concept of entropy appropriate in this case? If so, explain how one could get useful cooling by carrying out this process adiabatically.

31. Discuss the following comment of Panofsky and Phillips: "From the standpoint of formal physics there is only one concept which is asymmetric in the time, namely, entropy. But this makes it reasonable to assume that the second law of thermodynamics can be used to ascertain the sense of time independent of any frame of reference; that is, we shall take the positive direction of time to be that of statistically increasing disorder, or increasing entropy." (See, in this connection, "The Arrow of Time," by David Layzer, *Scientific American,* December 1975, p. 56.)

32. Explain the statement: "Cosmic rays continually *decrease* the entropy of the Earth on which they fall." Why does this not contradict the second law of thermodynamics?

33. When we put cards together in a deck or put bricks together to build a house, for example, we increase the order in the physical world. Does this violate the second law of thermodynamics? Explain.

34. Can one use terrestrial thermodynamics, which is known to apply to bounded and isolated bodies, for the whole universe? If so, is the universe bounded and from what is the universe isolated?

35. Temperature and pressure are examples of *intensive* properties of a system, their values for any sample of the system being independent of the size of the sample. However, entropy, like internal energy, is an *extensive* property, its value for any sample of a system being proportional to the size of the sample. Discuss.

36. The first and second laws of thermodynamics may be paraphrased, respectively, as follows: (1) You cannot win. (2) You cannot even break even. Explain in what sense these are permissible restatements.

EXERCISES

24-1 One-Way Processes

24-2 Defining Entropy Change

1. An ideal gas undergoes a reversible isothermal expansion at 132°C. The entropy of the gas increases by 46.2 J/K. How much heat is absorbed?

2. In Fig. 24-18, suppose that the change in entropy of the system in passing from state a to state b along path 1 is $+0.60$ J/K. What is the entropy change in passing (a) from state a to b along path 2 and (b) from state b to a along path 2?

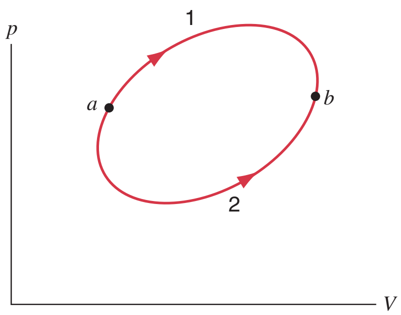


FIGURE 24-18. Exercise 2.

3. For the Carnot cycle shown in Fig. 24-19, calculate (a) the heat that enters and (b) the work done on the system.

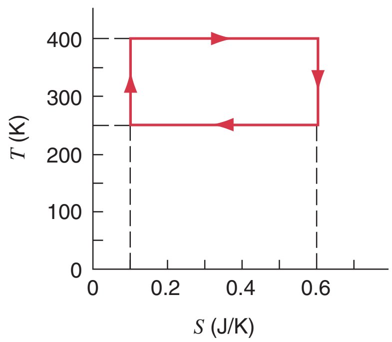


FIGURE 24-19. Exercise 3.

4. Four moles of an ideal gas are caused to expand from a volume V_1 to a volume $V_2 = 3.45V_1$. (a) If the expansion is isothermal at the temperature $T = 410$ K, find the work done on the expanding gas. (b) Find the change in entropy, if any. (c) If the expansion is reversibly adiabatic instead of isothermal, what is the entropy change?

5. Find (*a*) the heat absorbed and (*b*) the change in entropy of a 1.22-kg block of copper whose temperature is increased reversibly from 25.0 to 105°C.

6. Heat can be transferred from water at 0°C and atmospheric pressure without causing the water to freeze, if done with little disturbance of the water. Suppose the water is cooled to −5.0°C before ice begins to form. Find the change in entropy occurring during the sudden freezing of 1.0 g of water that then takes place.

24-3 Entropy Change for Irreversible Processes

7. An ideal gas undergoes an isothermal expansion at 77°C increasing its volume from 1.3 to 3.4 L. The entropy change of the gas is 24 J/K. How many moles of gas are present?

8. Suppose that the same amount of heat energy—say, 260 J—is transferred by conduction from a heat reservoir at a temperature of 400 K to another reservoir, the temperature of which is (*a*) 100 K, (*b*) 200 K, (*c*) 300 K, and (*d*) 360 K. Calculate the changes in entropy and discuss the trend.

9. A brass rod is in thermal contact with a heat reservoir at 130°C at one end and a heat reservoir at 24.0°C at the other end. (*a*) Compute the total change in the entropy arising from the process of conduction of 1200 J of heat through the rod. (*b*) Does the entropy of the rod change in the process?

24-4 The Second Law of Thermodynamics

10. A 50.0-g block of copper having a temperature of 400 K is placed in an insulating box with a 100-g block of lead having a temperature of 200 K. (*a*) What is the equilibrium temperature of this two-block system? (*b*) What is the change in the internal energy of the two-block system as it changes from the initial condition to the equilibrium condition? (*c*) What is the change in the entropy of the two-block system? (See Table 23-2.)

11. A mixture of 1.78 kg of water and 262 g of ice at 0°C is, in a reversible process, brought to a final equilibrium state where the water/ice ratio, by mass, is 1:1 at 0°C. (*a*) Calculate the entropy change of the system during this process. (*b*) The system is then returned to the first equilibrium state, but in an irreversible way (by using a Bunsen burner, for instance). Calculate the entropy change of the system during this process. (*c*) Show that your answer is consistent with the second law of thermodynamics.

12. In a specific heat experiment, 196 g of aluminum at 107°C is mixed with 52.3 g of water at 18.6°C. (*a*) Calculate the equilibrium temperature. Find the entropy change of (*b*) the aluminum and (*c*) the water. (*d*) Calculate the entropy change of the system. (Hint: See Sample Problem 24-3.)

24-5 Entropy and the Performance of Engines

13. A heat engine absorbs 52.4 kJ of heat and exhausts 36.2 kJ of heat each cycle. Calculate (*a*) the efficiency and (*b*) the work done by the engine per cycle.

14. A car engine delivers 8.18 kJ of work per cycle. (*a*) Before a tune-up, the efficiency is 25.0%. Calculate, per cycle, the heat absorbed from the combustion of fuel and the heat exhausted to the atmosphere. (*b*) After a tune-up, the efficiency is 31.0%. What are the new values of the quantities calculated in (*a*)?

15. Calculate the efficiency of a fossil-fuel power plant that consumes 382 metric tons of coal each hour to produce useful work at the rate of 755 MW. The heat of combustion of coal is 28.0 MJ/kg.

16. Engine A, compared to engine B, produces, per cycle, five times the work but receives three times the heat input and exhausts out twice the heat. Determine the efficiency of each engine.

17. In a Carnot cycle, the isothermal expansion of an ideal gas takes place at 412 K and the isothermal compression at 297 K. During the expansion, 2090 J of heat energy are transferred to the gas. Determine (*a*) the work performed by the gas during the isothermal expansion, (*b*) the heat rejected from the gas during the isothermal compression, and (*c*) the work done on the gas during the isothermal compression.

18. A Carnot engine has an efficiency of 22%. It operates between heat reservoirs differing in temperature by 75 C°. Find the temperatures of the reservoirs.

19. For the Carnot cycle illustrated in Fig. 24-8, show that the work done by the gas during process *BC* has the same absolute value as the work done on the gas during process *DA*.

20. (*a*) In a two-stage Carnot heat engine, a quantity of heat $|Q_1|$ is absorbed at a temperature T_1, work $|W_1|$ is done, and a quantity of heat $|Q_2|$ is expelled at a lower temperature T_2, by the first stage. The second stage absorbs the heat expelled by the first, does work $|W_2|$, and expels a quantity of heat $|Q_3|$ at a lower temperature T_3. Prove that the efficiency of the combination is $(T_1 - T_3)/T_1$. (*b*) A combination mercury–steam turbine takes saturated mercury vapor from a boiler at 469°C and exhausts it to heat a steam boiler at 238°C. The steam turbine receives steam at this temperature and exhausts it to a condenser at 37.8°C. Calculate the maximum efficiency of the combination.

21. In a steam locomotive, steam at a boiler pressure of 16.0 atm enters the cylinders, is expanded adiabatically to 5.60 times its original volume, and then is exhausted to the atmosphere. Calculate (*a*) the steam pressure after expansion and (*b*) the greatest possible efficiency of the engine.

22. One mole of an ideal monatomic gas is used as the working substance of an engine that operates on the cycle shown in Fig. 24-20. Calculate (*a*) the work done by the engine per cycle, (*b*) the heat added per cycle during the expansion stroke *abc*, and (*c*) the engine efficiency. (*d*) What is the Carnot efficiency of an engine operating between the highest and lowest temperatures present in the cycle? How does this compare to the efficiency calculated in (*c*)? Assume that $p_1 = 2p_0$, $V_1 = 2V_0$, $p_0 = 1.01 \times 10^5$ Pa, and $V_0 = 0.0225$ m^3.

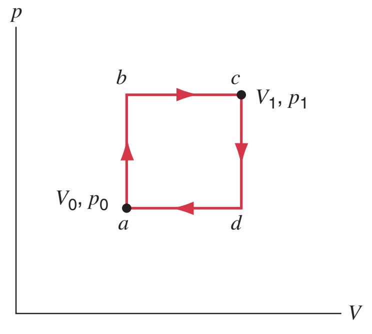


FIGURE 24-20. Exercise 22.

24-6 Entropy and the Performance of Refrigerators

23. To make some ice, a freezer extracts 185 kJ of heat at −12.0°C. The freezer has a coefficient of performance of 5.70. The room temperature is 26.0°C. (*a*) How much heat is delivered to the room? (*b*) How much work is required to run the freezer?

24. A refrigerator does 153 J of work to transfer 568 J of heat from its cold compartment. (*a*) Calculate the refrigerator's coefficient of performance. (*b*) How much heat is exhausted to the kitchen?

25. How much work must be done to extract 10.0 J of heat (*a*) from a reservoir at 7°C and transfer it to one at 27°C by means of a refrigerator using a Carnot cycle; (*b*) from one at −73°C to one at 27°C; (*c*) from one at −173°C to one at 27°C; and (*d*) from one at −223°C to one at 27°C?

26. Apparatus that liquefies helium is in a laboratory at 296 K. The helium in the apparatus is at 4.0 K. If 150 mJ of heat is transferred from the helium, find the minimum amount of heat delivered to the laboratory.

27. An air conditioner takes air from a room at 70°F and transfers it to the outdoors, which is at 95°F. For each joule of electrical energy required to run the refrigerator, how many joules of heat are transferred from the room?

28. An inventor claims to have created a heat pump that draws heat from a lake at 3.0°C and delivers heat at a rate of 20 kW to a building at 35°C, while using only 1.9 kW of electrical power. How would you judge the claim?

29. (*a*) A Carnot engine operates between a hot reservoir at 322 K and a cold reservoir at 258 K. If it absorbs 568 J of heat per cycle at the hot reservoir, how much work per cycle does it deliver? (*b*) If the same engine, working in reverse, functions as a refrigerator between the same two reservoirs, how much work per cycle must be supplied to transfer 1230 J of heat from the cold reservoir?

30. A heat pump is used to heat a building. The outside temperature is −5.0°C and the temperature inside the building is to be maintained at 22°C. The coefficient of performance is 3.8, and the pump delivers 7.6 MJ of heat to the building each hour. At what rate must work be done to run the pump?

31. In a refrigerator the low-temperature coils are at a temperature of −13°C and the compressed gas in the condenser has a temperature of 25°C. Find the coefficient of performance of a Carnot refrigerator operating between these temperatures.

32. The motor in a refrigerator has a power output of 210 W. The freezing compartment is at −3.0°C and the outside air is at 26°C. Assuming that the efficiency is 85% of the ideal, calculate the amount of heat that can be extracted from the freezing compartment in 15 min.

33. A Carnot engine works between temperatures T_1 and T_2. It drives a Carnot refrigerator that works between two different temperatures T_3 and T_4 (see Fig. 24-21). Find the ratio $|Q_3|/|Q_1|$ in terms of the four temperatures.

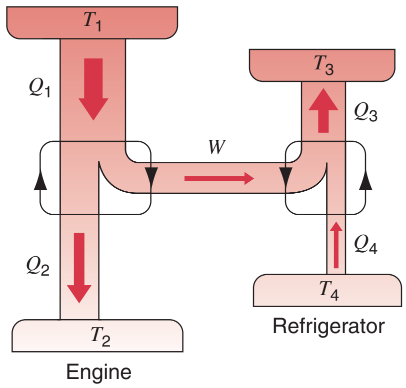


FIGURE 24-21. Exercise 33.

24-7 The Efficiencies of Real Engines
24-8 The Second Law Revisited
24-9 A Statistical View of Entropy

34. (*a*) Derive Stirling's approximation (Eq. 24-21) by substituting an integral for the sum in the expression

$$\ln N! = \sum_{x=1}^{N} \ln x \approx \int_1^N \ln x\, dx.$$

(*b*) For what values of N is the error in Stirling's approximation less than 1%, 0.1%, and 1×10^{-6}?

35. Consider a container that is divided into two sections. (*a*) Initially N molecules of a gas are in one section, and the other side is empty. Compute the multiplicity of this initial state. (*b*) After a hole is punched in the partition, the gas fills the entire container uniformly, with $N/2$ molecules on each side of the partition. Find the multiplicity of the final state. (*c*) Show that the change in entropy is $\Delta S = kN \ln 2$. (*d*) Compare this result with the result of Sample Problem 24-4 for the entropy change in a free expansion, and explain the similarities of the two results.

PROBLEMS

1. At very low temperatures, the molar specific heat of many solids is (approximately) proportional to T^3; that is, $C_V = AT^3$ where A depends on the particular substance. For aluminum, $A = 3.15 \times 10^{-5}$ J/mol·K^4. Find the entropy change of 4.8 mol of aluminum when its temperature is raised from 5.0 to 10 K.

2. An object of constant heat capacity C is heated from an initial temperature T_i to a final temperature T_f by being placed in contact with a reservoir at T_f. Represent the process on a graph of C/T versus T and show graphically that the total change in entropy ΔS (object plus reservoir) is positive and (*b*) show how the use of reservoirs at intermediate temperatures would allow the process to be carried out in a way that makes ΔS as small as desired.

3. One mole of an ideal monatomic gas is caused to go through the cycle shown in Fig. 24-22. (*a*) How much work is done on the gas in expanding the gas from a to c along path abc? (b) What is the change in internal energy and entropy in going from b to c? (c) What is the change in internal energy and entropy in going through one complete cycle? Express all answers in terms of the pressure p_0 and volume V_0 at point a in the diagram.

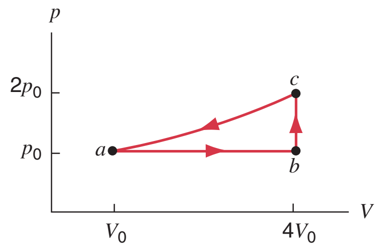


FIGURE 24-22. Problem 3.

4. One mole of an ideal diatomic gas is caused to pass through the cycle shown on the pV diagram in Fig. 24-23 where $V_2 = 3V_1$. Determine, in terms of p_1, V_1, T_1, and R: (a) p_2, p_3, and T_3; and (b) W, Q, ΔE_{int}, and ΔS for all three processes.

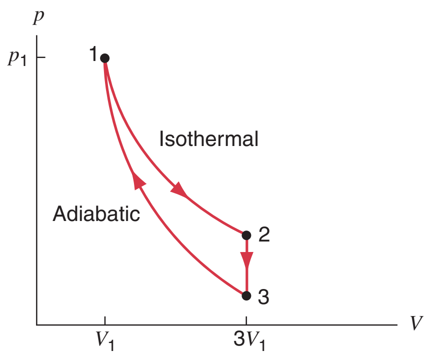


FIGURE 24-23. Problem 4.

5. One mole of a monatomic ideal gas is taken from an initial state of pressure p_0 and volume V_0 to a final state of pressure $2p_0$ and volume $2V_0$ by two different processes. (I) It expands isothermally until its volume is doubled, and then its pressure is increased at constant volume to the final state. (II) It is compressed isothermally until its pressure is doubled, and then its volume is increased at constant pressure to the final state. Show the path of each process on a pV diagram. For each process calculate in terms of p_0 and V_0: (a) the heat absorbed by the gas in each part of the process; (b) the work done on the gas in each part of the process; (c) the change in internal energy of the gas, $E_{\text{int, f}} - E_{\text{int, i}}$; and ($d$) the change in entropy of the gas, $S_f - S_i$.

6. A 12.6-g ice cube at −10.0°C is placed in a lake whose temperature is +15.0°C. Calculate the change in entropy of the system as the ice cube comes to thermal equilibrium with the lake. (Hint: Will the ice cube affect the temperature of the lake?)

7. A system consists of two objects that are allowed to come into thermal contact. Object 1 has mass m_1, specific heat capacity c_1, and is originally at temperature $T_{1,\text{i}}$. Object 2 has mass m_2, specific heat capacity c_2, and is originally at temperature $T_{2,\text{i}} < T_{1,\text{i}}$. As object 1 slowly cools, object 2 slowly warms. (a) Write an expression for the temperature of object 2, T_2, as a function of the temperature of object 1, T_1. (b) Find the change in entropy of the system ΔS as a function of T_1. (c) Show that ΔS is a maximum when both objects have the same temperature.

8. Two moles of a monatomic ideal gas are caused to go through the cycle shown in Fig. 24-24. Process bc is a reversible adiabatic expansion. Also, $p_b = 10.4$ atm, $V_b = 1.22$ m^3 and $V_c = 9.13$ m^3. Calculate (a) the heat added to the gas, (b) the heat leaving the gas, (c) the net work done by the gas, and (d) the efficiency of the cycle.

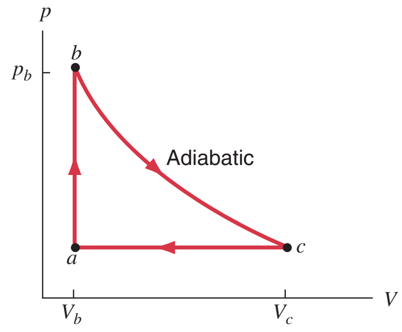


FIGURE 24-24. Problem 8.

9. One mole of a monatomic ideal gas initially at a volume of 10 L and a temperature 300 K is heated at constant volume to a temperature of 600 K, allowed to expand isothermally to its initial pressure, and finally compressed isobarically (that is, at constant pressure) to its original volume, pressure, and temperature. (a) Compute the heat input to the system during one cycle. (b) What is the net work done by the gas during one cycle? (c) What is the efficiency of this cycle?

10. A gasoline internal combustion engine can be approximated by the cycle shown in Fig. 24-25. Assume an ideal diatomic gas and use a compression ratio of 4:1 ($V_d = 4V_a$). Assume that $p_b = 3p_a$. (a) Determine the pressure and temperature of each of the vertex points of the pV diagram in terms of p_a and T_a. (b) Calculate the efficiency of the cycle.

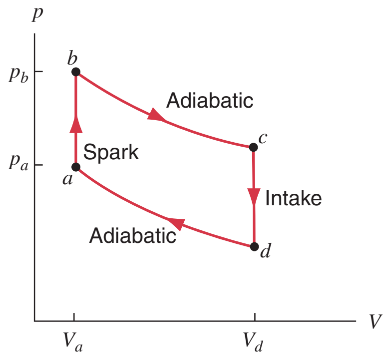


FIGURE 24-25. Problem 10.

11. (a) Plot accurately a Carnot cycle on a pV diagram for 1.00 mol of an ideal gas. Let point A (see Fig. 24-8) correspond to $p = 1.00$ atm, $T = 300$ K, and let point B correspond to 0.500 atm, $T = 300$ K; take the low-temperature reservoir to be at 100 K. Let $\gamma = 1.67$. (b) Compute graphically the work done in this cycle. (c) Compute the work analytically.

APPENDIX A

THE INTERNATIONAL SYSTEM OF UNITS (SI)*

The SI Base Units

Quantity	*Name*	*Symbol*	*Definition*
Length	meter	m	". . . the length of the path traveled by light in vacuum in 1/299,792,458 of a second." (1983)
Mass	kilogram	kg	". . . the mass of the international prototype of the kilogram." (1901)
Time	second	s	". . . the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom." (1967)
Electric current	ampere	A	". . . that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per meter of length." (1948)
Thermodynamic temperature	kelvin	K	". . . the fraction 1/273.16 of the thermodynamic temperature of the triple point of water." (1967)
Amount of substance	mole	mol	". . . the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon 12." (1971)
Luminous intensity	candela	cd	". . . the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and that has a radiant intensity in that direction of 1/683 watt per steradian." (1979)

* Adapted from "Guide for the Use of the International System of Units (SI)," National Bureau of Standards Special Publication 811, 1995 edition. The definitions above were adopted by the General Conference of Weights and Measures, an international body, on the dates shown. In this book we do not use the candela.

Some SI Derived Units

Quantity	*Name of Unit*	*Symbol*	*Equivalent*
Area	square meter	m^2	
Volume	cubic meter	m^3	
Frequency	hertz	Hz	s^{-1}
Mass density (density)	kilogram per cubic meter	kg/m^3	
Speed, velocity	meter per second	m/s	
Angular velocity	radian per second	rad/s	
Acceleration	meter per second squared	m/s^2	
Angular acceleration	radian per second squared	rad/s^2	
Force	newton	N	$kg \cdot m/s^2$
Pressure	pascal	Pa	N/m^2
Work, energy, quantity of heat	joule	J	$N \cdot m$
Power	watt	W	J/s
Quantity of electricity	coulomb	C	$A \cdot s$
Potential difference, electromotive force	volt	V	$N \cdot m/C$
Electric field	volt per meter	V/m	N/C
Electric resistance	ohm	Ω	V/A
Capacitance	farad	F	$A \cdot s/V$
Magnetic flux	weber	Wb	$V \cdot s$
Inductance	henry	H	$V \cdot s/A$
Magnetic field	tesla	T	Wb/m^2, $N/A \cdot m$
Entropy	joule per kelvin	J/K	
Specific heat capacity	joule per kilogram kelvin	$J/(kg \cdot K)$	
Thermal conductivity	watt per meter kelvin	$W/(m \cdot K)$	
Radiant intensity	watt per steradian	W/sr	

The SI Supplementary Units

Quantity	*Name of Unit*	*Symbol*
Plane angle	radian	rad
Solid angle	steradian	sr

APPENDIX B

FUNDAMENTAL PHYSICAL CONSTANTS*

Constant	Symbol	Computational Value	Best (1998) Value	
			Value[a]	Uncertainty[b]
Speed of light in a vacuum	c	3.00×10^{8} m/s	2.99792458	exact
Elementary charge	e	1.60×10^{-19} C	1.602176462	0.039
Electric constant (permittivity)	ϵ_0	8.85×10^{-12} F/m	8.85418781762	exact
Magnetic constant (permeability)	μ_0	1.26×10^{-6} H/m	1.25663706143	exact
Electron mass	m_e	9.11×10^{-31} kg	9.10938188	0.079
Electron mass[c]	m_e	5.49×10^{-4} u	5.485799110	0.0021
Proton mass	m_p	1.67×10^{-27} kg	1.67262158	0.079
Proton mass[c]	m_p	1.0073 u	1.00727646688	0.00013
Neutron mass	m_n	1.67×10^{-27} kg	1.67492716	0.079
Neutron mass[c]	m_n	1.0087 u	1.00866491578	0.00054
Electron charge-to-mass ratio	e/m_e	1.76×10^{11} C/kg	1.758820174	0.040
Proton-to-electron mass ratio	m_p/m_e	1840	1836.1526675	0.0021
Planck constant	h	6.63×10^{-34} J·s	6.62606876	0.078
Electron Compton wavelength	λ_e	2.43×10^{-12} m	2.426310215	0.0073
Molar gas constant	R	8.31 J/mol·K	8.314472	1.7
Avogadro constant	N_A	6.02×10^{23} mol^{-1}	6.02214199	0.079
Boltzmann constant	k	1.38×10^{-23} J/K	1.3806503	1.7
Molar volume of ideal gas at STP[d]	V_m	2.24×10^{-2} m^3/mol	2.2413996	1.7
Faraday constant	F	9.65×10^{4} C/mol	9.64853415	0.040
Stefan–Boltzmann constant	σ	5.67×10^{-8} W/m^2·K^4	5.670400	7.0
Rydberg constant	R_∞	1.10×10^{7} m^{-1}	1.0973731568549	0.0000076
Gravitational constant	G	6.67×10^{-11} m^3/s^2·kg	6.673	1500
Bohr radius	a_0	5.29×10^{-11} m	5.291772083	0.0037
Electron magnetic moment	μ_e	9.28×10^{-24} J/T	9.28476362	0.040
Proton magnetic moment	μ_p	1.41×10^{-26} J/T	1.410606633	0.041
Bohr magneton	μ_B	9.27×10^{-24} J/T	9.27400899	0.040
Nuclear magneton	μ_N	5.05×10^{-27} J/T	5.05078317	0.040
Fine structure constant	α	1/137	1/137.03599976	0.0037
Magnetic flux quantum	Φ_0	2.07×10^{-15} Wb	2.067833636	0.039
von Klitzing constant	R_K	25800 Ω	25812.807572	0.0037

[a] Same unit and power of ten as the computational value.
[b] Parts per million.
[c] Mass given in unified atomic mass units, where 1 u = $1.66053873 \times 10^{-27}$ kg.
[d] STP—standard temperature and pressure = 0°C and 1.0 bar.

* *Source:* Peter J. Mohr and Barry N. Taylor, *Journal of Physical and Chemical Reference Data,* vol. 28, no. 6 (1999) and *Reviews of Modern Physics,* vol. 72, no. 2 (2000). See also http://physics.nist.gov/constants.

APPENDIX C

ASTRONOMICAL DATA

The Sun, the Earth, and the Moon

Property	*Sun*[a]	*Earth*	*Moon*
Mass (kg)	1.99×10^{30}	5.98×10^{24}	7.36×10^{22}
Mean radius (m)	6.96×10^{8}	6.37×10^{6}	1.74×10^{6}
Mean density (kg/m^3)	1410	5520	3340
Surface gravity (m/s^2)	274	9.81	1.67
Escape velocity (km/s)	618	11.2	2.38
Period of rotation[c] (d)	26–37[b]	0.997	27.3
Mean orbital radius (km)	2.6×10^{17}[d]	1.50×10^{8}[e]	3.82×10^{5}[f]
Orbital period	2.4×10^{8} y[d]	1.00 y[e]	27.3 d[f]

[a] The Sun radiates energy at the rate of 3.90×10^{26} W; just outside the Earth's atmosphere solar energy is received, assuming normal incidence, at the rate of 1380 W/m^2.
[b] The Sun—a ball of gas—does not rotate as a rigid body. Its rotational period varies between 26 d at the equator and 37 d at the poles.
[c] Measured with respect to the distant stars.
[d] About the galactic center.
[e] About the Sun.
[f] About the Earth.

Some Properties of the Planets

	Mercury	*Venus*	*Earth*	*Mars*	*Jupiter*	*Saturn*	*Uranus*	*Neptune*	*Pluto*
Mean distance from Sun (10^6 km)	57.9	108	150	228	778	1,430	2,870	4,500	5,900
Period of revolution (y)	0.241	0.615	1.00	1.88	11.9	29.5	84.0	165	248
Period of rotation[a] (d)	58.7	243[b]	0.997	1.03	0.409	0.426	0.451[b]	0.658	6.39
Orbital speed (km/s)	47.9	35.0	29.8	24.1	13.1	9.64	6.81	5.43	4.74
Inclination of axis to orbit	$<28°$	$\approx 3°$	23.4°	25.0°	3.08°	26.7°	97.9°	29.6°	57.5°
Inclination of orbit to Earth's orbit	7.00°	3.39°	—	1.85°	1.30°	2.49°	0.77°	1.77°	17.2°
Eccentricity of orbit	0.206	0.0068	0.0167	0.0934	0.0485	0.0556	0.0472	0.0086	0.250
Equatorial diameter (km)	4,880	12,100	12,800	6,790	143,000	120,000	51,800	49,500	2,300
Mass (Earth = 1)	0.0558	0.815	1.000	0.107	318	95.1	14.5	17.2	0.002
Average density (g/cm^3)	5.60	5.20	5.52	3.95	1.31	0.704	1.21	1.67	2.03
Surface gravity[c] (m/s^2)	3.78	8.60	9.78	3.72	22.9	9.05	7.77	11.0	0.03
Escape speed (km/s)	4.3	10.3	11.2	5.0	59.5	35.6	21.2	23.6	1.3
Known satellites	0	0	1	2	16 + rings	19 + rings	15 + rings	8 + rings	1

[a] Measured with respect to the distant stars.
[b] The sense of rotation is opposite to that of the orbital motion.
[c] Measured at the planet's equator.

APPENDIX D

PROPERTIES OF THE ELEMENTS

Element	*Symbol*	*Atomic Number, Z*	*Molar Mass* (g/mol)	*Density* (g/cm^3) at 20°C	*Melting Point* (°C)	*Boiling Point* (°C)	*Specific Heat* (J/g · C°) at 25°C
Actinium	Ac	89	(227)	10.1 (calc.)	1051	3200	0.120
Aluminum	Al	13	26.9815	2.699	660	2519	0.897
Americium	Am	95	(243)	13.7	1176	2011	—
Antimony	Sb	51	121.76	6.69	630.6	1587	0.207
Argon	Ar	18	39.948	1.6626×10^{-3}	−189.3	−185.9	0.520
Arsenic	As	33	74.9216	5.72	817 (28 at.)	614 (subl.)	0.329
Astatine	At	85	(210)	—	302	337	—
Barium	Ba	56	137.33	3.5	727	1597	0.204
Berkelium	Bk	97	(247)	14 (est.)	1050	—	—
Beryllium	Be	4	9.0122	1.848	1287	2471	1.83
Bismuth	Bi	83	208.980	9.75	271.4	1564	0.122
Bohrium	Bh	107	(264)	—	—	—	—
Boron	B	5	10.81	2.34	2075	4000	1.03
Bromine	Br	35	79.904	3.12 (liquid)	−7.2	58.8	0.226
Cadmium	Cd	48	112.41	8.65	321.1	767	0.232
Calcium	Ca	20	40.08	1.55	842	1484	0.647
Californium	Cf	98	(251)	—	900 (est.)	—	—
Carbon	C	6	12.011	2.25	3550	—	0.709
Cerium	Ce	58	140.12	6.770	798	3424	0.192
Cesium	Cs	55	132.905	1.873	28.44	671	0.242
Chlorine	Cl	17	35.453	3.214×10^{-3} (0°C)	−101.5	−34.0	0.479
Chromium	Cr	24	51.996	7.19	1907	2671	0.449
Cobalt	Co	27	58.9332	8.85	1495	2927	0.421
Copper	Cu	29	63.54	8.96	1084.6	2562	0.385
Curium	Cm	96	(247)	13.5 (calc.)	1345	—	—
Dubnium	Db	105	(262)	—	—	—	—
Dysprosium	Dy	66	162.50	8.55	1412	2567	0.170
Einsteinium	Es	99	(252)	—	860 (est.)	—	—
Erbium	Er	68	167.26	9.07	1529	2868	0.168
Europium	Eu	63	151.96	5.244	822	1529	0.182
Fermium	Fm	100	(257)	—	1527	—	—
Fluorine	F	9	18.9984	1.696×10^{-3} (0°C)	−219.6	−188.1	0.824
Francium	Fr	87	(223)	—	27	677	—
Gadolinium	Gd	64	157.25	7.90	1313	3273	0.236

(Continued)

Element	*Symbol*	*Atomic Number,* Z	*Molar Mass* (g/mol)	*Density* (g/cm^3) at 20°C	*Melting Point* (°C)	*Boiling Point* (°C)	*Specific Heat* (J/g · C°) at 25°C
Gallium	Ga	31	69.72	5.904	29.76	2204	0.371
Germanium	Ge	32	72.61	5.323	938.3	2833	0.320
Gold	Au	79	196.967	19.3	1064.18	2856	0.129
Hafnium	Hf	72	178.49	13.31	2233	4603	0.144
Hassium	Hs	108	(269)	—	—	—	—
Helium	He	2	4.0026	0.1664×10^{-3}	−272.2	−268.9	5.19
Holmium	Ho	67	164.930	8.79	1474	2700	0.165
Hydrogen	H	1	1.00797	0.08375×10^{-3}	−259.34	−252.87	14.3
Indium	ln	49	114.82	7.31	156.6	2072	0.233
Iodine	I	53	126.9044	4.93	113.7	184.4	0.145
Iridium	Ir	77	192.2	22.4	2446	4428	0.131
Iron	Fe	26	55.845	7.87	1538	2861	0.449
Krypton	Kr	36	83.80	3.488×10^{-3}	−157.4	−153.2	0.248
Lanthanum	La	57	138.91	6.145	918	3464	0.195
Lawrencium	Lr	103	(260)	—	—	—	—
Lead	Pb	82	207.19	11.35	327.5	1749	0.129
Lithium	Li	3	6.941	0.534	180.5	1342	3.58
Lutetium	Lu	71	174.97	9.84	1663	3402	0.154
Magnesium	Mg	12	24.305	1.74	650	1090	1.02
Manganese	Mn	25	54.9380	7.43	1244	2061	0.79
Meitnerium	Mt	109	(268)	—	—	—	—
Mendelevium	Md	101	(258)	—	827	—	—
Mercury	Hg	80	200.59	13.55	−38.83	356.7	0.140
Molybdenum	Mo	42	95.94	10.22	2623	4639	0.251
Neodymium	Nd	60	144.24	7.00	1021	3074	0.190
Neon	Ne	10	20.180	0.8387×10^{-3}	−248.6	−246.0	1.03
Neptunium	Np	93	(237)	20.25	644	3902	1.26
Nickel	Ni	28	58.69	8.902	1455	2913	0.444
Niobium	Nb	41	92.906	8.57	2477	4744	0.265
Nitrogen	N	7	14.0067	1.1649×10^{-3}	−210.0	−195.8	1.04
Nobelium	No	102	(259)	—	—	—	—
Osmium	Os	76	190.2	22.57	3033	5012	0.130
Oxygen	O	8	15.9994	1.3318×10^{-3}	−218.8	−183.0	0.918
Palladium	Pd	46	106.4	12.02	1555	2963	0.246
Phosphorus	P	15	30.9738	1.82	44.15	280.5	0.769
Platinum	Pt	78	195.08	21.45	1768	3825	0.133
Plutonium	Pu	94	(244)	19.84	640	3228	0.130
Polonium	Po	84	(209)	9.32	254	962	—
Potassium	K	19	39.098	0.86	63.28	759	0.757
Praseodymium	Pr	59	140.907	6.773	931	3520	0.193
Promethium	Pm	61	(145)	7.264	1042	3000 (est.)	—
Protactinium	Pa	91	(231)	15.4 (calc.)	1572	—	—
Radium	Ra	88	(226)	5.0	700	1140	—
Radon	Rn	86	(222)	9.96×10^{-3} (0°C)	−71	−61.7	0.094
Rhenium	Re	75	186.2	21.02	3186	5596	0.137
Rhodium	Rh	45	102.905	12.41	1964	3695	0.243
Rubidium	Rb	37	85.47	1.53	39.31	688	0.363
Ruthenium	Ru	44	101.07	12.41	2334	4150	0.238
Rutherfordium	Rf	104	(261)	—	—	—	—
Samarium	Sm	62	150.35	7.52	1074	1794	0.197
Scandium	Sc	21	44.956	2.99	1541	2836	0.568
Seaborgium	Sg	106	(266)	—	—	—	—
Selenium	Se	34	78.96	4.79	221	685	0.321
Silicon	Si	14	28.086	2.33	1414	3265	0.705
Silver	Ag	47	107.68	10.49	961.8	2162	0.235
Sodium	Na	11	22.9898	0.971	97.72	883	1.23

(Continued)

Element	*Symbol*	*Atomic Number, Z*	*Molar Mass* (g/mol)	*Density* (g/cm^3) at 20°C	*Melting Point* (°C)	*Boiling Point* (°C)	*Specific Heat* (J/g·C°) at 25°C
Strontium	Sr	38	87.62	2.54	777	1382	0.301
Sulfur	S	16	32.066	2.07	115.2	444.6	0.710
Tantalum	Ta	73	180.948	16.6	3017	5458	0.140
Technetium	Tc	43	(98)	11.5 (calc.)	2157	4265	—
Tellurium	Te	52	127.60	6.24	449.5	988	0.202
Terbium	Tb	65	158.924	8.23	1356	3230	0.182
Thallium	Tl	81	204.38	11.85	304	1473	0.129
Thorium	Th	90	(232)	11.72	1750	4788	0.113
Thulium	Tm	69	168.934	9.32	1545	1950	0.160
Tin	Sn	50	118.71	7.31	231.93	2602	0.228
Titanium	Ti	22	4788	4.54	1668	3287	0.523
Tungsten	W	74	183.85	19.3	3422	5555	0.132
Ununnilium*	Uun	110	(271)	—	—	—	—
Unununium*	Uuu	111	(272)	—	—	—	—
Ununbium*	Uub	112	(277)	—	—	—	—
Ununquadium*	Uuq	114	(285)	—	—	—	—
Ununhexium*	Uuh	116	(289)	—	—	—	—
Ununoctium*	Uuo	118	(293)	—	—	—	—
Uranium	U	92	(238)	18.95	1135	4131	0.116
Vanadium	V	23	50.942	6.11	1910	3407	0.489
Xenon	Xe	54	131.30	5.495×10^{-3}	-111.75	-108.0	0.158
Ytterbium	Yb	70	173.04	6.966	819	1196	0.155
Yttrium	Y	39	88.905	4.469	1522	3345	0.298
Zinc	Zn	30	65.39	7.133	419.53	907	0.388
Zirconium	Zr	40	91.22	6.506	1855	4409	0.278

Values of molar masses correspond to a mole of *atoms* of the element. For diatomic gases (H_2, O_2, N_2, etc.) the mass of a mole of *molecules* is double the tabulated value.

The values in parentheses in the column of molar masses are the mass numbers of the longest-lived isotopes of those elements that are radioactive.

All the physical properties are given for a pressure of one atmosphere except where otherwise specified.

Except for the molar mass, the data for gases are valid only when these are in their usual molecular state, such as H_2, He, O_2, Ne, etc. The specific heats of the gases are the values at constant pressure.

* Temporary names for these elements.

Source: Handbook of Chemistry and Physics, 79th edition (CRC Press, 1998). See also http://www.webelements.com.

APPENDIX E

PERIODIC TABLE OF THE ELEMENTS

	ALKALI METALS (including hydrogen)																		NOBLE GASES
1	1 H																		2 He
2	3 Li	4 Be												5 B	6 C	7 N	8 O	9 F	10 Ne
3	11 Na	12 Mg												13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
4	19 K	20 Ca		21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
5	37 Rb	38 Sr		39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
6	55 Cs	56 Ba	57–70	71 Lu	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
7	87 Fr	88 Ra	88–102	103 Lr	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Uun*	111 Uuu*	112 Uub*	113	114 Uuq*	115	116 Uuh*	117	118 Uuo*

Lanthanide series	57 La	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb
Actinide series	89 Ac	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No

* Discovery of these elements has been reported but names for them have not yet been adopted. The symbols shown represent temporary names assigned to the elements. See http://www.webelements.com for recent information on the discovery and properties of the elements.

APPENDIX F

ELEMENTARY PARTICLES

1. THE FUNDAMENTAL PARTICLES

Leptons

Particle	*Symbol*	*Anti-particle*	*Charge (e)*	*Spin (h/2π)*	*Rest Energy* (MeV)	*Mean Life* (s)	*Typical Decay Products*
Electron	e^-	e^+	−1	1/2	0.511	∞	
Electron neutrino	ν_e	$\bar{\nu}_e$	0	1/2	<0.000015	∞	
Muon	μ^-	μ^+	−1	1/2	105.7	2.2×10^{-6}	$e^- + \bar{\nu}_e + \nu_\mu$
Muon neutrino	ν_μ	$\bar{\nu}_\mu$	0	1/2	<0.19	∞	
Tau	τ^-	τ^+	−1	1/2	1777	2.9×10^{-13}	$\mu^- + \bar{\nu}_\mu + \nu_\tau$
Tau neutrino	ν_τ	$\bar{\nu}_\tau$	0	1/2	<18	∞	

Quarks

Flavor	*Symbol*	*Antiparticle*	*Charge (e)*	*Spin (h/2π)*	*Rest Energy*[a] (MeV)	*Other Property*
Up	u	$\bar{u}$	+2/3	1/2	3	$C = S = T = B = 0$
Down	d	$\bar{d}$	−1/3	1/2	6	$C = S = T = B = 0$
Charm	c	$\bar{c}$	+2/3	1/2	1300	Charm (C) = +1
Strange	s	$\bar{s}$	−1/3	1/2	120	Strangeness (S) = −1
Top	t	$\bar{t}$	+2/3	1/2	174,000	Topness (T) = +1
Bottom	b	$\bar{b}$	−1/3	1/2	4300	Bottomness (B) = −1

Field Particles

Particle	*Symbol*	*Interaction*	*Charge (e)*	*Spin (h/2π)*	*Rest Energy* (GeV)
Graviton[b]		Gravity	0	2	0
Weak boson	W^+, W^-	Weak	±1	1	80.4
Weak boson	Z^0	Weak	0	1	91.2
Photon	γ	Electromagnetic	0	1	0
Gluon	g	Strong (color)	0	1	0

2. SOME COMPOSITE PARTICLES

Baryons

Particle	*Symbol*	*Quark Content*	*Anti-particle*	*Charge (e)*	*Spin (h/2π)*	*Rest Energy* (MeV)	*Mean Life* (s)	*Typical Decay*
Proton	p	uud	$\overline{p}$	+1	1/2	938	$>10^{33}$	$\pi^0 + e^+$ (?)
Neutron	n	udd	$\overline{n}$	0	1/2	940	887	$p + e^- + \overline{\nu}_e$
Lambda	Λ^0	uds	$\overline{\Lambda^0}$	0	1/2	1116	2.6×10^{-10}	$p + \pi^-$
Omega	Ω^-	sss	$\overline{\Omega^-}$	−1	3/2	1672	8.2×10^{-11}	$\Lambda^0 + K^-$
Delta	Δ^{++}	uuu	$\overline{\Delta^{++}}$	+2	3/2	1232	5.7×10^{-24}	$p + \pi^+$
Charmed lambda	Λ_c^+	udc	$\overline{\Lambda_c^+}$	+1	1/2	2285	1.9×10^{-13}	$\Lambda^0 + \pi^+$

Mesons

Particle	*Symbol*	*Quark Content*	*Anti-particle*	*Charge (e)*	*Spin (h/2π)*	*Rest Energy* (MeV)	*Mean Life* (s)	*Typical Decay*
Pion	π^+	$u\overline{d}$	π^-	+1	0	140	2.6×10^{-8}	$\mu^+ + \nu_\mu$
Pion	π^0	$u\overline{u} + d\overline{d}$	π^0	0	0	135	8.4×10^{-17}	$\gamma + \gamma$
Kaon	K^+	$u\overline{s}$	K^-	+1	0	494	1.2×10^{-8}	$\mu^+ + \nu_\mu$
Kaon	K^0	$d\overline{s}$	$\overline{K^0}$	0	0	498	0.9×10^{-10}	$\pi^+ + \pi^-$
Rho	ρ^+	$u\overline{d}$	ρ^-	+1	1	770	4.4×10^{-24}	$\pi^+ + \pi^-$
D-meson	D^+	$c\overline{d}$	D^-	+1	0	1869	1.1×10^{-12}	$K^- + \pi^+ + \pi^+$
Psi	ψ	$c\overline{c}$	ψ	0	1	3097	7.6×10^{-21}	$e^+ + e^-$
B-meson	B^+	$u\overline{b}$	B^-	+1	0	5279	1.6×10^{-12}	$D^- + \pi^+ + \pi^+$
Upsilon	Υ	$b\overline{b}$	Υ	0	1	9460	1.3×10^{-20}	$e^+ + e^-$

[a] The rest energies listed for the quarks are not those associated with free quarks; since no free quarks have yet been observed, measuring their rest energies in the free state has not yet been possible. The tabulated values are effective rest energies corresponding to quarks bound in composite particles.
[b] Particles expected to exist but not yet observed.

Source: "Review of Particle Properties," *European Physical Journal C,* vol. 15 (2000). Also see http://pdg.lbl.gov/.

APPENDIX G

CONVERSION FACTORS

Conversion factors may be read directly from the tables. For example, 1 degree = 2.778×10^{-3} revolutions, so $16.7° = 16.7 \times 2.778 \times 10^{-3}$ rev. The SI quantities are capitalized. Adapted in part from G. Shortley and D. Williams, *Elements of Physics,* Prentice-Hall, 1971.

Plane Angle

	°	′	″	RADIAN	rev
1 degree =	1	60	3600	1.745×10^{-2}	2.778×10^{-3}
1 minute =	1.667×10^{-2}	1	60	2.909×10^{-4}	4.630×10^{-5}
1 second =	2.778×10^{-4}	1.667×10^{-2}	1	4.848×10^{-6}	7.716×10^{-7}
1 RADIAN =	57.30	3438	2.063×10^{5}	1	0.1592
1 revolution =	360	2.16×10^{4}	1.296×10^{6}	6.283	1

Solid Angle

1 sphere = 4π steradians = 12.57 steradians

Length

	cm	METER	km	in.	ft	mi
1 centimeter =	1	10^{-2}	10^{-5}	0.3937	3.281×10^{-2}	6.214×10^{-6}
1 METER =	100	1	10^{-3}	39.37	3.281	6.214×10^{-4}
1 kilometer =	10^{5}	1000	1	3.937×10^{4}	3281	0.6214
1 inch =	2.540	2.540×10^{-2}	2.540×10^{-5}	1	8.333×10^{-2}	1.578×10^{-5}
1 foot =	30.48	0.3048	3.048×10^{-4}	12	1	1.894×10^{-4}
1 mile =	1.609×10^{5}	1609	1.609	6.336×10^{4}	5280	1

1 angström = 10^{-10} m
1 nautical mile = 1852 m
= 1.151 miles = 6076 ft
1 fermi = 10^{-15} m

1 light-year = 9.460×10^{12} km
1 parsec = 3.084×10^{13} km
1 fathom = 6 ft
1 Bohr radius = 5.292×10^{-11} m

1 yard = 3 ft
1 rod = 16.5 ft
1 mil = 10^{-3} in.
1 nm = 10^{-9} m

Area

	METER^2	cm^2	ft^2	in.^{2}
1 SQUARE METER =	1	10^4	10.76	1550
1 square centimeter =	10^{-4}	1	1.076×10^{-3}	0.1550
1 square foot =	9.290×10^{-2}	929.0	1	144
1 square inch =	6.452×10^{-4}	6.452	6.944×10^{-3}	1

1 square mile = 2.788×10^7 ft^2 = 640 acres
1 barn = 10^{-28} m^2
1 acre = 43,560 ft^2
1 hectare = 10^4 m^2 = 2.471 acre

Volume

	METER^3	cm^3	L	ft^3	in.^{3}
1 CUBIC METER =	1	10^6	1000	35.31	6.102×10^4
1 cubic centimeter =	10^{-6}	1	1.000×10^{-3}	3.531×10^{-5}	6.102×10^{-2}
1 liter =	1.000×10^{-3}	1000	1	3.531×10^{-2}	61.02
1 cubic foot =	2.832×10^{-2}	2.832×10^4	28.32	1	1728
1 cubic inch =	1.639×10^{-5}	16.39	1.639×10^{-2}	5.787×10^{-4}	1

1 U.S. fluid gallon = 4 U.S. fluid quarts = 8 U.S. pints = 128 U.S. fluid ounces = 231 in.^{3}
1 British imperial gallon = 277.4 in.^{3} = 1.201 U.S. fluid gallons

Mass

	g	KILOGRAM	slug	u	oz	lb	ton
1 gram =	1	0.001	6.852×10^{-5}	6.022×10^{23}	3.527×10^{-2}	2.205×10^{-3}	1.102×10^{-6}
1 KILOGRAM =	1000	1	6.852×10^{-2}	6.022×10^{26}	35.27	2.205	1.102×10^{-3}
1 slug =	1.459×10^4	14.59	1	8.786×10^{27}	514.8	32.17	1.609×10^{-2}
1 u =	1.661×10^{-24}	1.661×10^{-27}	1.138×10^{-28}	1	5.857×10^{-26}	3.662×10^{-27}	1.830×10^{-30}
1 ounce =	28.35	2.835×10^{-2}	1.943×10^{-3}	1.718×10^{25}	1	6.250×10^{-2}	3.125×10^{-5}
1 pound =	453.6	0.4536	3.108×10^{-2}	2.732×10^{26}	16	1	0.0005
1 ton =	9.072×10^5	907.2	62.16	5.463×10^{29}	3.2×10^4	2000	1

1 metric ton = 1000 kg
Quantities in the colored areas are not mass units but are often used as such. When we write, for example, 1 kg "=" 2.205 lb this means that a kilogram is a *mass* that *weighs* 2.205 pounds under standard condition of gravity (g = 9.80665 m/s^2).

Density

	slug/ft^3	KILOGRAM/METER^3	g/cm^3	lb/ft^3	lb/in.^{3}
1 slug per ft^3	1	515.4	0.5154	32.17	1.862×10^{-2}
1 KILOGRAM per METER^3 =	1.940×10^{-3}	1	0.001	6.243×10^{-2}	3.613×10^{-5}
1 gram per cm^3 =	1.940	1000	1	62.43	3.613×10^{-2}
1 pound per ft^3 =	3.108×10^{-2}	16.02	1.602×10^{-2}	1	5.787×10^{-4}
1 pound per in.^{3} =	53.71	2.768×10^4	27.68	1728	1

Quantities in the colored areas are weight densities and, as such, are dimensionally different from mass densities. See note for mass table.

Time

	y	d	h	min	SECOND
1 year =	1	365.25	8.766×10^{3}	5.259×10^{5}	3.156×10^{7}
1 day =	2.738×10^{-3}	1	24	1440	8.640×10^{4}
1 hour =	1.141×10^{-4}	4.167×10^{-2}	1	60	3600
1 minute =	1.901×10^{-6}	6.944×10^{-4}	1.667×10^{-2}	1	60
1 SECOND =	3.169×10^{-8}	1.157×10^{-5}	2.778×10^{-4}	1.667×10^{-2}	1

Speed

	ft/s	km/h	METER/SECOND	mi/h	cm/s
1 foot per second =	1	1.097	0.3048	0.6818	30.48
1 kilometer per hour =	0.9113	1	0.2778	0.6214	27.78
1 METER per SECOND =	3.281	3.6	1	2.237	100
1 mile per hour =	1.467	1.609	0.4470	1	44.70
1 centimeter per second =	3.281×10^{-2}	3.6×10^{-2}	0.01	2.237×10^{-2}	1

1 knot = 1 nautical mi/h = 1.688 ft/s 1 mi/min = 88.00 ft/s = 60.00 mi/h

Force

	dyne	NEWTON	lb	pdl	gf	kgf
1 dyne =	1	10^{-5}	2.248×10^{-6}	7.233×10^{-5}	1.020×10^{-3}	1.020×10^{-6}
1 NEWTON =	10^{5}	1	0.2248	7.233	102.0	0.1020
1 pound =	4.448×10^{5}	4.448	1	32.17	453.6	0.4536
1 poundal =	1.383×10^{4}	0.1383	3.108×10^{-2}	1	14.10	1.410×10^{-2}
1 gram-force =	980.7	9.807×10^{-3}	2.205×10^{-3}	7.093×10^{-2}	1	0.001
1 kilogram-force =	9.807×10^{5}	9.807	2.205	70.93	1000	1

Quantities in the colored areas are not force units but are often used as such. For instance, if we write 1 gram-force "=" 980.7 dynes, we mean that a gram-mass experiences a force of 980.7 dynes under standard conditions of gravity ($g = 9.80665$ m/s^2).

Energy, Work, Heat

	Btu	erg	ft · lb	hp · h	JOULE	cal	kW · h	eV	MeV	kg	u
1 British thermal unit =	1	1.055×10^{10}	777.9	3.929×10^{-4}	1055	252.0	2.930×10^{-4}	6.585×10^{21}	6.585×10^{15}	1.174×10^{-14}	7.070×10^{12}
1 erg =	9.481×10^{-11}	1	7.376×10^{-8}	3.725×10^{-14}	10^{-7}	2.389×10^{-8}	2.778×10^{-14}	6.242×10^{11}	6.242×10^{5}	1.113×10^{-24}	670.2
1 foot-pound =	1.285×10^{-3}	1.356×10^{7}	1	5.051×10^{-7}	1.356	0.3238	3.766×10^{-7}	8.464×10^{18}	8.464×10^{12}	1.509×10^{-17}	9.037×10^{9}
1 horsepower-hour =	2545	2.685×10^{13}	1.980×10^{6}	1	2.685×10^{6}	6.413×10^{5}	0.7457	1.676×10^{25}	1.676×10^{19}	2.988×10^{-11}	1.799×10^{16}
1 JOULE =	9.481×10^{-4}	10^{7}	0.7376	3.725×10^{-7}	1	0.2389	2.778×10^{-7}	6.242×10^{18}	6.242×10^{12}	1.113×10^{-17}	6.702×10^{9}
1 calorie =	3.969×10^{-3}	4.186×10^{7}	3.088	1.560×10^{-6}	4.186	1	1.163×10^{-6}	2.613×10^{19}	2.613×10^{13}	4.660×10^{-17}	2.806×10^{10}
1 kilowatt-hour =	3413	3.6×10^{13}	2.655×10^{6}	1.341	3.6×10^{6}	8.600×10^{5}	1	2.247×10^{25}	2.247×10^{19}	4.007×10^{-11}	2.413×10^{16}
1 electron volt =	1.519×10^{-22}	1.602×10^{-12}	1.182×10^{-19}	5.967×10^{-26}	1.602×10^{-19}	3.827×10^{-20}	4.450×10^{-26}	1	10^{-6}	1.783×10^{-36}	1.074×10^{-9}
1 million electron volts =	1.519×10^{-16}	1.602×10^{-6}	1.182×10^{-13}	5.967×10^{-20}	1.602×10^{-13}	3.827×10^{-14}	4.450×10^{-20}	10^{6}	1	1.783×10^{-30}	1.074×10^{-3}
1 kilogram =	8.521×10^{13}	8.987×10^{23}	6.629×10^{16}	3.348×10^{10}	8.987×10^{16}	2.146×10^{16}	2.497×10^{10}	5.610×10^{35}	5.610×10^{29}	1	6.022×10^{26}
1 unified atomic mass unit =	1.415×10^{-13}	1.492×10^{-3}	1.101×10^{-10}	5.559×10^{-17}	1.492×10^{-10}	3.564×10^{-11}	4.146×10^{-17}	9.32×10^{8}	932.0	1.661×10^{-27}	1

Quantities in the colored areas are not properly energy units but are included for convenience. They arise from the relativistic mass–energy equivalence formula $E = mc^2$ and represent the energy equivalent of a mass of one kilogram or one unified atomic mass unit (u).

Pressure

	atm	dyne/cm²	inch of water	cm Hg	PASCAL	lb/in.²	lb/ft²
1 atmosphere =	1	1.013×10^{6}	406.8	76	1.013×10^{5}	14.70	2116
1 dyne per cm² =	9.869×10^{-7}	1	4.015×10^{-4}	7.501×10^{-5}	0.1	1.405×10^{-5}	2.089×10^{-3}
1 inch of water° at 4°C =	2.458×10^{-3}	2491	1	0.1868	249.1	3.613×10^{-2}	5.202
1 centimeter of mercury[a] at 0°C =	1.316×10^{-2}	1.333×10^{4}	5.353	1	1333	0.1934	27.85
1 PASCAL =	9.869×10^{-6}	10	4.015×10^{-3}	7.501×10^{-4}	1	1.450×10^{-4}	2.089×10^{-2}
1 pound per in.² =	6.805×10^{-2}	6.895×10^{4}	27.68	5.171	6.895×10^{3}	1	144
1 pound per ft² =	4.725×10^{-4}	478.8	0.1922	3.591×10^{-2}	47.88	6.944×10^{-3}	1

[a] Where the acceleration of gravity has the standard value 9.80665 m/s².

1 bar = 10^6 dyne/cm² = 0.1 MPa   1 millibar = 10^3 dyne/cm² = 10^2 Pa   1 torr = 1 millimeter of mercury

Power

	Btu/h	ft · lb/s	hp	cal/s	kW	WATT
1 British thermal unit per hour =	1	0.2161	3.929×10^{-4}	6.998×10^{-2}	2.930×10^{-4}	0.2930
1 foot-pound per second =	4.628	1	1.818×10^{-3}	0.3239	1.356×10^{-3}	1.356
1 horsepower =	2545	550	1	178.1	0.7457	745.7
1 calorie per second =	14.29	3.088	5.615×10^{-3}	1	4.186×10^{-3}	4.186
1 kilowatt =	3413	737.6	1.341	238.9	1	1000
1 WATT =	3.413	0.7376	1.341×10^{-3}	0.2389	0.001	1

Magnetic Flux

	maxwell	WEBER
1 maxwell =	1	10^{-8}
1 WEBER =	10^{8}	1

Magnetic Field

	gauss	TESLA	milligauss
1 gauss =	1	10^{-4}	1000
1 TESLA =	10^{4}	1	10^{7}
1 milligauss =	0.001	10^{-7}	1

1 tesla = 1 weber/meter^2

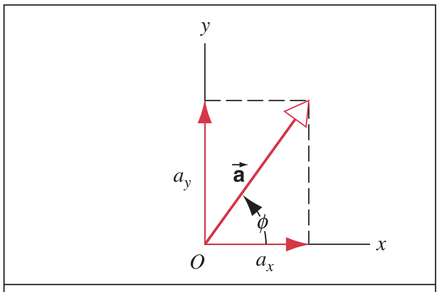
APPENDIX H

VECTORS

H-1 COMPONENTS OF VECTORS

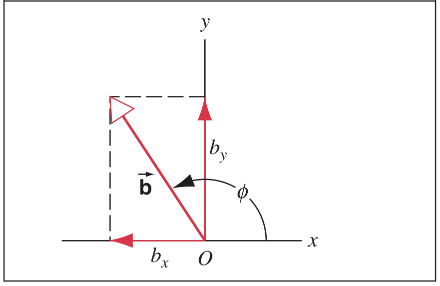
$a_x = a \cos \phi \qquad a_y = a \sin \phi$

$a = \sqrt{a_x^2 + a_y^2} \qquad \tan \phi = a_y/a_x$



$b_x = b \cos \phi \ (<0)$

$b_y = b \sin \phi \ (>0)$



$a_x = a \sin \theta \cos \phi$

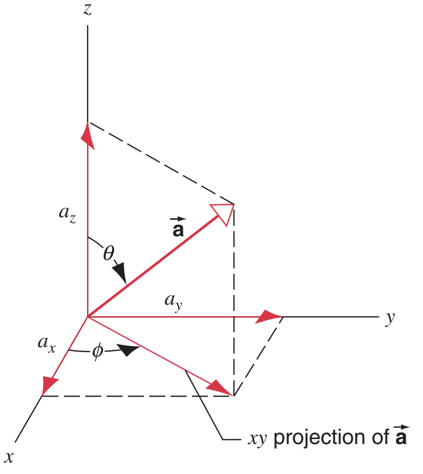
$a_y = a \sin \theta \sin \phi$

$a_z = a \cos \theta$

$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$

$\tan \phi = a_y/a_x$

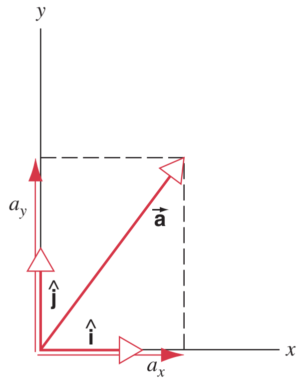
$\cos \theta = a_z/a$



H-2 UNIT VECTORS

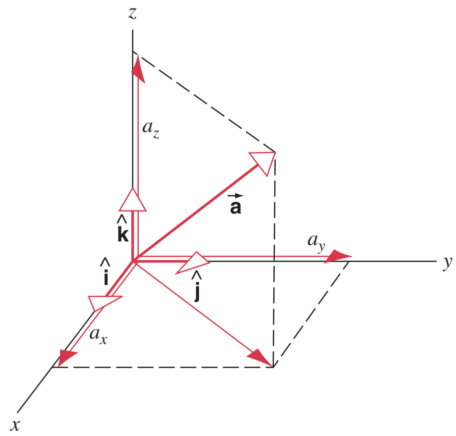
Two-dimensional Cartesian:

$\vec{\mathbf{a}} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}$



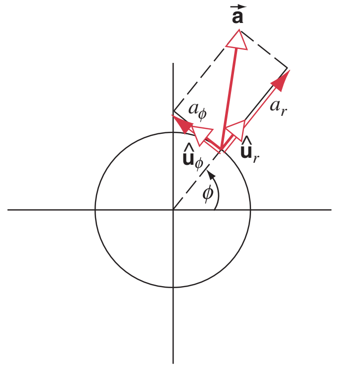
Three-dimensional Cartesian:

$\vec{\mathbf{a}} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$



Two-dimensional polar:

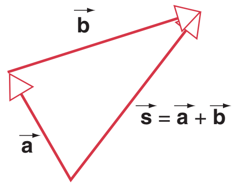
$\vec{\mathbf{a}} = a_r\hat{\mathbf{u}}_r + a_\phi\hat{\mathbf{u}}_\phi$



H-3 ADDING VECTORS

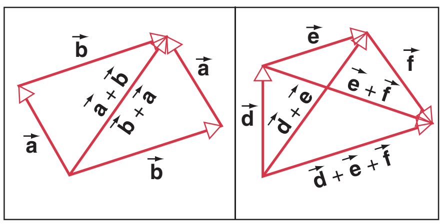
$\vec{\mathbf{s}} = \vec{\mathbf{a}} + \vec{\mathbf{b}}$

$s_x = a_x + b_x \qquad s_y = a_y + b_y$



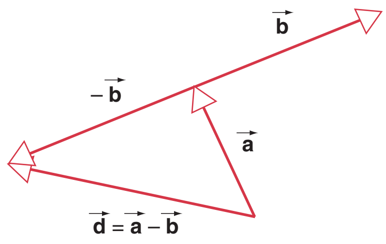
$\vec{\mathbf{a}} + \vec{\mathbf{b}} = \vec{\mathbf{b}} + \vec{\mathbf{a}}$ (commutative law)

$\vec{\mathbf{d}} + (\vec{\mathbf{e}} + \vec{\mathbf{f}}) = (\vec{\mathbf{d}} + \vec{\mathbf{e}}) + \vec{\mathbf{f}}$ (associative law)



$$\vec{d} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$

$$d_x = a_x - b_x \qquad d_y = a_y - b_y$$



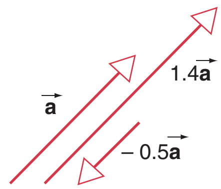
H-4 MULTIPLICATION OF VECTORS

Multiplication of a vector by a scalar:

$$\vec{b} = c\vec{a}$$

$$b_x = ca_x \qquad b_y = ca_y$$

$$b = |c|\,a$$



Dot product (or scalar product) of two vectors:

$$\vec{a}\cdot\vec{b} = ab\cos\phi = a(b\cos\phi) = b(a\cos\phi)$$

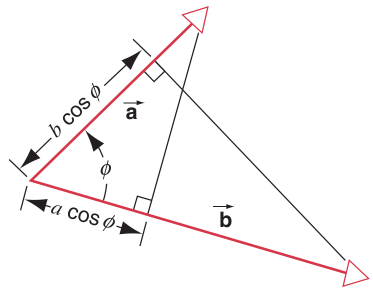
$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$$

$$\hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0$$

$$\vec{a}\cdot\vec{b} = a_xb_x + a_yb_y + a_zb_z$$

$$\vec{a}\cdot\vec{a} = a^2 = a_x^2 + a_y^2 + a_z^2$$

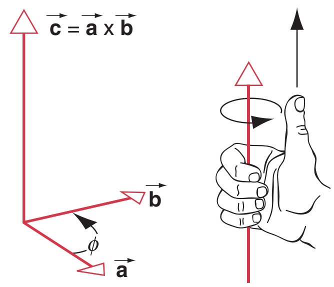


Cross product (or vector product) of two vectors:

$$\vec{c} = \vec{a}\times\vec{b}$$

$$|\vec{c}| = |\vec{a}\times\vec{b}| = ab\sin\phi$$

Direction of $\vec{c}$ is perpendicular to the plane of $\vec{a}$ and $\vec{b}$, determined by the right-hand rule.



$$\vec{b}\times\vec{a} = -\vec{a}\times\vec{b}$$

$$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$$

$$\hat{i}\times\hat{j} = \hat{k} \qquad \hat{j}\times\hat{k} = \hat{i} \qquad \hat{k}\times\hat{i} = \hat{j}$$

$$\vec{a}\times\vec{b} = (a_yb_z - a_zb_y)\hat{i} + (a_zb_x - a_xb_z)\hat{j} + (a_xb_y - a_yb_x)\hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})$$

$$(s\vec{a})\times\vec{b} = \vec{a}\times(s\vec{b}) = s(\vec{a}\times\vec{b}) \qquad (s = \text{a scalar}).$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

APPENDIX I

MATHEMATICAL FORMULAS

Geometry

Circle of radius r: circumference $= 2\pi r$; area $= \pi r^2$.
Sphere of radius r: area $= 4\pi r^2$; volume $= \frac{4}{3}\pi r^3$.
Right circular cylinder of radius r and height h:
area $= 2\pi r^2 + 2\pi r h$; volume $= \pi r^2 h$.
Triangle of base a and altitude h: area $= \frac{1}{2}ah$.

Quadratic Formula

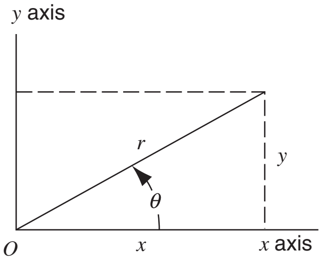
If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Trigonometric Functions of Angle θ

$\sin\theta = \dfrac{y}{r}$ $\quad \cos\theta = \dfrac{x}{r}$

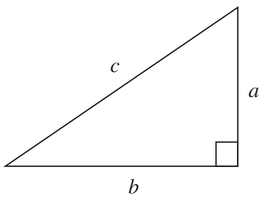
$\tan\theta = \dfrac{y}{x}$ $\quad \cot\theta = \dfrac{x}{y}$

$\sec\theta = \dfrac{r}{x}$ $\quad \csc\theta = \dfrac{r}{y}$



Pythagorean Theorem

$a^2 + b^2 = c^2$



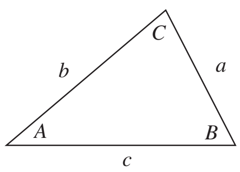
Triangles

Angles A, B, C
Opposite sides a, b, c

$A + B + C = 180°$

$\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

$c^2 = a^2 + b^2 - 2ab\cos C$



Mathematical Signs and Symbols

$=$ equals
$\approx$ equals approximately
$\neq$ is not equal to
$\equiv$ is identical to, is defined as
$>$ is greater than ($\gg$ is much greater than)
$<$ is less than ($\ll$ is much less than)
$\geq$ is greater than or equal to (or, is no less than)
$\leq$ is less than or equal to (or, is no more than)
$\pm$ plus or minus ($\sqrt{4} = \pm 2$)
$\propto$ is proportional to
Σ the sum of
$\bar{x}$ the average value of x (also x_{av})

Trigonometric Identities

$\sin(90° - \theta) = \cos\theta$
$\cos(90° - \theta) = \sin\theta$
$\sin\theta/\cos\theta = \tan\theta$
$\sin^2\theta + \cos^2\theta = 1 \qquad \sec^2\theta - \tan^2\theta = 1 \qquad \csc^2\theta - \cot^2\theta = 1$
$\sin 2\theta = 2\sin\theta\cos\theta$
$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
$\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$
$\sin\alpha \pm \sin\beta = 2\sin\frac{1}{2}(\alpha \pm \beta)\cos\frac{1}{2}(\alpha \mp \beta)$

Binomial Expansion

$$(1 \pm x)^n = 1 \pm \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \cdots \quad (x^2 < 1)$$

$$(1 \pm x)^{-n} = 1 \mp \frac{nx}{1!} + \frac{n(n+1)x^2}{2!} \mp \cdots \quad (x^2 < 1)$$

Exponential Expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Logarithmic Expansion

$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots \quad (|x| < 1)$

Trigonometric Expansions (θ in radians)

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots$$

Derivatives and Integrals

In what follows, the letters u and v stand for any functions of x, and a and m are constants. To each of the indefinite integrals should be added an arbitrary constant of integration. The *Handbook of Chemistry and Physics* (CRC Press Inc.) gives a more extensive tabulation.

1. $\frac{dx}{dx} = 1$
2. $\frac{d}{dx}(au) = a\frac{du}{dx}$
3. $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$
4. $\frac{d}{dx}x^m = mx^{m-1}$
5. $\frac{d}{dx}\ln x = \frac{1}{x}$
6. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
7. $\frac{d}{dx}e^x = e^x$
8. $\frac{d}{dx}\sin x = \cos x$
9. $\frac{d}{dx}\cos x = -\sin x$
10. $\frac{d}{dx}\tan x = \sec^2 x$
11. $\frac{d}{dx}\cot x = -\csc^2 x$
12. $\frac{d}{dx}\sec x = \tan x \sec x$
13. $\frac{d}{dx}\csc x = -\cot x \csc x$
14. $\frac{d}{dx}e^u = e^u\frac{du}{dx}$
15. $\frac{d}{dx}\sin u = \cos u\frac{du}{dx}$
16. $\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}$

1. $\int dx = x$
2. $\int au\,dx = a\int u\,dx$
3. $\int (u+v)\,dx = \int u\,dx + \int v\,dx$
4. $\int x^m\,dx = \frac{x^{m+1}}{m+1} \quad (m \neq -1)$
5. $\int \frac{dx}{x} = \ln|x|$
6. $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$
7. $\int e^x\,dx = e^x$
8. $\int \sin x\,dx = -\cos x$
9. $\int \cos x\,dx = \sin x$
10. $\int \tan x\,dx = -\ln\cos x$
11. $\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x$
12. $\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x$
13. $\int e^{-ax}\,dx = -\frac{1}{a}e^{-ax}$
14. $\int xe^{-ax}\,dx = -\frac{1}{a^2}(ax+1)e^{-ax}$
15. $\int x^2e^{-ax}\,dx = -\frac{1}{a^3}(a^2x^2 + 2ax + 2)e^{-ax}$
16. $\int x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$
17. $\int_0^\infty x^{2n}e^{-ax^2}\,dx = \frac{1\cdot3\cdot5\cdots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$
18. $\int \frac{dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}$

APPENDIX J

NOBEL PRIZES IN PHYSICS*

1901	Wilhelm Konrad Röntgen	1845–1923	for the discovery of x-rays
1902	Hendrik Antoon Lorentz Pieter Zeeman	1853–1928 1865–1943	for their researches into the influence of magnetism on radiation phenomena
1903	Antoine Henri Becquerel	1852–1908	for his discovery of spontaneous radioactivity
	Pierre Curie Marie Sklowdowska-Curie	1859–1906 1867–1934	for their joint researches on the radiation phenomena discovered by Professor Henri Becquerel
1904	Lord Rayleigh (John William Strutt)	1842–1919	for his investigations of the densities of the most important gases and for his discovery of argon
1905	Philipp Eduard Anton von Lenard	1862–1947	for his work on cathode rays
1906	Joseph John Thomson	1856–1940	for his theoretical and experimental investigations on the conduction of electricity by gases
1907	Albert Abraham Michelson	1852–1931	for his optical precision instruments and metrological investigations carried out with their aid
1908	Gabriel Lippmann	1845–1921	for his method of reproducing colors photographically based on the phenomena of interference
1909	Guglielmo Marconi Carl Ferdinand Braun	1874–1937 1850–1918	for their contributions to the development of wireless telegraphy
1910	Johannes Diderik van der Waals	1837–1923	for his work on the equation of state for gases and liquids
1911	Wilhelm Wien	1864–1928	for his discoveries regarding the laws governing the radiation of heat
1912	Nils Gustaf Dalén	1869–1937	for his invention of automatic regulators for use in conjunction with gas accumulators for illuminating lighthouses and buoys
1913	Heike Kamerlingh Onnes	1853–1926	for his investigations of the properties of matter at low temperatures, which led, *inter alia,* to the production of liquid helium
1914	Max von Laue	1879–1960	for his discovery of the diffraction of Röntgen rays by crystals
1915	William Henry Bragg William Lawrence Bragg	1862–1942 1890–1971	for their services in the analysis of crystal structure by means of X rays
1917	Charles Glover Barkla	1877–1944	for his discovery of the characteristic X rays of the elements
1918	Max Planck	1858–1947	for his discovery of energy quanta
1919	Johannes Stark	1874–1957	for his discovery of the Doppler effect in canal rays and the splitting of spectral lines in electric fields
1920	Charles-Édouard Guillaume	1861–1938	for the service he has rendered to precision measurements in Physics by his discovery of anomalies in nickel steel alloys
1921	Albert Einstein	1879–1955	for his services to theoretical physics, and especially for his discovery of the law of the photoelectric effect
1922	Neils Bohr	1885–1962	for the investigation of the structure of atoms, and of the radiation emanating from them
1923	Robert Andrews Millikan	1868–1953	for his work on the elementary charge of electricity and on the photoelectric effect

* See *Nobel Lectures, Physics,* 1901–1970, Elsevier Publishing Company, for biographies of the awardees and for lectures given by them on receiving the prize. For more information, see http://www.nobel.se/physics/laureates/index.html.

1924	Karl Manne Georg Siegbahn	1886–1978	for his discoveries and research in the field of X-ray spectroscopy
1925	James Franck	1882–1964	for their discovery of the laws governing the impact of an electron
	Gustav Hertz	1887–1975	on an atom
1926	Jean Baptiste Perrin	1870–1942	for his work on the discontinuous structure of matter, and especially for his discovery of sedimentation equilibrium
1927	Arthur Holly Compton	1892–1962	for his discovery of the effect named after him
	Charles Thomson Rees Wilson	1869–1959	for his method of making the paths of electrically charged particles visible by condensation of vapor
1928	Owen Willans Richardson	1879–1959	for his work on the thermionic phenomenon and especially for the discovery of the law named after him
1929	Prince Louis-Victor de Broglie	1892–1987	for his discovery of the wave nature of electrons
1930	Sir Chandrasekhara Venkata Raman	1888–1970	for his work on the scattering of light and for the discovery of the effect named after him
1932	Werner Heisenberg	1901–1976	for the creation of quantum mechanics, the application of which has, among other things, led to the discovery of the allotropic forms of hydrogen
1933	Erwin Schrödinger	1887–1961	for the discovery of new productive forms of atomic theory
	Paul Adrien Maurice Dirac	1902–1984	
1935	James Chadwick	1891–1974	for his discovery of the neutron
1936	Victor Franz Hess	1883–1964	for the discovery of cosmic radiation
	Carl David Anderson	1905–1991	for his discovery of the positron
1937	Clinton Joseph Davisson	1881–1958	for their experimental discovery of the diffraction of electrons by
	George Paget Thomson	1892–1975	crystals
1938	Enrico Fermi	1901–1954	for his demonstrations of the existence of new radioactive elements produced by neutron irradiation, and for his related discovery of nuclear reactions brought about by slow neutrons
1939	Ernest Orlando Lawrence	1901–1958	for the invention and development of the cyclotron and for results obtained with it, especially for artificial radioactive elements
1943	Otto Stern	1888–1969	for his contribution to the development of the molecular ray method and his discovery of the magnetic moment of the proton
1944	Isidor Isaac Rabi	1898–1988	for his resonance method for recording the magnetic properties of atomic nuclei
1945	Wolfgang Pauli	1900–1958	for the discovery of the Exclusion Principle (Pauli Principle)
1946	Percy Williams Bridgman	1882–1961	for the invention of an apparatus to produce extremely high pressures, and for the discoveries he made therewith in the field of high-pressure physics
1947	Sir Edward Victor Appleton	1892–1965	for his investigations of the physics of the upper atmosphere, especially for the discovery of the so-called Appleton layer
1948	Patrick Maynard Stuart Blackett	1897–1974	for his development of the Wilson cloud chamber method, and his discoveries therewith in nuclear physics and cosmic radiation
1949	Hideki Yukawa	1907–1981	for his prediction of the existence of mesons on the basis of theoretical work on nuclear forces
1950	Cecil Frank Powell	1903–1969	for his development of the photographic method of studying nuclear processes and his discoveries regarding mesons made with this method
1951	Sir John Douglas Cockcroft	1897–1967	for their pioneer work on the transmutation of atomic nuclei by artificially
	Ernest Thomas Sinton Walton	1903–1995	accelerated atomic particles
1952	Felix Bloch	1905–1983	for their development of new methods for nuclear magnetic precision methods
	Edward Mills Purcell	1912–1997	and discoveries of connection therewith
1953	Frits Zernike	1888–1966	for his demonstration of the phase-contrast method, especially for his invention of the phase-contrast microscope
1954	Max Born	1882–1970	for his fundamental research in quantum mechanics, especially for his statistical interpretation of the wave function
	Walther Bothe	1891–1957	for the coincidence method and his discoveries made therewith
1955	Willis Eugene Lamb	1913–	for his discoveries concerning the fine structure of the hydrogen spectrum
	Polykarp Kusch	1911–1993	for his precision determination of the magnetic moment of the electron
1956	William Shockley	1910–1989	for their research on semiconductors and their discovery of the transistor
	John Bardeen	1908–1991	effect
	Walter Houser Brattain	1902–1987	
1957	Chen Ning Yang	1922–	for their penetrating investigation of the parity laws, which has led to
	Tsung Dao Lee	1926–	important discoveries regarding the elementary particles
1958	Pavel Aleksejecič Ĉerenkov	1904–1990	for the discovery and the interpretation of the Cerenkov effect
	Il' ja Michajlovič Frank	1908–1990	
	Igor Yevgenyevich Tamm	1895–1971	

1959	Emilio Gino Segrè	1905–1989	for their discovery of the antiproton
	Owen Chamberlain	1920–	
1960	Donald Arthur Glaser	1926–	for the invention of the bubble chamber
1961	Robert Hofstadter	1915–1990	for his pioneering studies of electron scattering in atomic nuclei and for his thereby achieved discoveries concerning the structure of the nucleons
	Rudolf Ludwig Mössbauer	1929–	for his researches concerning the resonance absorption of γ-rays and his discovery in this connection of the effect that bears his name
1962	Lev Davidoviĉ Landau	1908–1968	for his pioneering theories of condensed matter, especially liquid helium
1963	Eugene P. Wigner	1902–1995	for his contribution to the theory of the atomic nucleus and the elementary particles, particularly through the discovery and application of fundamental symmetry principles
	Maria Goeppert Mayer	1906–1972	for their discoveries concerning nuclear shell structure
	J. Hans D. Jensen	1907–1973	
1964	Charles H. Townes	1915–	for fundamental work in the field of quantum electronics, which has led to the construction of oscillators and amplifiers based on the maser–laser principle
	Nikolai G. Basov	1922–	
	Alexander M. Prochorov	1916–	
1965	Sin-itiro Tomonaga	1906–1979	for their fundamental work in quantum electrodynamics, with deep-ploughing consequences for the physics of elementary particles
	Julian Schwinger	1918–1994	
	Richard P. Feynman	1918–1988	
1966	Alfred Kastler	1902–1984	for the discovery and development of optical methods for studying Hertzian resonance in atoms
1967	Hans Albrecht Bethe	1906–	for his contributions to the theory of nuclear reactions, especially his discoveries concerning the energy production in stars
1968	Luis W. Alvarez	1911–1988	for his decisive contribution to elementary particle physics, in particular the discovery of a large number of resonance states, made possible through his development of the technique of using hydrogen bubble chamber and data analysis
1969	Murray Gell-Mann	1929–	for his contribution and discoveries concerning the classification of elementary particles and their interactions
1970	Hannes Alfvén	1908–1995	for fundamental work and discoveries in magneto-hydrodynamics with fruitful applications in different parts of plasma physics
	Louis Néel	1904–	for fundamental work and discoveries concerning antiferromagnetism and ferrimagnetism, which have led to important applications in solid state physics
1971	Dennis Gabor	1900–1979	for his discovery of the principles of holography
1972	John Bardeen	1908–1991	for their development of a theory of superconductivity
	Leon N. Cooper	1930–	
	J. Robert Schrieffer	1931–	
1973	Leo Esaki	1925–	for his discovery of tunneling in semiconductors
	Ivar Giaever	1929–	for his discovery of tunneling in superconductors
	Brian D. Josephson	1940–	for his theoretical prediction of the properties of a super-current through a tunnel barrier
1974	Antony Hewish	1924–	for the discovery of pulsars
	Sir Martin Ryle	1918–1984	for his pioneering work in radioastronomy
1975	Aage Bohr	1922–	for the discovery of the connection between collective motion and particle motion and the development of the theory of the structure of the atomic nucleus based on this connection
	Ben Mottelson	1926–	
	James Rainwater	1917–1986	
1976	Burton Richter	1931–	for their (independent) discovery of an important fundamental particle
	Samuel Chao Chung Ting	1936–	
1977	Philip Warren Anderson	1923–	for their fundamental theoretical investigations of the electronic structure of magnetic and disordered systems
	Nevill Francis Mott	1905–1996	
	John Hasbrouch Van Vleck	1899–1980	
1978	Peter L. Kapitza	1894–1984	for his basic inventions and discoveries in low-temperature physics
	Arno A. Penzias	1926–	for their discovery of cosmic microwave background radiation
	Robert Woodrow Wilson	1936–	
1979	Sheldon Lee Glashow	1932–	for their unified model of the action of the weak and electromagnetic forces for their prediction of the existence of neutral currents
	Abdus Salam	1926–1996	
	Steven Weinberg	1933–	
1980	James W. Cronin	1931–	for the discovery of violations of fundamental symmetry principles in the decay of neutral K mesons
	Val L. Fitch	1923–	

1981	Nicolaas Bloembergen	1920–	for their contribution of the development of laser spectroscopy
	Arthur Leonard Schawlow	1921–1999	
	Kai M. Siegbahn	1918–1999	for his contribution of high-resolution electron spectroscopy
1982	Kenneth Geddes Wilson	1936–	for his method of analyzing the critical phenomena inherent in the changes of matter under the influence of pressure and temperature
1983	Subrehmanyan Chandrasekhar	1910–1995	for his theoretical studies of the structure and evolution of stars
	William A. Fowler	1911–1995	for his studies of the formation of the chemical elements in the universe
1984	Carlo Rubbia	1934–	for their decisive contributions to the large project, which led to the discovery of the field particles W and Z, communicators of the weak interaction
	Simon van der Meer	1925–	
1985	Klaus von Klitzing	1943–	for his discovery of the quantized Hall resistance
1986	Ernst Ruska	1906–1988	for his invention of the electron microscope
	Gerd Binnig	1947–	for their invention of the scanning-tunneling electron microscope
	Heinrich Rohrer	1933–	
1987	Karl Alex Müller	1927–	for this discovery of a new class of superconductors
	J. Georg Bednorz	1950–	
1988	Leon M. Lederman	1922–	for experiments with neutrino beams and the discovery of the muon neutrino
	Melvin Schwartz	1932–	
	Jack Steinberger	1921–	
1989	Hans G. Dehmelt	1922–	for their development of technique for trapping individual atoms
	Wolfgang Paul	1913–1993	
	Norman F. Ramsey	1915–	for his discoveries in atomic resonance spectroscopy, which led to hydrogen masers and atomic clocks
1990	Richard E. Taylor	1929–	for their experiments on the scattering of electrons from nuclei, which revealed the presence of quarks inside nucleons
	Jerome I. Friedman	1930–	
	Henry W. Kendall	1926–1999	
1991	Pierre-Gilles de Gennes	1932–	for discoveries about the ordering of molecules in substances such as liquid crystals, superconductors, and polymers
1992	George Charpak	1924–	for his invention of fast electronic detectors for high-energy particles
1993	Joseph H. Taylor	1941–	for the discovery and interpretation of the first binary pulsar
	Russell A. Hulse	1950–	
1994	Bertram N. Brockhouse	1918–	for the development of neutron-scattering techniques
	Clifford G. Shull	1915–	
1995	Martin L. Perl	1927–	for the discovery of the tau lepton
	Frederick Reines	1918–1998	for the detection of the neutrino
1996	David M. Lee	1931–	for their discovery of superfluidity in ^3He
	Douglas M. Osheroff	1945–	
	Robert C. Richardson	1937–	
1997	Steven Chu	1948–	for the development of methods to cool and trap atoms with laser light
	Claude Cohen-Tannoudji	1933–	
	William D. Phillips	1948–	
1998	Robert B. Laughlin	1950–	for their discovery of a new form of quantum fluid with fractionally charged excitations
	Horst L. Stormer	1949–	
	Daniel C. Tsui	1939–	
1999	Gerardus 't Hooft	1946–	for elucidating the quantum structure of electroweak interactions in physics
	Martinus J. G. Veltman	1931–	
2000	Zhores I. Alferov	1930–	for developing semiconductor heterostructures used in high-speed and opto-electronics
	Herbert Kroemer	1928–	
	Jacks S. Kilby	1923–	for his part in the invention of the integrated circuit

ANSWERS TO ODD-NUMBERED EXERCISES AND PROBLEMS

CHAPTER 1

Exercises

3. 52.6 min; 5.2%. **5.** -0.44%. **7.** 3.33 ft. **9.** 55 s.
11. 2.2 d. **13.** (*a*) 100 m; 8.56 m; 28.1 ft.
(*b*) 1 mi is longer by 109 m or 358 ft. **15.** 1.88×10^{22} cm^3.
17. (*a*) 4.00×10^4 km. (*b*) 5.10×10^8 km^2.
(*c*) 1.08×10^{12} km^3. **19.** 2.86×10^{-3} ly/century.
21. (*a*) 4.85×10^{-6} pc; 1.58×10^{-5} ly
(*b*) 9.48×10^{12} km; 3.08×10^{13} km.
23. 5.98×10^{26}. **25.** New York. **27.** 840 km.
29. 605.780211 nm. **31.** m/s.
33. $(ch/G)^{1/2} = 5.46 \times 10^{-8}$ kg.

Problems

1. 7 h 44 min 50 s, p.m. **3.** (*a*) 31 m. (*b*) 22 m.
(*c*) Lake Ontario. **5.** About 1 lb. **7.** 0.260 kg.
9. (*a*) 282 pm. (*b*) 416 pm.

CHAPTER 2

Exercises

1. (*a*) Parallel. (*b*) Antiparallel. (*c*) Perpendicular.
3. (*a*) 4.5, 52° N of E. (*b*) 8.4, 25° S of E. **5.** 4.76 km.
7. (*a*) $2\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$. (*b*) 5.4, at 68° with the $+x$ axis.
9. (*a*) 5.0, 323°. (*b*) 10.0, 53.1°. (*c*) 11.2, 26.6°.
(*d*) 11.2, 79.7°. (*e*) 11.2, 260°.
11. (*a*) 370 m, 57° E of N. (*b*) 370 m, 420 m.
13. (*a*) 16.0 cm, 45.0° clockwise from vertically down.
(*b*) 22.6 cm, vertically up. (*c*) Zero.
15. 33,900 ft, 0.288° below the horizontal.
17. (*a*) $(6\text{ m})\hat{\mathbf{i}} - (106\text{ m})\hat{\mathbf{j}}$. (*b*) $(19\text{ m/s})\hat{\mathbf{i}} - (224\text{ m/s})\hat{\mathbf{j}}$.
(*c*) $(24\text{ m/s}^2)\hat{\mathbf{i}} - (336\text{ m/s}^2)\hat{\mathbf{j}}$. **19.** (*a*) $-(18\text{ m/s}^2)\hat{\mathbf{i}}$. (*b*) 0.75 s.
(*c*) Never. (*d*) 2.2 s. **21.** (*a*) 11.5 h. (*b*) 5.5 h.
(*c*) North Atlantic Ocean. **23.** 31 km. **25.** 2 cm/y.
27. 1 h 13 min. **29.** 48 km/h. **31.** 100 m. **33.** -20 m/s^2.
35. (*a*) *OA*: +,0; *AB*: +,−; *BC*: 0,0; *CD*: −,+. (*b*) No.
39. (*a*) m/s^2; m/s^3. (*b*) 2 s. (*c*) 24 m. (*d*) -16 m.
(*e*) 3.0, 0.0, -9.0, -24 m. (*f*) 0.0, -6.0, -12, -18 m/s^2.
(*g*) -10 m/s. **41.** (*a*) 3.1×10^6 s. (*b*) 4.7×10^{13} m.
43. 10.4 cm. **45.** 21g. **47.** (*a*) 5.00 s. (*b*) 61.5 m.
49. (*a*) 34.7 ft. (*b*) 41.6 s. **51.** 183 m/s. **53.** (*a*) 29.4 m.
(*b*) 2.45 s. **55.** (*a*) 3.19 s. (*b*) 1.32 s.
57. (*a*) 27.4 m/s. (*b*) 5.33 m/s. (*c*) 1.45 m. **59.** 1.52 s.
61. 0.39 m.

Problems

1. (*a*) 28 m. (*b*) 13 m. **3.** (*a*) $(10\text{ ft})\hat{\mathbf{i}} + (12\text{ ft})\hat{\mathbf{j}} + (14\text{ ft})\hat{\mathbf{k}}$.
(*b*) 21 ft. (*c*) Equal to or greater than, not less than. (*d*) 26 ft.
5. (*a*) 45.0 mi/h. (*b*) 42.8 mi/h. (*c*) 43.9 mi/h.
7. (*a*) An infinite number. (*b*) 87 km. **9.** (*a*) 28.5 cm/s.
(*b*) 18.0 cm/s. (*c*) 40.5 cm/s. (*d*) 28.1 cm/s. (*e*) 30.3 cm/s.
11. (*a*) 14 m/s; 18 m/s^2. (*b*) 6 m/s, 12 m/s^2; 24 m/s, 24 m/s^2.
13. No, his speed was ≤ 24 mi/h. **15.** (*a*) 0.75 s. (*b*) 50 m.
17. (*a*) 3.40 s. (*b*) 16.2 m. **19.** 1.23, 4.90, 11.0, 19.6, 30.6 cm.
21. (*a*) 110 km. (*b*) 330 s. **23.** (*a*) 8.85 m/s. (*b*) 0.999 m.
25. 96g. **27.** 0.3 s. **29.** 20.4 m.
31. Approximately $3.6h^{1/2}$, with h in meters.

CHAPTER 3

Exercises

1. 6.3 y. **3.** 1.0×10^{-15} N. **5.** 0.080 m/s^2. **7.** 1.9 mm.
9. (*a*) 4.55 m/s^2. (*b*) 2.59 m/s^2. **11.** (*a*) 9.9 N. (*b*) 2.1 m/s^2.
13. (*a*) 646 kg, 6320 N. (*b*) 412 kg, 4040 N.
15. (*a*) 12.2 N; 2.65 kg. (*b*) Zero; 2.65 kg. **17.** 1600 lb.
19. 1.19×10^6 N. **21.** (*a*) 5400 N. (*b*) 5.5 s. (*c*) 15 m.
(*d*) 2.7 s. **23.** (*a*) 210 m/s^2. (*b*) 1.8×10^4 N.
25. Lower it with acceleration greater than 1.3 m/s^2.
27. 33 m/s. **29.** (*a*) 5.0×10^5 N. (*b*) 1.4×10^6 N.
31. (*a*) 2.2×10^5 N. (*b*) 5.0×10^4 N.

Problems

1. (*a*) 0.28 μm. (*b*) 37 μm. **3.** (*a*) 1.8 m/s^2. (*b*) 3.8 m/s^2.
(*c*) 4.0 m. **5.** (*a*) 3260 N. (*b*) 2720 kg. (*c*) 1.20 m/s^2.
7. (*a*) 0.97 m/s^2. (*b*) 1.2 N. (*c*) 3.5 N.
9. (*a*) 1.23 N, 2.46 N, 3.69 N, 4.92 N. (*b*) 6.15 N. (*c*) 0.250 N.
11. (*a*) $P/(m + M)$. (*b*) $PM/(m + M)$.

CHAPTER 4

Exercises

1. (*a*) 2.4 ns. (*b*) 2.7 mm. (*c*) 9600 km/s, 2300 km/s.
3. (*a*) $(2Bt)\hat{\mathbf{j}} + C\hat{\mathbf{k}} = (8.0\text{ m/s}^2)t\hat{\mathbf{j}} + (1.0\text{ m/s})\hat{\mathbf{k}}$.
(*b*) $2B\hat{\mathbf{j}} = (8.0\text{ m/s}^2)\hat{\mathbf{j}}$. (*c*) A parabola.
5. $(0.83\text{ m/s}^2)\hat{\mathbf{i}} + (0.71\text{ m/s}^2)\hat{\mathbf{j}}$. **7.** (*a*) 2.2 m/s^2.
(*b*) 120 N. (*c*) 21 m/s^2. **9.** 11 m.
11. 6800 N at 21° from the line of motion. **13.** (*a*) 0.514 s.
(*b*) 9.94 ft/s. **15.** (*a*) 0.18 m. (*b*) 1.9 m. **19.** (*a*) 11 m.
(*b*) 23 m. (*c*) 17 m/s, 63° below the horizontal.
21. 1 cm longer. **23.** 78 ft/s, 65°. **25.** (*a*) 0.20 m. (*b*) No.
27. 115 ft/s. **29.** 1.47 N. **31.** $(m/b)\ln 2$. **33.** (*a*) 257 kN.
(*b*) 1.06°. **35.** (*a*) 19 m/s. (*b*) 35 rev/min. **37.** (*a*) 130 km/s.

(*b*) 790 km/s^2. **39.** 36 s; no.
41. The wind blows due east at 55 mi/h. **43.** (*a*) 0.71 s.
(*b*) 2.3 ft. **45.** (*a*) 46.8° E of N. (*b*) 6 min 35 s.

Problems
1. 60°. **3.** (*a*) 8.44 km. (*b*) 59.0 km. **5.** (*a*) 1.16 s.
(*b*) 13.0 m. (*c*) 18.8 m/s; 5.56 m/s. (*d*) No. **7.** (*a*) 99 ft.
(*b*) 90 ft/s. (*c*) 180 ft. **9.** 31° to 63° above the horizontal.
13. 1.30 m/s. **15.** (*a*) g. (*b*) $(mg/b)^{1/2}$. (*c*) $0.75g$.
17. (*a*) $ge^{-bt/m}$;g;0. (*b*) $(mg/b)[t + (m/b)(e^{-bt/m} - 1)]$.
19. (*a*) 15 km. (*b*) 77 km/h. **21.** 220 m/s^2.
23. (*b*) Maximum: $v_x = 2\omega R$, $v_y = 0$; $a_x = 0$, $a_y = -\omega^2 R$.
Minimum: $v_x = v_y = 0$; $a_x = 0$, $a_y = \omega^2 R$.
25. (2.976 to 2.991) $\times 10^8$ m/s. **27.** 98.1 km/h, 15.1°.

CHAPTER 5

Exercises
1. (*a*) 0.0018 N. (*b*) 0.0033 N. **3.** (*a*) 7.3 kg. (*b*) 89 N.
5. (*a*) 6.8 m/s. (*b*) Climb the rope. **7.** 18 kN. **9.** 2°.
11. 9.3 m/s^2. **13.** 900 N. **15.** (*a*) 9.1 kN. (*b*) 9.0 kN.
17. (*b*) 219 N. (*c*) 81 N. **19.** 0.040; 0.026. **21.** 0.487.
23. (*a*) 3.2 m/s^2, down the plane. (*b*) 2.9 m. (*c*) Stays there.
25. (*a*) 70 lb. (*b*) 4.6 ft/s^2. **27.** 155 N. **29.** (*a*) Zero.
(*b*) 13.4 ft/s^2, down the plane. (*c*) 4.27 ft/s^2, up the plane.
31. (*a*) 7.6 m/s^2. (*b*) 0.87 m/s^2. **33.** (*a*) 730 lb (3200 N).
(*b*) 0.3. **35.** (*a*) 0.67 m/s. (*b*) 1.8 m/s^2. (*c*) 0.53 N.
37. (*a*) 2.2×10^6 m/s. (*b*) 9.1×10^{22} m/s^2. (*c*) 8.3×10^{-8} N.
39. $(Mgr/m)^{1/2}$. **41.** (*a*) 0.23. (*b*) 128 km/h.
43. 0.162; 0.295. **45.** (*a*) 9.5 m/s. (*b*) 20 m.
47. (*a*) 0.0337 N. (*b*) 9.77 N.

Problems
1. (*b*) -1.73 m/s^2; 23.4 N. (*c*) $m_2 < 2.60$ kg; $m_2 > 2.60$ kg;
$m_2 = 2.60$ kg. **5.** (*a*) 11.1 N. (*b*) 47.3 N. (*c*) 40.1 N.
7. (*a*) $\mu_k mg/(\sin\theta - \mu_k \cos\theta)$. (*b*) $\tan^{-1}\mu_s$. **9.** 490 N.
13. (*a*) 0.46. (*b*) 0.92. **15.** (*a*) 30 cm/s.
(*b*) 1.7 m/s^2, radially inward. (*c*) 2.9 mN. (*d*) 0.40.
17. (*a*) 8.74 N. (*b*) 37.9 N. (*c*) 6.45 m/s.
19. (*b*) 45°; 1.72 mrad. (*c*) Zero; zero.

CHAPTER 6

Exercises
1. (*a*) 52.0 km/h. (*b*) 178 km/h.
3. 205 kg·m/s; up, perpendicular to the plate. **5.** (*a*) $2mv/\Delta t$.
(*b*) 560 N. **7.** 3.29 kN. **9.** 930 N. **11.** 8.8 m/s.
13. 1.95×10^5 kg·m/s for any direction of thrust. **15.** 2.0 s.
17. Increases by 4.54 m/s. **19.** 3960 km/h.
21. 5.6 m/s to the left. **23.** 1.77 m/s.
25. $(4.0\text{ m/s})\hat{\mathbf{i}} + (5.0\text{ m/s})\hat{\mathbf{j}}$.
27. 3.43 m/s, deflected 17.3° to the left. **29.** 100 g.
31. 1.2 kg. **33.** 120°. **35.** 2.44 m/s, to the left.

Problems
1. $2\mu u$. **3.** (*a*) 2.20 N·s. (*b*) 212 N. **5.** (*a*) 0.480 g.
(*b*) 7.2 kN. **7.** $mgR[(2h/g)^{1/2} + t]$; 41.0 N. **9.** (*a*) 130 tons.
(*b*) 0.88 in. (*c*) elastic. **11.** 37.1 mi/h, 63.6° S of W.
13. 28.0°. (*b*) 7.44 m/s. **15.** (*a*) 74.4 m/s.
(*b*) 81.5 m/s; 84.1 m/s.
17. v_2 and v_3 will be at 30.0° to v_0 and have magnitude 6.93 m/s; v_1 is opposite to v_0 and has magnitude 2.00 m/s.
19. (*a*) 746 m/s. (*b*) 963 m/s.
21. (*a*) *A*: 4.57 m/s; *B*: 3.94 m/s. (*b*) 7.53 m/s.

CHAPTER 7

Exercises
1. 4640 km (1730 km beneath Earth's surface). **3.** 75.2 km/h.
5. (*a*) The center of mass does not move. (*b*) 1.23 m.
7. 14.5 ft. **9.** 33.4 m.
11. 6.75×10^{-12} m from the nitrogen atom on the axis of symmetry.
13. $L/5$ from the heavy rod, along the symmetry axis.
15. $x_{cm} = y_{cm} = 20$ cm; $z_{cm} = 16$ cm. **17.** 27.
19. (*a*) 3.2 m/s. (*b*) 3.2 m/s. **21.** (*a*) 2.72. (*b*) 7.39.
23. 1.33 km/s. **25.** 1.29 m/s.

Problems
1. (*a*) Down; $mv/(m + M)$. (*b*) The balloon is again stationary.
3. $g(1 - 2x/L)$. **5.** $(HM/m)[(1 + m/M)^{1/2} - 1]$.
7. (*a*) 540 m/s. (*b*) 40.4°. **9.** 60 N.
11. Fast barge: 49.5 N more; slow barge: no change.

CHAPTER 8

Exercises
1. $n(n + 1)/2$. **3.** (*a*) $a + 3bt^2 - 4ct^3$. (*b*) $6t(b - 2ct)$.
5. (*a*) $\omega_0 + at^4 - bt^3$. (*b*) $\omega_0 t + at^5/5 - bt^4/4$. **7.** 14.
9. (*a*) 4.8 m/s. (*b*) No. **11.** $1/T_S = 1/T_P - 1/T_E$.
13. (*a*) 8140 rev/min^2. (*b*) 425 rev. **15.** (*a*) -1.28 rad/s^2.
(*b*) 248 rad. (*c*) 29.5 rev. **17.** (*a*) 2.0 rev/s. (*b*) 3.8 s.
19. (*a*) 369 s. (*b*) -3.90×10^{-3} rad/s^2. (*c*) 108 s.
21. 0.132 rad/s. **23.** (*a*) 2.48×10^{-3} rad/s. (*b*) 19.7 m/s^2.
(*c*) Zero. **25.** (*a*) 7.27×10^{-5} rad/s. (*b*) 355 m/s.
(*c*) 7.27×10^{-5} rad/s; 463 m/s. **27.** (*a*) 310 m/s. (*b*) 340 m/s.
29. (*a*) $r\alpha^2 t^2$. (*b*) $r\alpha$. (*c*) 44.1°. **31.** Yes; +0.16.
33. (*a*) $(-26.2\text{ m/s})\hat{\mathbf{i}}$. (*b*) $(4.87\text{ m/s}^2)\hat{\mathbf{i}} - (375\text{ m/s}^2)\hat{\mathbf{j}}$.
(*c*) 1.83 m.

Problems
1. (*a*) 4.0 rad/s; 28 rad/s. (*b*) 12 rad/s^2.
(*c*) 6.0 rad/s^2; 18 rad/s^2. **3.** (*b*) 23 h 56 min. **5.** (*a*) 0.92 rev.
(*b*) 6.0 rad/s. **7.** (*a*) 1.99×10^{-7} rad/s. (*b*) 29.9 km/s.
(*c*) 5.94 mm/s^2. **9.** (*a*) 3800 rad/s. (*b*) 190 m/s.
11. (*a*) 22.4 rad/s. (*b*) 5.38 km. (*c*) 1.15 h.
13. (*a*) $\omega b/\cos^2\omega t$ in any direction in the plane perpendicular to $\vec{\omega}$.
(*b*) $\pi/2\omega$.

CHAPTER 9

Exercises
1. (*a*) 15 N·m. (*b*) 10 N·m. (*c*) 15 N·m.
5. 27 units, $+z$ direction. **7.** $a^2 b \sin\phi$, $\pi/2 - \phi$.
9. $(-4.8\text{ N}\cdot\text{m})\hat{\mathbf{i}} + (-0.85\text{ N}\cdot\text{m})\hat{\mathbf{j}} + (3.4\text{ N}\cdot\text{m})\hat{\mathbf{k}}$.
11. (*a*) 0.14 kg·m^2. (*b*) 91 rad/s^2. **13.** (*a*) 2.6×10^{-2} kg·m^2.
(*b*) No change. **15.** (*a*) 482 kN. (*b*) 11.2 kN·m.
17. $M(a^2 + b^2)/3$. **19.** $5mL^2 + (8/3)mL^2$. **23.** (*a*) 2.5 m.
(*b*) 7.3°. **25.** 10.4 m. **27.** 340 lb; 420 lb.
29. $W[h(2r - h)]^{1/2}/(r - h)$. **31.** (*a*) 47.0 lb.
(*b*) 21.3 lb; 10.9 lb. **33.** 7.63 rad/s^2, out of the page.
35. (*a*) 28.2 rad/s^2. (*b*) 338 N·m. **37.** 690 rad/s.
39. 1.73×10^5 g·cm^2. **41.** (*a*) 56.5 rad/s. (*b*) -8.88 rad/s^2.
(*c*) 69.2 m. **43.** (*a*) 1.13 s. (*b*) 13.6 m.

Problems

1. (*a*) Slides; 31°. (*b*) Tips; 34°. **3.** (*a*) $W(1 + r^2/L^2)^{1/2}$. (*b*) Wr/L. **5.** (*a*) 269 N. (*b*) 874 N, 10.7° above the ladder. **7.** $F_1 = W \sin\theta_2/\sin(\theta_2 - \theta_1)$; $F_2 = W\sin\theta_1/\sin(\theta_2 - \theta_1)$; normal to the planes. **9.** (*a*) $L/2$, $L/4$, $L/6$. (*c*) $N = n$. **15.** (*a*) $dm/M = 2r\,dr/R^2$. (*b*) $dI = 2Mr^3\,dr/R^2$. (*c*) $\frac{1}{2}MR^2$. **17.** 1.73 m/s²; 6.92 m/s². **19.** (*a*) $(2/3)\mu MgR$. (*b*) $(3/4)\omega_0 R/4\mu g$. **21.** (*a*) 10.7°. (*b*) $0.186g$. **25.** (*a*) The sphere. (*b*) No.

CHAPTER 10

Exercises

1. 0.62 kg · m²/s. **5.** 2.49×10^{11} kg · m²/s. **9.** (*a*) 0.521 kg · m²/s. (*b*) 4080 rev/min. **13.** $v_f = 2.90$ m/s in direction of impulse; $\omega_f = 10.7$ rad/s about center of mass. **15.** $R_1R_2I_1\omega_0/(R_1^2I_2 + R_2^2I_1)$. **17.** 3.0 min. **19.** 354 rev. **21.** 171 rev/min. **23.** (*a*) 9.66 rad/s, clockwise as viewed from above. (*b*) Final motion is the same as in (*a*). **25.** 0.739 rad/s. **27.** 1.90 min.

Problems

1. (*a*) 14.1 kg · m²/s, out of the page. (*b*) 1.76 N · m, out of the page. **7.** (*a*) 1.18 s. (*b*) 8.6 m. (*c*) 5.18 rev. (*d*) 6.07 m/s. **9.** Longer by 0.4 s. **11.** (*a*) $(I\omega - mRv)/(I + mR^2)$. (*b*) $\frac{1}{2}m(v + R\omega)^2/(1 + mR^2/I)$.

CHAPTER 11

Exercises

1. (*a*) 580 J. (*b*) 0. (*c*) 0. **3.** (*a*) 430 J. (*b*) − 400 J. (*c*) 0. **5.** (*a*) 2160 J. (*b*) − 1430 J. **7.** 22.2°. **9.** − 19. **11.** 720 W. **13.** 24 W. **15.** 25 hp. **17.** (*a*) 0.77 mi. (*b*) 71 kW. **19.** 2.66 hp. **21.** 800 J. **23.** (*a*) 23 mm. (*b*) 45 N. **27.** 1200 km/s. **29.** (*a*) 2.88×10^7 m/s. (*b*) 1.32 MeV. **31.** (*a*) 493 J. (*b*) 168 W. **33.** (*a*) 2.5×10^5 ft · lb (340 kJ). (*b*) 14 hp (10 kW). (*c*) 28 hp (20 kW). **35.** 6.75×10^{12} rad/s. **37.** 1.36 kW. **39.** (*a*) 2.57×10^{29} J. (*b*) 1.32 Gy. **41.** 12.9 tons.

Problems

1. 2.1×10^{-10} N. **3.** (*a*) 215 lb. (*b*) 10,100 ft · lb. (*c*) 48.0 ft. (*d*) 10,300 ft · lb. **5.** (*a*) 2.45×10^5 ft · lb. (*b*) 0.619 hp. **9.** (*a*) 10.0 kW. (*b*) 2.97 kW. **11.** $3F_0x_0/2$. **13.** $15W_0$. **17.** Man: 2.41 m/s; boy: 4.82 m/s. **19.** (*a*) 9.0×10^4 megatons of TNT. (*b*) 45 km. **23.** 0.0217. **25.** 0.792. **27.** $K_a = 97.5$ J; $K_b = 941$ J. **31.** (*a*) $mv_i/(m + M)$. (*b*) $M/(m + M)$. **33.** 700 J gained. **35.** (*a*) $m_1v_{1i}^2/2$. (*b*) $m_1^2v_{1i}^2/2(m_1 + m_2)$. (*c*) $m_2/(m_1 + m_2)$. (*d*) $m_1m_2v_{1i}^2/2(m_1 + m_2)$; zero; one; yes.

CHAPTER 12

Exercises

1. (*a*) $U(x) = -Gm_1m_2/x$. (*b*) $Gm_1m_2d/x_1(x_1 + d)$. **3.** $U(x) = -(\alpha/2\beta)e^{-\beta x^2}$ with $U(\infty) = 0$. **5.** 110 MN/m. **7.** 2.15 m/s. **9.** (*a*) v_0. (*b*) $(v_0^2 + gh)^{1/2}$. (*c*) $(v_0^2 + 2gh)^{1/2}$. **11.** (*a*) 7.63 N/cm. (*b*) 57.4 J. (*c*) 73.8 cm. **13.** (*a*) $K = \frac{1}{2}mg^2t^2$; $U = mg(h - \frac{1}{2}gt^2)$. (*b*) $K = mg(h - y)$; $U = mgy$. **15.** 4.24 m. **19.** 1.25 cm. **21.** $d[m_1/(m_1 + m_2)]^2$. **23.** $[2hmg/(m + I/r^2 + 2M/3)]^{1/2}$. **25.** (*a*) 3.43 m. (*b*) 2.65 s. (*c*) 23.1. **27.** $v_0(5/7)^{1/2}$. **29.** (*a*) 4.7 N. (*b*) $x = 1.2$ m to $x = 14$ m. (*c*) 3.7 m/s. **31.** (*a*) $F_x = -kx$; $F_y = -ky$; $\vec{\mathbf{F}}$ points toward the origin. (*b*) $F_r = -kr$; $F_\theta = 0$.

Problems

1. (*a*) $k/(z + l) - k/(z - l)$. **3.** (*a*) 105 cm. (*b*) 322 cm/s. **5.** (*a*) $8.06mg$, at 82.9° left of vertical. (*b*) $5R/2$. **7.** (*a*) 26.9 J. (*b*) 19.7 m/s. **13.** $(9g/4L)^{1/2}$. **15.** $(2gr \sec\theta_0)^{1/2}$.

CHAPTER 13

Exercises

1. 740 m. **3.** 0.41 m/s. **5.** (*a*) 3.86 m/s. (*b*) 0.143 J. **7.** (*a*) 2.56×10^{12} J. (*b*) 3.82×10^8 J. **9.** 54%. **11.** 4.19 m. **13.** 6.55 m/s. **15.** 1.34 m/s; 0.981 m/s. **17.** (*a*) 862 N. (*b*) 2.42 m/s. **19.** (*a*) 22.4 kN. (*b*) 12.5 kJ. **21.** $[2E(M + m)/Mm]^{1/2}$.

Problems

3. (*a*) 3.02 m/s. (*b*) 1.60 km/s. **5.** (*a*) $\frac{1}{2}ke^2(1/r_2 - 1/r_1)$. (*b*) $-ke^2(1/r_2 - 1/r_1)$. (*c*) $-\frac{1}{2}ke^2(1/r_2 - 1/r_1)$. **7.** (*a*) 0.298 J. (*b*) 0.008 J.

CHAPTER 14

Exercises

1. 2.16. **3.** 2.9×10^{-11} N. **5.** 1.6×10^{-2} lb. **7.** (*a*) 1.33×10^{12} m/s². (*b*) 1.79×10^6 m/s. **9.** (*a*) 0.05%. (*b*) 7×10^{-4} s. **13.** 2.2×10^5 m/s. **17.** 1.55×10^7 m. **19.** (*a*) 3.34×10^7 m/s. (*b*) 5.49×10^7 m/s. **21.** $(Gm/d)^{1/2}$. **23.** 6.5×10^{23} kg. **25.** 0.354 lunar months. **27.** (*a*) 1.68 km/s. (*b*) 108 min. **29.** 58.3 km/s. **33.** (*a*) $2\pi d^{3/2}/[G(4M + m)]^{1/2}$. (*b*) 2. (*c*) 2. **35.** (*a*) Yes. (*b*) Yes. **37.** South, 35.4° above horizon. **39.** (*a*) 5389 s. (*b*) 4.3 J/s.

Problems

7. (*a*) 2.63×10^6 m. (*b*) 5.3×10^9 J. **9.** (*b*) 250 m, 50 m. (*c*) 293 m, 7 m. **11.** $(GMm/d^2)[1 - 1/8(1 - R/2d)^2]$. **13.** (*a*) 9.83 m/s². (*b*) 9.84 m/s². (*c*) 9.79 m/s². **15.** (*b*) 200 MN/m². (*c*) 180 km. **19.** 98.4 pJ. **21.** (*a*) 1.02 y. (*b*) 87.6 km/s. **23.** (*a*) $3.32Gm^2/R^2$ (*b*) $2\pi(R^3/3.32Gm)^{1/2}$. **25.** (*a*) $-GMm/r$. (*b*) $-2GMm/r$. (*c*) It falls vertically. **29.** (*a*) 7.54 km/s. (*b*) 97.3 min. (*c*) 405 km; 7.68 km/s; 92.3 min. (*d*) 3.18 mN. **31.** $(GM/L)^{1/2}$.

CHAPTER 15

Exercises

1. 429 kPa. **3.** 27.5 kN. **5.** 6.0 lb/in.². **7.** 19.0 kPa. **9.** 55.2 kPa. **11.** 0.412 cm. **13.** (*a*) 8.52 km. (*b*) 17.0 km. **19.** (*a*) 35.6 kN. (*b*) $\Delta V = -0.0851$ m³. **21.** 1070 g. **23.** 2.0×10^{-4}. **25.** (*a*) 38.4 kN. (*b*) 40.5 kN. (*c*) 2.35 kN. (*d*) 2.08 kN. **27.** 4.74 MN. **29.** Four. **31.** 0.031. **33.** 78 m. **35.** 54.3 mN.

Problems

1. (*b*) 26.6 kN (= 6000 lb). **3.** (*a*) $\rho gWD^2/2$. (*b*) $\rho gWD^3/6$. (*c*) $D/3$, up from the base. **5.** 43.5 km. **7.** (*b*) *a*. **9.** (*b*) $p = \rho gh$. **13.** 56.1 cm. **15.** 0.190. **17.** 2.79 g/cm³. **21.** 3.71 mm.

CHAPTER 16

Exercises

1. 1 h 49 min. **3.** 3.9 m. **5.** 1.1 m/s. **7.** (*a*) 241 lb/in.^{2}. (*b*) 0.326 ft^2. **9.** (*a*) 0.81 mm^2. (*b*) 440 L/d. **11.** (*a*) 560 Pa. (*b*) 52 kN. **13.** 30.4 L/s. **15.** 63 m/s. **19.** (*a*) 87.5 N. (*b*) 172 m^3. **21.** $\frac{1}{2}\rho v^2 A$. **25.** 71.6 L/s.

Problems

1. 3.4 m/s. **3.** (*b*) $H - h$. (*c*) $\frac{1}{2}H$; H. **5.** (*c*) $\frac{1}{2}\rho(v_1 - v_2)^2$. **7.** 410 m/s; $v_{\text{sound}} = 340$ m/s. **9.** (*a*) $v_1 = 4.46$ m/s; $v_2 = 21.2$ m/s. (*b*) 9.47×10^{-3} m^3/s.

CHAPTER 17

Exercises

1. 0.289 s. **3.** $>$455 Hz. **5.** (*a*) 1.00 mm. (*b*) 75.4 cm/s. (*c*) 568 m/s^2. **7.** (*a*) 3.27 m. (*b*) 4.33 m/s. (*c*) -230 m/s^2. (*d*) 1.33 Hz. (*e*) 0.750 s. **9.** 7.73 m/s. **11.** 2.08 h. **13.** (*a*) 5.27 Hz. (*b*) 415 g. (*c*) 42.5 cm. **15.** (*a*) 1.07 Hz. (*b*) 4.73 cm. **17.** (*b*) 3.21 s. **19.** (*a*) 6.97 MN/m. (*b*) 48,500. **21.** (*a*) 3.04 ms. (*b*) 3.84 m/s. (*c*) 90.7 J. **23.** (*a*) 0.319 m. (*b*) 34.4°. **25.** 0.249 m. **27.** (*a*) 33°. (*b*) 9×10^{-4}. **29.** 9.78 m/s^2. **31.** 8.35 s. **33.** (*a*) 0.436 Hz. (*b*) 1.31 m. **35.** 12.1 s. **37.** $1.22f_0$. **41.** (*a*) Straight line, $y = \pm x$. (*b*) Ellipse, $y^2 - \sqrt{3}\,xy + x^2 = A^2/4$. (*c*) Circle: $x^2 + y^2 = A^2$. **43.** 5.22×10^{11} N/m. **51.** 1.9 in. **53.** 0.362 s.

Problems

1. 708 N/m. **3.** 0.119 m. **7.** (*a*) 7.20 N/cm. (*b*) 4.43 kg. **11.** $mv[k(m + M)]^{-1/2}$. **13.** (*a*) 5.60 J. (*b*) 2.80 J. **15.** 0.906 s. **17.** (*a*) $2\pi[(L^2 + 12d^2)/12gd]^{1/2}$. **19.** (*b*) R. **21.** (*a*) 2.00 s. (*b*) 18.5 N·m/rad. **23.** (*b*) $(GM/R^3)^{1/2}/2\pi$. (*c*) 10^{-16} Hz. **25.** (*a*) $k = 490$ N/cm; $b = 1100$ kg/s.

CHAPTER 18

Exercises

1. (*a*) 7.43 kHz. (*b*) 135 μs. **3.** (*a*) 0.712 s. (*b*) 1.40 Hz. (*c*) 1.93 m/s. **7.** (*a*) 6.0 cm. (*b*) 1.0 m. (*c*) 2.0 Hz. (*d*) 2.0 m/s. (*e*) $-x$ direction. (*f*) 0.75 m/s. **9.** 135 N. **11.** (*a*) 5.0 cm. (*b*) 40 cm. (*c*) 12 m/s. (*d*) 33 ms. (*e*) 9.4 m/s. (*f*) $y = (5.0\text{ cm}) \sin[(15.7\text{ rad/m})x + (190\text{ rad/s})t + 0.93\text{ rad}]$. **13.** 7.54 m from the end where the earlier pulse originated. **15.** 198 Hz. **17.** 4.0 kW. **19.** 68.8°; 1.20 rad. **25.** (*a*) 81.4 m/s. (*b*) 16.7 m. (*c*) 4.87 Hz. **27.** (*a*) -3.9 cm. (*b*) $y = (0.15\text{ m}) \sin[(0.79\text{ rad/m})x + (13\text{ rad/s})t]$. (*c*) -14 cm. **29.** (*a*) 1.25 m. (*b*) $y = (3.80\text{ mm}) \sin(10.1\text{ rad/m})x \cos(3910\text{ rad/s})t$. **31.** 7.47 Hz; 14.9 Hz; 22.4 Hz. **33.** 480 cm; 160 cm; 96 cm.

Problems

1. (*a*) 10.9 cm. (*b*) 199°. **3.** (*b*) 304 m/s. **7.** $[k\,\Delta L(L + \Delta L)/m]^{1/2}$. **11.** 8.5%. **15.** 1.18 m/s. **21.** 36.8 N.

CHAPTER 19

Exercises

1. (*a*) 600 cm/s. (*b*) $(0.30\text{ cm}) \sin[(0.26\text{ rad/cm})x + (160\text{ rad/s})t]$. **3.** (*a*) 76.2 μm. (*b*) 333 μm. **5.** (*a*) 57 nm. (*b*) 35. **7.** 170 m. **9.** 1800 km. **11.** 4.47 W. **13.** 0.0271 J. **17.** 51.9 nJ/m^3. **19.** 190 dB. **21.** 63 dB. **23.** 18.4 cm. **25.** 64.4 Hz, 129 Hz. **27.** (*a*) $L(1 - 1/r)$. (*b*) 13.3, 16.0, 20.0, 26.7 cm. **29.** 11.9 m. **31.** 188 Hz. **33.** (*a*) 405 m/s. (*b*) 611 N. **35.** 5.0 cm from one end. **37.** 387 Hz. **39.** 505, 507, 508 Hz or 501, 503, 508 Hz. **41.** 17.4 kHz. **43.** (*a*) 522 Hz. (*b*) 554 Hz. **45.** 570 m/s. **47.** 160 Hz. **49.** (*a*) 464 Hz. (*b*) 490 Hz. **51.** 41.2 kHz.

Problems

1. (*a*) $L(V - v)/Vv$. (*b*) 43.5 m. **3.** (*a*) 44.2 μW/m^2. (*b*) 164 nm. (*c*) 0.894 Pa. **5.** (*a*) $\propto r^{-1}$. (*b*) $\propto r^{-1/2}$. **7.** (*a*) 66.8 μW/m^2. (*b*) 5.02 nW. (*c*) 7.53 μJ. **9.** 346 m. **13.** 45.4 N. **15.** 2.65×10^8 m/s. **17.** 7.16 km. **19.** (*a*) 1050 Hz. (*b*) 1070 Hz. **21.** (*a*) 2000 Hz. (*b*) 2000 Hz.

CHAPTER 20

Exercises

1. (*a*) 710 ps. (*b*) 2.5×10^{-18} m. **3.** 1.30 m. **5.** 0.445 ps. **7.** (*a*) 87.4 m. (*b*) 394 ns. **9.** (*a*) 15.8 km/s. (*b*) 6.95×10^{-10}. **11.** 0.75. **13.** (*a*) $x' = 3.78 \times 10^7$ m; $t' = 2.26$ s. (*b*) 6.54×10^8 m; 3.14 s. **15.** (*a*) $v'_x = -u$; $v'_y = c(1 - u^2/c^2)^{1/2}$. **17.** (*a*) $0.347c$. (*b*) $0.619c$. **19.** (*a*) $0.933c$; 31.0° E of S. (*b*) $0.933c$; 59.0° W of N. **21.** 6.29 cm. **23.** 1.23 μs. **25.** (*a*) 26.3 y. (*b*) 52.3 y. (*c*) 4.06 y. **29.** (*a*) 0.999 165. (*b*) 0.0133. **31.** (*a*) 0.9988; 20.6. (*b*) 0.145; 1.01. (*c*) 0.073; 1.0027. **33.** 21.2 smu/y. **35.** (*b*) $v/c < 0.115$. **37.** $\sqrt{8}\,mc$. **41.** (*a*) 996 eV. (*b*) 1.05 MeV. **43.** $0.796c$.

Problems

1. (*a*) 2.60×10^8. (*b*) Two. **3.** (*a*) 25.8 μs. (*b*) The red flash (Doppler shifted). **7.** Seven. **9.** 2.43 μs. **11.** (*a*) 4.00 μs. (*b*) 2.50 μs. **13.** (*b*) $K = p^2/2m$. (*c*) $206m_e$. **15.** (*a*) $c[(\gamma - 1)/(\gamma + 1)]^{1/2}$. (*b*) $m[2(\gamma + 1)]^{1/2}$. **17.** (*b*) 202 GeV. (*c*) 49.1 GeV. **19.** $1 + (2\gamma_i + 1)^{1/2}$ where $\gamma_i = (1 - v_i^2/c^2)^{-1/2}$.

CHAPTER 21

Exercises

1. (*a*) $T_S = (9/5)T_C + 491.69$. (*b*) 671.69°S; 491.69°S. **3.** (*a*) $T_Q = T_C + 273.15$. (*b*) 373.15°Q; 273.15°Q. (*c*) Kelvin scale. **5.** No; 310 K = 98.6°F. **7.** 291.1 K. **9.** 31.2. **11.** 0.073 cm Hg; nitrogen. **13.** 0.038 in. **15.** 6.2 mm. **17.** (*a*) 13.9 cm^2. (*b*) 115 cm^3. **19.** (*a*) 1.6×10^{-4}/C°. (*b*) Zero. **21.** 2.3×10^{-5}/C°. **27.** 360°C. **29.** 909 g. **31.** (*a*) Zero. (*b*) -0.36%. (*c*) -0.36%. **33.** $+0.68$ s/h. **35.** 0.17 mm. **37.** (*a*) 22.5 L. **39.** (*a*) 113 mol. (*b*) 900 L. **41.** 26.9 lb/in.^{2}. **43.** 104 cm^3.

Problems

7. (*b*) Steel: 71 cm; brass: 41 cm. **9.** 998.4 kg/m^3. **11.** 66.4°C. **13.** 0.27 mm. **15.** (*a*) 2.25 ft. (*b*) 3.99 ft. **17.** 1.74 atm. **21.** 152 kPa.

CHAPTER 22

Exercises

1. (*a*) 0.0130 mol. (*b*) 7.23×10^{21}. **3.** (*a*) 39.9 L. (*b*) 74.4 g. **5.** 4.34×10^{-5}. **7.** (*a*) 531 m/s. (*b*) 0.472 mol/m^3. (*c*) 28 g/mol; N_2. **9.** (*a*) 2.69×10^{25}. (*b*) 0.171 nm. **11.** 3.86 GHz. **13.** -12°C. **15.** (*a*) 420 m/s; 458 m/s; yes.

17. 180 m/s. **21.** 1.5 cm/s. **23.** (*a*) 10,000 K; 160,000 K. (*b*) 440 K; 7000 K. **25.** 13.9 rev/s. **29.** 3.09×10^{-3}. **31.** 3.8×10^{-5} m^3/mol.

Problems

1. 0.76. **3.** 1/5. **5.** (*a*) 1.67. (*b*) 49.5×10^{-6} cm. (*c*) 7.87×10^{-6} cm. **7.** 4.71. **9.** (*a*) $3N/v_0^3$. (*b*) $0.750v_0$. (*c*) $0.775v_0$. **13.** (*a*) $(N_a + N_b)kT/V$. (*b*) $\frac{1}{2}$. **15.** 89.0°C. **17.** (*a*) $V/n = [a \pm (a^2 - 4abRT)^{1/2}]/2RT$. (*c*) 131 K.

Chapter 23

Exercises

1. 720°C. **3.** (*a*) 546 C°/m. (*b*) 394 kW. (*c*) 63.9°C. **5.** *b*. **7.** 1.15 m. **9.** (*a*) 1.8 W. (*b*) 0.025 C°. **11.** (*a*) − 6.0 J. (*b*) − 43 J. (*c*) 40 J. (*d*) 18 J; 18 J. **13.** 44.5 m^3. **15.** 107 g. **17.** (*a*) 75.4 kJ. (*b*) 4.46 kJ. (*c*) 757°C. **19.** (*a*) 117 s. (*b*) 718 s. **21.** 4.81 g. **23.** (*a*) 542 J/kg · K. (*b*) 0.722 mol. (*c*) 27.9 J/mol · K. **25.** $W_1 = 45$ kJ; $W_2 = -45$ kJ. **27.** 1.14 kJ. **29.** (*a*) 8.39 atm. (*b*) 544 K. (*c*) 966 J. **31.** 2480 J. **33.** (*a*) 1.20. (*b*) 105°C. (*c*) 628 mol. (*d*) 1.96 MJ; 2.96 MJ. (*e*) 0.813. **35.** 11.3 kJ. **37.** (*a*) 7880 J. (*b*) 5630 J. (*c*) 3380 J. **39.** $(n_1C_1 + n_2C_2 + n_3C_3)/(n_1 + n_2 + n_3)$. **41.** $nRT \ln V_f/V_i$. **43.** (*a*) 1090°C. (*b*) 460°C. **45.** (*a*) $-1.5nRT_1$. (*b*) $4.5nRT_1$. (*c*) $6nRT_1$. (*d*) $2R$. **47.** (*a*) 15.9 J. (*b*) 34.4 J/mol · K. (*c*) 26.1 J/mol · K.

Problems

1. (*a*) 24 kW. (*b*) 24 W. **7.** (*a*) 5.26°C; no ice left. (*b*) 0°C; 62.0 g of ice left. **9.** (*a*) 6.75×10^{-20} J. (*b*) 10.7. **11.** 1.2 kJ. **13.** 265 K. **15.** (*a*) 2.95 cm. (*b*) 2.11 cm. **17.** Diatomic. **19.** (*a*) *AB*: 3740 J, 3740 J, 0 J; *BC*: 0 J, − 1810 J, − 1810 J; *CA*: − 3220 J, − 1930 J, 1290 J; cycle: 520 J, 0 J, − 520 J. (*b*) $p_B = 2.00$ atm; $V_B = 0.0246$ m^3; $p_C = 1.00$ atm; $V_C = 0.0373$ m^3. **21.** 12.0 kW.

Chapter 24

Exercises

1. 18.7 kJ. **3.** (*a*) 200 J. (*b*) − 75 J. **5.** (*a*) 37.7 kJ. (*b*) 112 J/K. **7.** 3.0 mol. **9.** (*a*) 1.06 J/K. (*b*) No. **11.** (*a*) − 926 J/K. (*b*) 926 J/K. **13.** (*a*) 30.9%. (*b*) 16.2 kJ. **15.** 25.4%. **17.** (*a*) 2090 J. (*b*) 1510 J. (*c*) 1510 J. **21.** (*a*) 1.62 atm. (*b*) 43.7%. **23.** (*a*) 217 kJ. (*b*) 32.5 kJ. **25.** (*a*) 0.714 J. (*b*) 5.00 J. (*c*) 20.0 J. (*d*) 50.0 J. **27.** 21 J. **29.** (*a*) 113 J. (*b*) 305 J. **31.** 6.8. **33.** $(1 - T_2/T_1)/(1 - T_4/T_3)$. **35.** (*a*) 1 (*b*) $N!/[(N/2)!]^2$

Problems

1. 44 mJ/K. **3.** (*a*) $-3p_0V_0$. (*b*) $6p_0V_0$; $(3R/2) \ln 2$. (*c*) Both are zero. **5.** Path I: $Q_T = p_0V_0 \ln 2$, $Q_V = (9/2)p_0V_0$; Path II: $Q_T = -p_0V_0 \ln 2$, $Q_p = (15/2)p_0V_0$. (*b*) Path I: $W_T = -p_0V_0 \ln 2$, $W_V = 0$; Path II: $W_T = p_0V_0 \ln 2$, $W_p = 3p_0V_0$. (*c*) $(9/2)p_0V_0$ for each process. (*d*) $4R \ln 2$ for each process. **7.** (*a*) $T_2 = (m_1c_1T_{1i} + m_2c_2T_{2i} - m_1c_1T_1)/m_2c_2$. (*b*) $S = m_1c_1 \ln T_1/T_{1i} + m_2c_2 \ln [(m_1c_1T_{1i}/m_2c_2 + T_{2i} - m_1c_1T_1/m_2c_2)/T_{2i}]$. **9.** (*a*) 7200 J. (*b*) 960 J. (*c*) 13%. **11.** (*c*) 1.15 kJ.

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SOME PHYSICAL CONSTANTS*

Speed of light in vacuum	c	3.00×10^{8} m/s
Newtonian gravitational constant	G	6.67×10^{-11} N·m²/kg²
Avogadro constant	N_A	6.02×10^{23} mol^{-1}
Molar gas constant	R	8.31 J/mol·K
Mass-energy relation	c^2	8.99×10^{16} J/kg
		931.5 MeV/u
Electric constant (permittivity)	ϵ_0	8.85×10^{-12} F/m
Magnetic constant (permeability)	μ_0	1.26×10^{-6} H/m
Planck constant	h	6.63×10^{-34} J·s
		4.14×10^{-15} eV·s
Boltzman constant	k	1.38×10^{-23} J/K
		8.62×10^{-5} eV/K
Elementary charge	e	1.60×10^{-19} C
Electron mass	m_e	9.11×10^{-31} kg
Electron rest energy	$m_e c^2$	511.0 keV
Proton mass	m_p	1.67×10^{-27} kg
Proton rest energy	$m_p c^2$	938.3 MeV
Bohr radius	a_0	5.29×10^{-11} m
Bohr magneton	μ_B	9.27×10^{-24} J/T
		5.79×10^{-5} eV/T

*For a more complete list, showing also the best experimental values, see Appendix B.

SOME CONVERSION FACTORS*

Mass

1 kg = 1000 g = 6.02×10^{26} u
1 u = 1.66×10^{-27} kg

Length

1 m = 100 cm = 39.4 in. = 3.28 ft
1 mi = 1.61 km = 5280 ft
1 in. = 2.54 cm
1 light-year = 3.26 parsec = 9.46×10^{15} m
1 Å = 0.1 nm = 100 pm = 10^{-10} m

Time

1 d = 86,400 s
1 y = $365\frac{1}{4}$ d = 3.16×10^{7} s

Volume

1 L = 1000 cm³ = 10^{-3} m³ = 1.06 quart
1 gal (U.S.) = 231 in.³ = 3.79 L

Angular measure

1 rad = 57.3° = 0.159 rev
π rad = 180° = $\frac{1}{2}$ rev

Speed

1 m/s = 3.28 ft/s = 2.24 mi/h
1 km/h = 0.621 mi/h

Force and Pressure

1 N = 10^{5} dyne = 0.225 lb
1 Pa = 1 N/m² = 10 dyne/cm² = 1.45×10^{-4} lb/in.²
1 atm = 1.01×10^{5} Pa = 14.7 lb/in.² = 76 cm-Hg

Energy and Power

1 J = 10^{7} erg = 0.239 cal = 0.738 ft·lb
1 kW·h = 3.6×10^{6} J
1 cal = 4.19 J
1 eV = 1.60×10^{-19} J
1 horsepower = 746 W = 550 ft·lb/s

Electricity and magnetism

1 T = 1 Wb/m² = 10^{4} gauss

*See Appendix G for a more complete list.

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SOME PHYSICAL PROPERTIES

Air (dry, at 20 °C and 1 atm)	
Density	1.21 kg/m^3
Specific heat capacity at constant pressure	1010 $J/kg \cdot K$
Ratio of specific heat capacities	1.40
Speed of sound	343 m/s
Electrical breakdown strength	3×10^6 V/m
Effective molar mass	0.0289 kg/mol
Water	
Density	1000 kg/m^3
Speed of sound	1460 m/s
Specific heat capacity at constant pressure	4190 $J/kg \cdot K$
Heat of fusion (0 °C)	333 kJ/kg
Heat of vaporization (100 °C)	2260 kJ/kg
Index of refraction ($\lambda = 589$ nm)	1.33
Molar mass	0.0180 kg/mol
Earth	
Mass	5.98×10^{24} kg
Mean radius	6.37×10^6 m
Free fall acceleration at the Earth's surface	9.81 m/s^2
Standard atmosphere	1.01×10^5 Pa
Period of satellite at 100 km altitude	86.3 min
Radius of the geosynchronous orbit	42,200 km
Escape speed	11.2 km/s
Magnetic dipole moment	8.0×10^{22} $A \cdot m^2$
Mean electric field at surface	150 V/m, down
Distance to:	
Moon	3.82×10^8 m
Sun	1.50×10^{11} m
Nearest star	4.04×10^{16} m
Galactic center	2.2×10^{20} m
Andromeda galaxy	2.1×10^{22} m
Edge of the observable universe	$\sim 10^{26}$ m

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SOME MATHEMATICAL SYMBOLS

$=$	equals	∞	infinity
$\approx$	equals approximately	lim	the limit of
$\neq$	is not equal to	Σ	the sum of
$\equiv$	is identical to, is defined as	$\int$	the integral of
$>$	is greater than	Δx	the change or difference in x
$\gg$	is much greater than	$\lvert x \rvert$	the absolute value or magnitude of x
$\geq$	is greater than or equal to	x_{av}	the average value of x
$<$	is less than	$x!$	x factorial
$\ll$	is much less than	$\ln x$	natural logarithm of x
$\leq$	is less than or equal to	$f(x)$	a function of x
$\sim$	is of the order of magnitude of	df/dx	the derivative of f with respect of x
$\propto$	is proportional to	$\partial f/\partial x$	the partial derivative of f with respect to x

SI PREFIXES

Factor	*Prefix*	*Symbol*	*Factor*	*Prefix*	*Symbol*
10^{24}	yotta	Y	10^{-1}	deci	d
10^{21}	zetta	Z	10^{-2}	centi	c
10^{18}	exa	E	10^{-3}	milli	m
10^{15}	peta	P	10^{-6}	micro	μ
10^{12}	tera	T	10^{-9}	nano	n
10^{9}	giga	G	10^{-12}	pico	p
10^{6}	mega	M	10^{-15}	femto	f
10^{3}	kilo	k	10^{-18}	atto	a
10^{2}	hecto	h	10^{-21}	zepto	z
10^{1}	deka	da	10^{-24}	yocto	y

THE GREEK ALPHABET

Alpha	A	α	Iota	I	ι	Rho	P	ρ
Beta	B	β	Kappa	K	κ	Sigma	Σ	σ
Gamma	Γ	γ	Lambda	Λ	λ	Tau	T	τ
Delta	Δ	δ	Mu	M	μ	Upsilon	Υ	υ
Epsilon	E	ϵ	Nu	N	ν	Phi	Φ	ϕ, φ
Zeta	Z	ζ	Xi	Ξ	ξ	Chi	X	χ
Eta	H	η	Omicron	O	o	Psi	Ψ	ψ
Theta	Θ	θ	Pi	Π	π	Omega	Ω	ω

SOME SELECTED TABLES

SOME UNITS AND ABBREVIATIONS

ampere	A	kelvin	K
atmosphere	atm	light-year	ly
British thermal unit	Btu	liter	L
calorie (physical)	cal	meter	m
calorie (nutritional)	Cal	mile	mi
coulomb	C	minute	min
day	d	mole	mol
degree Celsius	°C	newton	N
degree Fahrenheit	°F	ohm	Ω
electron-volt	eV	pascal	Pa
farad	F	pound	lb
foot	ft	radian	rad
gauss	G	revolution	rev
gram	g	second	s
henry	H	tesla	T
hertz	Hz	unified atomic mass unit	u
horsepower	hp	volt	V
hour	h	watt	W
inch	in.	weber	Wb
joule	J	year	y